№1

$\xi \sim R[0,\theta]$ выборка объёма n оц. парам-а θ.

$\tilde{\theta}_1 = 2\bar{x}$ $\quad \tilde{\theta}_2 = x_{min}$ $\quad \tilde{\theta}_3 = x_{max}$ $\quad \tilde{\theta}_4 = x_1 + \frac{\sum_{k=2}^{n} x_k}{n-1}$

$p(x) = \frac{1}{\theta}\ \{[0,\theta]\}$ $\quad M[\xi] = \frac{\theta}{2}$ $\quad D[\xi] = \frac{\theta^2}{12}$

а) $\tilde{\theta}_1 = 2\bar{x}$ $\quad M[\tilde{\theta}_1] = \theta \Rightarrow$ несмещ.

$D[\tilde{\theta}_1] = D\left[\frac{2}{n}\sum_{i=1}^{n} x_i\right] = \frac{4}{n^2}\sum_{i=1}^{n} D[x_i] = \frac{\theta^2}{3n} \xrightarrow[n\to\infty]{} 0 \Rightarrow$ состоят.

$\tilde{\theta}_2 = x_{min}$ $\quad \Phi(y) = 1-(1-F(y))^n$

$\varphi(y) = n(1-F(y))^{n-1}\cdot p(y) = n\left(1-\frac{y}{\theta}\right)^{n-1}\cdot\frac{1}{\theta}\ \{[0,\theta]\}$

$M[\tilde{\theta}_2] = \int_0^\theta n\left(1-\frac{y}{\theta}\right)^{n-1}\frac{y}{\theta}\,dy = \left\{ \begin{array}{l} t = 1-\frac{y}{\theta} \\ dt = -\frac{dy}{\theta} \end{array} \right\} =$

$= -\int_1^0 n t^{n-1}(1-t)\,\theta\,dt = \int_0^1 n t^{n-1}\theta\,dt - \int_0^1 n t^n \theta\,dt =$

$= \theta\left(1-\frac{n}{n+1}\right) = \frac{\theta}{n+1} \Rightarrow$ смещ.

$\tilde{\theta}_2' = (n+1)x_{min}$

$D[\tilde{\theta}_2'] = (n+1)^2 D[\tilde{\theta}_2]$

$M[\tilde{\theta}_2^2] = \int_0^\theta n\left(1-\frac{y}{\theta}\right)^{n-1}\frac{y^2}{\theta}\,dy = \left\{ \begin{array}{l} t = 1-\frac{y}{\theta} \\ dt = -\frac{dy}{\theta} \end{array} \right\} =$

$= -\int_1^0 n t^{n-1}(1-t)^2\theta^2\,dt = \int_0^1 n\theta^2 t^{n+1}\,dt - 2\int_0^1 n\theta^2 t^n\,dt +$

$+ \int_0^1 n\theta^2 t^{n-1}\,dt = \theta^2\left[\frac{n}{n+2} - 2\frac{n}{n+1} + 1\right] =$

$= \theta^2\left[\frac{n(n+1) - 2n(n+2) + (n+1)(n+2)}{(n+1)(n+2)}\right] = \theta^2\left[\frac{2}{(n+1)(n+2)}\right] = \frac{2\theta^2}{(n+1)(n+2)}$

$D[\tilde{\theta}_2] = \frac{2\theta^2}{(n+1)(n+2)} - \frac{\theta^2}{(n+1)^2} = \frac{\theta^2(2n+2-n-2)}{(n+1)^2(n+2)} = \frac{n\theta^2}{(n+1)^2(n+2)} \xrightarrow[n\to\infty]{} 0$

$D[\tilde{\theta}_2'] = \frac{n\theta^2}{n+2} \underset{n\to\infty}{\not\to} 0$

$\tilde{\theta}_2'$ по опр.

$\forall\theta > 0\ \forall\varepsilon > 0\ \ P(|\tilde{\theta}_2' - \theta| \geq \varepsilon) \xrightarrow[n\to\infty]{} 0$

$P(|\tilde{\theta}_2' - \theta| \geq \varepsilon) \geq P(\tilde{\theta}_2' \geq \theta+\varepsilon) = P((n+1)x_{min} \geq \theta+\varepsilon) =$

$= P\left(x_{min} \geq \frac{\theta+\varepsilon}{n+1}\right) = 1 - P\left(x_{min} < \frac{\theta+\varepsilon}{n+1}\right) =$

$= 1 - \left(1 - \left(1 - F\left(\frac{\theta+\varepsilon}{n+1}\right)\right)^n\right) = \left(1 - \frac{\theta+\varepsilon}{\theta(n+1)}\right)^n \to e^{-\frac{\theta+\varepsilon}{\theta}} > 0 \Rightarrow$ не сост.

$\tilde{\theta}_2$ по опр.

$P(|\tilde{\theta}_2 - \theta| \geq \varepsilon) = P(\tilde{\theta}_2 \leq \theta - \varepsilon) + P(\tilde{\theta}_2 \geq \theta+\varepsilon) =$

$= P(x_{min} < \theta - \varepsilon) = \Phi(\theta-\varepsilon) = 1 - \left(1 - \frac{\theta-\varepsilon}{\theta}\right)^n =$

$= 1 - \left(\frac{\varepsilon}{\theta}\right)^n \xrightarrow[n\to\infty]{} 1 \Rightarrow$ не сост.

$\tilde{\theta}_3 = x_{max}$ $\quad M[\tilde{\theta}_3] = \int_0^\theta n\left(\frac{z}{\theta}\right)^n dz = \frac{n}{n+1}\theta \Rightarrow$ смещ.

$\tilde{\theta}_3' = \frac{n+1}{n}x_{max}$ $\quad D[\tilde{\theta}_3'] = \left(\frac{n+1}{n}\right)^2 D[\tilde{\theta}_3]$.

$D[\tilde{\theta}_3] = \frac{n}{n+2}\theta^2 - \left(\frac{n}{n+1}\right)^2\theta^2 = \theta^2\left[\frac{n(n+1)^2 - n^2(n+2)}{(n+1)^2(n+2)}\right] = \theta^2\left[\frac{n}{(n+1)^2(n+2)}\right] \xrightarrow[n\to\infty]{} 0$

$D[\tilde{\theta}_3'] = \frac{\theta^2}{n(n+2)} \xrightarrow[n\to\infty]{} 0 \Rightarrow$ сост.

$\tilde{\theta}_3$ по опр. $\forall \theta > 0 \quad \forall \varepsilon > 0$

$P(|\tilde{\theta}_3 - \theta| \geq \varepsilon) = P(X_{max} \leq \theta - \varepsilon) + P(X_{max} \geq \theta + \varepsilon) =$

$= F^n(\theta - \varepsilon) = \left(\frac{\theta - \varepsilon}{\theta}\right)^n \xrightarrow[n \to \infty]{} 0 \; ; \; \varepsilon \geq \theta \to (0)^n \xrightarrow[n \to \infty]{} 0$ — сост.

$\tilde{\theta}_3'$ по опр.

$\forall \theta > 0 \; \forall \varepsilon > 0 \hookrightarrow P(|\tilde{\theta}_3' - \theta| \geq \varepsilon) \xrightarrow[n \to \infty]{} 0.$

$P(|\tilde{\theta}_3' - \theta| \geq \varepsilon) = P(\tilde{\theta}_3' \leq \theta - \varepsilon) + P(\tilde{\theta}_3' \geq \theta + \varepsilon)$

P_1 $\qquad\qquad$ P_2

$P_1: P\left(X_{max} \leq \frac{\theta - \varepsilon}{1 + \frac{1}{n}}\right) = \left(F\left(\frac{\theta - \varepsilon}{1 + \frac{1}{n}}\right)\right)^n$

$\left.\begin{array}{l} \varepsilon \geq \theta : 0^n \to 0, n \to \infty \\ 0 < \varepsilon < \theta : \left(\frac{\theta - \varepsilon}{\theta(1 + \frac{1}{n})}\right)^n = \left(\frac{1 - \frac{\varepsilon}{\theta}}{1 + \frac{1}{n}}\right)^n \xrightarrow[n \to \infty]{} 0 \end{array}\right\} P_1 \xrightarrow[n \to \infty]{} 0$

$P_2: P\left(X_{max} \geq \frac{\theta + \varepsilon}{1 + \frac{1}{n}}\right) = 1 - \left(F\left(\frac{\theta + \varepsilon}{1 + \frac{1}{n}}\right)\right)^n = 1 - \left(\frac{1 + \frac{\varepsilon}{\theta}}{1 + \frac{1}{n}}\right)^n$

$\frac{\theta + \varepsilon}{1 + \frac{1}{n}} > \theta : (1 - 1^n) \xrightarrow[n \to \infty]{} 0. \Rightarrow P_2 \xrightarrow[n \to \infty]{} 0$

$\exists n_0 : \forall n \geq n_0 \hookrightarrow \frac{n(\theta + \varepsilon)}{n + 1} > \theta$

$\Rightarrow P_1 + P_2 \xrightarrow[n \to \infty]{} 0 \Rightarrow$ сост.

$\hat{\theta}_4 = x_1 + \frac{1}{n-1} \sum_{k=2}^{n} x_k$

$M[\hat{\theta}_4] = M[x_1 + \frac{1}{n-1} \sum_{k=2}^{n} x_k] = M[x_1] + \frac{1}{n-1} \sum_{k=2}^{n} M[x_k] = \frac{\theta}{2} + \frac{\theta}{2} = \theta \Rightarrow$ несмещ.

$D[\hat{\theta}_4] = D[x_1 + \frac{1}{n-1} \sum_{k=2}^{n} x_k] = D[\xi] + \frac{1}{(n-1)^2} (n-1) D[\xi] = \frac{\theta^2}{12} + \frac{\theta^2}{12(n-1)} =$

$= \frac{n\theta^2}{12(n-1)} \xrightarrow[n \to \infty]{} 0 \Rightarrow$ дост. усл. не раб.

$\xi_1 \ldots \xi_n$ — независ. и од. распр. $\Rightarrow$ использ. ЗБЧ Хинчина

$\frac{1}{n} \sum_{k=1}^{n} \xi_k \xrightarrow{P} M[\xi] \Rightarrow \frac{1}{n-1} \sum_{k=2}^{n} x_k \xrightarrow{P} M[\xi] = \frac{\theta}{2} \Rightarrow \hat{\theta}_4 = x_1 + \frac{1}{n-1} \sum_{k=2}^{n} x_k \xrightarrow{P}$

$\xrightarrow{P} x_1 + \frac{\theta}{2} \Rightarrow$ несост.

в) $\Rightarrow$ сост.: $\hat{\theta}_1 = 2\bar{x}$ и $\tilde{\theta}_3' = \frac{n+1}{n} x_{max}$.

$D[\hat{\theta}_1] = \frac{\theta^2}{3n} \qquad D[\tilde{\theta}_3'] = \frac{\theta^2}{n(n+2)}$.

$\frac{\theta^2}{n(n+2)} < \frac{\theta^2}{3n} \quad \forall \theta > 0. \quad \forall n > 1 \Rightarrow$

$\Rightarrow \tilde{\theta}_3'$ эфф-ее чем $\hat{\theta}_1$.