Final

1.

(a) N_b, N_g - number of each type of fish
S_1, S_2 - size of sample at time 1 and 2
C_b - number of blue fish caught (at time 1)
T_1 - Number of fish tagged (at time 1)
T_2 - number of tagged fish caught (at time 2)

(b) $\text{Beta}(\frac{N_b}{N_g}; 1, 1)$

(c) $P(N_b, N_g | S_1, S_2, T) \propto \text{Bin}(C_b; S_1, \frac{N_b}{N_g})\text{Bin}(T_2; S_2, \frac{T_1}{N_b, N_g})\text{Beta}(\frac{N_b}{N_g}; 1, 1)$

2.

(a) The posterior $p(a, b|x, y) \propto \prod_i (\frac{(bx_i)^a}{\Gamma(a)} y_i^{a-1} e^{-bx_i y_i}) \frac{1}{\sqrt{2\pi}\log(2)a} e^{-\frac{1}{2(\log 2)^2}(\log a - \log 5)^2} \frac{1}{\sqrt{2\pi}\log(10)b} e^{-\frac{1}{2(\log 10)^2}(\log b - (\log 0.1))^2}$
Therefore the $\log p(a, b|x, y) = \sum_i (a \log(bx_i) - \log(\Gamma(a)) + (a-1)\log(y_i) - bx_i y_i) - \log(\sqrt{2\pi}\log(2)) - \log(a) + \frac{1}{2(\log 2)^2}(\log a - \log 5)^2 - \log(\sqrt{2\pi}\log(10)) - \log(b) + \frac{1}{2(\log 10)^2}(\log b - \log 0.1)^2$
gradient:
$\frac{d \log p(a,b|x,y)}{da} = \sum_i (\log \beta x_i - \Gamma(\alpha)\digamma(\alpha) + \log(y_i)) - \frac{1}{a} + \frac{1}{2(\log 2)^2}(\frac{2\log\alpha}{\alpha} + \frac{\log(5)}{\log\alpha})$
$\frac{d \log p(a,b|x,y)}{db} = \sum_i (\frac{a}{b} - x_i y_i) - \frac{1}{b} + \frac{1}{2(\log 0.1)^2}(\frac{2\log\beta}{\beta} + \frac{\log(.1)}{\log\beta})$

(b)

3.

(a) Likelihood function: $P(y|\theta, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(y-\theta)^2}$

Posterior: $P(\theta|y, \sigma^2) = \frac{\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(y-\theta)^2}}{\int_0^1 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(y-\theta)^2} d\theta}$

(b)

(c) The mean squared error of the MLE is $(\theta_0)^2 \int_{-\infty}^0 N(y|\theta_0, \sigma^2)dy + (1-\theta_0)^2 \int_1^\infty N(y|\theta_0, \sigma^2)dy + \int_0^1 (y - \theta_0)^2 N(y|\theta_0, \sigma^2)dy$ The mean squared error of the posterior mean is