№4

$$p(x) = a\,\{(-1,1)\setminus\{0\}\} + b\,\{0\} + b\,\{2\}$$

$$\int_{-\infty}^{+\infty} p(x) = \int_{-1}^{1} a\,dx + 2b = 1 \Rightarrow 2a + 2b = 1 \Rightarrow a = \theta \quad b = \frac{1}{2} - \theta$$

Замена $\quad p(\theta, x) = \theta\{(-1,1)\setminus\{0\}\} + (\frac{1}{2}-\theta)\{0\} + (\frac{1}{2}-\theta)\{2\}.$

т.к. $\left.\begin{matrix}\theta > 0 \\ \frac{1}{2} - \theta > 0\end{matrix}\right\} \Rightarrow \theta \in (0, \frac{1}{2})$

Метод мом-ов

$$\alpha_1 = \int_{-\infty}^{+\infty} x\,p(\theta,x)dx = \cancel{\int_{-1}^{1} \theta x\,dx}^{\,0} + \cancel{0(\tfrac{1}{2}-\theta)}^{\,0} + 2(\tfrac{1}{2}-\theta) = 1 - 2\theta$$

$$\alpha_2 = \int_{-\infty}^{+\infty} x^2 p(\theta,x)dx = \theta\int_{-1}^{1} x^2 dx + 4(\tfrac{1}{2}-\theta) + 0^2(\tfrac{1}{2}-\theta) = \tfrac{2}{3}\theta + 2 - 4\theta = 2 - \tfrac{10}{3}\theta$$

$$D[\xi] = \mu_2 = \alpha_2 - \alpha_1^2 = 2 - \tfrac{10}{3}\theta - 1 + 4\theta - 4\theta^2 = 1 + \tfrac{2}{3}\theta - 4\theta^2$$

$$\alpha_1 = \bar{\alpha}_1 = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$$

$$1 - 2\theta = \bar{x} \Rightarrow \bar{\theta}_1 = \frac{1-\bar{x}}{2}$$

$$D[\bar{\theta}_1] = D[\tfrac{1}{2} - \tfrac{\bar{x}}{2}] = D[\tfrac{\bar{x}}{2}] = \tfrac{1}{4}D[\bar{x}] = \tfrac{1}{4n}D[\xi] = \tfrac{1}{4n}(1 + \tfrac{2}{3}\theta - 4\theta^2) \xrightarrow[n\to\infty]{} 0$$

состоят.

Эффективная.

а) Регулярная модель

1. непр. дифф.

2. $\int_{-\infty}^{+\infty} \frac{\partial}{\partial\theta} p(\theta, x)dx = 0.$

$\int_{-1}^{1} 1dx + (-1) + (-1) = 2 - 2 = 0$

3. $I(\theta) = \int_{-\infty}^{+\infty} \left(\frac{\partial \ln p(\theta,x)}{\partial\theta}\right)^2 p(x,\theta)dx = \int_{-1}^{1} \left(\frac{1}{\theta}\right)^2 \theta dx + \frac{\frac{1}{2}-\theta}{(\frac{1}{2}-\theta)^2} + \frac{\frac{1}{2}-\theta}{(\frac{1}{2}-\theta)^2} =$

$= \frac{2}{\theta} + 2 - \frac{1}{1-\theta} = \frac{2}{\theta(1-2\theta)} > 0.$

$I(\theta) > 0$, непр $\Rightarrow$ мод. рег.

б) Рег. оц.

$D[\bar{\theta}_1]$ огр. на $\forall$ компакте из $(0, \frac{1}{2})$

Нер. Крамера-Рао $\quad D[\bar{\theta}_1] \geq \frac{1}{nI(\theta)} = \frac{\theta(1-2\theta)}{2n}.$

$$\frac{1 + \frac{2}{3}\theta - 4\theta^2}{4n} \geq \frac{\theta(1-2\theta)}{2n}.$$

использ. метод макс. правдоподобия

$$L(\theta) = \prod_{i=1}^{n} p(x_i, \theta)$$

$\exists$ в выборке $\vec{x}$ $m$ раз встр. знач. из $\{0, 2\}$.

$$L(\theta) = \theta^{n-m}(\tfrac{1}{2} - \theta)^m$$

$$\ln L(\theta) = (n-m)\ln\theta + m\ln(\tfrac{1}{2} - \theta)$$

$$\frac{\partial \ln L(\theta)}{\partial\theta} = \frac{n-m}{\theta} + \frac{m}{\frac12-\theta}(-1) = \frac{(n-m)(\frac12-\theta)-m\theta}{\theta(\frac12-\theta)} = 0$$

$$\frac n2 - n\theta - \frac m2 = 0 \qquad \theta = \frac{n-m}{2n} = \frac12 - \frac12 \nu \qquad \nu = \frac mn$$

$$\frac{\partial^2 \ln L(\theta)}{\partial\theta^2} = \frac{m-n}{\theta^2} + \frac{m}{(\frac12-\theta)^2}(-1)^3 = \frac{(m-n)(\frac12-\theta)^2 - m\theta^2}{\theta^2(\frac12-\theta)^2} =$$

$$= \frac{m\cdot\frac12(\frac12-\theta) - n(\frac12-\theta)^2}{\theta^2(\frac12-\theta)^2} \le \frac{\frac m2(\frac12-\theta)}{\theta^2(\frac12-\theta)^2} < 0.$$

$$\hat\theta_2 = \frac12 - \frac12\nu$$

$M[\hat\theta_2] = \frac12 - \frac12 M[\nu] = \frac12 - \frac12 p = \frac12 - (\frac12-\theta) = \theta$ несмещ.

$D[\hat\theta_2] = D[\frac12-\frac12\nu] = \frac14 D[\nu] = \frac14\left(\frac{p(1-p)}{n}\right) = \frac14\left(\frac{2(\frac12-\theta)(2\theta)}{n}\right) = \frac{\theta(\frac12-\theta)}{n} = \frac{\theta(1-2\theta)}{2n} \to 0$, $n\to\infty$ $\Rightarrow$ сост.

$D[\hat\theta_2] = \frac{\theta(1-2\theta)}{2n} \le \frac{\theta(1-2\theta)}{2n} \Rightarrow \hat\theta_2$ эфф.

$\hat\theta_1$ — не эфф., т.к. $\exists!$ эфф. оц.