# Logic

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# 1 Propositional logic

Convention: In this course we write $T$ for true and $F$ for false.

**Definition. 1.1.** The language of propositional logic consists of following symbols: *propositional variables* denoted (mostly) by $p, q, \ldots$ or $p_1, p_2, \ldots, q_1, q_2, \ldots$ and the *connectives* $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$.

**Definition. 1.2.** A *propositional formula* is a string of symbols obatained in the following way

1. Any variable is a formula

2. If $\phi$ and $\psi$ are formulas then so are $(\phi \wedge \psi), (\phi \vee \psi), (\neg \phi), (\phi \rightarrow \psi), (\phi \leftrightarrow \psi)$

3. Any formula is obtained in this way.

**Definition. 1.3.** A *truth function* of $n$ variables is a function

$$f : \{T, F\}^n \rightarrow \{T, F\} \quad .$$

***Exercise.*** How many functions are there for $n$ variables?

**Definition. 1.4.** Suppose $\phi$ is a formula with variables $p_1, \ldots, p_n$ then we obtain a truth function $F_\phi : \{T, F\}^n \rightarrow \{T, F\}$ whose value at $(x_1, \ldots, x_n) \quad x_i \in \{T, F\}$ is the truth value of $\phi$ when $p_i$ has value $x_i$. THe function $F_\phi$ is the *truth function of* $\phi$.

***Example.*** What is the truth function of

$$(((p \rightarrow q) \wedge (q \rightarrow (\neg p))) \rightarrow (\neg p)) \quad ?$$

It is always true.

**Definition. 1.5.** A propositional formula $\phi$ whose truth function $F_\phi$ is always true is called *tautology*. Say that formulas $\phi, \psi$ are *logically equivalent* (l.e.) if they have the same truth function.

***Remark.*** $\phi, \psi$ are l.e. iff $(\phi \leftrightarrow \psi)$ is a tautology. Also, suppose that we got some formula $\phi$ with variables $p_1, \ldots, p_n$ and $\phi_1, \ldots, \phi_n$ are formulas with variables $q_1, \ldots, q_r$. For each $i \leq n$ substitute $\phi$ in place of $p_i$ in $\phi$. Then the result is a formula $\psi$ and if $\phi$ is a tautology, then so is $\psi$.

*Proof.* The first statement is easy. For the second remark that

$$F_\psi(q_1, \ldots, q_r) = F_\phi(F_{\phi_1}(q_1, \ldots, q_r), \ldots, F_{\phi_n}(q_1, \ldots, q_r))$$

by the induction on the number of connectives in $\phi$. $\square$

***Example.***

1. $(p_1 \wedge (p_2 \wedge p_3))$ is l.e. to $((p_1 \wedge p_2) \wedge p_3)$,

2. same with $\vee$,

3. $(p_1 \vee (p_2 \wedge p_3))$ is l.e. to $((p_1 \vee p_2) \wedge (p_1 \vee p_3))$

4. similar the other way around.

5. etc.

***Remark.*** Note that by the remark above, we can boost these equivalences bz substituting formulas for the variables.