Ex. 2 MLE α ; $f(x) = \frac{\alpha x_m^{\alpha}}{x^{\alpha+1}}$; $n=1500$

(S1) likelihood function

$$L(\alpha, x) = f(x_1) f(x_2) \dots f(x_n) = \left(\frac{\alpha x_m^{\alpha}}{x_1^{\alpha+1}}\right)\left(\frac{\alpha x_m^{\alpha}}{x_2^{\alpha+1}}\right)\dots$$

$$\dots \left(\frac{\alpha x_m^{\alpha}}{x_n^{\alpha+1}}\right) = \frac{\alpha^n x_m^{\alpha n}}{(x_1 x_2 \dots x_n)^{\alpha+1}}$$

(S2) log-likelihood function

$$l(\alpha) = \ln(L[\alpha]) = n\ln(\alpha) + n\alpha\ln(x_m) -$$

$$-(\alpha+1)\sum \ln(x_i)$$

(S3) The MLE is the solution of:

$$\frac{dl}{d\alpha}(\tilde{\alpha}) = 0 \iff \frac{n}{\tilde{\alpha}} + n\ln(x_m) - \sum \ln(x_i) = 0$$

$$\frac{n}{\tilde{\alpha}} = \sum \ln(x_i) - n\ln(x_m)$$

$$n = \tilde{\alpha}\left(\sum \ln(x_i) - n\ln(x_m)\right)$$

$$\tilde{\alpha} = \frac{n}{\sum \ln(x_i) - n\ln(x_m)}$$

Ex 3.

$$I(\alpha) = -E\left[\frac{d^2 \ln f(x)}{d\alpha^2}\right]$$ second derivative

p.d.f of Pareto distribution; $f(x) = \frac{\alpha x_m^{\alpha}}{x^{\alpha+1}}$

$$I(\alpha) = -E\left[-\frac{n}{\alpha^2}\right] = \frac{n}{\alpha^2}$$

Distribution of $\hat{\alpha}$

$$\hat{\alpha} \sim N\left(\alpha, \frac{1}{I_n(\alpha)}\right) \Leftrightarrow \hat{\alpha} \dot{\sim} N\left(\alpha, \frac{\alpha^2}{n}\right)$$

This means that the MLE $\hat{\alpha}$ of α is asymptotically normally distributed with mean(α) and variance $\left(\frac{\alpha^2}{n}\right)$

Ex 4.

$$E(X) = \frac{\alpha x_m}{\alpha - 1}$$

The MME is the solution of $E(x) = \bar{x}$

$$\frac{\alpha x_m}{\alpha - 1} = \frac{\bar{x}}{1}$$

$$\alpha x_m = \bar{x}\alpha - \bar{x}$$

$$\bar{x} = \alpha(\bar{x} - x_m) \Leftrightarrow \alpha = \frac{\bar{x}}{\bar{x} - x_m}$$

Ex.5

from Ex2. $\hat{\alpha} = \dfrac{n}{\sum \ln(x_i) - n\ln(x_m)}$

A 90% CI is given by:

$$\left(\hat{\alpha} \pm z_{5\%} \sqrt{\frac{1}{I(\hat{\alpha})}}\right)$$

$$I(\alpha) = \frac{n}{\alpha^2}$$

$$\hat{\alpha} \sim N\left(\alpha, \frac{1}{I(\alpha)}\right) \Leftrightarrow \hat{\alpha} \sim N\left(\alpha, \frac{\hat{\alpha}^2}{n}\right) \rightarrow ese(\hat{\alpha}) = \sqrt{\frac{\hat{\alpha}^2}{n}} =$$

$$= \frac{\hat{\alpha}}{\sqrt{n}}$$

A 90% CI for α is given by:

$$\left(\hat{\alpha} \pm 1.645 \cdot \frac{\hat{\alpha}}{\sqrt{n}}\right)$$