High Dimensional Linear Regression using Lattice Basis Reduction

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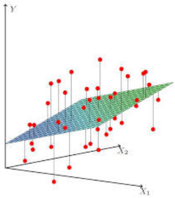
ORC, MIT

Cornell ORIE Young Researchers Workshop, 2018

Linear Regression

Let (unknown) $\beta^* \in \mathbb{R}^p$.
For **measurement matrix** $X \in \mathbb{R}^{n \times p}$, and **noise vector** $W \in \mathbb{R}^n$,
we **observe** n noisy linear samples of β^*, $Y = X\beta^* + W$.

Goal: Given (Y, X), **recover** β^*.



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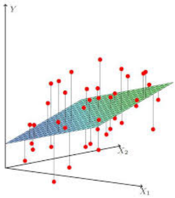

Distributional assumption:
X has iid $N(0,1)$ entries and W has iid $N(0,\sigma^2)$ entries.

Main Question:

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An immediate answer under full generality: at least p.

Reason: Even if $\mathrm{W} = 0$, we have $\mathrm{Y} = \mathrm{X}\beta^*$, a linear system with p unknowns and n equations!
To solve it, we need at least p equations, i.e. $\mathrm{n} \geq \mathrm{p}$.

Problem: A High Dimensional Reality

In many *real-life applications* of Linear Regression
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Question: Are we doomed to not use all the features or can we handle such a situation?

Structural Assumptions on β^*

-*Sparsity!* k $\leq$ p non zero coordinates.

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Success, but is that n sufficiently small for all applications?
Does this structure leads to the best algorithms?

Main Motivation

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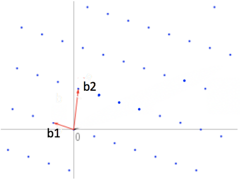
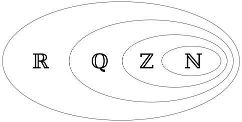
Generic and **natural** assumption that allows

- *efficient* recovery of β^*
- for significantly *small sample sizes.*

This talk

New efficient algorithm for recovering β^* from (Y, X)
based on LLL algorithm (short vectors on lattices)
under a new generic structural assumption (**Q-rationality assumption**).





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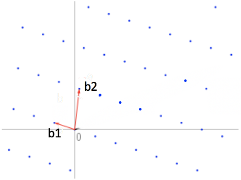

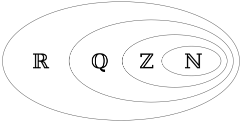

Guarantees: works for any n (even n = 1) given sufficiently small noise!

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Every entry of β^* is a **rational number** with denominator Q.

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The New Model

Setup: Let β^* be a **Q-rational** vector. For

- $X \in \mathbb{R}^{n \times p}$ consisting of entries i.i.d $N(0,1)$ **random variables**
- $W \in \mathbb{R}^n$ consisting of entries i.i.d. $N(0,\sigma^2)$ **random variables**

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$$\mathsf{Y} := \mathsf{X}\beta^* + \mathsf{W}.$$

Goal: Given (Y,X), recover efficiently β^* with n as small as possible. The recovery should happen with probability tending to 1 as p tend to infinity **(w.h.p.)**.

Brute-force with one sample

Any hope for $n = 1$? Recall $y_1 = \langle X_1, \beta^* \rangle + w_1$.

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Brute Force Algorithm

Check all Q-rational β for

$$y_1 = \langle X_1, \beta \rangle.$$

Termination Time $\underbrace{(Q+1)(Q+1)\dots(Q+1)}_{p \text{ terms}} = (Q+1)^p$**-not efficient!**

Brute-force with one sample (proof)

Lemma (Brute-force works!)

Suppose β^* Q-*rational and* $\sigma^2 = 0$.
There is no Q-*rational* $\beta \neq \beta^*$ *with* $y_1 = \langle X_1, \beta \rangle$, *almost surely.*

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Proof Sketch:

For any $\beta \neq \beta^*$,

$$\mathbb{P}\left(y_1 = \langle X_1, \beta \rangle\right) = \mathbb{P}\left(\langle X_1, \beta^* \rangle = \langle X_1, \beta \rangle\right) = \mathbb{P}\left(\langle X_1, \beta^* - \beta \rangle = 0\right) = 0,$$

since $\langle X_1, \beta^* - \beta \rangle \sim N\left(0, \|\beta^* - \beta\|_2^2\right)$.

Union bound over Q-rational β completes the proof.

An Efficient Algorithm

Theorem (informal, (Gamarnik,Z. NIPS '18))

Suppose you have $\mathsf{n} \ll \mathsf{p}$ *samples and* $0 \leq \sigma \leq \exp\left(-\frac{\mathsf{p}(\mathsf{p}+\log \mathsf{Q})}{2\mathsf{n}}\right)$.
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(1) An efficient algorithm which works for any sample size $n \ll p$.
(2) For $n = 1$, It works in time poly in $p, \log Q$, an exponential decrease from brute-force $(Q+1)^p$.

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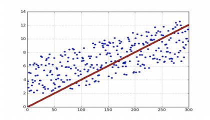
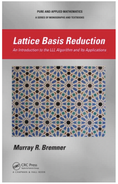
(1) The algorithm has optimal (exponentially small) noise tolerance in the 'high' Q regime!

(2) $\mathrm{Q} = 2^{\mathrm{p}}$ means p bits per coordinate of β^*, reasonable regime.

The Algorithm: Connecting HDLR with Lattices





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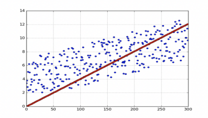

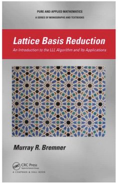

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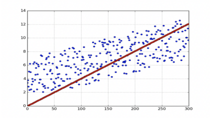

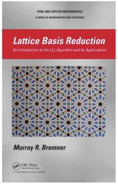

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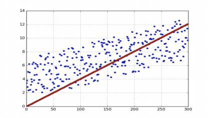

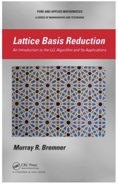

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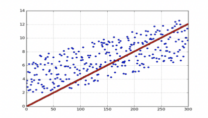

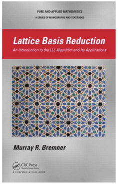

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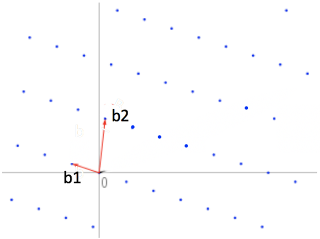
(3) Hint for the general case

Lattices

Let $b_1, \dots, b_m \in \mathbb{Z}^p$ linearly independent vectors.

Definition

The lattice $\mathcal{L}$ spanned by $b_1, \dots, b_m$ is the set of all integer combinations of the m vectors.



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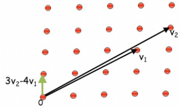
"The Shortest vector problem" $\min_{x \in \mathcal{L} \setminus \{0\}} \|x\|_2$



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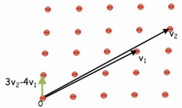

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Comments:

- Generally NP-hard but,
- A famous algorithm proposed by Lenstra-Lenstra-Lovasz (LLL) efficiently approximates it; find $\hat{x} \in \mathcal{L} \setminus \{0\}$ with

$$\|\hat{x}\|_2 \leq 2^{\frac{p}{2}} \min_{x \in \mathcal{L} \setminus \{0\}} \|x\|_2.$$

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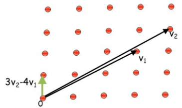

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  Seems *terrible* but it is **not!!**

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- Use LLL and recover an "approximation" of β^*.
- Recover β^* from approximation using the structure of β^*.

The Algorithm (special case, [F '84])

Assume

- $n = 1$, $\sigma = 0$, β^* binary: $y = \langle X_1, \beta^* \rangle$.
- $X_1 \in \mathbb{Z}^p$ with iid **uniform in** $[2^N]$ **entries** for large N (say $N = p^2$).

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(1) For M sufficiently large enough set $\mathcal{L}_M(y_1, X_1)$ produced by the columns of

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Intuition:

$$z = A_M \begin{bmatrix} \beta \\ \lambda \end{bmatrix} = \begin{bmatrix} M\langle X_1, \beta \rangle - M\lambda y_1 \\ \beta \end{bmatrix} = \begin{bmatrix} M\langle X_1, \beta - \lambda\beta^* \rangle \\ \beta \end{bmatrix},$$

$\mathbb{P}\,(\text{Lemma is false}) \leq \mathbb{P}\,(\exists \beta \neq \lambda\beta^* : \|\beta\|_2 < M, \langle X_1, \beta - \lambda\beta^* \rangle = 0) \to 0.$

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- We know $A_M \begin{bmatrix} \beta^* \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \beta^* \end{bmatrix} \in \mathcal{L}$.
- LLL gives $\hat{x}$ with norm at most $2^{\frac{p^2}{2}} \| \begin{bmatrix} 0 \\ \beta^* \end{bmatrix} \|_2 \leq 2^{\frac{p^2}{2}}\sqrt{p} < M$.
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(3) Rescale to get β^*.

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- Reminiscent of the big M-method in LP, but for lattices!

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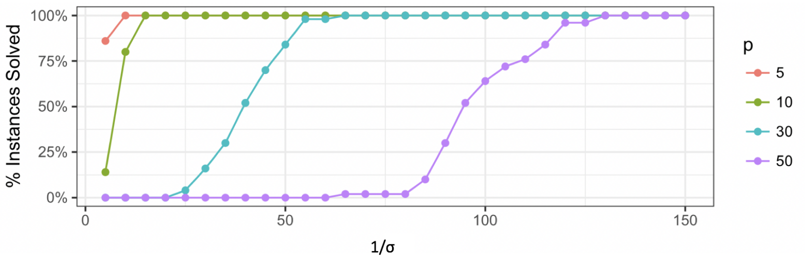
- Reminiscent of the big M-method in LP, but for lattices!
- Generalizes (after quite some work)
  - From noiseless to noisy measurements.
  - From iid uniform in $[2^N]$ to iid Gaussian ("truncate and multiply")
  - From $n = 1$ to multiple n
  - From binary to Q-rational β.

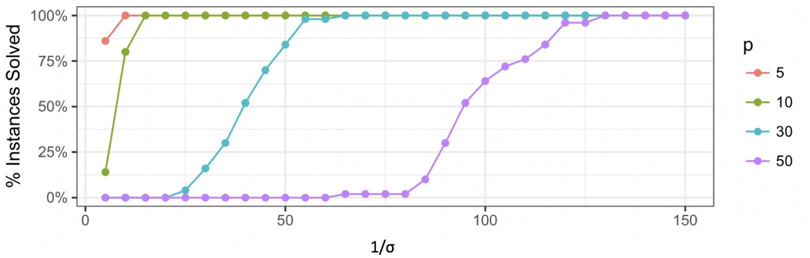
Experimental Results (small p)

(Julia Code by Andrew Zheng and Patricio Foncea (MIT ORC))



Experimental Results (small p)

(Julia Code by Andrew Zheng and Patricio Foncea (MIT ORC))



- Show that algorithm works even for small p!
- Runtime at most 8 minutes (even for $p = 50$)

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(3) The algorithm is built on an innovative **algorithmic connection** with Lattice theory and Linear Regression.

(4) **Synthetic experiments** suggest the algorithm works for small p.

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Thank you!!