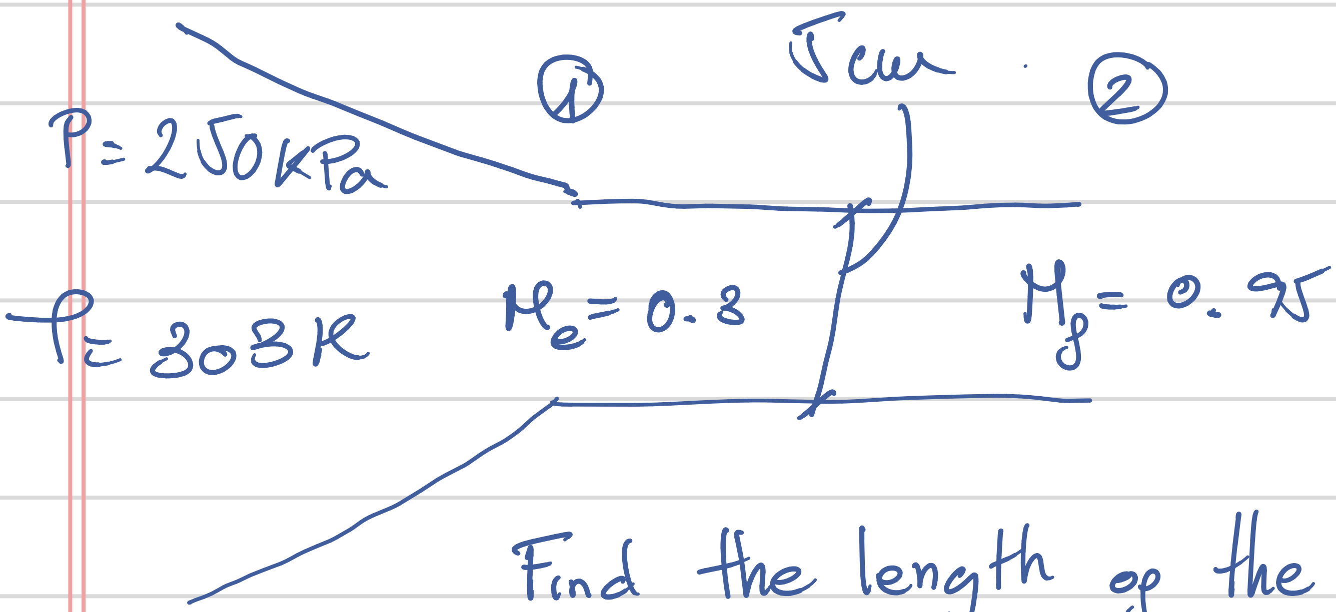
Task 3 - Tasks corresponding to Chapters 11 and 12

Gas Dynamics (Technische Universiteit Delft)

Problem 11.1



Find the length of the pipe and the pressure at the end.

Assumptions: Isentropic flow.

$f = 0.005$

Constant area channel flow:

$$\frac{4\gamma f}{D_h} x = \frac{M^2(x) - M_0^2}{M^2(x)\, M_0^2} + \frac{\gamma+1}{2} \ln \frac{M_0^2\left(1+\frac{\gamma-1}{2}M^2(x)\right)}{M^2(x)\left(1+\frac{\gamma-1}{2}M_0^2\right)}$$

$$\frac{4\cdot\frac{7}{5}\cdot 0.005}{0.05} x = \frac{0.95^2 - 0.3^2}{0.95^2\cdot 0.3^2} + \frac{\frac{7}{5}+1}{2} \ln \frac{0.3^2\left(1+\frac{\frac{7}{5}-1}{2}\cdot 0.95\right)^2}{0.95\left(1+\frac{\frac{7}{5}-1}{2}\,0.3^2\right)}$$

$$0.56\, x = 7.414 \rightarrow \boxed{x = 13.239\ \text{m}}$$

Let's define P_2 as:

$$P_2 = \frac{P_2}{P^*} \cdot \frac{P^*}{P_1} \cdot \frac{P_1}{P_0} \cdot P_0$$

$$\frac{P_2}{P^*} = \frac{1}{M_2}\sqrt{\frac{\gamma+1}{2+(\gamma-1)M_2^2}} = 1.061$$

$$\frac{P_*}{P_1} = \frac{1}{\frac{1}{M_1^2}\sqrt{\frac{\gamma+1}{2+(\gamma-1)M_1^2}}} = \frac{1}{3.619}$$

$$\frac{P_1}{P_0} = \frac{1}{\left(1+\frac{\gamma-1}{2}M_1^2\right)^{\frac{\gamma}{\gamma-1}}} = \frac{1}{1.064}$$

$$P_2 = 1.061 \cdot \frac{1}{3.619} \cdot \frac{1}{1.064} \cdot 250$$

$$= \boxed{68.88\ kPa}$$

b) If $L' = 0.75L$, what's the value of M_2' and P_2'?

We know that,

$$\frac{4\gamma f}{D_h}x = \frac{M^2(x) - M_0^2}{M^2(x)M_0^2} + \frac{\gamma+1}{2}\ln\frac{M_0^2\left(1+\frac{\gamma-1}{2}M^2(x)\right)}{M^2(x)\left(1+\frac{\gamma-1}{2}M_0^2\right)}$$

where x is $0.75 \cdot 13.239 = 9.929\ m$. Now, solving for $M(x)$, we get that $M(L') = 0.471$

$$P_2' = \frac{P_2'}{P^*} \frac{P^*}{P_1} \frac{P_1}{P_0} \cdot P_0$$

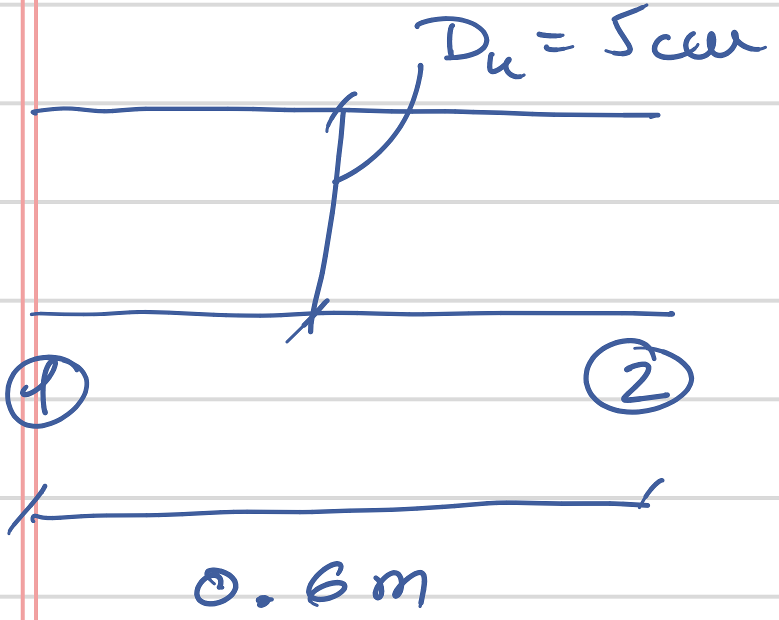
$$\frac{P_2'}{P^*} = \frac{1}{M_2}\sqrt{\frac{\gamma+1}{2+(\gamma-1)M_2^2}} = 2.271$$

$$\frac{P_*}{P_1} = \frac{1}{\frac{1}{M_1^2}\sqrt{\frac{\gamma+1}{2+(\gamma-1)M_1^2}}} = \frac{1}{3.619}$$

$$\frac{P_1}{P_0} = \frac{1}{\left(1+\frac{\gamma-1}{2}M_1^2\right)^{\frac{\gamma}{\gamma-1}}} = \frac{1}{1.064}$$

$$P_2' = 2.271 \cdot \frac{1}{3.619} \cdot \frac{1}{1.064} \cdot 250 = 147.44\,kPa$$

Problem 11.2



① $M_1 = 2$
$T_1 = 293\,K$
$P = 80\,kPa$
$f = 0.005$

The Mach number can be obtained from:

$$\frac{4\gamma f}{D_h} x = \frac{M^2(x) - M_0^2}{M^2(x) M_0^2} + \frac{\gamma+1}{2} \ln \frac{M_0^2\left(1 + \frac{\gamma-1}{2} M^2(x)\right)}{M^2(x)\left(1 + \frac{\gamma-1}{2} M_0^2\right)}$$

$$\boxed{M_2 = 1.3}$$

$$P_2 = \frac{P_2}{P^*} \cdot \frac{P^*}{P_1} \cdot P_1$$

$$\frac{P_2}{P^*} = \frac{1}{M_2}\sqrt{\frac{\gamma+1}{2+(\gamma-1)M_2^2}} = 0.728$$

$$\frac{P^*}{P_1} = \frac{1}{\frac{1}{M_1^2}\sqrt{\frac{\gamma+1}{2+(\gamma-1)M_1^2}}} = \frac{1}{0.408}$$

$$\boxed{P_2 = 0.728 \cdot \frac{1}{0.408} \cdot 80 = 142.66\,kPa}$$

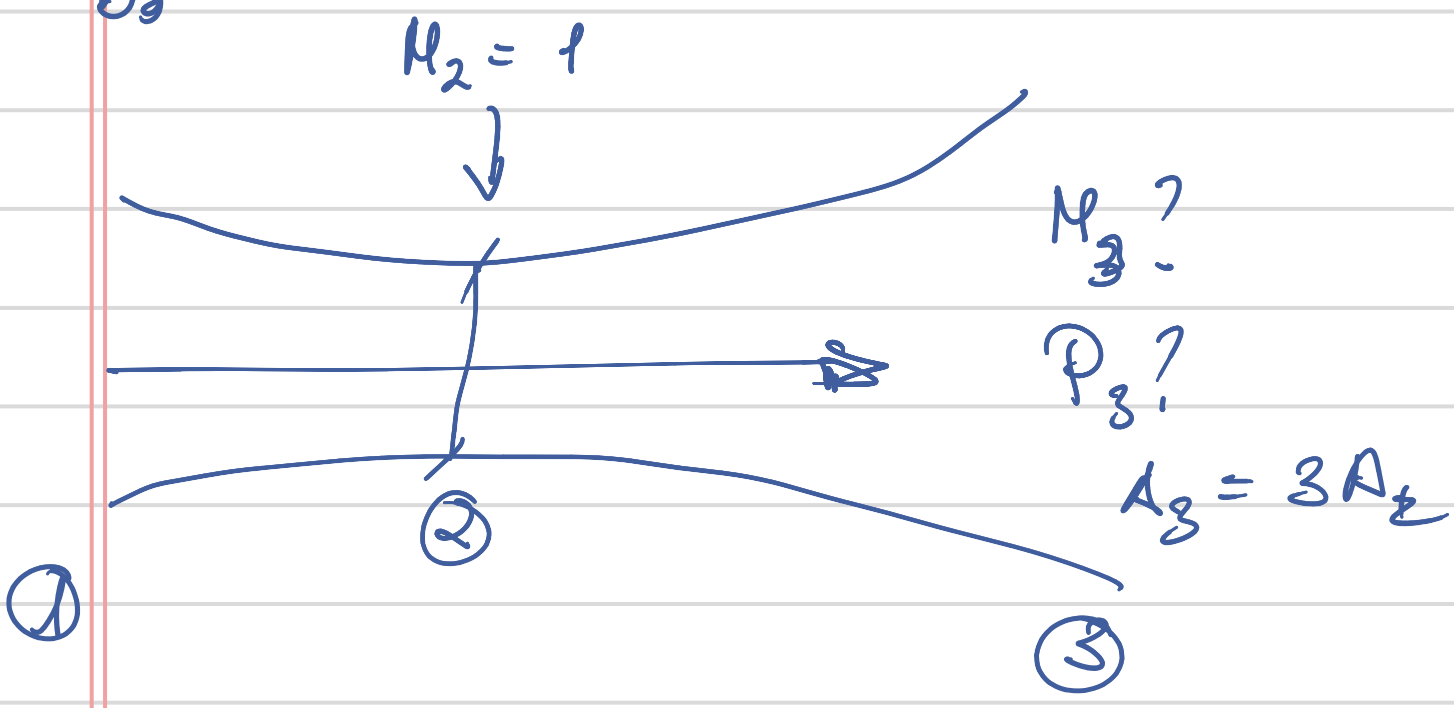
$$T_2 = \frac{T_2}{T^*} \cdot \frac{T^*}{T_1} \cdot T_1$$

$$\frac{T_2}{T^*} = \frac{\gamma+1}{2+(\gamma-1)M^2} = 0.897$$

$$\frac{T^*}{T_1} = \frac{1}{\frac{\gamma+1}{2+(\gamma+1)M^2}} = \frac{1}{0.66}$$

$$T_2 = 0.897 \cdot \frac{1}{0.66} \cdot 293 = 394.23\,K$$

b)



$M_1 = 1.3$

$P_1 = 142.66\,KPa$

As the flow is isentropic, we can use the following relation:

$$\frac{A_3}{A_1} = \frac{1}{M_3} \cdot \left(\frac{2}{\gamma+1} \cdot \left(1 + \frac{\gamma-1}{2} M^2\right)\right)^{\frac{\gamma+1}{2(\gamma-1)}}$$

For the previous equation, I assume that M at the throat is 1. Then, solving for M:

$$\boxed{M_3 = 0.197}$$

$$P_3 = \frac{P_3}{P^*} \frac{P^*}{P_2} \frac{P_2}{P_1} P_1$$

$$\frac{P_3}{P^*} = \frac{1}{M} \sqrt{\frac{\gamma+1}{2+(\gamma-1)M^2}} = 5.539$$

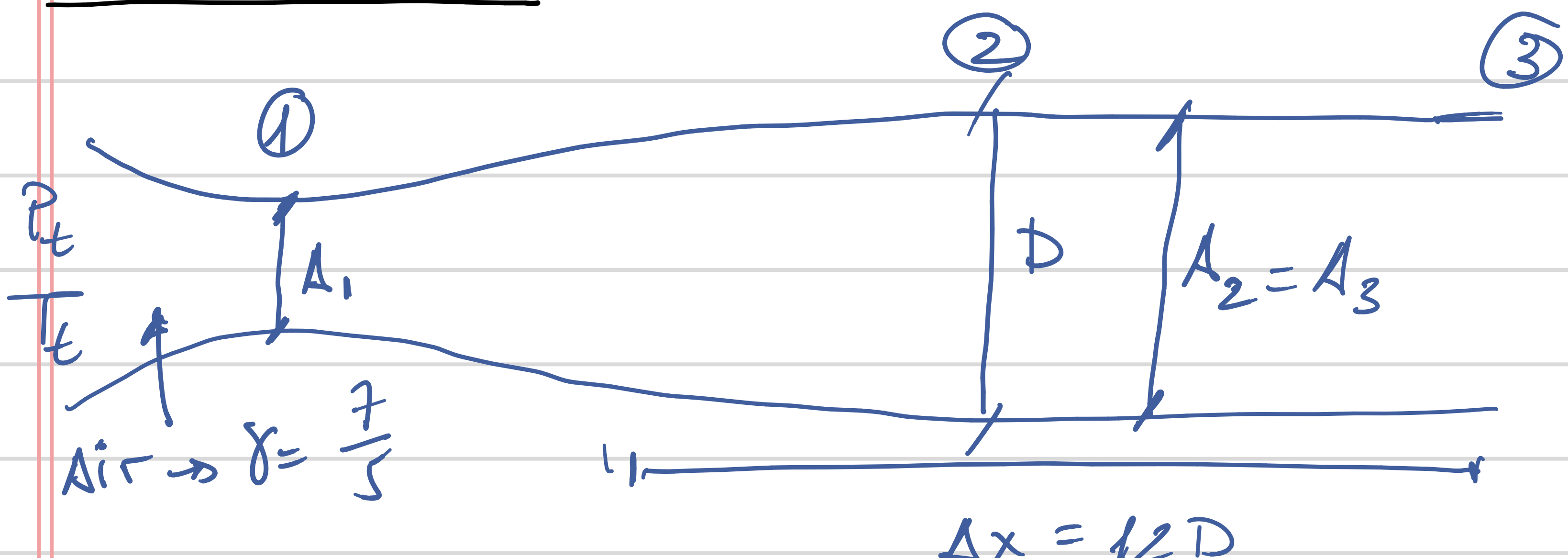
$\frac{P^*}{P_2} = 1$ as P_2 is P when $M = 1 \Rightarrow P^*$

$$\frac{P_2}{P_1} = \frac{1}{\frac{1}{M}\sqrt{\frac{\gamma+1}{2+(\gamma-1)M^2}}} = \frac{1}{0.728}$$

$$P_3 = 5.539 \cdot 1 \cdot \frac{1}{0.728} \cdot 142.66 =$$

$$= \boxed{1085.431 \text{ kPa}}$$

Problem 11.3



$P_t = 10$ bar ; $T_t = 300K$; $f = 0.0025$; $\frac{A_3}{A_1} = 3$

a) Compute the receiver pressure that would place a shock

I) in the nozzle throat:

I will assume that the flow is isentropic, so the following equation can be applied:

$$\frac{A_2}{A_1} = \frac{1}{M_2}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M_2^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

Solving for M_2: $M_2 = 0.197$

$\frac{4fL_2^*}{D} = 15.134$ → From Fanno line charts.

$$\frac{4fL^*_{2-3}}{D} = \frac{4 \cdot 0.0025 \cdot 12D}{D} = 0.12$$

$$\frac{4fL^*_{2-3}}{D} = \frac{4fL^*_2}{D} - \frac{4fL^*_3}{D} \rightarrow \frac{4fL^*_3}{D} = 15.134 - 0.12 = $$
$$= 15.014$$

$\frac{4fL_3^*}{D} = 15.014$ → $M_3 = 0.198$ → From Fanno line charts.

$$P_3 = \frac{P_3}{P^*} \cdot \frac{P^*}{P_2} \cdot \frac{P_2}{P_0} \cdot P_0$$

$$\frac{P_3}{P^*} = \frac{1}{M} \sqrt{\frac{\gamma+1}{2+(\gamma-1)M^2}} = 5.511$$

$$\frac{P^*}{P_2} = \frac{1}{\frac{1}{M}\sqrt{\frac{\gamma+1}{2+(\gamma-1)M^2}}} = \frac{1}{5.539} = 0.181.$$

$$\frac{P_2}{P_0} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{-\frac{\gamma}{\gamma-1}} = 0.97$$

$$\boxed{P_3 = 5.511 \cdot 0.181 \cdot 0.97 \cdot 10 = 9.67 \text{ bar.}}$$

II) at the nozzle exit

In the case where the shock is formed at the nozzle exit we know that;

$$P_3 = \frac{P_3}{P^*} \cdot \frac{P^*}{P_{PS}} \cdot \frac{P_{PS}}{P_2} \cdot \frac{P_2}{P_0} \cdot P_0$$

Now, $M_2 = 2.637.$

$$\frac{P_2}{P_0} = \left(1 + \frac{\gamma-1}{2} M_2^2\right)^{-\frac{\gamma}{\gamma-1}} = 0.0473$$

From the normal shock wave equations:

$$\frac{P_{PS}}{P_2} = \frac{2\gamma M_2^2 - (\gamma-1)}{\gamma+1} = 7.946$$

From the normal shock wave equations:

$$M_{PS}^2 = \frac{(\gamma-1)M_2^2 + 2}{2\gamma M_2^2 - (\gamma-1)} = 0.2507 \rightarrow M_{PS} = 0.5$$

From Fanno charts for $M_{ps} = 0.5$

- $\frac{4\gamma L^*_{ps}}{D} = 1.0691$
- $\frac{P_{ps}}{P^*} = 2.138 \rightarrow \frac{P^*}{P_{ps}} = \frac{1}{2.138}$

$\frac{4f L^*_{ps-3}}{D} = 0.12$ (From case I)

$$\frac{4f L^*_3}{D} = \frac{4f L^*_{ps}}{D} - \frac{4f L^*_{ps-3}}{D} = 0.9491$$

↳ From Fanno charts:

- $M_3 = 0.516$
- $\frac{P_3}{P^*} = 2.052$

$$\boxed{P_3 = 2.052 \cdot \frac{1}{2.138} \cdot 7.946 \cdot 0.0473 \cdot 10 = 3.61 \text{ bar}}$$

III) at the duct exit

In this case $P_r = P_{ps}$, thus:

$$P_{ps} = \frac{P_{ps}}{P_3} \cdot \frac{P_3}{P^*} \cdot \frac{P^*}{P_2} \cdot \frac{P_2}{P_0} \cdot P_0 \; ; \quad \text{For } M_2 = 2.687$$

↳ $M_3 = 2.124$

From the Fanno charts.

$$\frac{P_3}{P^*} = \frac{1}{M_3}\sqrt{\frac{\gamma+1}{2+(\gamma-1)M_3^2}} = 0.374$$

$$\frac{P^*}{P_2} = \frac{1}{\frac{1}{M_2}\sqrt{\frac{\gamma+1}{2+(\gamma-1)M_2^2}}} = 3.722$$

$\frac{P_2}{P_0} = 0.0473 \rightarrow$ From case II

From the normal shock wave equations:

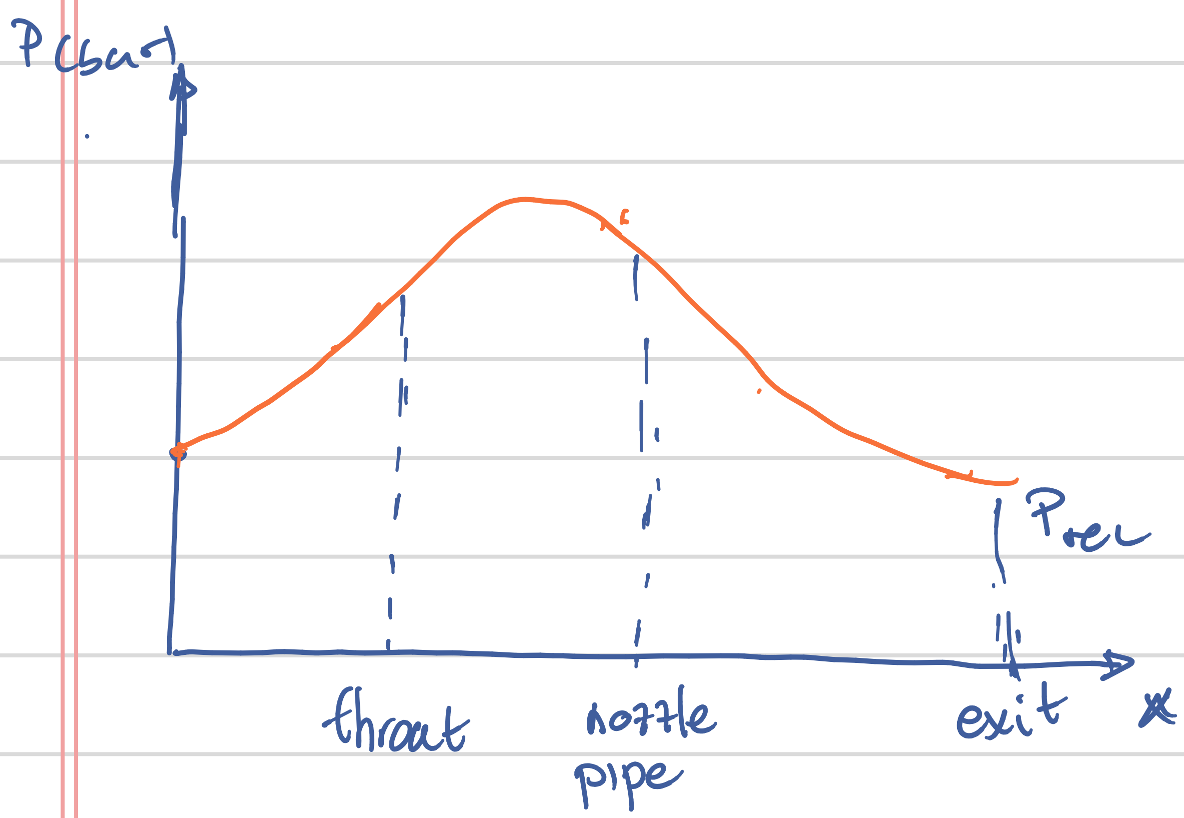
$$\frac{P_{ps}}{P_3} = \frac{2\gamma M_3^2 - (\gamma-1)}{\gamma+1} = 5.097$$

$$\boxed{P_{ps} = 5.097 \cdot 0.374 \cdot 3.722 \cdot 0.0473 \cdot 10 = 3.356 \text{ bar}}$$

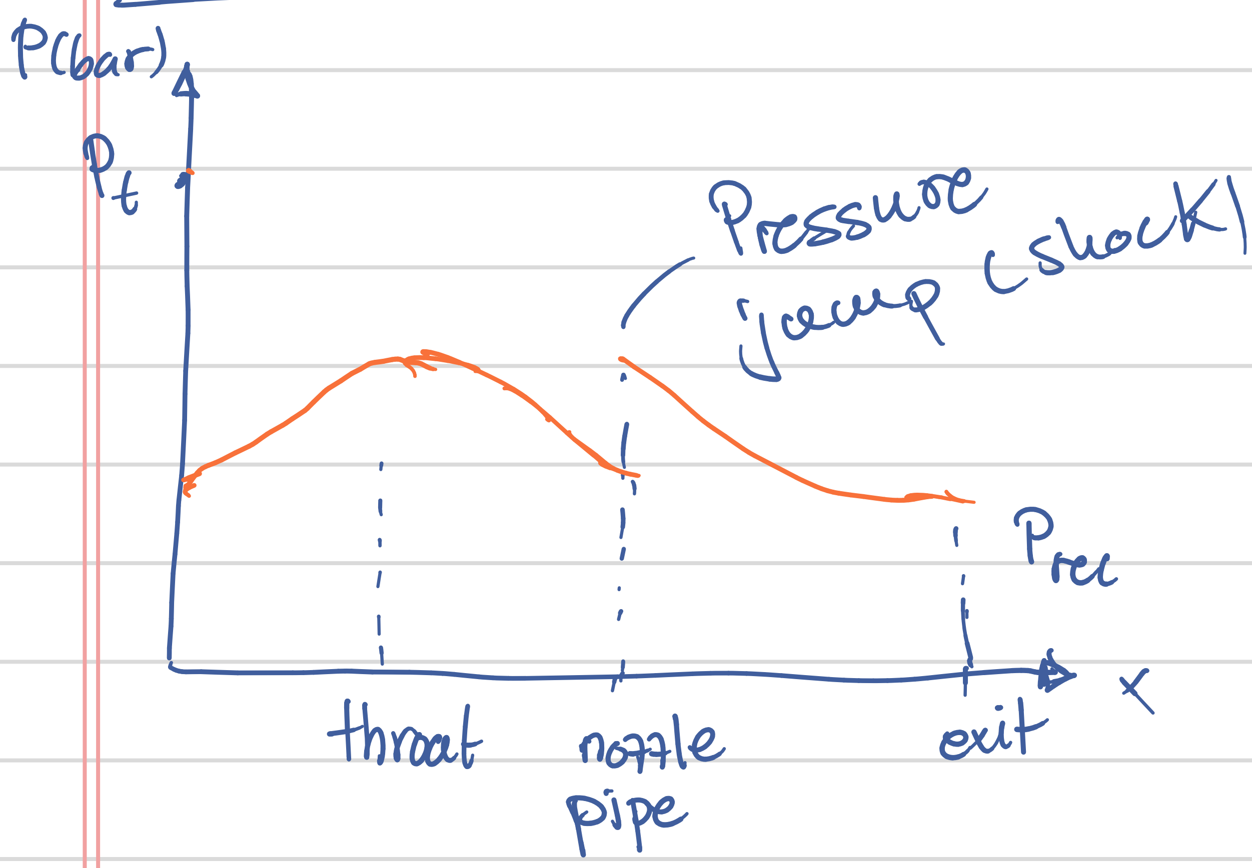
b) According to section A, we see that the closer the shock to the nozzle throat, the higher the pressure. As the shock is moved forward respect the nozzle, the pressure is lower. Then, for a $P_{rec} < 3.356$ (Case III), the shock may appear ahead, but definitely out of the system.

c)

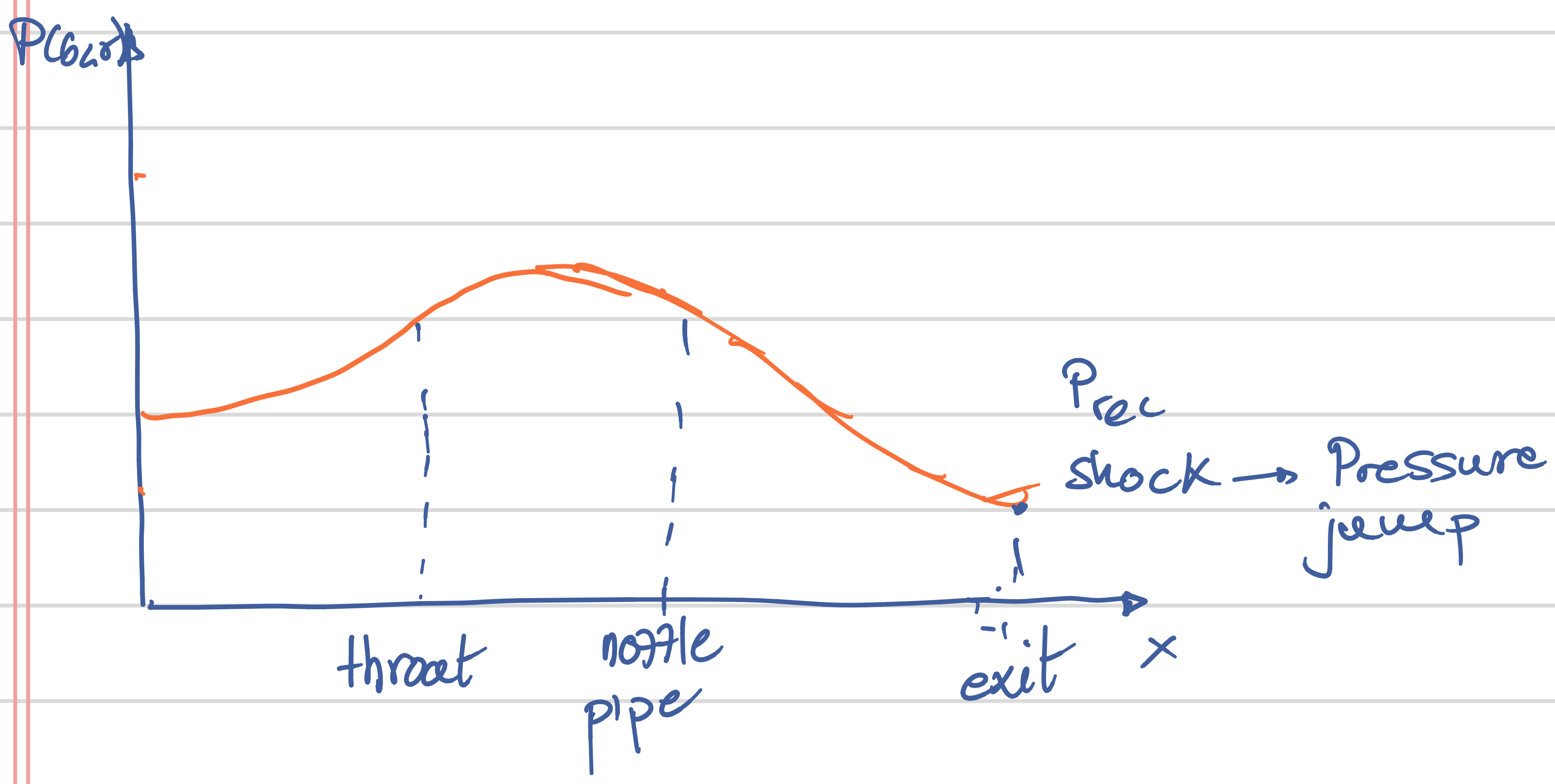
Case I



Case II

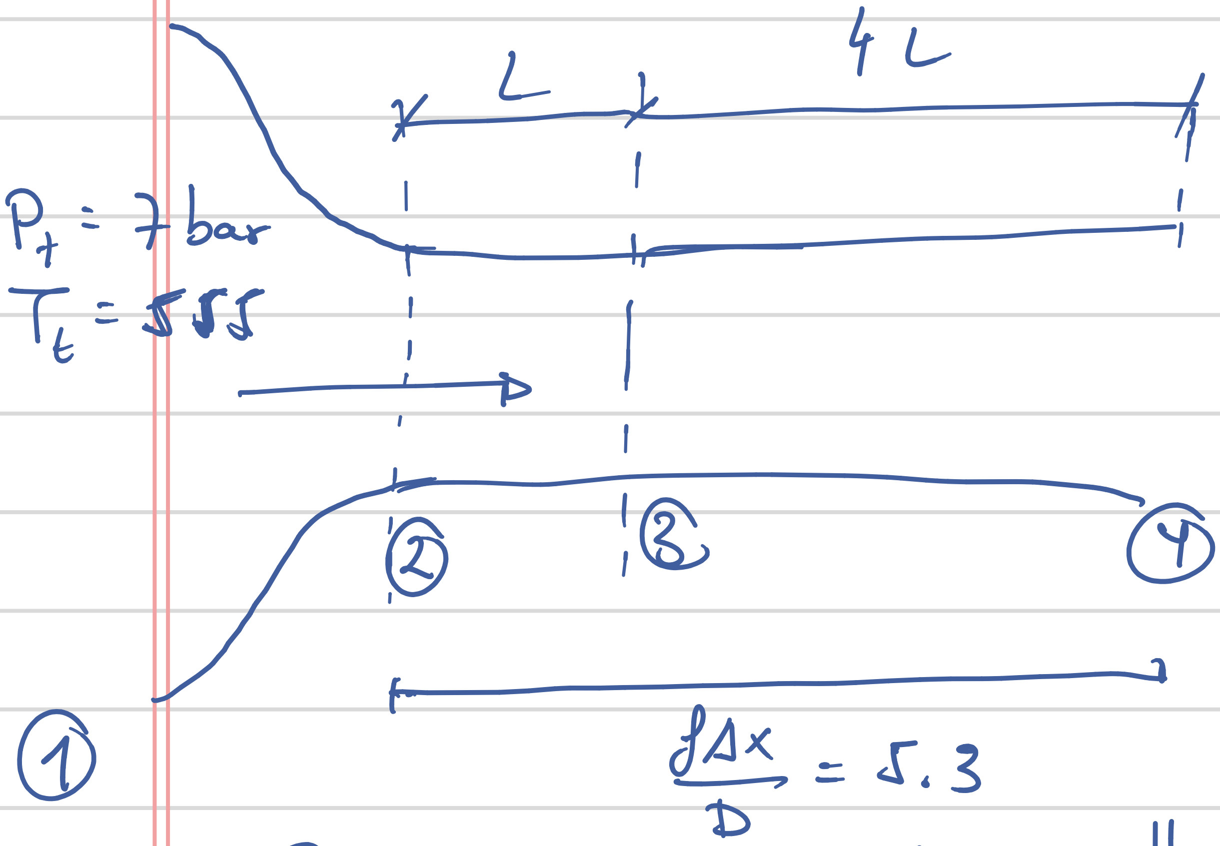


Case III



Problem 11.4

Gas: Oxygen → $\gamma = 1.4$



a) Pressure at the end of the duct?

$$P_4 = \frac{P_4}{P^*} \cdot \frac{P^*}{P_2} \cdot \frac{P_2}{P_1} \cdot P_1$$

$$\frac{P_4}{P^*} = 1 \quad \text{as} \quad M_4 = 1 \longrightarrow \frac{4 f \Delta x_4}{D} = 0 \rightarrow \frac{4 f \Delta x_2}{D} = \frac{4 f \Delta x_{2-3}}{D}$$

$$\frac{4 \cdot f \cdot \Delta x_2}{D} = 4 \cdot 5.3 = 21.2$$

From Fanno charts: $M_2 = 0.17$

$$\frac{P^*}{P_2} = \frac{1}{\frac{1}{M}\sqrt{\frac{\gamma+1}{2+(\gamma-1)M^2}}} = 0.1556$$

$$\frac{P_2}{P_1} = \left(1 + \frac{\gamma-1}{2} M_2^2\right)^{-\frac{\gamma}{\gamma-1}} = 0.98$$

$$\boxed{P_4 = 1 \cdot 0.1556 \cdot 0.98 \cdot 7 = 1.067 \text{ bar}}$$

b) Now the duct length is L

$$P_3 = \frac{P_3}{P^*} \cdot \frac{P^*}{P_2} \cdot \frac{P_2}{P_1} \cdot P_1$$

M_3 is 1 as at the exit the flow is choked.

Then $\rightarrow \frac{P_3}{P^*} = 1$

$$\frac{4 \cdot f \cdot \Delta x}{D} = 4 \cdot \frac{5.3}{5} = 4.24.$$

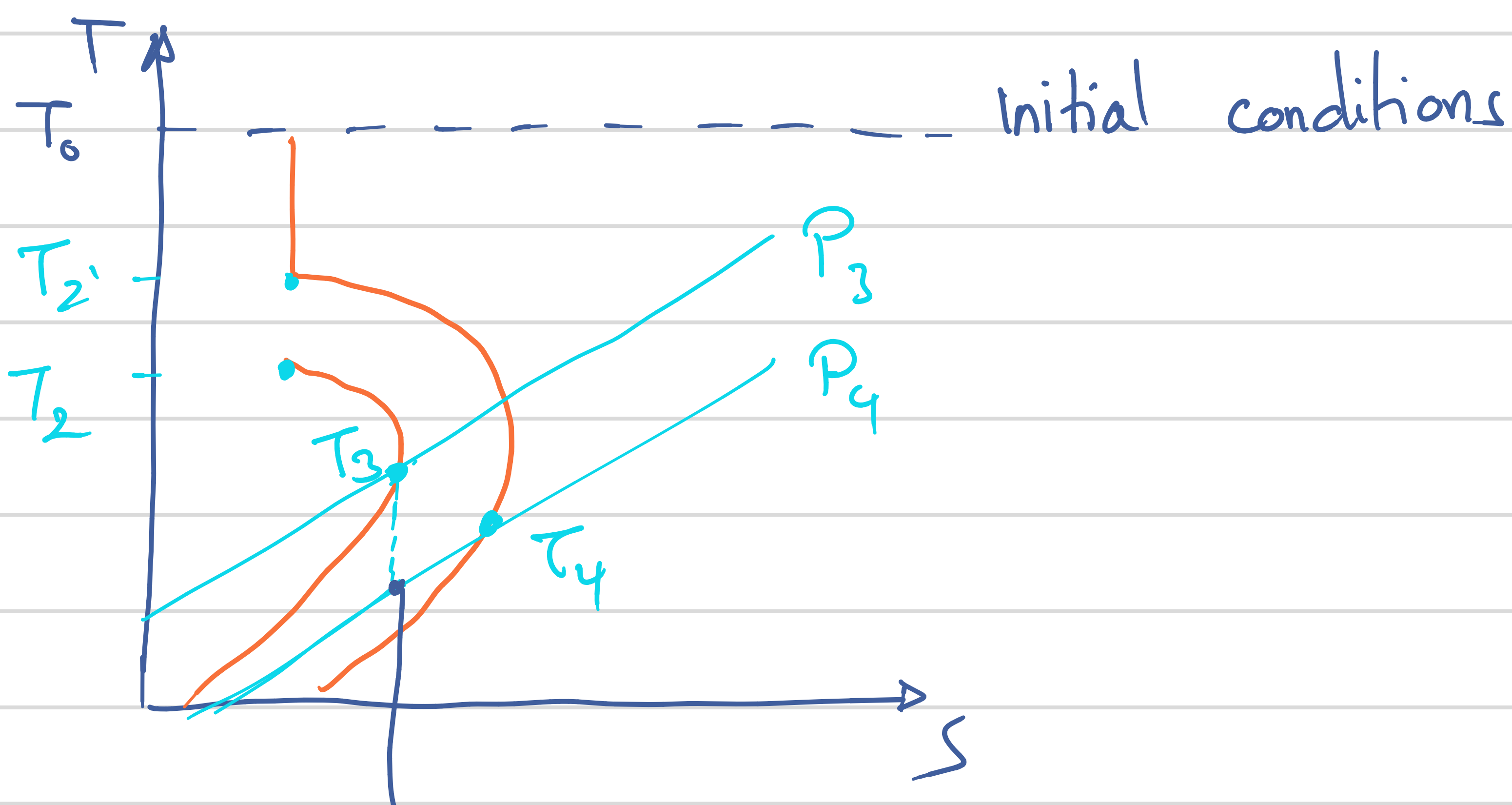
$\rightarrow$ From Fanno chart $\rightarrow$ $M_2 = 0.326$

$\hookrightarrow \frac{P^*}{P_2} = 0.3$

$$\frac{P_2}{P_1} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-\frac{\gamma}{\gamma-1}} = 0.929$$

$$\boxed{P_3 = 1 \cdot 0.3 \cdot 0.929 \cdot 7 = 1.951 \text{ bar}}$$

c)



Need to expand until P_{cc} on the second case.

Chapter 12

Problem 12.1

① ②

$P_1 = 50\, kPa$ $\qquad$ P_2 ? $\quad M_2$?

$T_1 = 30°C\ (308\, K)$ $\qquad$ T_2^* ?

$V_1 = 80\, m/s$

Heating value of the fuel = 40 MJ/kg

$$a_1 = \sqrt{\gamma R T_1} = \sqrt{1.4 \cdot 287 \cdot 303} = 348.92\, m/s$$

$$\hookrightarrow M_1 = \frac{V_1}{a_1} = \frac{80}{348.92} = 0.2293$$

Stagnation temperature: $T_t = T\left(1 + \frac{\gamma - 1}{2} M^2\right)$

$$\hookrightarrow \text{In point } 1 \rightarrow T_t = 303\left(1 + \frac{1.4-1}{2} \cdot 0.2293^2\right) =$$

$$= 306.186\, K$$

$$AF = \frac{\text{Heat fuel}}{\text{Heat air}} \rightarrow \text{Heat air} = \frac{40000\, kJ/kg}{40}$$

$$= 1000\, kJ/kg$$

$$q = c_p \left(T_{t2} - T_{t1}\right)$$

$$c_{p_{Air \to T = 303K}} = 1.0065$$

$$\boxed{T_{t_2} = \frac{q}{c_p} + T_{t_1} = 1299.73K}$$

$$\frac{T_{t1}}{T_t^*} = \frac{2(1+\gamma)M^2}{(1+\gamma M^2)^2} \cdot \left(1 + \frac{\gamma - 1}{2} M^2\right)$$

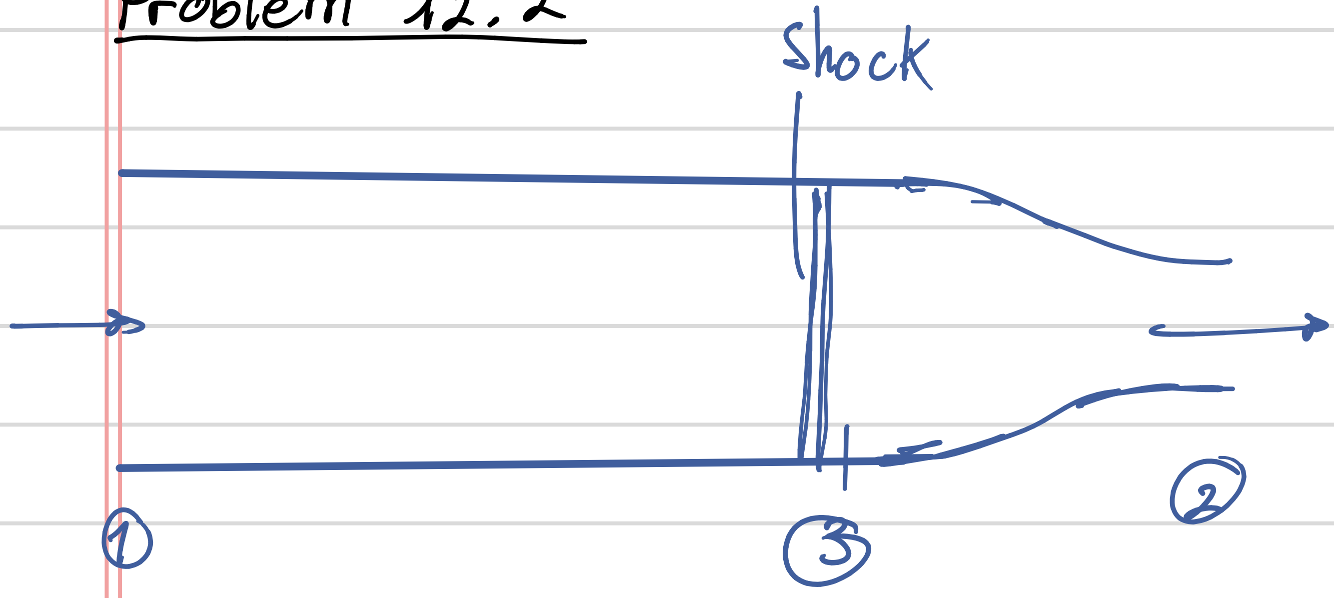
$$\frac{T_{t_1}}{T_t^*} = 0.221 \rightarrow T_t^* = \frac{T_{t_1}}{0.221} = 1383.84\ K$$

$$\frac{T_{t_2}}{T_t^*} = 0.939$$

Solving the $\frac{T_{t_2}}{T_t^*}$ for $M_2 \rightarrow \boxed{M_2 = 0.748}$

$$\frac{P_2}{P_1} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2} \rightarrow \boxed{P_2 = 30.1kPa}$$

Problem 12.2



$$\frac{A_2}{A_3} = 0.98$$

$T_1 = 300K$ $\qquad$ $M_2 = 1$

$M_1 = 1.5$

$$\frac{A_2}{A_3} = \left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}} \frac{\left(1+\frac{\gamma-1}{2}M_3^2\right)^{\frac{\gamma+1}{2(\gamma-1)}}}{M_3}$$

$\rightarrow$ Solving for $\boxed{M_3 = 0.851}$ $(M_3 = M_{post})$

The other solution for M_3 is 1.16, but it can't be as after the shock the flow is subsonic.

Normal shock equation for M:

$$M_{post}^2 = \frac{(\gamma-1)M_{pre}^2 + 2}{2\gamma M_{pre}^2 - (\gamma-1)} \rightarrow M_{pre} = 1.186$$

$$q = c_p(T_{t_2} - T_{t_1})$$

$$T_{t_1} = T\left(1 + \frac{\gamma-1}{2}M^2\right) = 435$$

$$\frac{T_{t_1}}{T_t^*} = \frac{2(1+\gamma)M_1^2}{(1+\gamma M_1^2)^2} \cdot \left(1 + \frac{\gamma-1}{2}M_1^2\right)$$

$\rightarrow T_t^* = 478.403K$

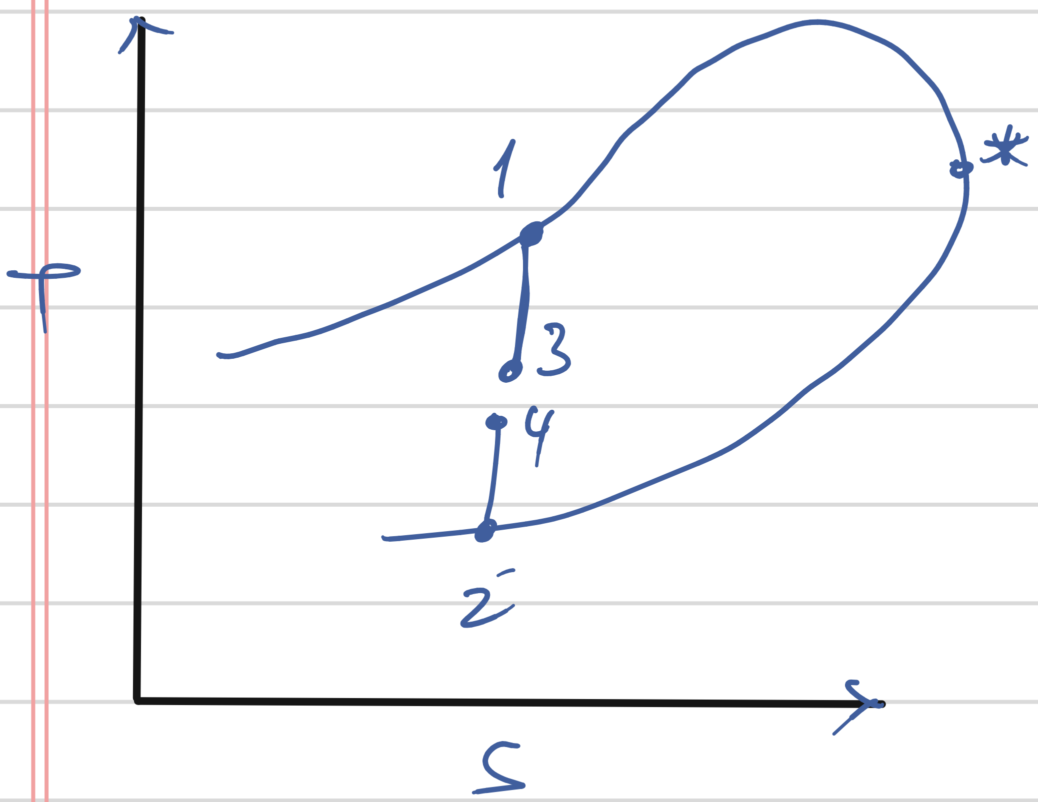
$$\frac{T_{t2}}{T_t^*} = \frac{2(1+\gamma)M^2}{(1+\gamma M^2)^2} \cdot \left(1 + \frac{\gamma-1}{2} M^2\right)$$

$\hookrightarrow$ Solving for $T_{t2} \rightarrow T_{t2} = 469.43\,K$

$$\boxed{q = \underbrace{c_p}_{1.007} (T_{t2} - T_{t1}) = 84.67\ kJ/kg}$$

$T_{t2} > T_{t1}$, thus, heat has been added to the system.

Problem 12.3



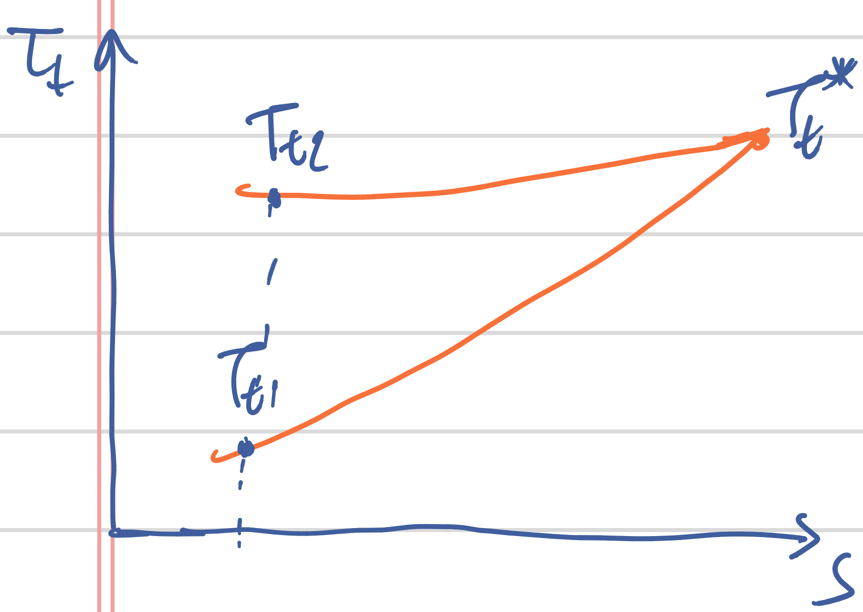
a) If 3 and 4 coincide, A_3 is greater, equal or less than A_4.

From energy equation → $H_1 + q = H_2$ → $\partial H = \partial q$,

$ds = \frac{\partial q}{T}$ → $T = \frac{\partial H}{\partial s}$

From the graph, $T_1 > T_2$ so H_1 to H^* curve has a greater slope than H_2 to H^* curve.

↳ For a total temperature curve the same annalogy can be made (Perfect gas consideration)



→ As both must get to the same T_t^* → $T_{t1} < T_{t2}$

As the expansion in 1 and the compression in 2 are isentropic process, we end up in this equation knowing that $T_3 = T_4$:

$$T_{t3} - T_{t4} = \frac{1}{2c_p}(V_3^2 - V_4^2)$$

From it, as $T_3 = T_4$ (from the statement) and $T_{t4} > T_{t3}$ (isentropic process) $\rightarrow$ $V_4 > V_3$

We also know from mass conservation that:

$$\rho \cdot A V = \text{const.} \rightarrow \cancel{\rho_3} A_3 V_3 = \cancel{\rho_4} A_4 V_4$$

$\rho_3 = \rho_4$ and A_3 must be greater than A_4 in order to fulfil the above equation. $\boxed{A_3 > A_4}$

b) Points 3 and 4 are no longer coincident.

$M_3 = M_4 = 1$.

- V_3 relation with V_4?
- A_3 relation with A_4?

We know that $\frac{T_t}{T_t^*} = \frac{2(1+\gamma)M^2}{(1+\gamma M^2)^2}\left(1 + \frac{\gamma-1}{2}M^2\right)$

As $M = 1$

$$\frac{T_t}{T_t^*} = \frac{2\cancel{(1+\gamma)}}{(1+\gamma)^{\cancel{2}}}\left(1 + \frac{\gamma-1}{2}\right) = \frac{2}{1+\gamma} + \frac{\gamma-1}{\gamma+1}$$

Computed $\rightarrow \dfrac{T_t}{T_t^*} = \dfrac{2}{1+\gamma} + \dfrac{\gamma-1}{\gamma+1} \rightarrow T_t = K\,T_t^*$

From section a $T_{t2} > T_{t1}$

$$\cancel{K}\,T_{t2}^* > \cancel{K}\,T_{t1}^* \rightarrow T_{t2}^* > T_{t1}^*$$

$$\hookrightarrow T_{t4}^* > T_{t3}^*$$

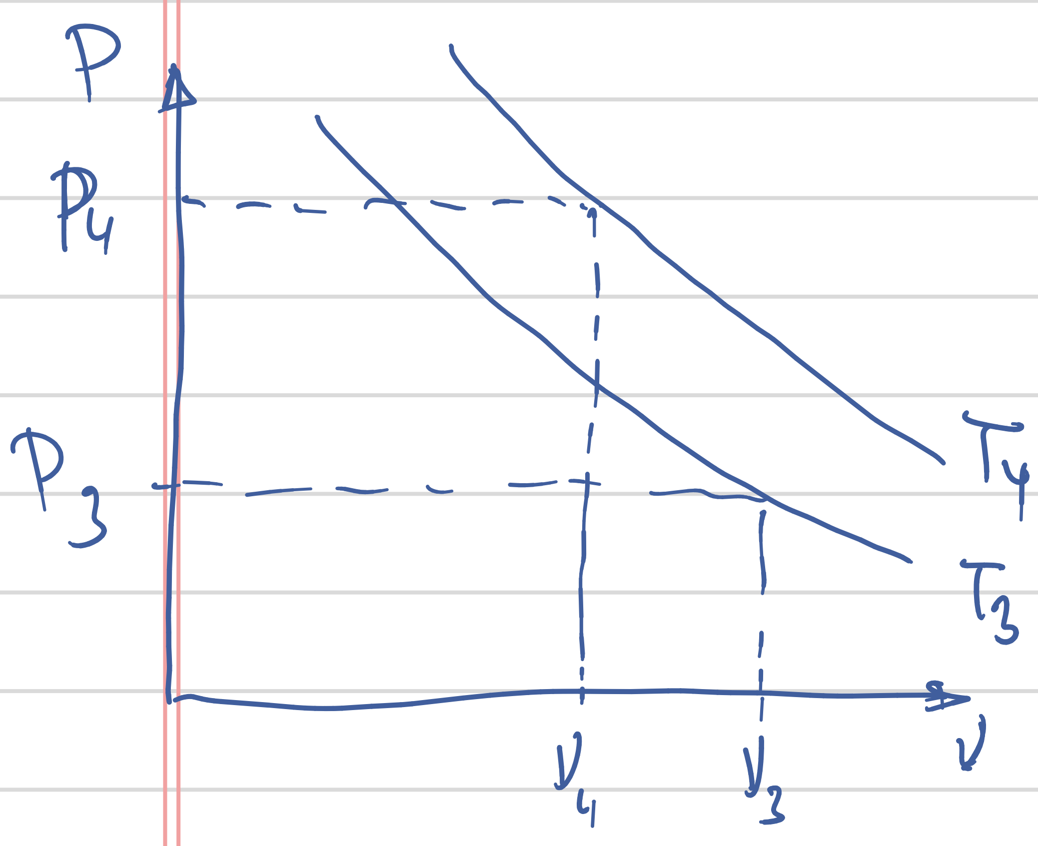
Mach number definition:

$M = \dfrac{V}{a} = \dfrac{V}{\sqrt{\gamma R T}}$ (perfect gas condition)

$$M^* = \frac{V}{\sqrt{\gamma R T_3^*}} \rightarrow \frac{V_3}{\sqrt{\gamma R T_3^*}} = \frac{V_4}{\sqrt{\gamma R T_4^*}}$$

As T_4^* is higher than T_3^*, to keep the ratio $\rightarrow \boxed{V_4 > V_3}$

Regarding area relations:



As $T_4 > T_3$

For Rayleigh flows:

$$P_3 A_3 + \rho_3 A_3 V_3^2 = P_4 A_4 + \rho_4 A_4 V_4^2 \rightarrow$$

$$\rightarrow A_3 (P_3 + \rho_3 V_3^2) = A_4 (P_4 + \rho_4 V_4^2) \rightarrow$$

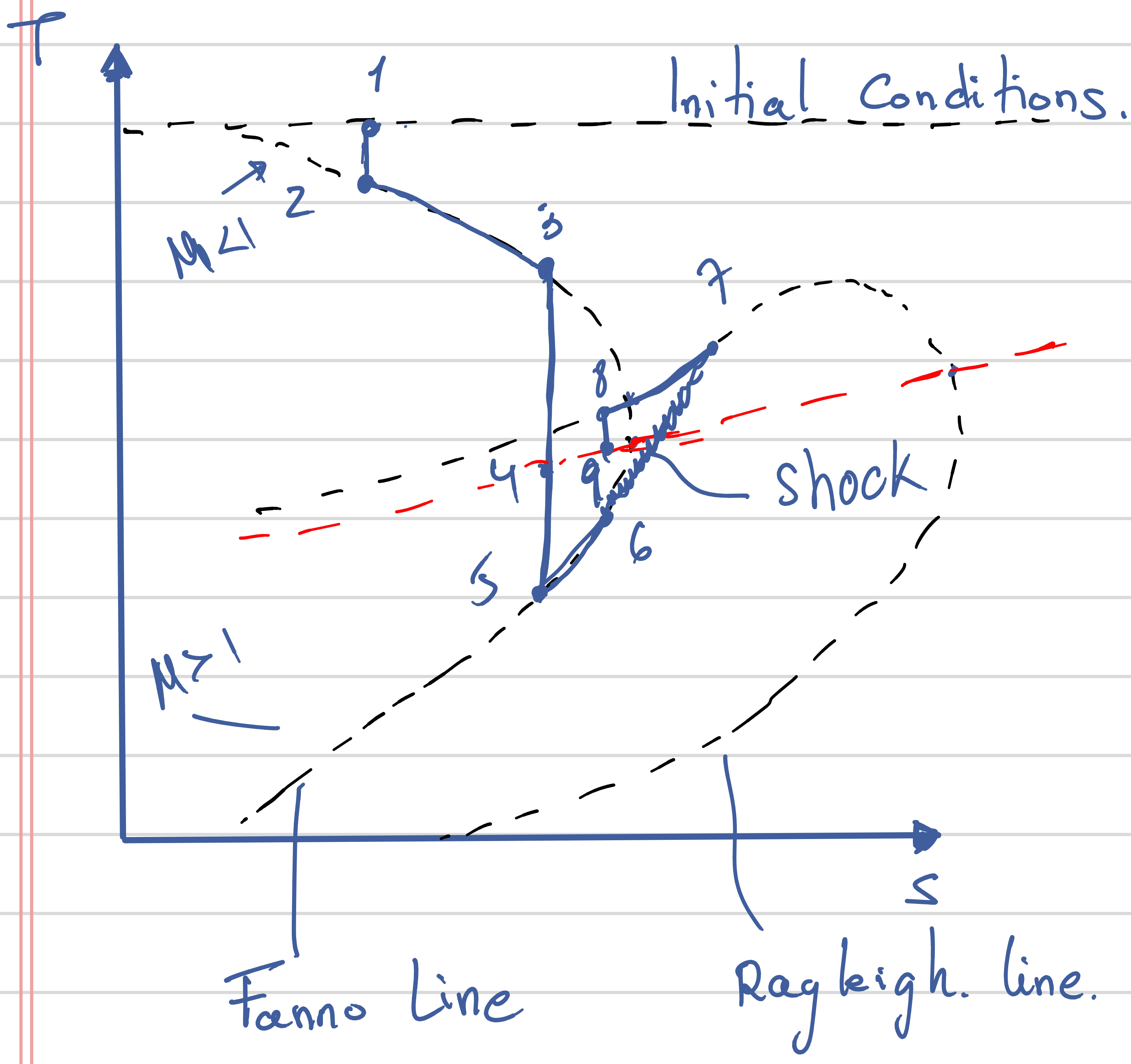
$$\rightarrow \frac{A_3}{A_4} = \frac{P_4 + \rho_4 V_4^2}{P_3 + \rho_3 V_3^2} = \frac{P_4 + \rho_4 \gamma R T_4 \underbrace{M_4^2}_{=1}}{P_3 + \rho_3 \gamma R T_3 \underbrace{M_3^2}_{1}} =$$

$$= \frac{P_4 + \rho_4 \gamma R T_4}{P_3 + \rho_3 \gamma R T_3} = \frac{P_4 + \gamma P_4}{P_3 + \gamma P_3} =$$

$$= \frac{P_4 \cancel{(1+\gamma)}}{P_3 \cancel{(1+\gamma)}} = \frac{P_4}{P_3} > 1$$ As $P_4 > P_3$ from P-V graph.

Then $\frac{A_3}{A_4} > 1 \rightarrow \boxed{A_3 > A_4}$

Problem 12.4



1-2 → Isentropic expansion.

2-3 → Non-isentropic expansion (Friction plays a role - - Fanno line Subsonic region).

3-4 → Isentropic expansion (Subsonic)

4 → $M = M^* = 1$ → Flow choked.

4-5 → Isentropic expansion (supersonic).

5-6 → Non-simple expansion (Friction plays a role, Fanno line Supersonic region).

7 → Intersection between the Fanno and the Rayleigh lines → Subsonic.

7-8 → Q removed through Rayleigh line

8-9 → Isentropic expansion.