Task 2 - Tasks corresponding to Chapters 6, 7

Gas Dynamics (Technische Universiteit Delft)

Delft University of Technology

Gasdynamics

AE4140

Assignment: Task 2 - Method of Characteristics

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November 7, 2023

Contents

1 Introduction

The task proposes an underexpanded jet. In order to analyse the development of the jet flow downstream, the Method Of Characteristics (M.O.C) will be applied. By doing this, the flow properties will be correctly solved until the development of a shock wave. A MATLAB code has been developed to reach the task's primary objective.

2 Assumptions and known variables

To simplify the calculation process, some assumptions are made:

- The flow is 2D
- The flow is steady
- The flow is homentropic, so there is no entropy variation between regions

In addition to these assumptions, the flow properties at the nozzle exit are given in the statement:

- The Mach number at the exit is 2 ($M_e = 2$)
- The flow is horizontal, so the angle that sets the flow direction is 0 ($\varphi_e = 0$)
- The pressure relation between the nozzle exit and the ambient is: $P_e = 2 \cdot P_{Ambient}$

3 Method of Characteristics (M.O.C)

In order to define each characteristic line, we need to define first the flow direction angle (φ) and the Prandtl-Mayer angle (υ). Given the previous assumptions, we know that the homentropic relations for the Riemann invariants (V^- or V^+) are:

- $V^- = \upsilon + \varphi$ are constant along the Γ^- with a slope of $\frac{\partial y}{\partial x} = tan(\varphi - \mu)$
- $V^+ = \upsilon - \varphi$ are constant along the Γ^+ with a slope of $\frac{\partial y}{\partial x} = tan(\varphi + \mu)$

Also, thanks to the Prandlt-Meyer function, υ can be obtained from the Mach number by using the following equation:

$$\upsilon = \int_1^M \sqrt{\frac{M^2-1}{q}} dq = \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1} \sqrt{\frac{\gamma+1}{\gamma-1}(M^2-1)} - \tan^{-1} \sqrt{M^2-1} \tag{1}$$

To obtain μ we can also use M:

$$sin(\mu) = \frac{1}{M} \tag{2}$$

$$\mu = \arcsin\left(\frac{1}{M}\right) \tag{3}$$

Once υ and φ are defined, the other flow properties can be calculated as they can be derived from υ and φ.

4 Methodology

After the brief theory introduction where the fundamental relations are set, the approach followed in this task will be explained.

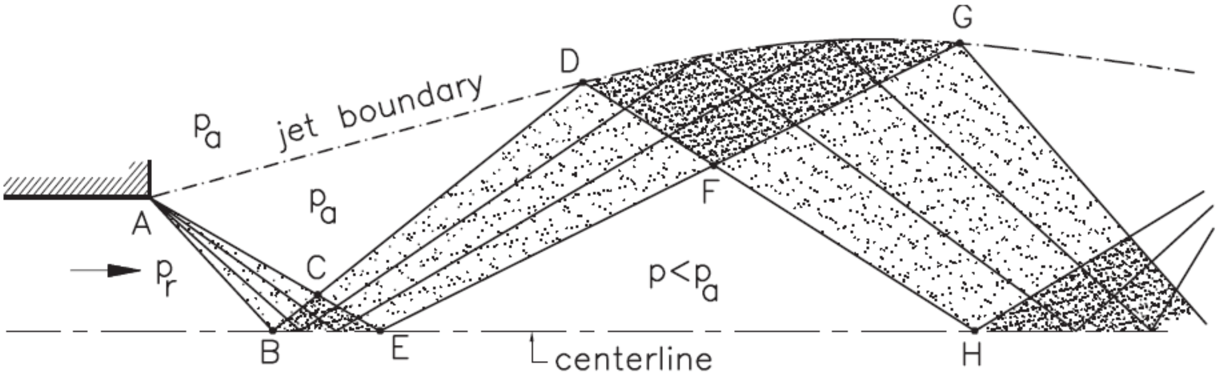

Figure 1: Unexpanded Jet

As the known variables set the flow properties are the nozzle exit, we can calculate the flow properties in region ACD, that have the same value as the ambient flow properties. We know the pressure ratio between the nozzle exit and region ACD. Then, by using the isentropic flow equations, $P_{Ambient}$ can be obtained from:

$$\frac{P_T}{P} = (1 + \frac{\gamma - 1}{2}M^2)^{(\frac{\gamma}{\gamma-1})} \tag{4}$$

where P is the pressure at one region an P_T is constant. Thus, as we know the properties at the nozzle exit we can obtain P_T as a function of P_e:

$$P_T = P_e(1 + \frac{\gamma - 1}{2}M_e^2)^{(\frac{\gamma}{\gamma-1})} \tag{5}$$

Here, we can calculate $M_{Ambient}$ from substituting equation 5 in equation 4 and considering P as $P_{Ambient}$ and M as $M_{Ambient}$. Once $M_{Ambient}$ is obtained, $\upsilon_{Ambient}$ can be calculated by using equation 1.

At this point, $M_{Ambient}$ and $\upsilon_{Ambient}$ are calculated and the only flow property left is $\varphi_{Ambient}$. To calculate it, the Riemann's invariant V^+ so $\upsilon_e - \varphi_e = \upsilon_{Ambient} - \varphi_{Ambient}$. Then, all flow properties in region ACD are defined.

Before calculating any simple wave region or non-simple wave region, I obtained the flow properties at regions EDH and GIJ. For the EDH region, as the flow is horizontal, the $\varphi_{EDH} = 0$ as the flow close to the centre line must fulfil the previous requirement. Then, taking a Riemann's invariant V^- going from ACD to EFH, we get the following expression:

$$\upsilon_{Ambient} + \varphi_{Ambient} = \upsilon_{EFH} \tag{6}$$

Now, to obtain all the flow properties at region EFH we need to calculate the Mach number at this region. To do so, equation 1 is used but instead of solving it for υ, it is solved for the Mach number (M_{EDH}) and then all the flow properties for this region are defined. Finally, for the region GIJ the process is similar to the one followed in region ACD.

Now, once the flow properties for all uniform regions are defined, I calculate the flow properties for all simple wave regions (ABC, CDEF and FGHI). For regions ABC and FGHI the process to follow is almost the same, so I will explain the one I followed in region ABC.

For a simple wave region, φ for the first and the last characteristic line is known. In addition to this, $\Delta\varphi$ is the same for each characteristic line. Then, $\Delta\varphi$ is obtained from the following expression:

$$\Delta\varphi = \frac{\varphi_{post} - \varphi_{pre}}{N-1} \tag{7}$$

where φ_{post} in this case is $\varphi_{Ambient}$, φ_{pre} is φ_e and N is the number of characteristics.

Once how φ varies along a simple wave region is calculated, the value of φ for each characteristic line can be obtained from:

$$\varphi_i = \varphi_{pre} + \Delta\varphi \tag{8}$$

where again, φ_{pre} is φ_e and i defines which characteristic line is being calculated. Given the value of φ of each characteristic line, the corresponding υ can be obtained as from Rieman's invariant V^+, $\upsilon - \varphi$ is constant. Thus,

$$\upsilon_i = \upsilon_{pre} - \varphi_{pre} + \varphi_i \tag{9}$$

Now, by using equation 1, the Mach number for each characteristic line can be obtained and all flow properties are calculated for region ABC. The process for the region FGHI is the same, but φ_{post} is φ_{GIJ} and φ_{pre} is φ_{EFH}.

At this point, there is only one simple wave region left, region CDEF. As it happens in the other two simple wave regions, the value of φ for the first and the last characteristic line is known, so equation 7 can be used here to obtain $\Delta\varphi_{CDEF}$. After computing $\Delta\varphi_{CDEF}$, φ_i corresponding to region CDEF can be obtained from equation 8, where φ_{pre} is $\varphi_{amb}(\varphi_{ACD})$.

For regions ABC and FGHI, the Rieman's invariant V^+ is used to calculate υ, however, in this case the Rieman's invariant V^- is used, so $\upsilon + \varphi$ is constant. Then, equation 9 would be

$$\upsilon_i = \upsilon_{pre} + \varphi_{pre} - \varphi_i \tag{10}$$

where υ_{pre} is $\upsilon_{Ambient}$, φ_{pre} is $\varphi_{Ambient}(\varphi_{ACD})$. Knowing the values of φ and υ, the Mach number can be computed from equation 1 so all the flow properties for region CDEF are calculated.

Flow properties for almost all the regions are calculated. Nevertheless, the properties for regions BCE and DFG are still unknown. These regions are more complicated than the previous ones calculated given that regions BCE and DFG are non-simple wave regions. In non-simple wave regions Γ^- and Γ^+ intersect each other, so the way that flow properties can be obtained vary from the simple wave regions.

As the first non-simple wave region to encounter is region BCE, I will explain first how to calculate the flow properties for this region. In the case of this region, we can obtain the flow properties along the central line that joins point B and point E as there, $\varphi_{BE} = 0$ so υ can be calculated from the Rieman's invariant V^- where

$$\upsilon_{BE} = \upsilon_{ABC} + \varphi_{ABC} \tag{11}$$

The flow properties at the boundary that joins B and C are also known, as their value is the same as in region ABC. The rest of the flow properties are obtained from the following expressions

$$\varphi_{BCE} = \frac{1}{2}(\varphi_{BCE}(i-1,j) + \varphi_{BCE}(i,j-1)) + \frac{1}{2}(\upsilon_{BCE}(i-1,j) - \upsilon_{BCE}(i,j-1)) \tag{12}$$

$$\upsilon_{BCE} = \frac{1}{2}(\upsilon_{BCE}(i-1,j) + \upsilon_{BCE}(i,j-1)) + \frac{1}{2}(\varphi_{BCE}(i-1,j) - \varphi_{BCE}(i,j-1)) \tag{13}$$

From equations 12 and 13, a diagonal matrix for φ and υ is obtained following this structure

$$BCE = \begin{bmatrix} BCE_{11} & BCE_{12} & \cdots & BCE_{1n} \\ 0 & BCE_{22} & \cdots & BCE_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & BCE_{nn} \end{bmatrix} \tag{14}$$

The Mach number for the BCE region can be computed by taking υ_{BCE} values and substituting them in equation 1.

In region DFG, the process to follow is similar, however, the points that are in the jet boundary have a $\varphi \neq 0$. In this case, the Mach number is constant along the jet boundary, which means that the Prandlt-Mayer angle is constant. Then, the value of φ along the jet boundary is obtained from

$$(V^+) : \varphi_{jetboundary} = \upsilon_{Ambient} - \upsilon_{CDEF} + \varphi_{CDEF} \tag{15}$$

Flow properties at the boundary that joins points D and F are also known as they are the same as in region CDEF. Therefore, the rest of the flow properties can be obtained from

$$\varphi_{DFG} = \frac{1}{2}(\varphi_{DFG}(i-1,j) + \varphi_{DFG}(i,j-1)) + \frac{1}{2}(\upsilon_{DFG}(i-1,j) - \upsilon_{DFG}(i,j-1)) \tag{16}$$

$$\upsilon_{DFG} = \frac{1}{2}(\upsilon_{DFG}(i-1,j) + \upsilon_{DFG}(i,j-1)) + \frac{1}{2}(\varphi_{DFG}(i-1,j) - \varphi_{DFG}(i,j-1)) \tag{17}$$

Resulting in a matrix similar to the BCE:

$$DFG = \begin{bmatrix} DFG_{11} & 0 & \cdots & 0 \\ DFG_{21} & DFG_{22} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ DFG_{n1} & DFG_{n2} & \cdots & DFG_{nn} \end{bmatrix} \tag{18}$$

Again, the Mach number for the DFG region can be computed by taking υ_{DFG} values and substituting them in equation 1. Now, all the flow properties are fully defined.

At this point, all the flow properties (Mach number, υ, μ and φ) have been defined along the unexpanded jet. Now, to complete the task, the geometry needs to be defined. To do so, the region limits and characteristic lines directions will be defined. To carry out this, M.O.C theory will be considered in order to define the slope of both positive and negative characteristic lines:

Γ^- with slope of $\frac{\partial y}{\partial x} = \tan(\varphi - \mu)$

Γ^+ with slope of $\frac{\partial y}{\partial x} = \tan(\varphi + \mu)$

Knowing the value of φ and μ for each region, the slope of the characteristic line (positive and negative one) is then defined at each node. Furthermore, by utilizing these slopes, the points of intersection between characteristic lines can be determined through the system of equations:

$$y_A - y_P = m_{\Gamma^-} \cdot (x_A - x_P) y_B - y_p = m_{\Gamma^+} \cdot (x_B - x_P) \tag{19}$$

The internal flow geometry within the jet is subsequently defined, except for determining the precise shape of the overall jet boundary. This is addressed by applying a similar method, which involves identifying intersection points between characteristic lines and flow direction paths (streamlines), as opposed to calculating intersections between both positive and negative characteristic lines:

$$y_A - y_P = m_{\varphi} \cdot (x_A - x_P) y_B - y_p = m_{\Gamma^+} \cdot (x_B - x_P) \tag{20}$$

5 Results

5.1 Mach along the unexpanded jet

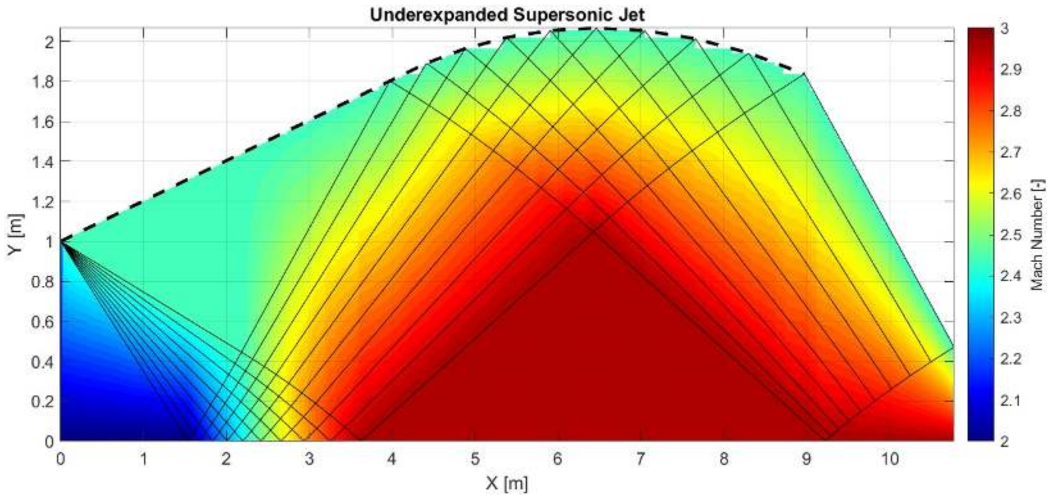

Figure 2: Mach along Unexpanded Jet

Figure 2 shows the Mach distribution along the unexpanded jet. Analysing the Mach number distribution, it can be inferred if the region is uniform, simple-wave or non-simple wave regions. For those in which the Mach number is constant (uniform), it can be stated that the Riemann invariants (V^- and V^+) are constant as well and thus, the region is uniform. This happens in regions ACD, EFH and GIK.

For those regions that are affected by an expansion or compression wave, one Riemann invariant change (V^- in the case of characteristic lines with a positive slope, V^+ in the opposite case) so the region is a simple-wave region. Then, the Mach number along the region varies as can be seen in regions ABC, CDEF and FGHI. Finally, those regions affected by two waves have variable Riemann invariants and, as expected, a non-uniform Mach distribution leading into non-simple wave regions. This last can be seen in regions BCE, DFG and HIJ.

Regarding the nature of the waves, two different types can be distinguished: compression and expansion waves. In the case of compression waves, the pressure along it increases. Therefore, the Mach needs to decrease. On the other hand, for expansion waves, the pressure decreases which means that the Mach number increases. Following this reasoning, it can be stated that the first characteristic and those reflected in the symmetry boundary are expansion waves whereas those reflected in the jet boundary line are compression waves.

5.2 Pressure along the unexpanded jet

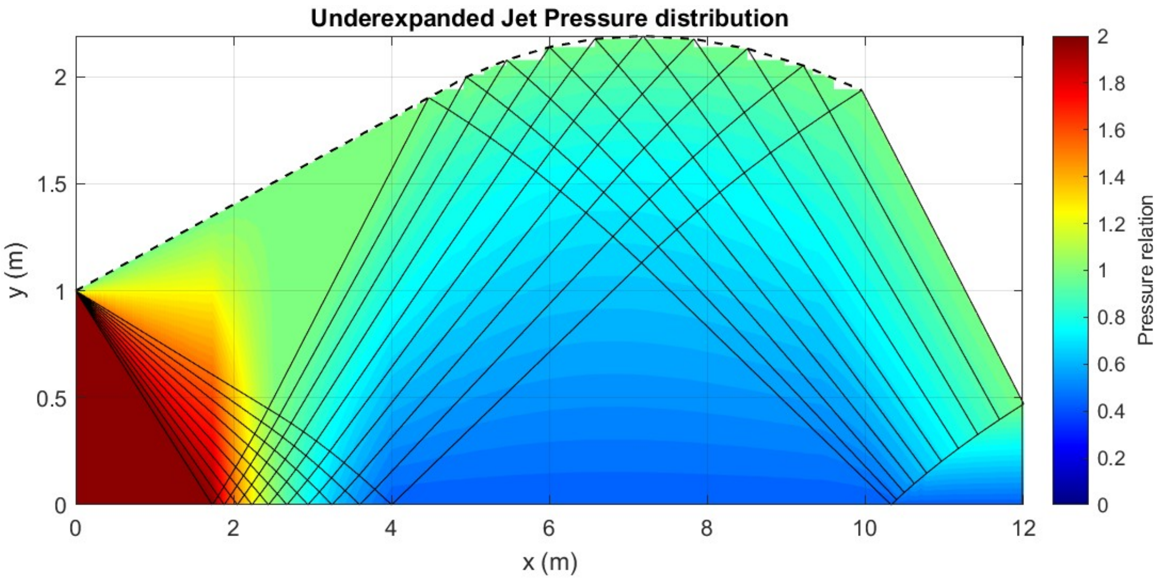

Figure 3: Mach along Unexpanded Jet

Figure 3 shows the pressure distribution along the unexpanded jet. By studying the Mach number as in section 5.1, the nature of each wave can be stated. However, by studying how the pressure varies along through each wave it is more obvious the nature of the wave.

Therefore, by analysing the pressure through region ABC it can be seen how it is decreased. Then, the wave starting at point A must be an expansion wave. Moving to the next wave, going from the centreline to the jet boundary, the pressure is also decreased when a particle travels from ACD region to EFH region. Thus, again, the wave must be an expansion one. Finally, the last wave going from the jet boundary to the centreline is different. In this case, the pressure increases as you go from region EFH to region GIJ. Thus, the wave must be a compression wave.

Also, the behaviour of the different regions matches the behaviour stated in section 5.1. As happens with the Mach number, for regions before the first wave, region ACD, EFH and GIK the pressure is constant which means that those regions have a uniform behaviour. For regions ABC, CDEF and FGHI the pressure varies from one characteristic line to another one, which represents a simple wave behaviour. To finalise, the pressure along regions BCE and DFG shows a behaviour typical of non-simple waves.

6 Source code

In this section, I will display the source code without the plotting routines.

6.1 Main script

In this main script, the parameters given in the main statement have been define. Once the initial parameters are defined, first I calculate the flow properties at each uniform region by calling different functions that will be defined after. Similarly, I do the same process for the single-wave regions and the non-uniform regions.

```
clc
clear
close all
Known variables of the problem
From the main statement, the following variables are known.
M_e=2;
P_a=1;
P_e=2*P_a;
Phi_e=0;
gamma=7/5;

Characteristics variables
mu=asin(1/M_e);
ve=sqrt((gamma+1)/(gamma-1))*atan(sqrt(((gamma-1)/(gamma+1))*((M_e^2)-1)))
-atan(sqrt((M_e^2)-1));

Ambient Characteristics (ACD Region)
P_ra=2*(1+((gamma-1)/2)*M_e^2)^(gamma/(gamma-1));
M_amb=sqrt((P_ra^((gamma-1)/gamma)-1)*(2/(gamma-1)));
v_amb=vfromM(M_amb,gamma);
Phi_amb=v_amb-ve;

Characteristics at EFH Region
v_EFH=v_amb+Phi_amb;
Phi_EFH=0;
M_EFH = Mfromv(v_EFH,gamma);
P_EFH=PfromM(M_EFH,M_e,P_e,gamma);

Region GIJ
Phi_GIJ=Phi_amb;
v_GIJ=v_amb;
M_GIJ=Mfromv(v_GIJ,gamma);
P_GIJ=PfromM(M_GIJ,M_e,P_e,gamma);
Ask for Number of Characteristics
prompt="Number of Characteristics "
NoC=input(prompt);

Simple Wave (Region ABC)
As in region ABC, we need a V+ to calculate the M and v along
the expansion wave.
[Phi_ABC, v_ABC, M_ABC, mu_ABC, P_ABC] =
SimpleWaveRegion(NoC,Phi_amb, Phi_e, ve, gamma, M_e, P_e);
```

```
Simple Region Region CEFG
[Phi_CEFD, v_CEFD, M_CEFD, mu_CEFD] =
SimpleWaveRegionMinus(NoC, Phi_EFH, Phi_amb, v_amb, gamma, M_e, P_e);

Simple Region Region FGHI
[Phi_FGHI, v_FGHI, M_FGHI, mu_FGHI, P_FGHI] =
SimpleWaveRegion(NoC,Phi_EFH, Phi_GIJ, v_EFH, gamma, M_e, P_e);

Non-Simple waves
Region BCE
Close to the centre line, the expansion wave from the bottom side encounters
the one from point A. Then, there is an area where a non-simple
wave is formed, region BCE.

[Phi_BCE, v_BCE, M_BCE, mu_BCE, P_BCE] =
NonSimpleWaveBCE(NoC,Phi_ABC,v_ABC,gamma, M_e, P_e);

Region DFG
[Phi_DFG, v_DFG, M_DFG, mu_DFG, P_DFG] =
NonSimpleWave_DFG(NoC,M_amb,Phi_CEFD,v_CEFD,gamma, M_e, P_e);

Region HIK
[Phi_HIK, v_HIK, M_HIK, mu_HIK, P_HIK] =
NonSimpleRegionHIK(NoC,Phi_FGHI,v_FGHI,gamma, M_e, P_e);
```

6.2 Functions for M, v and P

In order to calculate the Mach number from v, the P from the Mach number and the v for the first region, the following functions have been developed. These functions have been used to obtain the Mach number and the pressure at each region (uniform, simple-wave and non-uniform regions)

```
function [P] = PfromM(M, M_e, p_e, gamma)
 ptot=p_e*(1+(gamma-1)/2*M_e^2)^(gamma/(gamma-1));
 P=ptot*(1+(gamma-1)/2*M^2)^(-gamma/(gamma-1));
end

function [M] = Mfromv(v,gamma)
syms x
solx=vpasolve(v==sqrt((gamma+1)/(gamma-1))*
atan(sqrt(((gamma-1)/(gamma+1))*(x^2-1)))-atan(sqrt(x^2-1)),x,2);
M=solx;
end

function [v] = vfromM(M,g)
gamma=g;
v=sqrt((gamma+1)/(gamma-1))*atan(sqrt(((gamma-1)/
(gamma+1))*(M^2-1)))-atan(sqrt(M^2-1));
```

```
end
```

6.3 Simple wave regions

By using these two functions, the first one for the characteristic lines going to the centreline and the second one for the characteristic lines going to the jet boundary, the flow properties along the expansion and compression waves can be computed.

```
function [Phi, v, M, mu, P] =
SimpleWaveRegion(N, Phipost, Phipre, v_pre, g, M_e,P_e)
gamma=g;
NoC=N;
dPhi=((Phipost-Phipre)/(NoC-1));
Phi=zeros(1,NoC);
v=zeros(1,NoC);
M=zeros(1,NoC);
mu=zeros(1,NoC);
P=zeros(1,NoC);
for i=1:(NoC)
    Phi(1,i)=Phipre+dPhi*(i-1);
    v(1,i)=v_pre+Phi(1,i)-Phipre;
    M(1,i)=Mfromv(v(1,i),gamma);
    mu(1,i)=mufromM(M(1,i));
    P(1,i)=PfromM(M(1,i),M_e,P_e,gamma);
end
end

function [Phi, v, M, mu, P] =
SimpleWaveRegionMinus(N, Phiamb, Phi0, v_pre, g, M_e, P_e)
gamma=g;
NoC=N;
dPhi=((Phiamb-Phi0)/(NoC-1));
Phi=zeros(1,NoC);
v=zeros(1,NoC);
M=zeros(1,NoC);
mu=zeros(1,NoC);
P=zeros(1,NoC);
for i=1:NoC
    Phi(1,i)=Phiamb-dPhi*(i-1);
    v(1,i)=v_pre+Phi(1,i);
    M(1,i)=Mfromv(v(1,i),gamma);
    mu(1,i)=mufromM(M(1,i));
    P(1,i)=PfromM(M(1,i), M_e, P_e, gamma);
end
end
```

6.4 Non-uniform wave regions

These three functions have been used to determine the flow properties along the non-uniform regions where both waves coincide.

```
function [Phi_ij, v_ij, M_ij, mu_ij, P_ij] =
NonSimpleWaveBCE(N,Phi_ABC,v_ABC,g, M_e, P_e)
Phi_ij=zeros(N);
v_ij=zeros(N);
M_ij=zeros(N);
mu_ij=zeros(N);
P_ij=zeros(N);
for i=1:N
    for j=i:N
        if i==1
            Phi_ij(i,j)=Phi_ABC(1,j);
            v_ij(i,j)=v_ABC(1,j)+Phi_ABC(1,j)-Phi_ij(i,j);%Along g-
        else
            if i==j
                Phi_ij(i,j)=0;
                v_ij(i,j)=v_ABC(1,i)+Phi_ABC(1,i)-Phi_ij(i,j);
            else
                Phi_ij(i,j)=1/2*(Phi_ij(i-1,j)+Phi_ij(i,j-1))+
                1/2*(v_ij(i-1,j)-v_ij(i,j-1));%Intersection between g+ and g-
                v_ij(i,j)=1/2*(v_ij(i-1,j)+v_ij(i,j-1))+
                1/2*(Phi_ij(i-1,j)-Phi_ij(i,j-1));
            end
        end
        M_ij(i,j)=Mfromv(v_ij(i,j),g);
        mu_ij(i,j)=mufromM(M_ij(i,j));
        P_ij(i,j)=PfromM(M_ij(i,j), M_e, P_e, g);
    end
end
end

function [phi_ij, v_ij, M_ij, mu_ij, P_ij] =
NonSimpleWave_DFG(N,M_a,phi_CEFD,v_CEFD,g, M_e, P_e)
c=1:N;
    phi_ij=zeros(N);
    v_ij=zeros(N);
    M_ij=zeros(N);
    mu_ij=zeros(N);
    P_ij=zeros(N);
    for i=c
        for j=1:i
            if i==j
                v_ij(i,j)=vfromM(M_a,g);
                if i==1
                    phi_ij(i,j)=v_ij(i,j)-v_CEFD(1,j)+phi_CEFD(1,j); %Along a
```

```
            else
                phi_ij(i,j)=v_ij(i,j)-v_ij(i,j-1)+phi_ij(i,j-1); %Along a
            end
        else
            if j==1
                v_ij(i,j)=v_CEFD(i);
                phi_ij(i,j)=phi_CEFD(i); %Along a g+
            else
                phi_ij(i,j)=0.5*(phi_ij(i-1,j)+phi_ij(i,j-1))+
                0.5*(v_ij(i-1,j)-v_ij(i,j-1));%g+ g- intersections
                v_ij(i,j)=0.5*(v_ij(i-1,j)+v_ij(i,j-1))+
                0.5*(phi_ij(i-1,j)-phi_ij(i,j-1));
            end
        end
        M_ij(i,j)=Mfromv(v_ij(i,j),g);
        mu_ij(i,j)=mufromM(M_ij(i,j));
        P_ij(i,j)=PfromM(M_ij(i,j), M_e, P_e, g);
    end
  end
end

function [phi_ij, v_ij, M_ij, mu_ij, p_ij]=
NonSimpleRegionHIK(N,phi_FGHI,v_FGHI,g, M_e, P_e)
  c=1:N;
  phi_ij=zeros(N);
  v_ij=zeros(N);
  M_ij=zeros(N);
  mu_ij=zeros(N);
  p_ij=zeros(N);
  for i=c
    for j=i:N
        if i==1
            phi_ij(i,j)=phi_FGHI(1,j);
            v_ij(i,j)=v_FGHI(1,j)+phi_FGHI(1,j)-phi_ij(i,j); %Along a g-
        else
            if i==j
                phi_ij(i,j)=0;
                v_ij(i,j)=v_FGHI(1,i)+phi_FGHI(1,i)-phi_ij(i,j);
            else
                phi_ij(i,j)=(1/2)*(phi_ij(i-1,j)+phi_ij(i,j-1))
                +(1/2)*(v_ij(i-1,j)-v_ij(i,j-1));%g+ g- intersections
                v_ij(i,j)=1/2*(v_ij(i-1,j)+v_ij(i,j-1))+
                1/2*(phi_ij(i-1,j)-phi_ij(i,j-1));
            end
        end
        M_ij(i,j)=Mfromv(v_ij(i,j),g);
        mu_ij(i,j)=mufromM(M_ij(i,j));
        p_ij(i,j)=PfromM(M_ij(i,j), M_e, P_e, g);
```

```
        end
    end
end
```