№3

$$p(x) = \begin{cases} \dfrac{e^{-\frac{x}{\theta}}}{\theta}, & x \geq 0 \\ 0, & x < 0. \end{cases} \quad \theta > 0 \quad F(x) = \int_{-\infty}^{x} p(x)\,dx = \begin{cases} 1 - e^{-\frac{x}{\theta}}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

$n = 3 \quad \theta: \tilde{\theta}_1 = \bar{x}, \ \tilde{\theta}_3 = x_{(2)}.$

$$M[\xi] = \int_0^{+\infty} x e^{-\frac{x}{\theta}} \cdot \frac{1}{\theta}\,dx = \frac{1}{\theta}\left(-\theta x e^{-\frac{x}{\theta}}\Big|_0^{+\infty} + \theta \int_0^{+\infty} e^{-\frac{x}{\theta}}\,dx\right) =$$

$$= \int_0^{+\infty} e^{-\frac{x}{\theta}}\,dx = \theta e^{-\frac{x}{\theta}}\Big|_0^{+\infty} = \theta$$

$$M[\xi^2] = \int_0^{+\infty} x^2 e^{-\frac{x}{\theta}} \cdot \frac{1}{\theta}\,dx = \frac{1}{\theta}\int_0 x^2 e^{-\frac{x}{\theta}}\,dx = \frac{1}{\theta}\left(-\theta x^2 e^{-\frac{x}{\theta}}\Big|_0^{+\infty} + \right.$$

$$\left. + 2\theta \int_0^{+\infty} x e^{-\frac{x}{\theta}}\,dx\right) = 2\theta^2$$

$$D[\xi] = M[\xi^2] - M^2[\xi] = \theta^2$$

1) $\tilde{\theta}_1$

$$M[\tilde{\theta}_1(\bar{X}_n)] = M\left[\sum_{i=1}^{3} X_i\right] \cdot \frac{1}{3} = \frac{1}{3} \cdot 3M[\xi] = \theta \text{ - несмещ.}$$

$\tilde{\theta}_3$:

$$\varphi(y) = np(y) C_{n-1}^{k-1} (1 - F(y))^{n-k} (F(x))^{k-1} \underset{\substack{k=2\\n=3}}{=} (6e^{-\frac{2x}{\theta}} - 6e^{-\frac{3x}{\theta}})\frac{1}{\theta}$$

$$M[\tilde{\theta}_3] = M[X_{(2)}] = \int_0^{+\infty} y\varphi(y)\,dy = \int_0^{+\infty} \frac{6}{\theta}(ye^{-\frac{2y}{\theta}} - ye^{-\frac{3y}{\theta}})\,dy =$$

$$= \frac{6}{\theta}\left(-\frac{\theta}{2} y e^{-\frac{2y}{\theta}}\Big|_0^{+\infty} + \frac{\theta}{3} y e^{-\frac{3y}{\theta}}\Big|_0^{+\infty} + \frac{\theta}{2}\int_0^{+\infty} e^{-\frac{2y}{\theta}}\,dy - \frac{\theta}{3}\int_0^{+\infty} e^{-\frac{3y}{\theta}}\,dy\right) =$$

$$= 6\left(\frac{1}{2}\frac{\theta}{2}(-e^{-\frac{2y}{\theta}})\Big|_0^{+\infty} - \frac{1}{3}\frac{\theta}{3}(-e^{-\frac{3y}{\theta}})\Big|_0^{+\infty}\right) = \frac{5\theta}{6} \Rightarrow \tilde{\theta}_3 \text{ - смещ.}$$

$$\tilde{\theta}_3' = \frac{6}{5}\tilde{\theta}_3 \text{ - несмещ.}$$

2) $\tilde{\theta}_1$:

$D[\tilde{\theta}_1] = D[\frac{1}{3}\sum_{i=1}^{3} X_i] = \frac{1}{9}\cdot 3D[\xi] = \frac{\theta^2}{3}$

$\tilde{\theta}_3 = M[\tilde{\theta}_3^2] = M[X_{(2)}^2] = \frac{6}{\theta}\int_0^{+\infty}(x^2 e^{-\frac{2x}{\theta}} - x^2 e^{-\frac{3x}{\theta}})dx =$

$= \frac{6}{\theta}(-\frac{\theta}{2}x^2 e^{-\frac{2x}{\theta}}\Big|_0^{+\infty} + \frac{\theta}{3}x^2 e^{-\frac{3x}{\theta}}\Big|_0^{+\infty} + 2\cdot\frac{\theta}{2}\int_0^{+\infty} x e^{-\frac{3x}{\theta}}dx -$

$- 2\cdot\frac{\theta}{3}\int_0^{+\infty} x e^{-\frac{3x}{\theta}}dx) = \frac{6}{\theta}(\int_0^{+\infty} x e^{-\frac{2x}{\theta}}dx - \frac{2}{3} - \frac{2}{3}\int_0^{+\infty} x e^{-\frac{3x}{\theta}}dx) =$

$= 6(\frac{\theta^2}{4} - \frac{2\theta^2}{27}) = \frac{6(27\theta^2 - 8\theta^2)}{2\cdot 3\cdot 2\cdot 9} = \frac{19}{18}\theta^2$

$D[\tilde{\theta}_3] = M[\tilde{\theta}_3^2] - M^2[\tilde{\theta}_3] = \frac{13}{36}\theta^2$

$D[\tilde{\theta}_3'] = \frac{36}{25}\cdot\frac{13}{36}\theta^2 = \frac{13}{25}\theta^2$

$D[\tilde{\theta}_1] < D[\tilde{\theta}_3] \Rightarrow \tilde{\theta}_1$ – более эфф. оц.

3) $I(\theta) = M[(\frac{\partial \ln p(x,\theta)}{\partial\theta})^2] = M[(\frac{\partial \ln p(x,\theta)}{\partial\theta})^2] =$

$= ~~M[\frac{\partial}{\partial\theta}(\ln\frac{e^{-\frac{x}{\theta}}}{\theta}...)]~~ = M[(\frac{\partial}{\partial\theta}(\ln\frac{e^{-\frac{x}{\theta}}}{\theta}))^2] =$

$= M[(\frac{\partial}{\partial\theta}(-\frac{x}{\theta} - \ln\theta))^2] = M[(\frac{x}{\theta^2} - \frac{1}{\theta})^2] = \frac{1}{\theta^4}M[\xi^2] - \frac{2}{\theta^3}M[\xi] +$

$+ \frac{1}{\theta^2} = \frac{2}{\theta^2} - \frac{2}{\theta^2} + \frac{1}{\theta^2} = \frac{1}{\theta^2}$

$\hat{\theta}_1$:

$D[\hat{\theta}_1] = \frac{\theta^2}{3}$ – огр. на $\forall$ компакте из $(0;+\infty)$

$(\forall [a,b] \subset (0;+\infty) \Rightarrow \forall\theta \in [a,b] \hookrightarrow \frac{\theta^2}{3} \le \frac{b^2}{3}) \Rightarrow$

$\Rightarrow \hat{\theta}_1$ – рег. по дост. усл.

$\hat{\theta}_3$:

$D[\tilde{\theta}_3'] = \frac{13}{25}\theta^2$ – огр. на $\forall$ компакте из $(0;+\infty)$

$(\forall [a,b] \subset (0;+\infty) \Rightarrow \forall\theta \in [a,b] \hookrightarrow \frac{13}{25}\theta^2 \le \frac{13}{25}b^2) \Rightarrow$

$\Rightarrow \tilde{\theta}_3'$ – рег. по дост. усл.

$p(x,\theta) = p(x,\theta)$ – непр. дифф. по θ на $(0;+\infty)$.

$\frac{\partial}{\partial\theta}p(x,\theta) = \frac{xe^{-\frac{x}{\theta}}}{\theta^3} - \frac{e^{-\frac{x}{\theta}}}{\theta^2}$

$\int_0^{+\infty}\frac{\partial}{\partial\theta}p(x,\theta)dx = \int_0^{+\infty}(\frac{xe^{-\frac{x}{\theta}}}{\theta^3} - \frac{e^{-\frac{x}{\theta}}}{\theta^2})dx = (-\frac{x}{\theta^2}e^{-\frac{x}{\theta}}\Big|_0^{+\infty} +$

$+ \int_0^{+\infty}\frac{x}{\theta^2}e^{-\frac{x}{\theta}} - \int_0^{+\infty}\frac{x}{\theta^2}e^{-\frac{x}{\theta}} = 0$.

$\frac{\partial}{\partial\theta}\int_0^{+\infty}p(x,\theta)dx = \int_0^{+\infty}\frac{\partial}{\partial\theta}p(x,\theta)dx$

$I(\theta) = \frac{1}{\theta^2} \in C[(0;+\infty)]$.

$I(\theta) > 0 \ \forall\theta \in (0;+\infty)$.

$\Rightarrow$ модель рег. $\Rightarrow$ можно использовать н-во Крамера-Рао

$\forall\theta \in (0;+\infty) \hookrightarrow D[\tilde{\theta}_1] \ge \frac{1}{nI(\theta)}$

$\frac{\theta^2}{3} \ge \frac{\theta^2}{3}$, $D[\tilde{\theta}_1] = \frac{1}{nI(\theta)} \Rightarrow$ по дост. усл. $\tilde{\theta}_1$ – эфф.

$\tilde{\theta}_3' \ne \tilde{\theta}_1 \Rightarrow \tilde{\theta}_3$ – не эфф. т.к. может быть только одна эф. оц.