1 Introduction to numerical methods for radiation in astrophsyics

For basic introduction, see pdf-slides of ”lesson 2” on toledo.

The below note are now organized according to the following sections:

2) is about the basic *radiation quantities* used during class.

3) is about the *radiative transfer equation* including the limit of *radiative diffusion.*

4) is about the equations of *radiation-hydrodynamics.*

5) is about numerical techniques for solving the *radiative diffusion* equation.

6) is about computing the *radiative force* density in supersonic flows.

In addition, Appendix:

A) is about equilibrium *black-body* radiation.

B) is about simple solutions to the *radiative transfer equation.*

C) is about the *random walk and diffusion picture* of photons in optically thick media.

Additional background references are: The radiation-hydrodynamics book by Castor and the lecture-notes on radiative processes in astronomy by Sundqvist (both available on toledo).

2 Radiation quantities

2.1 Specific intensity

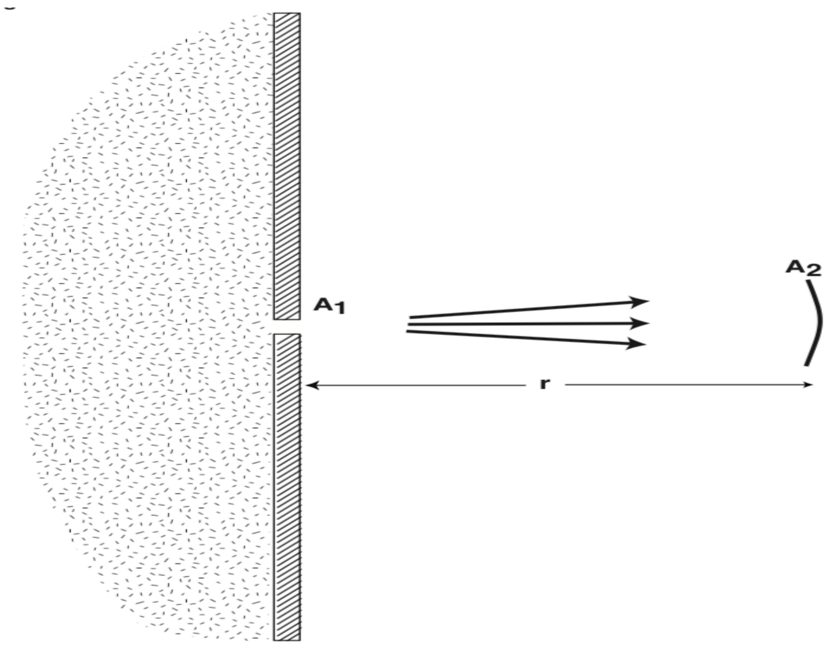

Figure 1: Sketch of experiment defining specific intensity. See text. Figure adapted from Castor (2003).

In general, the radiation field is a function of position and time, and at any given such position and time it has a certain distribution in both frequency and angle. Astrophysical descriptions of radiation (almost always) begin with a quantity called *specific intensity*, or simply just the intensity.

Following Castor (2003) (Fig. 1), we can set up a simple experiment demonstrating the definition of the specific intensity: Consider a screen with a small hole, an aperture, with area A_1, through which we let radiation flow through and travel freely (we assume we have managed to clear the right-hand-side of the screen from all matter) to a detector with area A_2, at a distance r from A_1 and oriented just like it. Let's assume now we have a device with which we can open and close A_1, and that also let radiation go through only in the frequency interval $\Delta\nu$. If we open this for a time Δt, then the definition of specific intensity I_ν is that the energy collected by the detector is

$$\Delta\epsilon = I_\nu A_1 A_2/r^2 \Delta\nu\Delta t \tag{1}$$

where $A_2/r^2 = \Delta\Omega$ is the solid angle subtended by A_2 as seen from A_1, defined (using differentials) as $d\Omega = \sin\theta d\theta d\phi$ for polar angle θ and azimuthal angle ϕ. Note that $\oint d\Omega = 4\pi$.

Exercise: Invariance of specific intensity. Use the above definition to show that, in the absence of interaction with matter, the specific intensity is constant along a ray.

In CGS, I_ν has units of [ergs /cm^2/sr/Hz/s]. So in general, I_ν is a function of three spatial coordinates, time, frequency, and two angles specifying its direction into space, i.e. in full generality it is seven-dimensional; the time and spatial coordinates tell when and where we're measuring the intensity, the angles in which direction it's going, and the frequency coordinate says we're measuring in a small spectral band around a given frequency in the spectrum. The total intensity is found from the monochromatic one by integration over the full frequency-band, i.e. $I = \int I_\nu d\nu$.

From a macroscopic viewpoint, the specific intensity provides a complete description of (unpolarized) radiation, and it will be the basic fundamental quantity used in most parts of this course.[1]

2.2 Angular moments of the intensity

Angular moments of the intensity are both mathematically useful and have deep physical significance. We will work below with the following key quantities, which, just like the intensity above, may be defined either per frequency (or wavelength) band or as frequency-(wavelength-)integrated quantities; i.e. for quantity X, X_ν or $X = \int X_\nu d\nu$:

- The radiation energy density (a scalar) E
- The radiation flux (formally a vector) $\mathbf{F}$
- The radiation pressure (formally a tensor) $\mathbf{P}$
- The mean intensity J, the Eddington flux H, and Eddington's K.

2.3 Radiation energy density

Using the above, the volume occupied by photons that slipped through the aperture A_1 during the time Δt it was open is $\Delta V = A_1 c \Delta t$, so that the energy per volume is

$$\frac{\Delta \epsilon}{\Delta V} = \frac{I_\nu}{c} \Delta\Omega \Delta\nu \tag{2}$$

Integrating over all solid angles and frequencies then gives for the total *radiation energy density*

$$E = \frac{1}{c} \int \oint I_\nu \, d\Omega d\nu \tag{3}$$

which can also be defined per frequency band as

$$E_\nu = \frac{1}{c} \oint I_\nu \, d\Omega \tag{4}$$

[1]As an fyi, you can also look into Castor's Ch. 4 for a neat and simple way of identifying the specific intensity ($\times c^2/h^4\nu^3$) as the basic unknown in *Boltzmann's transport equation*; indeed, the radiation transport equation introduced here can be shown to be a Boltzmann transport equation for photons.

such that

$$E = \int E_\nu \, d\nu \tag{5}$$

2.4 Radiation flux vector

The energy density transport with speed c in unit-vector direction $\mathbf{n}$ is from the above

$$\frac{\Delta\epsilon}{\Delta V} c\mathbf{n} \tag{6}$$

Then the flux (net-rate of energy transport) of radiation transported across a surface parallell to this flux is

$$\mathbf{F} = \int\int I_\nu \mathbf{n} \, d\nu d\Omega \tag{7}$$

And as before we also have for the monochromatic (frequency dependent) flux $\mathbf{F} = \int \mathbf{F}_\nu d\nu$.

Exercise: Compute the net rate of radiant energy flow L (energy/time) in the radial direction for a spherical surface with radius R and radial flux F.

In Cartesian coordinates the unit vector components are

$$n_x = \sin\theta\cos\phi \qquad n_y = \sin\theta\sin\phi \qquad n_z = \cos\theta \tag{8}$$

such that for solid angle $d\Omega = \sin\theta d\theta d\phi$ the components of the radiation flux vector are

$$F_x = \int_0^{2\pi}\int_0^{\pi} I_\nu(\theta,\phi)\sin^2\theta\cos\phi \; d\theta d\phi \tag{9}$$

$$F_y = \int_0^{2\pi}\int_0^{\pi} I_\nu(\theta,\phi)\sin^2\theta\sin\phi \; d\theta d\phi \tag{10}$$

$$F_z = \int_0^{2\pi}\int_0^{\pi} I_\nu(\theta,\phi)\sin\theta\cos\theta \; d\theta d\phi \tag{11}$$

The above emphasizes the vectorial nature of the radiative flux.

In a medium where physical quantities vary only in one dimension (e.g. cartesian in, say, z but also for spherical coordinates in r), I_ν is independent of ϕ and only the last component in the flux vector above is non-zero. In astrophysical literature, this is often referred to as the vertical flux, the radial flux (in spherical coordinates), or simply "the flux". That is, the scalar flux

$$F_\nu = F_{\nu,z} = 2\pi\int_0^{\pi} I_\nu(\theta)\sin\theta\cos\theta d\theta = 2\pi\int_{-1}^{1} I_\nu(\mu)\mu d\mu \tag{12}$$

for $\mu \equiv \cos\theta$.

Exercise1: Show that $F_x = F_y = 0$ for a radiation field (I_ν) independent of ϕ.

Exercise2: Show that eqn. 12 above holds for the variable substitution $\mu = \cos\theta$.

Exercise3: Show that for an isotropic radiation field $F_\nu = 0$.

Exercise4: Solve for F_ν in the case of a angle-independent "outward" flowing radiation, i.e. for $I_\nu = I_\nu^- = 0$ for $\mu = -1 \ldots 0$ and $I_\nu = I_\nu^+$ for $\mu = 0 \ldots 1$.

The small exercises above tell us that an isotropic radiation field indeed gives a zero net flux, as it should since in such a situation equal amounts of radiation are flowing in all directions. Moreover, from the last exercise we obtain the important result that in the case of zero incoming intensity I_ν^- (like, for example, at the surface of an isolated star) $F_\nu = \pi I_\nu^+$ if I_ν^+ is independent of angle.

2.5 Radiation pressure tensor

The *monochromatic radiation pressure tensor* (or radiation energy stress tensor) is defined as

$$\mathbf{P}_\nu = \frac{1}{c}\int I_\nu \mathbf{nn} d\Omega, \tag{13}$$

and is analogous to gas pressure, i.e. it is the *pressure of a photon gas.* Note here that throughout this course we will (hopefully at least quite consistently...) denote radiation pressure with capital P, and gas pressure with lower-case p. A formal definition of the radiation stress tensor can be realized by considering the net-rate of momentum transfer in direction i, per unit-area of a surface parallel to to direction j; the photon-flow in direction j is $(\psi_\nu c) n_j$ and each photon carries a momentum $(h\nu/c) n_i$ in direction i. Summing over all solid-angles, this gives for the i, j component of the monochromatic radiation pressure $P_{\nu,ij} = 1/c \int I_\nu n_i n_j d\Omega$, where $I_\nu = h\nu c \psi_\nu$ has been used.

As you can see above, formally the pressure tensor is represented by a 3×3 matrix, and the trace of that matrix defined as the sum of its diagonal elements. Inspection of eqn. 13 then gives for this trace $\mathbf{n} \cdot \mathbf{n}$. But since $\mathbf{n}$ is a unit vector, we thus have

$$Tr(\mathbf{P}_\nu) = E_\nu. \tag{14}$$

Consider now a very opaque, gaseous astrophysical object (or any gaseous object, really). Deep inside such a system, all directions (almost) look the same, since material far away with perhaps different properties (e.g. different density, temperature) are hidden from us by the intervening material. That is, here the radiation field tends to be (very nearly) *isotropic*, i.e. the same in all directions. The radiation pressure tensor then becomes a 'scalar tensor', given by a factor times the unit matrix. Following Castor, we can understand this as follows, keeping in mind that since the intensity is isotropic, nothing should happen when performing the following operations: i) the diagonal elements should all be the same, since we can turn y into x by rotating about z, and vice versa, and ii) the off-diagonal elements need to be zero, since otherwise a reflection, e.g. x $\rightarrow$ -x, would flip the signs of P_{xy} and P_{xz}. Moreover, if the pressure tensor is a scalar tensor $\mathbf{P}_\nu = P_\nu \mathbf{I}$

with $\mathbf{I}$ the unit matrix, then the trace is $Tr(\mathbf{P}_\nu) = 3P_\nu = E_\nu$.

In analogy with the radiation flux vector discussion above, often in astrophysics literature the scalar version of the radiation pressure P_ν is used, as defined from the equation above (which really is P_{zz}, see exercise below). In full generally, though, $\mathbf{P}$ can have 6 components (it is a symmetrical tensor), though often the off-diagonal terms (representing "radiation viscosity") are negligible.

Exercise: For a grey (=frequency-independent) radiation field with total intensity $I(\theta)$ (i.e., I independent of ϕ), compute the 9 elements of the radiation pressure tensor $\mathbf{P}$ in Cartesian coordinates, and express them in terms of a "scalar" pressure $P = P_{zz}$ and the radiation energy density E. Verify that the trace $Tr(\mathbf{P}) = E$.

2.6 Moments of the intensity

Collecting the above we thus have the following neat-to-have summary:

The radiation energy density (a scalar) is (in cgs units)

$$E_\nu = \frac{1}{c} \int I_\nu d\Omega \quad [\mathrm{erg/cm^3/Hz}]. \tag{15}$$

The radiation flux (formally a vector) is:

$$\mathbf{F}_\nu = \int I_\nu \, \mathbf{n} \, d\Omega \quad [\mathrm{erg/cm^2/Hz/s}]. \tag{16}$$

And the radiation pressure (formally a tensor) is:

$$\mathbf{P}_\nu = \frac{1}{c} \int I_\nu \, \mathbf{nn} \, \mathbf{d\Omega} \quad [\mathrm{erg/cm^3/Hz}]. \tag{17}$$

Noting now the symmetry in the above expressions we next connect these physical quantities with the direct $0^{th} - 2^{th}$ order angular moments of the intensity, i.e. with the *mean intensity* J_ν, the Eddington flux $\mathbf{H}_\nu$ and Eddington's $\mathbf{K}_\nu$:

$$J_\nu \equiv \frac{1}{4\pi} \int I_\nu d\Omega \; = \frac{c}{4\pi} \, E_\nu \tag{18}$$

$$\mathbf{H}_\nu \equiv \frac{1}{4\pi} \int I_\nu \, \mathbf{n} \, d\Omega \; = \frac{1}{4\pi} \, \mathbf{F}_\nu \tag{19}$$

$$\mathbf{K}_\nu \equiv \frac{1}{4\pi} \int I_\nu \, \mathbf{nn} \, d\Omega \; = \frac{c}{4\pi} \, \mathbf{P}_\nu \tag{20}$$

where all of the above are defined analogously for the corresponding frequency-integrated quantities, i.e. $X = \int X_\nu d\nu$. Often in radiative transfer theory, Eddington's angular moments J, H, K are used to rid us of some c's and π's; vice versa, in radiation-dynamics theory it is E, F, P that typically enter the equations. Regardless of which, it will also be useful in the following to define the Eddington factor

$$f_\nu = \frac{K_\nu}{J_\nu} = \frac{P_\nu}{E_\nu}. \tag{21}$$

Exercise: Compute the Eddington factor f for i) an isotropic radiation field, and ii) one that is very strongly peaked in the radial (vertical in cartesian) direction $\mu_0 = 1$ such that $I_\nu = I_0\delta(\mu - \mu_0)$.

Again quite analogous to the discussions above, in full generality the Eddington factor f_ν is a tensor, but for 1D systems it reduces to a scalar.

3 Radiation transport v. diffusion v. equilibrium

Having introduced basic radiation quantities in the previous section, this section now turns to the subject of actually *computing* that radiation when it interacts with matter. In particular, it will be important here to distinguish between situations of *equilibrium* (black-body) versus *transport* of radiation, with the *diffusion* of radiation acting as sort of a bridge between the two.

Typically, we think of radiation as something that has escaped from somewhere and reaches us somewhere else (like for example light hitting the earth from the surface of the sun or the light from a lamp in your room). But let's first now imagine instead that we have a closed, hollow box (a german "hohlraum") filled with a dilute gas that radiates. The radiation emitted by the particles in that box will have nowhere to escape, and instead keep bouncing around, getting absorbed by the particles and emitted again. After some time, both the colliding particles and the radiation will have reached their *equilibrium distribution values.* In contrast to the cases studied later in this chapter, such equilibrium radiation has *no transport* (or flux) of radiation and its integrated properties, as detailed below and in Appendix A, depends only on the (constant) equilibrium temperature T inside the box. If we now could look into this radiation box (e.g., through a very tiny hole without disturbing the equilibrium by letting too much radiation escape), what we would see is called a *black body.*

Under these conditions, the radiation field becomes isotropic (angle independent), and so the intensity equals the mean intensity and is given by the so-called *Planck function* B_ν, i.e. $I_\nu = J_\nu = B_\nu$. Properties of this Planck function are given in some detail in Appendix A.

However, this is an idealization (though a very useful such) of real astrophysical objects; for example, the surface of a star constitutes a considerable leak of photons into outer space (during a few lucky days per year we even see this type of photon leak from here in Belgium!), and as such a stellar surface does not have an isotropic radiation field.

In general, when a beam of radiation travels through a medium, it will interact with the medium so that energy may be added or subtracted from the beam. The intensity will then not remain constant: the global reduction effect is called *extinction* which can be either absorption (photon destruction; a photon is *absorbed* by the matter and its energy given to it), or *scattering* (the photon is scattered out of the beam) of photons. The amount of extinction per unit distance is dependent on the strength of the incoming intensity (if there is no intensity present in the beam, there is nothing to absorb) and on the material properties. On the other hand, when intensity is added to the beam it is called *emission.* This can be that a photon is scattered into the beam or that collisions between the particles create a photon along into the beam (adding energy to the radiation field). Since this can happen independently of wether there already is intensity along the beam or not, in contrast to extinction such emission is not itself directly dependent on the intensity along the beam.

3.1 Radiation transport equation

The net effect from extinction and emission along a path-length Δs in the direction of the radiation beam is

$$\Delta I_\nu = j_\nu \Delta s - k_\nu I_\nu \Delta s \tag{22}$$

or when put in differential form

$$\frac{dI_\nu}{ds} = j_\nu - k_\nu I_\nu. \tag{23}$$

This is a *basic equation for radiative transfer* along a beam ('ray').

A note on notation: The above and below will try to follow Castor's notation. Keep in mind though, that different authors use different notations for the extinction and emission coefficients; for example Mihalas & Mihalas (1984) and Hubeny & Mihalas (2014) (the authorative text-books on the subject) use one, Rob Rutten's popular lecture-notes on radiative transfer a second, and whereas my own (Sundqvist) lecture-notes here follow Castor's this is not the case for my notes on "radiative processes in astronomy" (where I use Rutten's α to denote extinction). This quite confusing mess is unfortunate, of course, but just the way it is; best to simply learn the concepts, then the notations will be quite easy no matter what.

The extinction is given here in terms of a coefficient k_ν, which is defined at every frequency ν and which can vary in space. Inspection of the above equations reveal directly that its dimension is $1/length$, or $1/cm$ in cgs-units. The corresponding emission coefficient is denoted here by j_ν; its dimensions are *not* the same as for k_ν, but are instead in cgs [ergs/cm^3/sr/Hz/s]. Note that this is equivalent to defining the emission coefficient as the energy addition per unit *volume*

$$j_\nu = \frac{d\epsilon_\nu}{dV dt d\nu d\Omega}, \tag{24}$$

since the volume element $dV = dAds$, whereby we recover $dI_\nu = j_\nu ds$ from the definition of the specific intensity in the previous chapter. Somewhat similar, one may view the extinction coefficient as sort of a cross-section per unit volume ($cm^2 cm^{-3} = cm^{-1}$). For some detailed discussions on how to obtain expressions and numerical values for these two radiation-matter interaction coefficients for different kinds of *radiative processes in astronomy*, I will refer to my lecture-notes on the subject. For now, let us take them as "known" so that we can easier start to inspect and analyse the properties of eqn. 23 itself.

Dividing through eqn. 23 by the extinction coefficient gives

$$\frac{dI_\nu}{k_\nu\, ds} = \frac{dI_\nu}{d\tau_\nu} = \frac{j_\nu}{k_\nu} - I_\nu = S_\nu - I_\nu. \tag{25}$$

where we have defined two critical quantities in the field of radiative transfer, the *source function*

$$S_\nu = \frac{j_\nu}{k_\nu} \tag{26}$$

and the *optical depth*

$$d\tau_\nu = k_\nu ds \tag{27}$$

or

$$\tau_\nu = \int k_\nu ds. \tag{28}$$

The extinction coefficient k_ν is further composed of

$$k_\nu = \sigma_\nu n = \kappa_\nu \rho \tag{29}$$

for interaction cross-section σ $[cm^2]$, number density n $[cm^{-3}]$, mass absorption coefficient κ $[cm^2 g^{-1}]$ (sometimes also called the opacity), and mass density ρ $[gcm^{-3}]$.

Appendix B goes through some of the standard simple examples for solutions to the radiative transfer equation, and also connects it physically to a random walk of photons. Here, we will move directly to the key aim of the section, namely obtaining equations for the coupling of the radiation with the dynamical equations. For this, we need to study a bit the

3.2 Geometry in radiative transfer equation

The section above discussed the radiative transfer along a ray ("beam") of radiation, but never specified how to evaluate the derivative of I with respect to path length s for a given geometry of the system. The absolute derivative for the general, seven-dimensional transport equation is then by the chain-rule (e.g., Hubeny & Mihalas 2014)

$$\frac{dI_{\mathbf{n},\nu}}{ds} = \frac{\partial I}{\partial t}\frac{dt}{ds} + \sum_{i=1}^{3}\frac{\partial I}{\partial q^i}\frac{dq^i}{ds} + \frac{\partial I}{\partial \theta}\frac{d\theta}{ds} + \frac{\partial I}{\partial \phi}\frac{d\phi}{ds} + \frac{\partial I}{\partial \nu}\frac{d\nu}{ds} = j_{\mathbf{n},\nu} - k_{\mathbf{n},\nu} I_{\mathbf{n},\nu}, \tag{30}$$

valid in any inertial frame for coordinates $\mathbf{q}$; e.g. for cartesian $\mathbf{q} = (x, y, z)$ and for spherical $\mathbf{q} = (r, \Theta, \Phi)$. Note here that for the latter case we must be careful to distinguish between the *radiation angles* θ, ϕ and the *coordinate angles* Θ, Φ, and further that in order to retain full generality the intensity, extinction, and emission coefficients now all are allowed to be functions of location, frequency and radiation angle.

Cartesian coordinates Here $d\theta/ds = d\phi/ds = 0$, for photons $ds = cdt$, and for static material, $dI/d\nu = 0$, and we can write eqn. 30 in a somewhat simpler, dyadic form

$$\frac{\partial I_{\mathbf{n},\nu}}{\partial t}\frac{1}{c} + \mathbf{n}\cdot\nabla \mathbf{I}_{\mathbf{n},\nu} = \mathbf{j}_{\mathbf{n},\nu} - \mathbf{k}_{\mathbf{n},\nu}\mathbf{I}_{\mathbf{n},\nu}. \tag{31}$$

where the propagation vector in Cartesian coordinates $\mathbf{n} = n_x\hat{x} + n_y\hat{y} + n_z\hat{z}$ is again

$$n_x = \sin\theta\cos\theta \qquad n_y = \sin\theta\sin\phi \qquad n_z = \cos\theta \tag{32}$$

Let us now reduce this to 1-D, with variations only along the z-axis, and further assume a steady-state ($\partial/\partial t = 0$). For $\mu \equiv \cos\theta$ we then obtain directly

$$\mu\frac{dI_{\mu,\nu}}{dz} = j_{\mu,\nu} - k_{\mu,\nu} I_{\mu,\nu}. \tag{33}$$

Spherical geometry For spherical coordinates, the unit vector is $\mathbf{n} = n_r\hat{r} + n_\theta\hat{\theta} + n_\phi\hat{\phi}$. From vector calculus we have

$$n_r = \cos\theta \qquad n_\theta = \sin\theta\cos\phi \qquad n_\phi = \sin\theta\sin\phi \tag{34}$$

and

$$\hat{r} = \sin\Theta\cos\Phi n_x + \sin\Theta\sin\Phi n_y + \cos\Theta n_z \tag{35}$$

$$\hat{\theta} = \cos\Theta\cos\Phi n_x + \cos\Theta\sin\Phi n_y - \sin\Theta n_z \tag{36}$$

$$\hat{\phi} = -\sin\Theta\cos n_x + \cos\Theta n_y + \cos\Theta n_z. \tag{37}$$

In general, this now means quite complicated expressions for the variation of the unit vector with both radiation and coordinate angles. But great simplification is obtained if we assume spherical symmetry and, e.g., set $\Phi = \Theta = 0$. Then the above is reduced to $\mathbf{n} = \cos\theta\hat{z}$, so that finally $I = I(r,\theta)$. But the difference to the cartesian case above is that now also the path lengths vary, according to $\partial r/\partial s = \cos\theta$ and $\partial\Theta/\partial s = -\sin\theta/r$. Thus we finally obtain

$$\mu\frac{\partial I_{\mu,\nu}}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial I_{\mu,\nu}}{\partial\mu} = j_{\mu,\nu} - k_{\mu,\nu}I_{\mu,\nu}. \tag{38}$$

which is valid for spherical symmetry, and often applied in e.g. studies of extended systems like the solar corona and for stellar wind outflows.

But note here that in the case we're interested to solve the equation for a spherically symmetric system over some length-scale δr that is significantly smaller than r, $\delta r << r$, $\partial/\partial\mu = 0$ and we obtain

$$\mu\frac{dI_{\mu,\nu}}{dr} = j_{\mu,\nu} - k_{\mu,\nu}I_{\mu,\nu}. \tag{39}$$

i.e. an equation on the same form as above.

Substituting the radial coordinate for a vertical axis z, we obtain the *plane-parallel* approximation for the equation of radiative transfer, which is quite widely used in astrophysics. The great advantage of this approximation is that the angle μ is now constant throughout the computational domain.

Appendix B goes through some standard examples of the transfer equation in this plane-parallell approximation.

3.3 The radiation energy and moment equations

Just like the angular moments of intensity are useful and physically meaningful, so are the angular moments of the transfer equation. We start with the general form for the Cartesian transfer equation

$$\frac{\partial I_{\mathbf{n},\nu}}{\partial t}\frac{1}{c} + \nabla\cdot(\mathbf{n}I_{\mathbf{n},\nu}) = j_{\mathbf{n},\nu} - k_{\mathbf{n},\nu}I_{\mathbf{n},\nu}. \tag{40}$$

where we've now taking advantage of the constancy of $\mathbf{n}$ to write it inside the parenthesis. Integrating over solid angle and frequency now gives the 0^{th} moment equation of radiative transfer

$$\frac{\partial E}{\partial t} + \nabla \cdot \mathbf{F} = \int (j_{\mathbf{n},\nu} - k_{\mathbf{n},\nu} I_{\mathbf{n},\nu}) d\nu d\Omega. \tag{41}$$

This equation expresses conservation of energy for the radiation field, where the left-hand-side takes the standard form for a conservation law (cmp. our equations of hydrodynamics) with an energy density and flux. The right hand side then represents the gain and loss of radiation energy per unit volume; emissivity is a gain for the radiation, and so a loss for the material, and vice versa for the absorption. It is the *radiation energy equation.*

Multiplying now the transfer equation with a factor $\mathbf{n}/c$ and integrating over solid angle and frequency yields

$$\frac{\partial \mathbf{F}}{\partial t}\frac{1}{c^2} + \nabla \cdot \mathbf{P} = \frac{1}{c}\int (j_{\mathbf{n},\nu} - k_{\mathbf{n},\nu} I_{\mathbf{n},\nu}) \mathbf{n} d\nu d\Omega, \tag{42}$$

which is the *radiation momentum equation.*

Assuming static material, emission and extinction coefficients are isotropic the above becomes

$$\nabla \cdot \mathbf{F} = \int (4\pi j_\nu - k_\nu c E_\nu) d\nu. \tag{43}$$

$$\nabla \cdot \mathbf{P} = -\frac{1}{c}\int k_\nu \mathbf{F}_\nu d\nu. \tag{44}$$

Exercise: Derive the radiation energy and momentum equations for a static, 1D, Cartesian ("plane-parallell") geometry assuming grey (=frequency-independent) and isotropic extinction and emission.

3.4 Radiation force and radiation equilibrium

In our every-day lives, we typically think of photons as carriers of energy (would you know a photon if it hit you in the eye?). But since photons also carry momentum, of magnitude $h\nu/c$, they can, in addition to exchange energy, also exert a *radiative force* on the material with which they interact (actually, from analogy with gas pressure, this might have been clear to you already from the definition of radiation pressure above).

We've already discussed just here above (in above section on moment equations) how the right-hand-side of the transfer equation, $j_\nu - k_\nu I_\nu$, is the net energy gain/loss by the radiation field at the loss/gain of the matter (per unit volume/time/bandwidth/solid angle). Therefore, we can identify (e.g., Castor 2004)

$$g_{\mathrm{r}}^0 = \int\int (j_\nu - k_\nu I_\nu) d\nu d\Omega \tag{45}$$

and

$$\mathbf{g}_{\mathrm{r}} = \frac{1}{c}\int\int (j_\nu - k_\nu)\mathbf{n}I_\nu d\nu d\Omega \tag{46}$$

as the correct *energy and momentum exchange rates* for the radiation field with the material. *These now represent the radiation source terms that will need to be added to the equations of hydrodynamics, see further below.*

Before we do that though, let us look at an example: the temperature structure in a static stellar atmosphere (for example the Sun) is essentially determined by so-called *radiative equilibrium*, i.e. by requiring $g_r^0 = 0$ above (or equivalently that the flux in a static, 1D, plane-parallell atmosphere be constant so that $dF/dz = 0$ in the radiation energy equation). This reflects the important fact that radiation is the dominating source of energy transport in many astrophysical situations.

Moreover, since extinciton/emission coefficients are isotropic in such a static situation the *radiative acceleration* $\mathbf{g}_{\mathrm{rad}}$ [cm/s^2] is

$$g_{\mathrm{rad}} = -\frac{g_{\mathrm{r}}}{\rho} = \frac{1}{c}\int F_\nu \kappa_\nu d\nu = \frac{F\kappa_{\mathrm{F}}}{c}, \tag{47}$$

where the opacity $\kappa_\nu = k_\nu/\rho$ as before. This equation shows how g_r above represents the *force density* (force per unit volume) exerted on material by radiation, and that it is equivalent to the right-hand-side of the radiation momentum equation (44).

The last equality in eqn. 47 defines a mean opacity, namely the *flux weighted mean*

$$\kappa_{\mathrm{F}} \equiv \frac{\int F_\nu \kappa_\nu d\nu}{\int F_\nu d\nu}, \tag{48}$$

and we note also that we've here again assumed a 1-D system, meaning g_{rad} is directed vertically (in the same direction as the 1-D flux above), typically in the opposite direction of e.g. a gravitational field. The radiation force density is further given by the gradient in radiation pressure (see eqn. 44, again quite analogous to gas pressure), which can be realized also by simple inspection of the flux equation in the so-called *radiative diffusion limit.*

3.5 The radiative diffusion approximation

As seen above, depending on the characteristics of our system and our region(s) of interest, the radiation field can take quite different shapes; e.g. in very deep regions with $\tau >> 1$ and very little leakage, the radiation field becomes very nearly isotropic and approaches the theoretical abstractum black-body radiation and perfect TE. But as we move closer to the surface, an increasing amount of photons start to leak out from the system and radiation can be transported over large distances; in this case we have to deal with the full radiative transfer equation in order to get a handle on the nature of the radiation field. The *radiative diffusion approximation* bridges these limits, and is valid at great optical depth $\tau >> 1$ but where the transport of energy via radiation still plays an important role for the overall system. For example, this approximation is the standard way of dealing with radiative transfer in models that aim to simulate the structure of a star all the way from the core to the surface.

Let's first remind ourselves of our equilibrium values, valid for black-body radiation in perfect equilibrium. There the basic (frequency-integrated) radiation quantities are (see Appendix A):

$$I = J = S = B = \sigma T^4/\pi \qquad E = 4\sigma T^4/c = aT^4 \qquad F = 0 \qquad P = aT^4/3 = E/3 \tag{49}$$

where B is the frequency-integrated Planck function. In the opposite, "near surface" limit discussed above, these quantities have instead to be computed from solving the full radiative transfer equation, repeated here in its most general form for cartesian coordinates

$$\frac{\partial I_{\mathbf{n},\nu}}{\partial t}\frac{1}{c} + \mathbf{n} \cdot \nabla I_{\mathbf{n},\nu} = j_{\mathbf{n},\nu} - k_{\mathbf{n},\nu} I_{\mathbf{n},\nu}. \tag{50}$$

Though the diffusion approximation below can be derived also for this general, multi-D equation (see e.g. Castor 2003), let us here simplify by assuming a static steady-state and 1D, Cartesian ("plane-parallel") geometry (see above). We further rearrange the transfer equation slightly, writing

$$I_\nu = S_\nu - \mu \frac{dI_\nu}{k_\nu dz} \tag{51}$$

Inspection of this reveals that the equilibrium value is simply given by the limit of no transport,

$$I_\nu^0 = S_\nu = j_\nu/k_\nu, \tag{52}$$

where we know that for equilibrium TE $I_\nu^0 = B_\nu$. Inserting I_ν^0 now on the right-hand-side of eqn. 51 gives the first-order equation

$$I_\nu = B_\nu - \mu \frac{dB_\nu}{k_\nu dz} \tag{53}$$

For the standard radiative diffusion approximation, we stop here. But, in principle, higher order equations could be obtained by simply repeating the same iterative procedure.

Let's now analyze eqn. 53. The radiation energy density in this limit is

$$E = \frac{1}{c}\int\int I_\nu d\Omega d\nu = \frac{2\pi}{c}\int_\nu \int_{-1}^{1} I_\nu d\mu d\nu = \frac{4\pi}{c}\int B_\nu d\nu = aT^4, \tag{54}$$

where the "trick" is to realize that since B_ν and k_ν are isotropic (the latter because we assume static material), the second term on the right-hand-side of eqn. 53 is an un-even function of μ and so vanishes in the integral above. *This shows that in the static diffusion limit the radiation energy density is still given by its equilibrium value.*

Similarly for the radiation pressure

$$P = \frac{1}{c}\int\int I_\nu \mu^2 d\Omega d\nu = aT^4/3 = E/3, \tag{55}$$

again recovering the equilibrium value.

Exercise: Show this.

However, now for the (first monochromatic) radiation flux

$$F_\nu = 2\pi \int_{-1}^{1} I_\nu \mu d\mu = -2\pi \int_{-1}^{1} \mu^2 \frac{dB_\nu}{k_\nu dz} d\mu = -\frac{4\pi}{3} \frac{dB_\nu}{k_\nu dz} \tag{56}$$

it's the second term in the intensity-equation that now remains instead (since flux is an un-even angular moment of the intensity), showing that the radiative flux now is non-zero (unlike its equilibrium value) and given by the gradient of the source (Planck) function. The total flux is then

$$F = \int F_\nu d\nu = -\frac{4\pi}{3} \int_0^\infty \frac{dB_\nu}{dz} \frac{1}{k_\nu} d\nu = -\frac{4\pi}{3\rho\kappa_\mathrm{R}} \frac{dT}{dz} \frac{dB}{dT} = -\frac{16\sigma}{3} \frac{T^3}{\rho\kappa_\mathrm{R}} \frac{dT}{dz}, \tag{57}$$

where the second-to-third equality then also defines the so-called *Rosseland Mean Opacity* according to

$$\frac{1}{\kappa_\mathrm{R}} = \frac{\int_0^\infty \frac{dB_\nu}{dT} \frac{1}{\kappa_\nu} d\nu}{\int_0^\infty \frac{dB_\nu}{dT} d\nu} \tag{58}$$

where the denominator is readily evaluated by moving d/dT outside the frequency integral. This Rosseland mean opacity plays an important role in quite many aspects of astrophysics, and is discussed a bit further below. The flux equation 57 above shows directly why this limit of radiation transport is called the *diffusion* approximation; the flux of radiation is proportional to the local gradient in temperature (in analogy with heat-flux in thermal diffusion theory for matter).

Writing this now in the form of the radiation energy density/pressure also gives the following alternative forms for (static) radiative diffusion flux

$$F = -\frac{c}{3\rho\kappa_\mathrm{R}} \frac{dE}{dz} = -\frac{c}{\kappa_\mathrm{R}\rho} \frac{dP}{dz}, \tag{59}$$

Exercise: Show that in the diffusion limit, the flux-weighted and Rosseland mean opacities are the same.

Exercise: Assuming radiative equilibrium in a 1D, static, plane-parallell medium, and further that the diffusion limit is valid throughout, derive an expression for the temperature structure as function of Rosseland optical depth $\tau_R = -\int \kappa_R \rho dz$, defined in the opposite direction of a coordinate axis z, and in terms of the object's "effective temperature" T_eff, i.e., for $T = T(\tau_R, T_\mathrm{eff})$. For this, you may assume as outer boundary condition (top of medium) $\tau_{R,0} = 0$ and the temperature $T_0 = T_\mathrm{eff}/2^{1/4}$. Under these approximations, at which τ_R does the temperature equal the effective temperature?

3.5.1 Time-dependent radiative diffusion

As already mentioned above, the radiative diffusion equation bears its name because of the analogy with the classical material thermal diffusion equation for matter (the "heat equation"). Actually, it is even so that just like the classical diffusion equation can be derived as the continuum limit of a random walk of particles, the radiative diffusion equation can be derived as the continuum limit of a random walk of photons (the ”particles” of radiation); this is done in Appendix C.

This Appendix C also introduces the concept of the *mean-free-path of photons* $\ell = 1/k = 1/(\kappa\rho)$. To now close the connection between radiative and material diffusion, we i) retain the time-dependent term in the radiation energy equation, ii) assume the contribution from the sources and sinks cancel (radiative equilibrium), and iii) use the derived diffusive flux in the equation for radiation energy conservation:

$$\frac{\partial E}{\partial t} = \nabla \cdot \mathbf{F} = \nabla \cdot \frac{c}{3\rho\kappa_R}\nabla E = \nabla \cdot D\nabla E \tag{60}$$

where the last equality now introduces the *diffusion coefficient* $D = \ell c/3$ for photon mean-free-path $\ell = 1/(\kappa_R\rho)$.

From you course in mathematical physics / partial differential equations, you may now notice that, indeed, this *equilibrium radiative diffusion* equation is mathematically identical to the *heat equation* for diffusion of matter.

4 The equations of radiation-hydrodynamics

4.1 Radiation energy and momentum equations

As analyzed in detail in the previous section, the *radiative energy and momentum equations* are

$$\frac{\partial E}{\partial t} + \nabla \cdot \mathbf{F} = \int (j_{\mathbf{n},\nu} - k_{\mathbf{n},\nu} I_{\mathbf{n},\nu}) d\nu d\Omega. \tag{61}$$

$$\frac{\partial \mathbf{F}}{\partial t}\frac{1}{c^2} + \nabla \cdot \mathbf{P} = \frac{1}{c}\int (j_{\mathbf{n},\nu} - k_{\mathbf{n},\nu} I_{\mathbf{n},\nu}) \mathbf{n} d\nu d\Omega, \tag{62}$$

These equations have the general structure (cmp. Euler equations for hydrodynamics):

$$\frac{\partial}{\partial t}(density\ of\ quantity) + (divergence\ of\ its\ flux) = sources - sinks \tag{63}$$

They are thus *dynamical equations for the radiation field* and should, in principle, always be considered in dynamical calculations of radiating fluids.

The equations here are written in the observer's, i.e. the Eulerian, frame, and we may note also that since photons are massless, we don't have any equivalent of the conservation of (material) mass for radiation.

4.2 Material energy, momentum, and mass equations

The hydrodynamical material equations in conservation form (see previous lectures) are

$$\frac{\partial \rho}{\partial t} \ + \ \nabla \cdot (\rho \mathbf{v}) \ = \ \mathbf{0} \tag{64}$$

$$\frac{\partial (\rho \mathbf{v})}{\partial t} \ + \ \nabla \cdot (\rho \mathbf{v}\mathbf{v} + \mathrm{p_g}\mathbf{I}) = \ \mathbf{f} \tag{65}$$

$$\frac{\partial e}{\partial t} \ + \ \nabla \cdot (e\mathbf{v} + \mathrm{p_g}\mathbf{v}) \ = \ \epsilon \ + \ \mathbf{v} \cdot \mathbf{f} \tag{66}$$

which use total (kin.+int.) energy density $e = \frac{1}{2}\rho v^2 + \frac{p_g}{(\gamma-1)} = \frac{1}{2}\rho v^2 + e_i$.

The momentum source $\mathbf{f}$ can be, e.g.: external gravity $\mathbf{S_g} = \rho \mathbf{g} = -\rho \frac{\mathbf{GM_*}}{\mathbf{r^2}}$, for a constant source mass M_*, and/or including radiation effects as in: $\rho \mathbf{g} + \rho \mathbf{g}_{rad}$.

Similarly, the energy sink/source terms ϵ (appear next to work done by forces $\mathbf{v} \cdot \mathbf{f}$) can be $\epsilon = -g_r^0$ as defined for radiation previously.

4.3 Equations of radiation-hydrodynamics

Radiation as source terms The above gives the full system of equations

$$\frac{\partial \rho}{\partial t} \ + \ \nabla \cdot (\rho \mathbf{v}) \ = \ \mathbf{0} \tag{67}$$

$$\frac{\partial(\rho\mathbf{v})}{\partial t} \;+\; \nabla\cdot(\rho\mathbf{v}\mathbf{v}+\mathrm{p_g}\mathbf{I}) \;=\; \mathbf{f} \tag{68}$$

$$\frac{\partial e}{\partial t} \;+\; \nabla\cdot(e\mathbf{v}+\mathrm{p_g}\mathbf{v}) \;=\; \epsilon \;+\; \mathbf{v}\cdot\mathbf{f} \tag{69}$$

$$\frac{\partial E}{\partial t}+\nabla\cdot\mathbf{F}=\int(j_{\mathbf{n},\nu}-k_{\mathbf{n},\nu}I_{\mathbf{n},\nu})d\nu d\Omega. \tag{70}$$

$$\frac{\partial\mathbf{F}}{\partial t}\frac{1}{c^2}+\nabla\cdot\mathbf{P}=\frac{1}{c}\int(j_{\mathbf{n},\nu}-k_{\mathbf{n},\nu}I_{\mathbf{n},\nu})\mathbf{n}d\nu d\Omega, \tag{71}$$

with $\mathbf{f}=\rho\mathbf{g}+\rho\mathbf{g}_{rad}$ (if gravity is included) and $\epsilon=-g_r^0$ (neglecting conduction and thermonuclear energy). These are now coupled equations for matter and radiation.

Radiating fluid equations Combining the matter and radiation equations, we can equivalently write the momentum and energy equations in conservative form for a radiating fluid in the Eulerian frame (still inow including gravity and work-terms as sources)

$$\frac{\partial}{\partial t}(\rho\mathbf{v}+\mathbf{F}/c^2) \;+\; \nabla\cdot(\rho\mathbf{v}\mathbf{v}+\mathbf{P}+p_g\mathbf{I})=\rho\mathbf{g} \tag{72}$$

$$\frac{\partial}{\partial t}(e+E) \;+\; \nabla\cdot(e\mathbf{v}+p_g\mathbf{v}+\mathbf{F}) \;=\mathbf{v}\cdot\mathbf{f} \tag{73}$$

Depending on the application, one may thus choose to include radiation either as source terms or as parts of the conserved variables.

4.4 The closure problem

The above gives us 4 additional equations, but in general we have $1+3+6$ new variables for the radiation energy density E, vector flux $\mathbf{F}$, and pressure tensor (stress tensor) $\mathbf{P}$. This reflects the general closure problem of moment equations (cmp. how the equation of state, e.g. the ideal gas law, is used to close the Euler equations). For radiation, the six unknowns are related through the *Eddington tensor* $\mathbf{f}_\nu$

$$\mathbf{f}_\nu E_\nu=\mathbf{P}_\nu. \tag{74}$$

which is a six-component tensor and a function of both space and frequency. In equilibrium, it however reduces to a scalar and is $f=1/3$ (see previous section on radiation quantities).

But (unlike the ideal gas law), the equilibrium relation for radiation cannot generally be said to be a very good assumption. Rather, to be consistent the Eddington tensor has to be computed from a full solution of the transfer equation to get I_ν and its moments as function of space, time and frequency.

4.5 Some very brief general considerations

So, why is radiation-hydrodynamics then so difficult in full generality? We can realize this by inspection of the equations above, illustrating that we generally need:

i) to solve a double set of integrals over both angle and frequency (which will depend heavily on what processes are important for our problem; is it Thomson scattering?, a significant radiation line-force?, can we live with the Rosseland mean opacity?, do we have radaitive cooling of a shock-heated plasma?, etc, etc,),

ii) consider the non-local nature of the transfer of radiation. Regarding this, let us consider the transport equation

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \vec{n} \cdot \nabla I_\nu = j_\nu - k_\nu I_\nu \tag{75}$$

To solve this, you need to know extinction k and emission j coefficients (motting now the frequency dependence for simplicity). But to know these, you need to know number densities n and source functions S ($k = n\sigma$, $S = j/k$). And to know these, you need to know temperature, mass density, AND (in general, for scattering) the radiation energy density E. But to know E, you need to know intensity I in all angles... And so the loop continues (here we also see the need for *iterative methods* in methods of full generality, as discussed during class).

iii) Recall further that the above equation is written in the Eulerian frame. In this frame, as soon as velocity fields start to become significant the emission and extinction terms can no longer be taken as isotropic, but are instead angle-dependent. This introduces cumbersome material interaction properties for the transfer equation (e.g. formally the Rosseland mean opacity can for example no longer be used). However, whiile a transformation of the transfer equation to the Lagrangian frame (the "co-moving" frame, see Castor's book for derivation) rids us of the angle-dependence in the material interaction terms, it on the other hand introduces additional (sometimes quite complicated) terms in the transfer equation and its moments. So to sum up, there isn't any obvious general way to choose frame.

Indeed, the full general problem has not been solved; simplifications are always needed. In the sections below, we will consider some such simplifications and their numerical techniques, applicable for simulations of various types of astrophysical systems.

5 Numerical techniques for the radiative diffusion approximation

5.1 The radiative diffusion approximation

The (arguably) most important, and definitely the most widely used, approximation for including radiation in dynamical simulations is the diffusion approximation. So we start there. Recalling from above

$$\frac{\partial E}{\partial t} = \nabla \cdot \mathbf{F} = \nabla \cdot \frac{c}{3\rho\kappa_R}\nabla E = D\nabla^2 E \tag{76}$$

for diffusion coefficient $D = c\ell/3$. For most research applications, this will be a function of space as well as time. But for the illustration of the numerical techniques involved, we will for now assume it to be constant (allowing us to bring it outside the second gradient in the right-hand-side of the equation above).

5.2 Numerical schemes for solving the diffusion equation

5.2.1 1D explicit FTCS

A straightforward forward-time centered-space (FTCS) finite-difference discretization scheme (see overview in previous lectures) gives for the 1D cartesian diffusion equation

$$\frac{E_i^{n+1} - E_i^n}{\Delta t} = D\frac{E_{i+1}^n - 2E_i^n + E_{i-1}^n}{\Delta x^2} \tag{77}$$

for time-index n and spatial index i. It is often convenient to rearrange this to

$$E_i^{n+1} = \alpha(E_{i+1}^n + E_{i-1}^n) + (1 - 2\alpha)E_i^n, \tag{78}$$

for the parameter $\alpha = D\Delta t/\Delta x^2$.

A key-point here is that while such a FTCS scheme is unconditionally un-stable for the linear advection equation (see again previous lectures), for the diffusion equation it is instead conditionally stable. This can be seen through a Von-Neumann analysis, which gives for a perturbation of E_0 on the form $E_i^n = G^n E_0 \exp(ikx)$ the amplification factor

$$|G^{n+1}/G^n| = |1 - 4\alpha \sin^2 k\Delta x/2| \tag{79}$$

implying a time-constraint for stability $\alpha \leq 1/2$, or

$$\Delta t \leq \frac{\Delta x^2}{2D}. \tag{80}$$

Class-room exercise: During class, we will code a very simple explicit explicit solver for the diffusion equation with a constant $D = 1$, using the initial condition $E(x, 0) = \sin(\pi x)$ together with fixed boundary conditions $x(0) = x(1) = 0$. This set-up condition has an analytic solution $E(x, t) = \sin(\pi x)\exp(-D\pi^2 t)$. How does your numerical solution compare to the analytic one?

How does the convergence behave when your decrease/increase the spatial/time resolution of your mesh? And what happens when you increase the time-step to be above the time-constraint limit?

5.2.2 The time-constraint problem for explicit solvers of radiative diffusion

In general, the Δx^2 time-constraint dependence above causes severe issues for explicit solvers of the diffusion equation in high resolution. For *radiative* diffusion, this becomes even more problematic due to the high velocities of photons (the speed of light). To illustrate this, let us first consider a radiative diffusion model of energy transport through a star like the Sun. To make an order of magnitude estimate of the photon mean-free-path, we will take an average solar mass density $\rho = M_\odot/(4\pi R_\odot^3)$ and assume a typical value $\kappa = 1\,cm^2/g$. This gives then $\ell \approx 2$ cm and thus a diffusion coefficient on order c. Let's further say we need at least 100 grid-points to resolve $R_\odot$, so that Δt will be an the order of a few years or so. At first glance, that may seem like a reasonably big time-step, but considering that the evolution time-scale of the Sun is on order billions (!) of years, it is easy to realize that an explicit diffusion solver in this case will be extremely unpractical. The time-scale constraint difference between radiative diffusion and advection of matter (for a low velocity flow) can be estimated by considering $t_{ad} \sim \Delta x/v$, $t_{diff} \sim \Delta x^2/(c\ell)$, so that $t_{diff}/t_{ad} \sim (a/c)\,(\Delta x/\ell)$ for sound speed a. In this respect, let's consider now instead some layers close to the surface of a star like the Sun, where the densities are much lower than in the centre, on order $\rho \sim 10^{-9}\,g/cm^2$. For the same assumed opacity then, we get mean free-paths on order $\ell \sim 10^9$ cm. But the sound speed in the solar surface layers is on order $a \sim 10\,km/s$ and the typical resolution needed set by the so-called pressure scale height H_p, say $\Delta x \sim 0.1 H_p = 0.1 a^2/g_*$ for solar surface gravity acceleration $g_* = 10^{4.44}$ (cgs). This results in $t_{diff}/t_{ad} \sim 10^{-7}$ or so, illustrating that for every hydro advection time-step, we will have to take some ten million steps for our diffusion solver. And estimates for other typical astrophysical conditions often give similar results. This naturally leads us into considering *implicit* solvers instead, which as we will see below typically are stable quite independently of the time-step.

5.2.3 1D implicit BTCS

The implicit backward time centered space (BTCS) scheme for the 1D diffusion equation is obtained by simply taking the discrete space-derivative at time-step n+1 instead of n

$$\frac{E_i^{n+1} - E_i^n}{\Delta t} = D\frac{E_{i+1}^{n+1} - 2E_i^{n+1} + E_{i-1}^{n+1}}{\Delta x^2} \tag{81}$$

Again it is often convenient to reformulate this as

$$E_i^n = -\alpha(E_{i+1}^{n+1} + E_{i-1}^{n-1}) + (1+2\alpha)E_i^{n+1}, \tag{82}$$

where again $\alpha = D\Delta t/\Delta x^2$ and you may note that the left-hand-side now expresses the (known) energy density values at time n. A Von-Neuman analysis now yields for the amplification factor

$$|G^{n+1}/G^n| = |\frac{1}{1 + 4\alpha \sin^2 k\Delta x/2}| \leq 1 \tag{83}$$

for all α; that is, the scheme is unconditionally stable.

Class-room exercise: Extend your explicit 1D solver above to an implicit one. For this, you will need to invert a tri-diagonal matrix, for which standard built-in routines exist in programming languages like python, IDL, MATLAB, etc. If you are coding in a low-level language such as FORTRAN, C, etc, you may consult e.g. "Numerical recipes" by Press et al. for standard algorithms. To solve the equation for time-step n+1, you set up your tri-diagonal matrix A with diagonal elements $1 + 2\alpha$ and sub- and super-diagonal elements $-\alpha$. Then you solve $p = A^{-1}r$ where p is your updated E_{rad} vector at $n + 1$ and r is E_{rad} at n. For the same example as above, verify that your solution now is stable for time-steps larger than the limit for the explicit solver.

5.2.4 2D alternating direction implicit (ADI)

To motivate the ADI technique, let us first consider a fully implicit BTCS scheme in 2D, for directions x and y with associated discrete points x_i and y_j. This gives

$$\frac{E^{n+1}_{i,j} - E^n_{i,j}}{\Delta t} = D\left(\frac{E^{n+1}_{i+1,j} - 2E^{n+1}_{i,j} + E^{n+1}_{i-1,j}}{\Delta x^2} + \frac{E^{n+1}_{i,j+1} - 2E^{n+1}_{i,j} + E^{n+1}_{i,j-1}}{\Delta y^2}\right) \tag{84}$$

or

$$E^n_{i,j} = -\alpha(E^{n+1}_{i+1,j} + E^{n-1}_{i-1,j}) + (1 + 2\alpha + 2\beta)E^{n+1}_i - \beta(E^{n+1}_{i,j+1} + E^{n-1}_{i,j-1}), \tag{85}$$

where now $\alpha = D\Delta t/(2\Delta x^2$ and $\beta = D\Delta t/(2\Delta y^2$. As compared to the 1D case, the matrix that needs to be inverted now generally contains terms that lie (much) farther away from the diagonal than on the sub/super-diagonal. As an illustration, let us consider a simple 3×3 system (i.e. $n_x = n{+}y = 3$). Then the solution vector would contain 9 elements, and inspection of, e.g., the $E_{2,2}$ term reveals that we have a 5-band matrix; $-\beta$ and $(1{+}2\alpha{+}2\beta)$ lie on the sub/super diagonal and on the diagonal, respectively, but now we also have $-\alpha$ on the centre of the two "blocks" to the right and left of the central block-matrix. And while this might be ok for such a small 3×3 system, for larger simulation spaces it may soon become quite difficult and unpractical to invert the full matrix.

This brings to the ADI technique, for which we split the problem into two 1D implicit ones, keeping explicit one direction at the time. In step 1 we solve implicitly in x but keep the y-direction explicit (for half the time-step):

$$\frac{E^{n+1/2}_{i,j} - E^n_{i,j}}{\Delta t/2} = D\left(\frac{E^{n+1/2}_{i+1,j} - 2E^{n+1/2}_{i,j} + E^{n+1/2}_{i-1,j}}{\Delta x^2} + \frac{E^{n}_{i,j+1} - 2E^{n}_{i,j} + E^{n}_{i,j-1}}{\Delta y^2}\right) \tag{86}$$

and in step two, we solve implicitly in y but keep the y-direction explicit (for the other half time-step):

$$\frac{E^{n+1}_{i,j} - E^{n+1/2}_{i,j}}{\Delta t/2} = D\left(\frac{E^{n+1/2}_{i+1,j} - 2E^{n+1/2}_{i,j} + E^{n+1/2}_{i-1,j}}{\Delta x^2} + \frac{E^{n+1}_{i,j+1} - 2E^{n+1}_{i,j} + E^{n+1}_{i,j-1}}{\Delta y^2}\right) \tag{87}$$

Class-room exercise: Show that each sub-step in the 2D ADI method now consists of solving a tri-diagonal matrix system for the time-updated energy vector. Then try to extend your previous implicit 1D solver above to a simple 2D ADI one.

5.3 The problem of flux-limiting

Not yet; for now, see your notes from black-board lecture in class.

6 Applications and approximations for a dynamically important radiative force in supersonic flows

As discussed in earlier sections, the radiative acceleration vector is

$$\mathbf{g}_{rad} = \frac{1}{\rho c} \int \oint \mathbf{n} k_\nu I_\nu d\Omega d\nu \tag{88}$$

Note that this gives a 3D radiation force in general, and that we also now kept the possibility of an angle-dependent extinction k_ν (due to the Doppler shift when velocity-effects are non-negligible).

As a simple example, consider first the extinction from continuum Thomson scattering (electron scattering, see Ch. 5 in Sundqivst's lecture-notes on "radiative processes"). Then $k_\nu = k_e$ is both frequency and isotropic (the latter true at least in a fore-aft sense), so that

$$\mathbf{g}_e = \frac{\kappa_e \mathbf{F}}{c} \tag{89}$$

for $\kappa_e \rho = k_e$.

On the other hand, consider now instead a spectral line with extinction

$$k_\nu^l = k_L \phi(\nu - \nu_0) = \sigma_{cl} f_{lu} n \phi_\nu \tag{90}$$

where $\phi(\nu - \nu_0) = \phi_\nu$ is the line-profile function peaked at line-centre frequency ν_0, which is normalized such that $\int \phi_\nu d\nu = 1$, n the number density of the lower level in the considered transition, σ_{cl} the classical frequency-integrated line cross-section, and f_{lu} the quantum mechanical oscillator-strength correction for the line. Again, you can read more about all these quantities in Chapters 5-6 of Sundqivst's lecture-notes on "radiative processes".

Neglecting the impact of any additional continuum extinction, this gives for the line-force

$$\mathbf{g}_{line} = \frac{k_L}{\rho c} \int \oint \mathbf{n} \phi_\nu I_\nu d\Omega d\nu \tag{91}$$

Assume now a spherically symmetric situation where the intensity I_ν streams from a central continuum source with constant values in the positive direction $\mu \geq 0$ and zero intensity for incoming directions $\mu \leq 0$ (think, e.g., of the surface of a star; radiation only flows outwards, escaping the star). If we then further assume that i) this I_ν does not vary much over the extent of the line-profile (a good assumption since continuum radiation varies much more slowly than line radiation), and ii) the central source does not get attenuated (i.e. the medium is optically thin), we obtain for the radial line force

$$g_{line}^{thin} = \frac{\pi I_\nu^0 k_L}{\rho c} = \frac{F_\nu^0 k_L}{\rho c} \tag{92}$$

where the latter equality uses $F = \pi I$ for this case (see exercise in previous section on radiation quantities) and the ^{0} signals that the intensity and flux are taken at the line-centre frequency.

It is now instructive to re-write the optically thin line acceleration as

$$g_{line}^{thin} = \left(\frac{F_\nu^0 \nu_0}{F}\right)\left(\frac{\kappa_e F}{c}\right)\left(\frac{k_L}{k_e \nu_0}\right) = w_\nu g_e q \tag{93}$$

where w_ν weights the line's position in the flux spectrum ($w_\nu \sim 1$ near the peak of the flux-spectrum), g_e is the previously considered force due to electron scattering, and

$$q = \frac{k_L}{k_e \nu_0} = \frac{n}{n_e} f_{lu} \frac{\sigma_{cl}}{\sigma_{Th}\nu_0} = \frac{n}{n_e} f_{lu} \frac{\pi Q}{2} \tag{94}$$

where $Q = 2\pi\nu_0/\gamma$ is the so-called $Q - value$ for the spectral line viewed as a driven, weakly damped, harmonic oscillator with classical damping coefficient γ (for derivation and some discussions of this, see again Sundqvist's Ch. 5 in radiative processes notes).

The key point to realize now is that the "quality" Q of the resonance is extremely large, on order $\sim 10^7$. Assuming an ionized plasma dominated by hydrogen, $n_e \sim n_H$ and for, say, a strong resonance carbon line with abundance $n/n_H \sim 10^{-3...-4}$ and $f_{lu} \sim 1$, we then get $q \sim 10^{3...4}$.

This thus means that the radiative acceleration from only one single optically thin line can exceed that of continuum electron scattering by more than a factor of thousand. Indeed, it is this high quality of the line resonance that makes it possible for the accumulative force from a myriad of spectral lines to overcome gravity and drive supersonic outflows from astrophysical objects such as hot, massive stars, cataclysmic variables, and AGN disks.

6.1 The line force in the Sobolev approximation

The analysis above neglects any attenuation of I_ν. More generally, I_ν needs to be computed from the radiative transfer equation, whose formal integral is

$$I_\nu(\tau_\nu) = I_\nu^0 e^{-\tau_\nu} + \int S(t_\nu) e^{-|\tau_\nu - t_\nu|} dt_\nu = I_{dir} + I_{diff} \tag{95}$$

To simplify a bit, we will now first make use of some knowledge we already have about the soon-to-be-introduced "Sobolev approximation". Namely, in such a case the above $I_{diff}(\mu) = I_{diff}(-\mu)$, so that this component vanishes when computing the line force in this Sobolev approximation. Thus we write

$$I_\nu(\tau_\nu) = I_\nu^0 e^{-\tau_\nu} \tag{96}$$

for line optical depth

$$\tau_\nu = \int k_L \phi_\nu dl \tag{97}$$

along some direction l. Using the Doppler formula

$$\frac{\nu - \nu_0}{\nu_0} = \frac{v_l}{c} \tag{98}$$

we can transform this from a spatial to a frequency integral

$$\tau_\nu = \int k_L \phi_\nu (dl/d\nu) d\nu = \tau_\nu = \int \phi_\nu \frac{k_L c}{\nu_0 dv_l/dl} d\nu. \tag{99}$$

Making now the *Sobolev assumption* that the line-of-sight velocity gradient dv_l/dl and the line extinction k_L are approximately constant over the line resonance region, the above can be recast as

$$\tau_\nu = \frac{k_L c}{\nu_0 |dv_l/dl|} \int \phi_\nu d\nu = \tau_S \int \phi_\nu d\nu \tag{100}$$

where the last equality introduces the *Sobolev optical depth*

$$\tau_S = \frac{k_L c}{\nu_0 |dv_l/dl|} = \frac{q \kappa_e \rho c}{|dv_l/dl|} = qt \tag{101}$$

where the last equality casts this in the form of the line-strength parameter q defined above and a characteristic optical-depth variable t.

Note i) that since the line profile is normalized, the optical depth integrated over the line becomes equal to this Sobolev optical depth, and ii) τ_S differs from the standard optical depth in being a *local* quantity. The projected line-of-sight velocity gradient is in general 3D

$$\frac{dv_l}{dl} = \mathbf{n} \cdot \nabla (\mathbf{n} \cdot \mathbf{v}) \tag{102}$$

for spherical symmetry

$$\frac{dv_l}{dl} = \mu^2 \frac{dv_r}{dr} + (1 - \mu^2) \frac{v_r}{r} \tag{103}$$

and for the radial streaming approximation $\mu = 1$,

$$\frac{dv_l}{dl} = \frac{dv_r}{dr}. \tag{104}$$

Inserting this into our line acceleration expression now gives

$$\mathbf{g}_{line} = \frac{k_L}{\rho c} \oint \int \mathbf{n} I_\nu^0 \phi_\nu e^{-\tau_S \int \phi_\nu d\nu} d\nu \, d\Omega \tag{105}$$

Assuming (for simplicity, it is possible to keep the full vector expression, see Castor's Ch. 6.8) again now spherical symmetry and that I_ν^0 is slowly varying in frequency and non-zero only for a radial direction $\mu = 1$, we can use the fundamental theorem of calculus to perform the frequency-integral analytically, yielding finally

$$g_{line} = \frac{k_L \pi I_\nu^0}{\rho c} \left(\frac{1 - e^{-\tau_S}}{\tau_S} \right) = g_{line}^{thin} \left(\frac{1 - e^{-\tau_S}}{\tau_S} \right) \tag{106}$$

where the latter expression uses the expression obtained above for the optically thin case, and where τ_S is understood to now be in the radial direction.

As it should, this key equation recovers the optically thin limit for $\tau_S \ll 1$, but also shows that for an optically thick line with $\tau_S \gg 1$, the thin line force is reduced by

$$g_{line}^{thick} = \frac{g_{line}^{thin}}{\tau_S} = w_\nu \frac{F}{c^2} \frac{dv_r/dr}{\rho}. \tag{107}$$

This now shows that the radiation force per unit mass is *independent* of the atomic opacity, and instead proportional to the spatial velocity gradient in the flow; a neat discussion on the physics of this quite remarkable property can be found toward the end of Castor's Ch. 6.8.

Note that the above analysis never needs to specify explicitly how the line profile behaves within the line resonance zone. In order to evaluate the validity of the Sobolev approximation, however, one needs to do so. For example, for a Doppler profile of characteristic width $v_{th} = \sqrt{2kT/m}$, the opacity in a radial streaming model needs to be roughly constant over a few radial *Sobolev lengths* $\Delta r \approx v_{th}/(dv/dr) = L_{Sob}$.

Exercise: Compute the radial Sobolev length L_{Sob} in the wind of a massive star with $v_{th} \approx 10$ km/s and typical velocity gradient $dv/dr \approx v/r$ for $v \approx 1000$ km/s and $r \approx R_*$. Express your answers in units of the stellar radius R_*.

6.2 CAK theory for accumulative line-force from an ensemble of lines

To evaluate the line-force stemming from a large ensemble of spectral lines, we sum over all contributions

$$g_{line}^{tot} = \sum_{lines} g_{line}. \tag{108}$$

The Castor-Abbott-Klein (1975) model (CAK) now assumes this sum can be replaced by a distribution function over the number of lines of certain line-strength

$$g_{line}^{tot} = \int g_{line} \frac{dN}{dq} dq \tag{109}$$

where the flux-weighted line-strength distribution function is assumed to follow a power-law of index $\alpha - 2$

$$\frac{dN}{dq} = \frac{1}{\Gamma(q)\bar{Q}} \left(\frac{q}{\bar{Q}}\right)^{\alpha-2} \tag{110}$$

where $\Gamma(\alpha)$ now represents the Gamma function (NOT to be confused with Eddington's gamma representing the ratio of radiation to gravitational acceleration!) and the line-strength normalization constant $\bar{Q}$ used here is due to Gayley (1995). Using this distribution, the total line force can be evaluated analytically to yield

$$g_{line}^{CAK} = \frac{g_e \bar{Q}}{(1-\alpha)(\bar{Q}t)^\alpha} \tag{111}$$

where the physical interpretation of $\bar{Q}$ as the line-force enhancement factor as compared to the Thomson scattering force in the case all line are optically thin ($\alpha = 0$) becomes apparent. Due to the blending of optically thin and thick lines, however, this maximum enhancement is reduced by a typical factor ${\tau_S}^\alpha$, evaluated for line-strength $\bar{Q}$ with α representing the ratio of optically thick to total number of contributing lines.

During class, we will use this line-force (and some extensions of it, to see how an incorporation of it into the radiation-hydrodynamics equations provide a good rationale for computations of supersonic wind outflows from stars that are much more massive than our Sun.

7 Appendix A: Properties of equilibrium black-body radiation

The shape of the emitted spectrum of a black-body was actually found empirically before Planck made his derivation based on the experimentally found curve and a (back then pretty strange) assumption of *quantum atomic oscillators*. Planck's black-body distribution is:

$$B_\nu(T) = I_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1} \tag{112}$$

$$B_\lambda(T) = I_\lambda(T) = \frac{2\,h\,c^2}{\lambda^5}\frac{1}{e^{hc/\lambda\,kT}-1}. \tag{113}$$

Note here that the frequency and wavelength curves are only equivalent if one takes proper care of the difference between spectral bandwidth and frequency bandwidth. Fig. 2 gives an example of the black-body distribution for a few different temperatures.

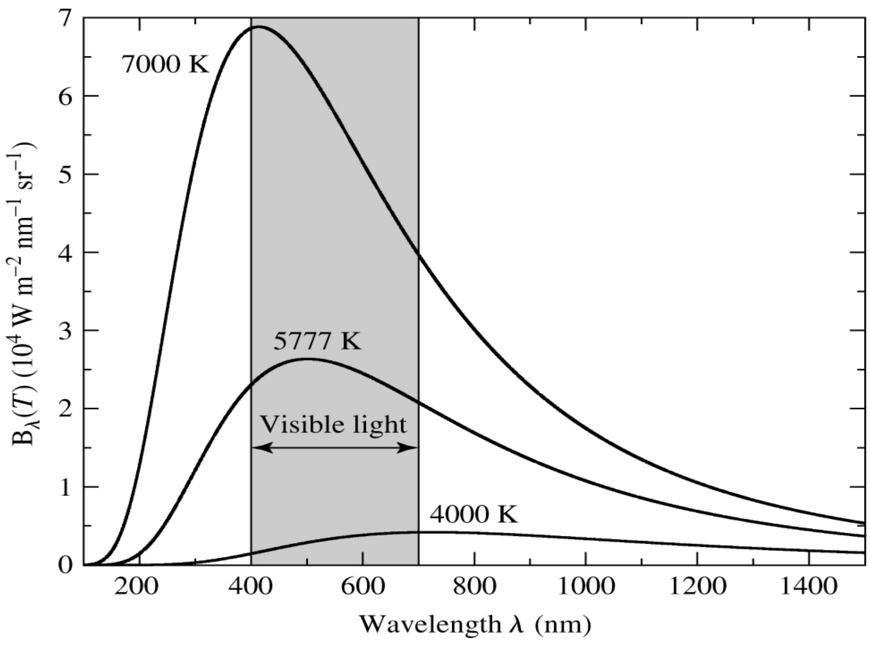

Figure 2: The Black-Body spectrum (B_λ) for different temperatures. For stars that are "hot" resp. "cool" on the surface (some astronomy jargon language, one may of course argue all stars are actually pretty hot...), the peak of the energy distribution falls outside the visible range.

Exercise: Derive B_λ from B_ν using the relations $c = \lambda\nu$ and $B_\nu d\nu = -B_\lambda d\lambda$.

7.1 Rayleigh-Jeans tail of BB radiation

When the frequency is small compared to the frequency of the peak of the distribution ($h\,\nu \ll kT$) we can stop the exponential expansion at the linear term:

$$\exp\left(\frac{h\nu}{kT}\right) = 1 + \frac{h\nu}{kT} + ... \tag{114}$$

The Rayleigh-Jeans tail of the distribution is linear in temperature and gives a quadratic slope of the distribution in frequency units, and the 4rth power in wavelength units :

$$B_\nu(T) = \frac{2\,\nu^2}{c^2}\,k\,T \tag{115}$$

$$B_\lambda(T) = \frac{2\,c\,k\,T}{\lambda^4} \tag{116}$$

Note here that these classical expressions (Planck's constant h doesn't appear) lead to an "ultraviolet catastrophe" where the radiation of the black-body continues to increase as we move towards shorter and shorter wavelengths (for the rest of the story, see Ch. **??**).

7.2 Wien's limit

In the opposite limit when h$\nu \gg$ k,T, the exponential is much larger than 1 so that:

$$B_\nu \sim \frac{2h\nu^3}{c^2}\,e^{-h\nu/kT}. \tag{117}$$

Note how this "cures" the ultraviolet catastrophe mentioned above, as the intensity drops fast with increasing frequency (the exponential wins over the ν^3).

7.3 Peak of the Planck curve – Wien's displacement law

The frequency at which the peak of the energy distribution occurs, can be found by solving for

$$\frac{\partial B_\nu}{\partial \nu} = 0 \tag{118}$$

$$\frac{\partial B_\lambda}{\partial \lambda} = 0 \tag{119}$$

When solving the first equation we get:

$$h\nu_{max} = 2.82kT \tag{120}$$

$$\frac{\nu_{max}}{T} = 5.8789\,\times\,10^{10}\mathrm{Hz\,K} \tag{121}$$

So the frequency of the peak of B_ν shifts linearly with temperature and this peak determined the full spectral energy distribution of the object.

Wien's displacement law is more known in solving for the peak of B_λ which gives:

$$\lambda_{max} T = 2897.8\,\mu\mathrm{m\,K} \tag{122}$$

Note that B_λ and B_ν are fundamentally different: for a given T will the peak of the distribution of B_ν fall on a different wavelength as the peak of the distribution of B_λ. For the Sun (T_{eff} = 5777 K) the peak of B_ν falls at 883 nm, while the peak of B_λ is at 502 nm. Normally, we will here work with this standard wavelength version of Wien's law.

Exercise: Derive eqn. 122 above. Hint: Change variable to $x = hc/(\lambda kT)$.

Exercise: Have a look at the observed spectra of the unknown stars in Fig. **??**. What effective temperatures would you guess these stars have? Which one do you think is hottest on the surface? (Note: stars, of course, don't really radiate like perfect black-bodies; nonetheless, just using Wien's displacement law as a very simple "0th order approximation", we can still learn quite a lot as you can see!)

Having now described and discussed some basics of the frequency distribution of a black-body, let us next derive what the radiation quantities introduced above look like for such black-body radiation.

7.4 Radiation energy density

Using the definitions from previous section, the radiative energy density is

$$E_\nu = \frac{4\,\pi\,I_\nu}{c} \tag{123}$$

so that for BB radiation

$$E_\nu = \frac{8\pi h\nu^3}{c^3}\,\frac{1}{e^{h\nu/kT} - 1}. \tag{124}$$

The total energy density (integral over frequency):

$$E(T) = \int_0^\infty E_\nu(T)\,d\nu = \frac{4}{c}\,\sigma\,T^4 = a\,T^4 \tag{125}$$

which also can be obtained directly from the Planck function:

$$E(T) = \frac{4\pi}{c}\int_0^\infty B_\nu(T)\,d\nu = \frac{4\sigma\,T^4}{c} = a\,T^4 \tag{126}$$

The associated number density of photons is

$$\int \psi_\nu d\Omega = \frac{E_\nu d\nu}{h\,\nu} = \frac{8\pi h\nu^2}{c^3}\frac{1}{e^{h\nu/kT}-1} \tag{127}$$

and for the total number density

$$\psi = \int_0^\infty \frac{E_\nu d\nu}{h\,\nu} = 16\pi\left(\frac{kT}{hc}\right)^3 \times 1.202 = 2.029\times 10^7 T^3 m^{-3} \tag{128}$$

This gives finally for the average photon energy of black-body radiation

$$h\nu_{aver} = \frac{E}{\psi} = 2.70\,k\,T \tag{129}$$

7.5 Radiation pressure

In analogy, the total radiation pressure is

$$P(T) = \frac{a}{3}T^4 = \frac{1}{3}E, \tag{130}$$

where the last equality is consistent with the reasoning in the last section (when first introducing radiation pressure).

Exercise: Show that $3P = E$ for any isotropic radiation field.

7.6 Radiation flux

Exercise: Show that the radiative flux is 0 for an isotropic radiation field given by BB radiaton.

7.7 Effective temperature

Even though the radiative flux for the idealized case of isotropic BB radiation is zero, such BB radiation is still used to define an important "flux temperature" very often used in astrophysics, the *effective temperature* $T_{\rm eff}$. Close to the surface of an object (e.g. a star), we may assume that all radiation streams outwards so that $I_\nu = I_\nu^+ + I_\nu^- = I_\nu^+$, where I_ν^+ is defined on the interval $\mu = 0\ldots1$. In this case (see previous section)

$$F_\nu = F_\nu^+ = \pi I_\nu^+ = \pi B_\nu \tag{131}$$

where the last equality applies here for the case of BB radiation. The frequency integrated flux is thus

$$F^+ = \pi\int B_\nu d\nu = \sigma T^4 \tag{132}$$

The *effective temperature* $T_{\rm eff}$ is then defined from this flux as the temperature the object *would have* if it were a black-body

$$F = 2\pi \int_0^\infty \int_{-1}^1 I_\nu(\mu)\mu d\mu d\nu \equiv \sigma T_{\rm eff}^4. \tag{133}$$

That is, for a given calculated or observed flux F, the "flux proxy" effective temperature is $(F/\sigma)^{1/4}$. The physical temperature of the object can deviate from this, of course, since e.g. stars are not perfect black-bodies (they are leaking a lot of photons from their surface!), but the effective temperature is actually a very good proxy for the overall distribution of surface temperatures of stars.

Moreover, consider the luminosity (radiative energy output per second) of a spherical star $L = FA = F4\pi R^2 = \sigma T_{\rm eff}^4 4\pi R^2$. It is convienient to work with quantities scaled to the solar ones,

$$\frac{L}{L_\odot} = \left(\frac{R}{R_\odot}\right)^2 \left(\frac{T_{\rm eff}}{T_{\rm eff,\odot}}\right)^4, \tag{134}$$

a relation from which we can learn something, e.g., about the sizes of stars depending on their relative location in the HR-diagram.

Exercise: Do stars in the upper-right corner of the HR-diagram have larger or smaller radii than the Sun?

7.8 More temperatures

7.8.1 Radiation temperature

Using similar arugments as for the effective temperature definition above (however now not using the flux, but rather the intensity itself), we define here the *radiation temperature* of an object as the temperature it *would have* it it were a black-body shining with intensity I_ν

$$I_\nu \equiv B_\nu(T_{rad}) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT_{rad}} - 1}. \tag{135}$$

Since astrophysical objects quite often (but certainly not always) have intensities fairly close to black-bodies, radiation temperatures typically gives a neat scaling for the intensities involved; if the object shines like a black-body, then of course $T_{rad} = T$ for gas temperature T. Note also that sometimes the radiation temperature is defined in terms of the mean intensity J_ν instead of I_ν as above.

Taking the Rayleigh-Jeans limit of this then gives the

7.8.2 Brightness temperature

T_b of an object with intensity I_ν

$$I_\nu \equiv B_\nu^{RJ}(T_b) = \frac{2\nu^2}{c^2}kT_b. \tag{136}$$

Unlike for the general radiation temperature definition above, in this regime the intensity is directly proportional to the brightness temperature.

8 Appendix B: Simple examples of solutions to the radiative transfer equation

8.1 Radiation through homogeneous slab

Consider a ray of radiation through a slab of thickness D in which the material properties do not vary with spatial location, i.e. through a so-called *homogeneous slab*. For simplicity here, we will also assume that opacities do not vary with frequency (that the material is "grey") and work with frequency-integrated intensities $I = \int I_\nu d\nu$.

Consider first the case of no interaction ($\alpha = \eta = 0$) so that

$$\frac{dI}{ds} = 0 \tag{137}$$

and we recover the previous result that the intensity is constant along a ray.

Next consider a case of only extinction ($\alpha \neq 0$, $\eta = 0$), which gives upon integration

$$I(D) = I(0)e^{-\tau(D)}. \tag{138}$$

where $I(0)$ is the incoming intensity at the begin of the slab and $\tau(D) = \int_0^D \alpha ds = \alpha D$, where the last equality applies here for the case of a homogeneous slab (non-varying α).

Finally for the general case ($\alpha \neq 0$, $\eta \neq 0$), we can use an integrating factor e^τ to obtain

$$I(D) = I(0)e^{-\tau(D)} + S(1 - e^{-\tau(D)}). \tag{139}$$

This equation shows how the intensity emergent at the "end" of the slab depends on the slab total optical depth and the source function; it recovers the previously obtained results: $I(D) = I(0)$ for $\tau(D) = 0$ and $I(D)/I(0) = e^{-\tau(D)}$ for $S = 0$. Moreover, it illustrates that $I(D) = S$ when $\tau(D) >> 1$, i.e. for very optically thick media the emergent intensity is given by the source function (making the meaning of S's name clear). This important result is quite useful to keep in mind also when we (later) are dealing with more complicated situations in which S varies with location in the material.

Exercise: Derive eqns. 138 and 139 above.

Exercise: Compute $I(D)$ in the cases $\tau(D) \gg 1$ and $\tau(D) \ll 1$.

8.2 Plane-parallel, semi-infinite medium

To (often) a quite good approximation, stars are spherical. But if you consider only a small domain around the region where (visible) light actually escape from the star – the so-called (optical) *photosphere* – then you can often use the plane-parallel approximation, and so deal with a simpler version of the transfer equation.

Exercise: Estimate the extent of the solar photosphere to its total radius by equating it with the so-called *pressure scale height* $H_p = a^2/g$; for a typical photospheric speed of sound a you can assume 10 km/s and the surface gravity acceleration g can be find in standard astronomy tables. Is the plane-parallell approximation valid?

Let us define a *fixed* optical depth scale that increases inwards from being 0 outside the star's atmosphere at $z \to \infty$ (where we as observers are located), i.e.

$$d\tau_\nu = -\alpha_\nu dz. \tag{140}$$

It is important to realize that this optical depth scale is not the same as the one we discussed before, which was defined along the ray with direction cosine μ. The plane-parallel transfer equation then becomes

$$\mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu. \tag{141}$$

Beware of the sign-difference between this equation – the standard equation in classical studies of radiative transfer in stellar atmospheres – and the one above defined along the ray. The optical depth between two positions along z ($\to 0$ at stellar center, $\to \infty$ at the observer) is then

$$\tau_\nu(z) = \int_z^{z_{\max}} \alpha_\nu(z')dz' \tag{142}$$

for integration dummy variable z'. Like above, we can write down a *formal solution* of this equation by using an integrating factor, now $e^{-\tau_\nu/\mu}$,

$$I_\nu(\tau_1, \mu) = I_\nu(\tau_2, \mu)e^{-(\tau_2-\tau_1)/\mu} + \int_{\tau_1}^{\tau_2} S_\nu(t_\nu)e^{-(t_\nu-\tau_1)/\mu}dt_\nu/\mu \tag{143}$$

which is generally valid; for $\mu > 0$ when $\tau_1 \leq \tau_2$ and for $\mu < 0$ when $\tau_2 \leq \tau_1$.

Exercise: Derive eqn. 143

A special case of interest for us here regards the limit when $\tau_1 = 0$ and $\tau_2 \to \infty$. This gives the *surface intensity* in a semi-infinite, plane-parallel atmosphere, where $\tau_1 = 0$ then represents the optical depth at the stellar surface and $\tau_2 \to \infty$ deep inside the core of the star. Eqn. 143 then becomes

$$I_\nu(0, \mu) = \int_0^\infty S_\nu(t_\nu)e^{-t_\nu/\mu}dt_\nu/\mu. \tag{144}$$

This equation shows that the *emergent intensity* at the surface is given by (an exponentially weighted) average of the source function. Let's consider two simple examples to illustrate this:

i) For a vertical (radial) ray $\mu = 1$ and constant source function $S_\nu = Const.$, we get $I_\nu(0, 1) = S_\nu$. This makes sense, since in case no "averaging" has to be done, and the emergent intensity is simply given by the constant source function (cmp. solution along a ray in last section).

ii) For a source function depending linearly on optical depth, $S_\nu = a + bt_\nu$,

$$I_\nu(0, \mu) = a + b\mu = S_\nu(\tau_\nu = \mu). \tag{145}$$

This is the famous *Eddington-Barbier approximation* for the emergent intensity in a semi-infinite atmosphere.

Exercise: Derive the two examples i) and ii) above.

The Eddington-Barbier relation shows that the emergent intensity viewed from angle μ reflects the optical depth along the *fixed, vertical* scale (see above) at that angle. For example, for $\mu = 1$, $I_\nu(0, 1) = S_\nu(\tau_\nu = 1)$, i.e. the surface intensity is given by the source function at optical depth unity. But for e.g. $\mu = 0.4$, $I_\nu(0, 1) = S_\nu(\tau_\nu = 0.4)$, i.e. I_ν reflects S_ν at a different optical depth along this *fixed* scale.

This can be readily understood by considering instead the optical depth *along the ray*

$$\tau_{\mu,\nu}(z) = \int_z^{z_{\max}} \alpha_\nu(z')dz'/\mu = \tau_\nu(z)/\mu, \tag{146}$$

where the last equality compares to the *fixed* optical depth scale along the axis z, as defined above. We note directly that, indeed, *along the ray* $I_\nu(0)$ always reflects $\tau_{\mu,\nu} = 1$ (cmp. also discussion about mean-free-path of photons in last section..).

Although approximate, the Eddington-Barbier relation is very useful e.g. to explain astrophysical phenomena such as limb-darkening/limb-brightening, i.e. the fact that the surface intensity of an astrophysical object (e.g. the Sun) is not constant over the observed surface.

Exercise: For the Sun, observations show that the solar disk at a given frequency in the optical waveband has lower intensity close to the limb than at the center of the disk. Using Eddington-Barbier, what does this tell you about the source function as function of height in the solar atmosphere? (Does it increase inwards, outwards, or remain constant?)

Exercise: Assuming $S_\nu = B_\nu$ (essentially "local thermodynamic equilibrium", LTE, see Chapter **??**), what does this further tell you about the temperature in the solar atmosphere? Does it increase inwards, outwards, or remain constant?

9 Appendix C: Connecting random walk of photons with radiative diffusion model

9.1 More on optical depth and mean-free-path of photons

The cross-section of a particle can be seen as the effective area of a particle; this really should be given at any wavelength or frequency, but for simplicity we will continue to use the grey (frequency-independent) assumption from above. Assume then that a beam impinges on a thin slab of area A (cm^2) in which the number density of particles is n (particles cm^{-3}). The number of particles in the slab is then $n\,A\,ds$ and the effective area blocked by all particles is $\sigma\,n\,A\,ds$. We assume this quantity is much lower then A. The fractional area is then $\sigma\,n\,ds$ which gives the fractional diminishing of the radiation. Or put in terms of photons; assume that N photons are shot randomly upon the thin slab, and are absorbed whenever they hit a particle. The fractional diminishing of the number of photons is then

$$\frac{dN}{N} = -\,\sigma\,n\,ds \tag{147}$$

Integrating over ds gives the exponential decay of the number of photons (just like when we obtained the exponential decrease of the intensity in the last section). At a distance of $s = (\sigma\,n)^{-1}$ the number of photons has been reduced to 1/e of the initial value.

A computation of the mean-free path shows that this is where the average destruction has this 1/e value so the mean free path l in a homogeneous material is

$$l = \frac{1}{n\sigma} = \frac{1}{\kappa\rho} = \frac{1}{k} \tag{148}$$

This gives the average geometrical length of a path of a photon, before it gets destroyed ("absorbed").

We can connect this to the last section by considering the "mean optical depth" traveled by a photon

$$< \tau > \equiv \frac{\int\limits_0^\infty \tau e^{-\tau} d\tau}{\int\limits_0^\infty e^{-\tau} d\tau}, \tag{149}$$

where we remind the reader that the optical depth here is computed *along the ray* of radiation (see last section, and cmp. e.g. with next section). The "mean free path" l is then the physical distance traveled in a homogeneous medium

$$l = \frac{< \tau >}{k} = \frac{1}{k}, \tag{150}$$

the last equality illustrating directly that we recover the above by $< \tau(s) >= 1$; in other words, the mean-free-path of photons corresponds to the location along the ray where the optical depth has reached unity (e.g. Fig. 3).

Exercise: Estimate the average photon mean-free-path ℓ inside the Sun. For this, you may assume a mass absorption coefficient $\kappa = 1\,\mathrm{cm}^2/\mathrm{g}$, and a solar average density ρ given simply by $M_\odot/V$, with solar mass $M_\odot$ and volume V.

How long does it then take the photon to move from the the core (where it is created by nuclear fusion) of the Sun to the surface where it can freely escape into space (and perhaps even hit us here on earth some $t = AU/c \approx 8$ minutes later)?

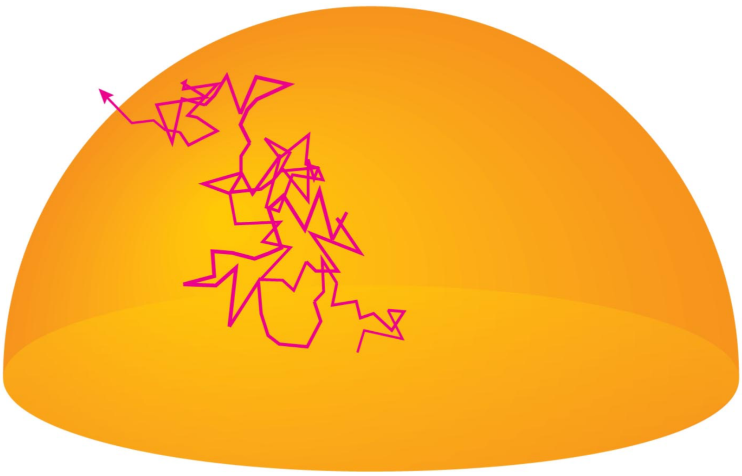

Figure 3: A random walk of photons. See text.

To get an order-of-magnitude estimate of this, let's consider the simple example where in each event the photon can only be scattered upwards a distance l or downwards a distance l Then at the N's event the total path length d traveled from the centre is either $d_{N,A} = d_{N-1} + l$ or $d_{N,B} = d_{N-1} - l$. Since the two options A and B are equally possible, squaring this we get for the *square average*

$$\langle d_N^2 \rangle = \frac{d_{N,A}^2 + d_{N,B}^2}{2} = d_{N-1}^2 + l^2. \tag{151}$$

But now we can recognize that since $\langle d_0^2 \rangle = 0$, $\langle d_1^2 \rangle = l^2$, $\langle d_2^2 \rangle = 2l^2$, and so forth, we can immediately write down $\langle d_N^2 \rangle = Nl^2$ or for the more commonly used quantity the root-mean-square-average of the distance from centre $\sqrt{\langle d_N^2 \rangle} = l\sqrt{N}$.

If we now have an object of thickness D through which the photons want to get out, this will thus take some $N = D^2/l^2$ steps, so that the associated random-walk time scale is $t_{RW} = l_{tot}/c = lN/c = (D/l)(D/c)$. Comparing this with the photon *free-flight* time scale $t_{ff} = D/c$, we see that the former is effectively enhanced with a factor D/l, reflecting the scale of the system over the mean free path (if the mean free path were equal to D, the photon would just escape as if it were free).

For an object like the Sun, this essentially means it takes a rather long time before photons produced in the core to reach the solar surface, whereas it takes rather little time for them to reach us here at earth once they have actually escaped the bound sun and are free-streaming through space.

Exercise: Estimate the random-walk time (also called the *diffusion* time) for photons going from the centre to the surface of the Sun. Compare this to the photon free-flight time for the same distance. For the mean-free-path of a solar photon, you can use 1/100 of the one you computed above for the solar mean density.

9.2 Radiative diffusion as continuum limit of random walk of photons

As already mentioned above, the radiative diffusion equation bears its name because of the analogy with the classical material thermal diffusion equation for matter (the "heat equation"). And since the classical diffusion equation can be derived as the continuum limit of a random walk of particles, one may ask if this is true also in the case of photons (the "particles" of radiation)? And indeed, this is the case.

If you recall, we already introduced the concept of the *mean-free-path of photons* $\ell = 1/k = 1/(\kappa\rho)$ in the section above, and also then calculated the *random walk time* t_{RW} it takes photons to travel from the deep core to the surface of the Sun. To make the connection between radiative diffusion and such a random walk of photons, we thus i) retain the time-dependent term, ii) assume the contribution from the sources and sinks cancel (radiative equilibrium), and iii) use the derived diffusive flux in 70 for radiation energy conservation:

$$\frac{\partial E}{\partial t} = \nabla \cdot \mathbf{F} = \nabla \cdot \frac{c}{3\rho\kappa_R} \nabla E = D\nabla^2 E \tag{152}$$

where the last equality now introduces the *diffusion coefficient* $D = \ell c/3$ for an (assumed here for simplicity, though generally not the case) spatially constant photon mean-free-path $\ell = 1/(\kappa_R \rho)$. (From you course in mathematical physics / partial differential equations, you may already now notice that this equation is identical to the *heat equation* for diffusion of matter.)

Assuming now we are interested in the number density of photons ψ at time $t+\Delta t$ and position z, that were released at time t and position $z-\Delta z$. Recall first that in Sect. 3.3., we let our random photon walkers move only vertically with $\Delta z = \pm\ell$ in each step. More generally in 3D space, the photons are free to move (here with equal probability) in all directions; this means they will now reach a distance $\Delta z = \mu\ell$ in each step. If we then (appropriate for diffusion) demand the photon density changes on a scale much slower than the mean free path, and Taylor expand around the initial position $z - \Delta z$:

$$\psi(t + \Delta t, z) \approx \psi(t, z - \Delta z) \approx \psi(t, z) - \frac{\partial \psi}{\partial z}\Delta z + \frac{\partial^2 \psi}{\partial z^2}\frac{\Delta z^2}{2}, \tag{153}$$

or

$$\frac{\psi(t + \Delta t, z) - \psi(t, z)}{\Delta t} = \frac{\partial \psi}{\partial t} = \frac{\partial \psi}{\partial z}\frac{\Delta z}{\Delta t} + \frac{\partial^2 \psi}{\partial z^2}\frac{\Delta z^2}{2\Delta t}. \tag{154}$$

Averaging this over all photon directions then gives

$$\frac{\partial \psi}{\partial t} = -\frac{\partial \psi}{\partial z}\frac{\langle \Delta z \rangle}{\Delta t} + \frac{\partial^2 \psi}{\partial z^2}\frac{\langle \Delta z^2 \rangle}{2\Delta t}. \tag{155}$$

The averages now give $\langle \Delta z \rangle = \ell \int_{-1}^{1} \mu d\mu = 0$ but $\langle \Delta z^2 \rangle = \ell^2 \int_{-1}^{1} \mu^2 d\mu = \ell^2 2/3$, and for one photon step $\Delta t = \ell/c$. Translating from photon density to energy density $E = h\nu\psi$, we thus finally obtain

$$\frac{\partial E}{\partial t} = \frac{\ell c}{3}\frac{\partial^2 E}{\partial z^2} = D\frac{\partial^2 E}{\partial z^2} \tag{156}$$

which is identical to the diffusion equation derived above directly from the radiative transfer equation.

Recall now how we previously estimated the time it took for photons to randomly walk through the Sun $t_{RW} = \ell N/c$, by computing the root-mean-square distance they reach after N steps. To make a direct comparison with the diffusion equation above, the only thing we need to change in that analysis is to now allow the photons to go in all possible directions, such that now $\langle d^2 \rangle = \ell^2 N 2/3 = R_\odot^2$ and thus $t_{RW} = t_{ff}(R_\odot/\ell)3/2$ with again photon free-flight time $t_{ff} = R_\odot/c$.

Now, since the diffusion analysis above has proceeded quite analogous to how the "standard" diffusion equation is derived as the continuum limit of a random walk of matter particles, we may now also use some further insights from standard analysis of the diffusive "heat equation". Namely, for an initial condition given by

$$E(z,0) = Const \times \delta(z-0), \tag{157}$$

corresponding to all photons (or energy) released from the "core" ($z = 0$) at $t = 0$, we know from studies of the heat equation during our course in mathematical physics / partial differential equations that the fundamental solution to the diffusion equation becomes

$$E(z,t) = \frac{Const}{\sqrt{4\pi D t}} e^{-z^2/(4Dt)}. \tag{158}$$

This is a Gaussian which spreads out with time according to a standard deviation $\sigma = \sqrt{\langle d^2 \rangle} = \sqrt{2Dt}$. Thus for the associated diffusion time

$$t_{diff} = \frac{\langle d^2 \rangle}{2D} = \frac{3R_\odot^2}{2c\ell} = \frac{3t_{ff}R_\odot}{2\ell} = t_{RW}. \tag{159}$$

Physically, this shows that radiative diffusion really can be seen as a random walk of a huge collection of photons each having a mean free path $\ell = 1/\alpha$. Recalling further that radiative diffusion is valid in very optically thick media with $\tau \gg 1$, we note that the above analysis is consistent with this, since for a constant ℓ over some distance d, $\tau = kd = d/\ell$ and the condition $\tau \gg 1$ becomes equivalent to requiring $\ell \ll d$. Finally, since $\int_{-\infty}^{\infty} E(z,t)dz = Const$, the analysis also demonstrates that the total diffusive radiative energy put into the system is preserved.

9.3 Monte Carlo radiative transfer

The above picture of randomly walking photons naturally leads the mind toward radiative transfer calculations by means of so-called Monte Carlo simulations. In such calculations, one follows the paths of a large number of photons (really packages of photons) as they travel through and interact with the medium of interest. One then builds up physical quantities by various counting procedures of the simulated photons.

Building on the previous section, a very simple Monte Carlo model (requiring only a few lines of computer code) is then to release n_γ number of photons from $x = 0$ and let them travel $\Delta x \pm \ell$ (with equal probabilities for $\pm$) for N number of steps. Note here that we, just for simplicity, have gone back to our original example with photons moving only in one spatial direction. Collecting the endpoints x of photons after N steps, our random photon-walkers should then build up the same diffusive picture as analyzed above. Fig. XX illustrates this, plotting results from such a Monte-Carlo simulation with $n_\gamma = 10^4$, $N = 10^3$, and $\ell = 1$. The histograms to the left show the simulated photonic end-points in x, illustrating that while most photons end up close to $x = 0$, a few of them indeed find their way to higher positions. Quantifying this, the computed mean value $\langle\rangle$ and standard deviation σ of the end-points of all 10^4 photons are $\langle x \rangle = 0.33$ and $\sigma = 31.3$; this agrees well with the analytic continuum values $\langle x \rangle = 0$ and $\sigma = \sqrt{N}\ell = 31.6$.

Actually, the field of Monte Carlo radiative transfer is much more general than the above example, and can be applied not only in the diffusion limit but also for full transport calculations as an alternative to solving the radiative transfer equation directly.

For example, the below is a set-up computing the *limb darkening* of the Sun by means of such a Monte Carlo simulation. NOT YET; to be included... This could also be a nice exercise for us to code up during class, giving a simple hands-on introduction to Monte-Carlo radiative transfer.

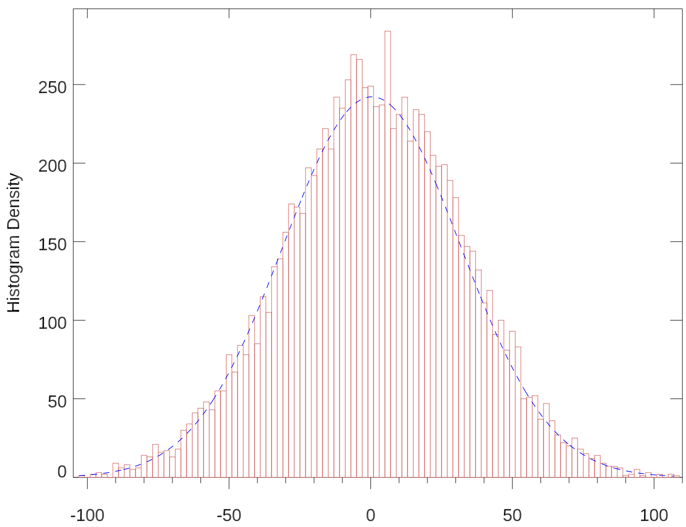

Figure 4: