№5

$\xi \sim R[\theta, 2\theta]$

$p(x) = \frac{1}{\theta}\{\theta \le x \le 2\theta\} + 0\{x \in \mathbb{R} \setminus [\theta; 2\theta]\}$

метод моментов

$$\alpha_1 = \int_\theta^{2\theta} \frac{x}{\theta}\,dx = \frac{x^2}{2\theta}\Big|_\theta^{2\theta} = 2\theta - \frac{1}{2}\theta = \frac{3}{2}\theta$$

$$\alpha_2 = \int_\theta^{2\theta} \frac{x^2}{\theta}\,dx = \frac{x^3}{3\theta}\Big|_\theta^{2\theta} = \frac{8}{3}\theta^2 - \frac{1}{3}\theta^2 = \frac{7}{3}\theta^2$$

$$D[\xi] = \frac{7}{3}\theta^2 - \frac{9}{4}\theta^2 = \frac{1}{12}\theta^2$$

$\tilde{\alpha}_1 = \bar{x}$

$\frac{3}{2}\hat{\theta}_1 = \bar{x} \quad \hat{\theta}_1 = \frac{2}{3}\bar{x}$

$\hat{\theta}_1 : M[\hat{\theta}_1] = \frac{2}{3}M[\bar{x}] = \frac{2}{3n}\sum_{i=1}^{n} M[x_i] = \theta \Rightarrow$ несмещ.

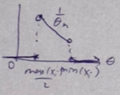
$D[\hat{\theta}_1] = D[\frac{2}{3}\bar{x}] = \frac{4n}{9n^2} \cdot \frac{1}{12}\theta^2 = \frac{\theta^2}{27n} \xrightarrow[n\to\infty]{} 0 \Rightarrow$ по т.о дост. усл. сост.

$\hat{\theta}_1$ – сост. оц.

метод макс. правдоподобия:

$$L(\theta) = \prod_{i=1}^{n} p(x_i, \theta) = \frac{1}{\theta^n}\left\{\frac{\max(x_i)}{2} < \theta,\ \min(x_i) > \theta\right\}$$

$$\left\{\frac{\max(x_i)}{2} < \theta < \min(x_i)\right\}$$




$\hat{\theta}_2 = \frac{x_{max}}{2}$

$$M[\hat{\theta}_2] = \frac{1}{2}M[x_{max}] = \frac{1}{2}\int_\theta^{2\theta} \frac{nz}{\theta}\left(\frac{z}{\theta} - 1\right)^{n-1} dz = \left\{t = \frac{z}{\theta}\right\} = \frac{\theta}{2}\int_1^2 nt(t-1)^{n-1}dt =$$

$$= \frac{\theta}{2}\left(t(t-1)^n\Big|_1^2 - \int_1^2 (t-1)^n dt\right) = \frac{\theta}{2}\left(2 - \frac{1}{n+1}\right) = \theta - \frac{\theta}{2(n+1)} = \frac{\theta(2n+1)}{2(n+1)}$$

$\hat{\theta}_2' = \frac{2n+2}{2n+1}\frac{x_{max}}{2} = \frac{n+1}{2n+1}x_{max}$ – несмещ.

$D[\hat{\theta}_2] = M[\hat{\theta}_2^2] - M^2[\hat{\theta}_2]$

$$M[\hat{\theta}_2^2] = \frac{1}{4}\int_\theta^{2\theta} \frac{nz^2}{\theta}\left(\frac{z}{\theta} - 1\right)^{n-1} dz = \left\{\frac{z}{\theta} = t\right\} = \frac{\theta^2}{4}\int_1^2 nt^2(t-1)^{n-1}dt = \left\{\begin{matrix} f = t^2\ df = 2t \\ dg = n(t-1)^{n-1}dt\ g = (t-1)^n \end{matrix}\right\} =$$

$$= \frac{\theta^2}{4}\left(t^2(t-1)^n\Big|_1^2 - 2\int_1^2 t(t-1)^n dt\right) = \frac{\theta^2}{4}\left(4 - \frac{2}{n+1}\int_1^2 (n+1)t(t-1)^n dt\right) =$$

$$= \left\{\begin{matrix} f = t\ df = dt \\ dg = (n+1)(t-1)^n dt\ g = (t-1)^{n+1} \end{matrix}\right\} = \frac{\theta^2}{4}\left(4 - \frac{2}{n+1}\left(t(t-1)^{n+1}\Big|_1^2 - \frac{(t-1)^{n+2}}{n+2}\Big|_1^2\right)\right) =$$

$$= \frac{\theta^2}{4}\left(4 - \frac{2}{n+1}\left(2 - \frac{1}{n+2}\right)\right) = \frac{\theta^2}{4}\left(4 - \frac{4}{n+1} + \frac{2}{(n+1)(n+2)}\right) = \frac{\theta^2}{4}\left(\frac{4n^2 + 12n + 8 - 4n - 8 + 2}{(n+1)(n+2)}\right) =$$

$$= \frac{\theta^2(4n^2 + 8n + 2)}{4(n+1)(n+2)}$$

$$D[\hat{\theta}_2] = \frac{\theta^2(4n^2 + 8n + 2)}{4(n+1)(n+2)} - \frac{\theta^2(4n^2 + 4n + 1)}{4(n+1)^2} = \frac{n\theta^2}{4(n+1)^2(n+2)}$$

$$D[\hat{\theta}_2'] = \frac{n\theta^2}{(n+2)(2n+1)^2} \xrightarrow[n\to\infty]{} 0 \Rightarrow \text{сост.}$$

$$D[\tilde{\theta}_1] = \frac{\theta^2}{27n} \geq \frac{n\theta^2}{(n+2)(2n+1)^2} = D[\tilde{\theta}_2'] \text{ при } n \geq 3 \quad \tilde{\theta}_2' \text{ эфф. ее } \tilde{\theta}_1.$$

Точный довер. интервал $\beta$

$\vec{X}_n$ выборка $\xi \sim R[\theta, 2\theta]$

$$f(\vec{X}_n, \theta) = \frac{X_{max}}{\theta} - 1.$$

$$P(f < t) = P(X_{max} < \theta t + \theta) = (F(\theta t + \theta))^n = \left\{ F(x) = \begin{cases} 0, & x \leq \theta \\ \frac{x}{\theta} - 1, & \theta < x \leq 2\theta \\ 1, & x > 2\theta \end{cases} \right\} =$$

$$= \begin{cases} 0, & t \leq 0 \\ t^n, & 0 < t \leq 1 \\ 1, & t > 1 \end{cases}$$

$$t_1 = q_{\frac{\alpha}{2}} = \sqrt[n]{\frac{\alpha}{2}} \quad t_2 = q_{1-\frac{\alpha}{2}} = \sqrt[n]{1 - \frac{\alpha}{2}} = \sqrt[n]{\frac{1+\beta}{2}} \quad \alpha + \beta = 1. \quad = \sqrt[n]{\frac{1-\beta}{2}}$$

$$P\left(t_1 < \frac{X_{max}}{\theta} - 1 < t_2\right) = \beta. \Rightarrow t_1 + 1 < \frac{X_{max}}{\theta} < t_2 + 1$$

$$\frac{1}{1+t_2} < \frac{\theta}{X_{max}} < \frac{1}{1+t_1}.$$

$$P\left(\frac{X_{max}}{1 + \sqrt[n]{\frac{1+\beta}{2}}} < \theta < \frac{X_{max}}{1 + \sqrt[n]{\frac{1-\beta}{2}}}\right) = \beta.$$

Асимптот. довер. интервал $\beta$

омм $\tilde{\theta}_1 = \frac{2}{3}\bar{X} = \frac{2}{3}\hat{\alpha}_1 = g(\hat{\alpha}_1).$

$g(\alpha_1) = \frac{2}{3}\alpha_1 = \theta \quad g' = \frac{2}{3}.$

$\mu_{11} = \alpha_2 - \alpha_1^2$

$\hat{\mu}_2 = \hat{\alpha}_2 - \hat{\alpha}_1^2 = S^2\frac{(n-1)}{n}.$

$$\frac{\sqrt{n}(\tilde{\theta}_1 - \theta)}{\frac{2}{3} S \sqrt{\frac{n-1}{n}}} = \frac{3n(\tilde{\theta}_1 - \theta)}{2S\sqrt{n-1}} \sim N(0,1)$$

$$P\left(t_1 < \frac{3n(\tilde{\theta}_1 - \theta)}{2S\sqrt{n-1}} < t_2\right) = \beta \qquad t_{1,2} = \mp 1.96.$$

$$\frac{2St_1\sqrt{n-1}}{3n} < \tilde{\theta}_1 - \theta < \frac{2St_2\sqrt{n-1}}{3n}$$

$$P\left(\tilde{\theta}_1 - \frac{2St_2\sqrt{n-1}}{3n} < \theta < \tilde{\theta}_1 - \frac{2St_1\sqrt{n-1}}{3n}\right) = \beta.$$