# Revealing Network Structure, Confidentially (Improved Rates for Node-Private Graphon Estimation)

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joint work with Christian Borgs[2], Jennifer Chayes[2] and Adam Smith[3]

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# Introduction

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- the disease transmission graph
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**Statistical Analysis of Networks:**
Important across *scientific fields* (sociology, medicine etc)
rich in *theory* (random graphs, graph algorithms etc)

# The privacy issue

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Cambridge
Analytica

### Natural Question

How can we **analyze** network data with human users,
but **respect individual privacy**?

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Restricting the class of algorithms may cause **significant accuracy loss!**

**Main Motivation: Quantify the trade-off**

How much accuracy is necessarily **sacrificed**
if we **restrict** ourselves to differentially private algorithms?

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(2) **Erdos-Renyi-Estimation of probability** p:

- Compute (almost) tightly **the optimal estimation rate**
- Uses a **novel extension lemma**, potentially of broad use

# Outline

(1) Node Differential Privacy and Stochastic Block Model

(2) The Statistical Task

(3) Main Results

- Upper Bound for k-SBM (optimal in many regimes)
- Lower Bound for k-SBM, $k \geq 2$
- The case $k = 1$ (Erdos-Renyi case)-an almost tight optimal rate

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**D.P. algorithms: General Idea**

If two input datasets " *differ only on the data of one user*",
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- **(Edge-DP)** Two graphs are close if they differ in an **edge**.
  (a *simple "local"* notion, protects *"relationships of individuals"*.)
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**This work is for node-DP!**

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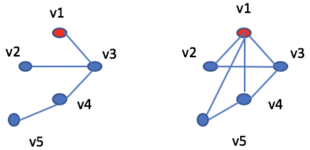

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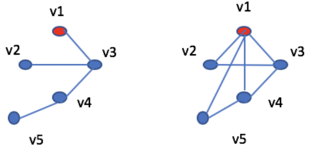


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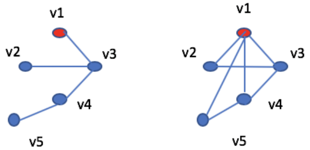


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**Definition**

A randomized $\mathcal{A}$ on n-vertex graphs is $\epsilon$-**node-DP** if for **all** node-neighbors $G, G'$and v in the output space,

$$\exp(-\epsilon)\,\mathbb{P}\left(\mathcal{A}(G') = v\right) \leq \mathbb{P}\left(\mathcal{A}(G) = v\right) \leq \exp(\epsilon)\,\mathbb{P}\left(\mathcal{A}(G') = v\right).$$

# k-Stochastic Block Model for Large Networks

Parameters

- n **nodes (users)**
- k **types** (based on characteristics such as *social status*, *cultural background*, *political identity*.)
- $B \in [0, 1]^{k \times k}$ symmetric **probability (frequency) matrix** between the k types.

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(1) Each node v chooses type(v) from [k] iid **u.a.r.**.

(2) Nodes v, w connect with an edge **w.p.** $B_{\text{type}(v),\text{type}(w)}$ independently.

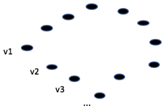
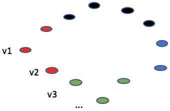



Figure: $n = 12$

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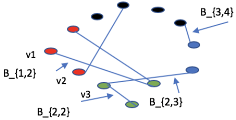


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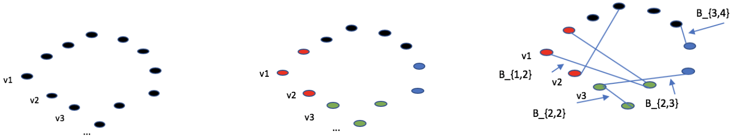
If $k = 1$, simple **Erdos-Renyi** model $G(n,p)$!



# Modeling Large Networks: k-Stochastic Block Model

**k-SBM**, $G(n, B)$, for sym. $B \in [0,1]^{k \times k}$:
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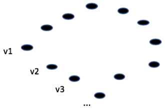
each edge between $v, w$ **with probability** $B_{type(v),type(w)}$ **independently.**

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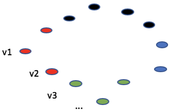


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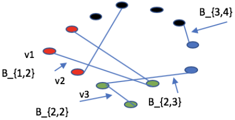


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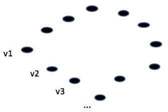


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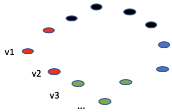


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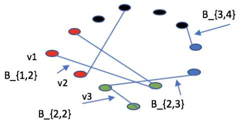


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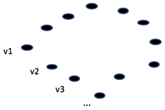


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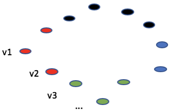


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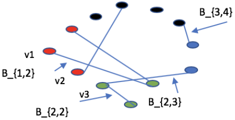


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*Vast literature* (without privacy): connections with
**community detection (gene expressions, webpage sorting), planted bisection problem, statistical physics models**.

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(2) **The Statistical Task**

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- Upper Bound for k-SBM (optimal in many regimes)
- Lower Bound for k-SBM, $k \geq 2$
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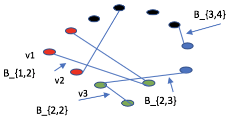
# The Statistical Setting

Let (unknown) $B \in [0, \rho]^{k \times k}$ for (known) k.



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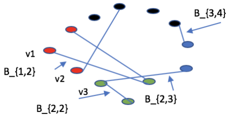


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We observe **one** n-vertex sample G from $G(n, B)$.
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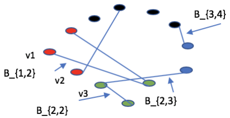


## Task:

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## Metric (types-order invariant) for fixed B

$$\delta_2(\mathcal{A}(G), B) = \min_{\pi:[k]\to[k]} \frac{1}{k} \|\mathcal{A}(G)_\pi - B\|_2,$$

where $\mathcal{A}(G)_\pi = \left(\mathcal{A}(G)_{\pi(i),\pi(j)}\right)_{i,j}$.
For $G \sim G(n, B)$, focus on **MSE** $\mathbb{E}_{G\sim G(n,B)}\left[\delta_2\left(\mathcal{A}(G), B\right)^2\right]$.

# The Statistical Question

## Performance of Algorithm for general B

Each $\mathcal{A}$ has (worst-case over B) **error**

$$\mathrm{err}(\mathcal{A}) = \max_{B \in [0,\rho]^{k \times k}} \mathbb{E}_{G \sim G(n,B)} \left[ \delta_2 \left( \mathcal{A}(G), B \right)^2 \right]$$

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## The Estimation Rate

$$R_k(\epsilon) = \min_{\mathcal{A}\ \epsilon\text{–node-DP}} \mathrm{err}(\mathcal{A}).$$

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- Note that we assume G is generated from k-SBM.
- In paper, we generalize to the **agnostic setting** to fitting k-SBM to a $k'$-SBM for unknown $k' > k$.

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# Upper Bound for k-SBM

**Theorem (informal, (BCSZ FOCS '18))**

*For any* $\epsilon > 0$,

$$\mathcal{R}_k(\epsilon) = O\left(\rho\left(\frac{k^2}{n^2} + \frac{\log k}{n}\right)\right) + O\left(\frac{\rho^2 (k-1)^2 \log n}{n\epsilon} + \frac{1}{n^2\epsilon^2}\right)$$

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- *Intuition:* $\frac{k^2}{n^2}$ parametric rate for B, $\frac{\log k}{n} = \frac{\log k^n}{n^2}$ combinatorial rate
- Via a new detailed analysis of an $\epsilon$-node-DP algorithm proposed in (BCS '15).

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Comments:

- (GLZ'14), (MS'17), (KTV'17): Optimal $\epsilon$-independent part.
- Many regimes (e.g. $\epsilon, k$ constant and $\frac{1}{n} < \rho < \frac{1}{\log n}$):
  - **No additional accuracy loss** by imposing privacy!
  - (BCS'15) algorithm, optimal accuracy loss over **all** algorithms!

# Upper Bound: Proof Idea

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(3) Adjust the analysis from (KTV'17)- a delicate net argument- to show that it **is not much affected** by additive errors.

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**What about the $\epsilon$-dependent parts?**

We prove $\frac{1}{n^2\epsilon^2}$ is (almost) necessary if $k \geq 2$.

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# A lower bound for $k \geq 2$: A variant of SBM

We prove that the term $\frac{1}{n^2\epsilon^2}$ is **necessary** under a small model change.

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Recall for some underlying probability matrix B:

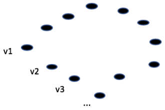


Figure: $n = 12$

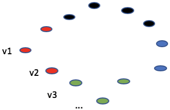


Figure: $k = 4$ u.a.r.

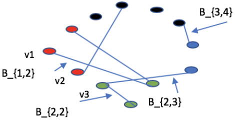


Figure: Assign Edges

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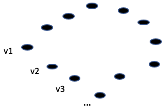
# A lower bound for $k \geq 2$: A variant of SBM

We prove that the term $\frac{1}{n^2\epsilon^2}$ is **necessary** under a small model change.

Recall for some underlying probability matrix B:

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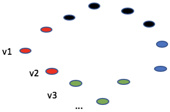


Figure: $k = 4$
**close to** u.a.r.

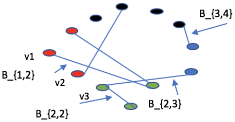


Figure: Assign Edges

.

## New k-SBM

Suppose each node $i \in [n]$ chooses it's type in a **close to** uniform way. (Say each type has probability in $[\frac{1}{4k}, \frac{4}{k}]$.)

# A lower bound for $k \geq 2$: Result

$$\mathcal{R}_k(\epsilon) = O\left(\rho\left(\frac{k^2}{n^2} + \frac{\log k}{n}\right)\right) + O\left(\frac{\rho^2(k-1)^2 \log n}{n\epsilon} + \frac{1}{n^2\epsilon^2}\right).$$

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**Proposition (informal, (BCSZ FOCS '18) )**

*For* k $\geq$ 2 *and any* $\epsilon > 0$,

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If $k = 1$, the rate can be improved to $\frac{1}{n^3\epsilon^2}$.

# Outline

(1) Node Differential Privacy and Stochastic Block Model

(2) The Statistical Task

(3) **Main Results**

- Upper Bound for k-SBM (optimal in many regimes)
- Lower Bound for k-SBM, $k \geq 2$
- **The case** $k = 1$ **(Erdos-Renyi case)-an almost tight optimal rate**

(4) The Extension Lemma

# The case $k = 1$: Learning privately Erdos Renyi graphs

**A fundamental open problem**

Observe simply a $G(n, p)$: estimate **privately** $p \in [0, 1]$

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## Task for $k = 1$

Compute

$$\mathcal{R}_1(\epsilon) = \min_{\mathcal{A}\ \epsilon\text{–node-DP}} \max_{p \in [0,1]} \mathbb{E}_{G \sim G_{n,p}} \left[ |\mathcal{A}(G) - p|^2 \right].$$

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What is the true $\epsilon$-dependent rate?! **(Almost tight) answer:** $\frac{1}{n^3 \epsilon^2}$

# The case $k = 1$: $\frac{1}{n^4\epsilon^2} \leq \epsilon-$dep. $\leq \frac{1}{n^2\epsilon^2}$

**Theorem (BCSZ FOCS '18)**

*For* $\epsilon > \frac{\log n}{n}$,

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*Furthermore, if* G *is sampled u.a.r. from graphs with a fixed number of edges (conditional Erdos Renyi) for* $\epsilon$ *constant,*

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For any f, $f(G) + \mathrm{Lap}(\frac{\Delta}{\epsilon})$ is $\epsilon$-node-DP for

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**Same upper bound**

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- Recall: $e(G) + \mathrm{Lap}(\frac{\Delta}{\epsilon})$ for $\Delta = \max_{G,G':d_v(G,G')=1} |e(G) - e(G')| = \Theta(\frac{1}{n})$.

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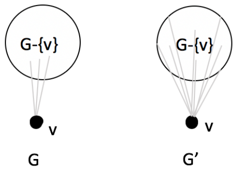

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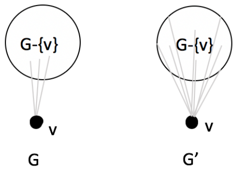

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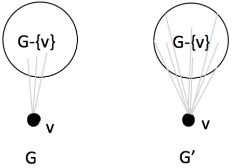
  $\Rightarrow$ $G$ or $G'$ very **atypical** for any Erdos-Renyi graph.
- How to exclude **atypical graphs**?
  (Challenge: need to be **private for all pairs** of graphs: )



# Upper Bound: Improving the Laplace estimator

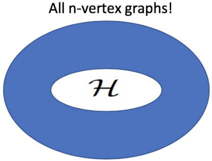
We construct a **subset** $\mathcal{H}$ of all n-vertex graphs

- **typical** for ER graphs

$$\max_{p\in[0,1]} \mathbb{P}_{G\sim G(n,p)}(G \notin \mathcal{H}) = O(\frac{1}{n^2}),$$

- with **lower sensitivity**

  $\max_{G,G'\in\mathcal{H}:d(G,G')=1} |e(G)-e(G')| = O(\sqrt{n\log n}/\binom{n}{2}) = O(\frac{\sqrt{\log n}}{n^{\frac{3}{2}}}).$



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- (Privacy in $\mathcal{H}$): Let $\hat{\mathcal{A}}(G) = e(G) + \mathrm{Lap}(\frac{2\sqrt{\log n}}{n^{\frac{3}{2}}\epsilon})$ ,$G \in \mathcal{H}$.

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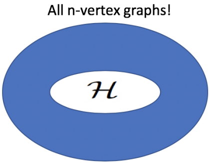

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- (Privacy in the whole space+ same accuracy:)
  **Extension lemma** which **extends** $\hat{\mathcal{A}}$ to $\mathcal{A}$
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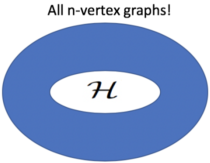


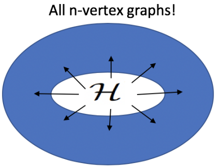

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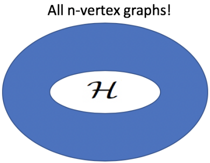


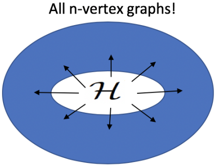

## The case $k = 1$: Main Result and Extension Lemma

### Theorem (BCSZ FOCS '18 )

*For* $\epsilon > \frac{\log n}{n}$,

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Special Importance for Upper Bound

**Extension Lemma**

*Extended private algorithm from typical instances to private algorithm on the whole space.*

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(4) **The Extension Lemma**

# The extension lemma: beyond networks

Technical challenge with *designing* differential private algorithms:

- **Privacy** constraint should hold for **any** pair of datasets
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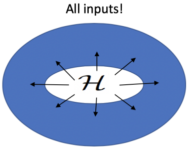
**Proposition ("Extending any DP Algorithm, (BCSZ FOCS'18) )**

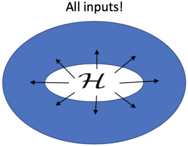
*Let $\hat{\mathcal{A}}$ $\epsilon$-DP on a subset of the input space $\mathcal{H} \subseteq \mathcal{M}$. Then there exists $\mathcal{A}$ defined on $\mathcal{M}$ which is 1) $2\epsilon$-DP on $\mathcal{M}$ and 2) $\forall \mathrm{D} \in \mathcal{H}$, $\mathcal{A}(\mathrm{D}) \overset{d}{=} \hat{\mathcal{A}}(\mathrm{D})$.*

Generalizes "extensions": (KNRS'13), (BBDS'13), (CZ'13), (RS'15).
**Note** on arXiv: *"Private Algorithms Can Always Be Extended"*



# Proof Ideas of Extension Lemma

- Differential-privacy can be translated into an $\epsilon$-**Lipschitz condition**. (small input changes leads to small output changes)



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- Differential-privacy can be translated into an $\epsilon$-**Lipschitz condition**. (small input changes leads to small output changes)
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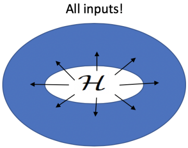

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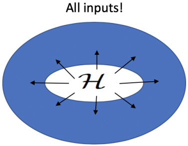

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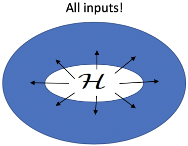

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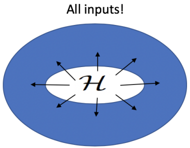
  Yet similar proof (alongside with measure-theory techniques) works by doubling the Lip constant.



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(4) Proved an **extension lemma** - potentially of broad use.

# Open Questions

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# Thank you!!

# The case $k = 1$: Lower Bound Sketch

**Theorem (BCSZ FOCS '18 )**

*For* $\epsilon > \frac{\log n}{n}$,

$$\mathcal{R}_1(\epsilon) = O\left(\frac{1}{n^2} + \frac{\log n}{n^3 \epsilon^2}\right).$$

*Furthermore, if* $G$ *is sampled u.a.r. from graphs with a fixed number of edges (conditional Erdos Renyi) for* $\epsilon$ *constant,*

$$\mathcal{R}_1'(\epsilon) = \Omega\left(\frac{1}{n^3 \epsilon^2}\right).$$

# Lower Bound: Estimation to Testing

## Goal

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## Random Graphs Question

For which $\alpha_n$, $G_{n,p}$ and $G_{n,p+\alpha_n}$ have node-distance $O\left(\frac{1}{\epsilon}\right)$?

# Lower Bound: Coupling Random Graphs

## Goal and an Easy Coupling

Need couple $G_{n,p}$ and $G_{n,p+\alpha_n}$ with *node-distance* $O\left(\frac{1}{\epsilon}\right)$.

Each edge $\alpha_n$-probability slack, easy to couple with $O\left(\alpha_n\binom{n}{2}\right)$ new edges.

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- **Harder** $\alpha_n = \frac{1}{n^{\frac{3}{2}} \epsilon}$ $\left(\Rightarrow \mathcal{R}_1(\epsilon) = \Omega\left(\frac{1}{n^3 \epsilon^2}\right)\right)$
  $O\left(\frac{\sqrt{n}}{\epsilon}\right)$ new edges, can we assign $\sqrt{n}$-edges per vertex?

## Proposition (Key Step)

*For appropriate choice of* $m = \Theta(n^2)$, *there is a coupling between* $G(n, m)$ *and* $G(n, m + o(\sqrt{n}))$ *where instances are always node-neighbors.*