Logic

Luka Ilić

October 15, 2018

Contents

Preface

The following notes, are to be regarded as such – notes. They should contain most of what is written down in the logic lecture at Imperial College London (2018) by professor Evans. More likely then not, there will be a considerable amount of spelling errors (–please report everything to email down below or in Github comments–) that hopefully do not alter any important meaning. These notes will be constantly reread (by you the readers as well as myself) so I hope that at the end of the term most errors will be corrected so that anybody reading this will find good lecture notes for the exam.

At the moment I still have problems with my labels so if you need to jump to a reference, just click on it

Anybody willing to help me, can write me an email at

luka.ilic18@imperial.ac.uk

Help will only consist of being able to edit errors yourself. (So no need for any texing, except if you really want to.)

This is a project for my fellow students, so I hope it will be appreciated and used. I wish everybody reading this a lot of fun with the following content.

1 Propositional logic

1.1 Truth functions

Convention: In this course we write T for true and F for false.

Definition. 1.1. The alphabet of propositional logic consists of following symbols: *propositional variables* denoted (mostly) by $p, q, \ldots$ or $p_1, p_2, \ldots, q_1, q_2, \ldots$ and the *connectives* $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$.

Definition. 1.2. A *propositional formula* is a string of symbols obtained in the following way:

1. Any variable is a formula.

2. If ϕ and ψ are formulas then so are $(\phi \wedge \psi), (\phi \vee \psi), (\neg\phi), (\phi \rightarrow \psi), (\phi \leftrightarrow \psi)$.

3. Any formula is obtained in this way.

Definition. 1.3. A *truth function* of n variables is a function

$$f : \{T, F\}^n \rightarrow \{T, F\} \quad .$$

Exercise. How many functions are there for n variables?

Definition. 1.4. Suppose ϕ is a formula with variables $p_1, \ldots, p_n$ then we obtain a truth function $F_\phi : \{T, F\}^n \rightarrow \{T, F\}$ whose value at $(x_1, \ldots, x_n) \quad x_i \in \{T, F\}$ is the truth value of ϕ when p_i has value x_i. The function F_ϕ is the *truth function of* ϕ.

Remark. The truth tables for the connectives are the following:

p	$\neg p$
T	F
F	T

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

.

Example. What is the truth function of

$$(((p \rightarrow q) \wedge (q \rightarrow (\neg p))) \rightarrow (\neg p)) \quad ?$$

Definition. 1.5. A propositional formula ϕ whose truth function F_ϕ is always true is called *tautology*. Say that formulas ϕ, ψ are *logically equivalent* (l.e.) if they have the same truth function.

Remark. ϕ, ψ are l.e. iff $(\phi \leftrightarrow \psi)$ is a tautology. Also, suppose that we got some formula ϕ with variables $p_1, \ldots, p_n$ and $\phi_1, \ldots, \phi_n$ are formulas with variables $q_1, \ldots, q_r$. For each $i \leq n$ substitute ϕ in place of p_i in ϕ. Then the result is a formula ψ and if ϕ is a tautology, then so is ψ.

Proof. The first statement is easy. For the second remark that

$$F_\psi(q_1, \dots, q_r) = F_\phi(F_{\phi_1}(q_1, \dots, q_r), \dots, F_{\phi_n}(q_1, \dots, q_r))$$

by induction on the number of connectives in ϕ. □

Example.

1. $(p_1 \wedge (p_2 \wedge p_3))$ is l.e. to $((p_1 \wedge p_2) \wedge p_3)$,
2. same with $\vee$,
3. $(p_1 \vee (p_2 \wedge p_3))$ is l.e. to $((p_1 \vee p_2) \wedge (p_1 \vee p_3))$
4. similar the other way around.
5. etc.

Remark. Note that by the remark above, we can boost these equivalences by substituting formulas for the variables.

Definition. 1.6. Say that a set of connectives is *adequate* if for every $n \geq 1$, every truth function of n variables is the truth function of some formula which involves only connectives from the set and variables $p_1, \dots, p_n$.

Theorem. 1.7. *The set* $\{\neg, \wedge, \vee\}$ *is adequate.*

Proof. Let $G : \{T, F\}^n \to \{T, F\}$

1. $G(v) = F$ for all $v \in \{T, F\}$. Take ϕ to be $(p_1 \wedge (\neg p_1))$ then $G = F_\phi$
2. (*Disjunctive Normal Form* List the $v \in \{T, F\}^n$ with $G(v) = T$ as $v_1, \dots, v_r$. Write $v_i = (v_{i1}, \dots, v_{in})$ Define

   $$q_{ij} = \begin{cases} p_j & \text{if } v_{ij} = T \\ (\neg p_j) & \text{if } v_{ij} = F \end{cases}$$

   So q_{ij} has value T iff p_j has value v_{ij}. Let ψ_i be

   $$(q_{i1}, \dots, q_{in})$$

   Then $F_{\psi_i}(v) = T$ iff each q_{ij} has value T iff $v = v_i$.

   Let θ be $(\phi_1 \vee, \dots, \vee \phi_r)$. Then $F_\theta(v) = T$ iff $F_{\psi_i}(v) = T$ for some i which is equivalent to $v = v_i$ for some $i \leq r$. Thus $F_\theta(v) = T$ iff $G(v) = T$ i.e. $F_\theta = G$. As θ was constructed using only $\neg, \vee, \wedge$ the statement follows.

□

Definition. 1.8. A formula in the form as θ in the proof above (1.7) is said to be in *disjunctive normal form (dnf).*

Remark. Apart from the very intuitive and useful dnf, having a small adequate set at our disposal is useful for the following reason. It shortens induction proofs over the structure of formulas by a considerable amount, as the reader will surely experience in due time.

Corollary. 1.9. *Suppose* χ *is a formula which truth function is not always false. Then* χ *is l.e. to a formula in dnf.*

Proof. Take $G = F_\chi$ and apply the second case from the proof above. □

Example. For

$$\chi : \quad ((p_1 \to p_2) \to (\neg p_2))$$

the truth function $F_\chi(v)$ is true precisely when $v = \{T, F\}$ or $v = \{F, F\}$. Hence the dnf is:

$$((p_1 \wedge (\neg p_2)) \vee ((\neg p_1) \wedge (\neg p_2))).$$

Corollary. 1.10. *The following sets of connectives are adequate:*

1. $\neg, \vee$
2. $\neg, \wedge$
3. $\neg, \to$.

Proof.

1. By 1.7 we just need to show, that $\wedge$ can be expressed using $\neg, \vee$. $(p \wedge q)$ is l.e. to $(\neg((\neg p) \vee (\neg q))$.
2. similar to the approach above. $(p \vee q)$ is l.e. to $(\neg((\neg p) \wedge (\neg q))$.
3. Due to the cases above, it suffices to express either $\wedge$ or $\vee$ using $\neg, \to$. $(p \vee q)$ is l.e. to $((\neg p) \to q)$.

□

Example. Some sets of connectives that are not adequate are:

1. $\wedge, \vee$
2. $\neg, \leftrightarrow$

Proof.

1. If ϕ is build using $\wedge, \vee$ then $F_\phi(T, \dots, T) = T$ as proven by induction over number of connectives.
2. exercise.

□

Example. The *NOR* connective $\downarrow$ has truth table:

p	q	$(p \downarrow q)$
T	T	F
T	F	F
F	T	F
F	F	T

It is adequate on its own.

(exercise - express $\neg, \wedge$)

1.2 A formal system for propositional logic

Idea: Try to generate all tautologies from certain basic assumptions (axioms) using appropriate deduction rules.

Definition. 1.11. This is important!

A *formal deduction system* Σ has the following ingredients:

1. An *alphabet* A of symbols ($A \neq \emptyset$).
2. A non empty set $\mathcal{J}$ of the set of all finite sequences ('strings') of the elements of A: the *formulas* of Σ.
3. A subset $\mathcal{A} \subseteq \mathcal{J}$ called the *axioms* of Σ.
4. A collection of *deduction rules.*

Definition. 1.12. A *proof* in Σ us a finite sequence of formulas in $\mathcal{J}$

$$\phi_1, \dots, \phi_n$$

such that each ϕ_i is either an axiom *or* is obtained from $\phi_1, \dots, \phi_{i-1}$ using one of the deduction rules. The last (or any) formula in a proof is a *theorem* of Σ. Write $\vdash_\Sigma \phi$ for 'ϕ is a theorem of Σ'.

Remark. 1. If $\phi \in \mathcal{A}$ then $\vdash_\Sigma \phi$.

2. We should have an algorithm to test whether a string of symbols really is a formula and whether it is an axiom. Then someone who is willing to follow an algorithm precisely (computer) should be able to generate all possible proofs in σ and check whether something is a proof. (We say Σ is *recursive* in this case.)

Definition. 1.13. The formal system L for propositional logic consists of:

- **Alphabet**: variables $p_1, p_2, p_3 \dots$ connectives $\neg, \rightarrow$ punctuation),(.
- **Formulas**: as defined in 1.2 and will be called *L-formulas.*
- **Axioms**: Suppose ϕ, ψ, χ are L-formulas. The following are axioms of L:

  A1 $(\phi \rightarrow (\psi \rightarrow \phi))$

  A2 $((\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\psi \rightarrow \chi)))$

  A3 : $(((\neg\psi) \rightarrow (\neg\phi)) \rightarrow (\phi \rightarrow \psi))$
- **deduction rule**: *Modus Ponens (MP)* from $\phi, (\phi \rightarrow \psi)$ deduce ψ.

Example. Suppose ϕ is an L-formula. Then $\vdash_L (\phi \rightarrow \phi)$. A proof in L could be as follows:

1. $(\phi \rightarrow ((\phi \rightarrow \phi) \rightarrow \phi))$ use A1
2. $(\chi \rightarrow (\phi \rightarrow ((\phi \rightarrow \phi) \rightarrow \phi)))$ use A1 and MP
3. $((\phi \rightarrow (\phi \rightarrow \phi)) \rightarrow (\phi \rightarrow \phi))$
4. $(\phi \rightarrow (\phi \rightarrow \phi))$

5. $(\phi \to \phi)$.

Definition. 1.14. Suppose Γ is a set of L- formulas. A deduction from Γ is a finite sequence of formulas of L-formulas $\phi_1, \dots, \phi_n$ such that each one of these ϕ_i is either

- an axiom
- a formula in Γ or
- is obtained from previous formulas $\phi_1, \dots, \phi_{i-1}$ using the deduction rule (MP).

Write $\Gamma \vdash_L \phi$ if there is a deduction from Γ ending in ϕ.

Say Γ is a *consequence* of Γ.

Remark. Being a consequence of the empty set ($\Gamma = \emptyset$) is the same as being a theorem of L. ($\emptyset \vdash_L \phi \Leftrightarrow \vdash_L \phi$)

Theorem. 1.15 (Deduction Theorem.)**.** *Suppose Γ is a set of L-formulas and phi, ψ are L-formulas. Suppose*

$$\Gamma \cup \{\phi\} \vdash \psi$$

then

$$\Gamma \vdash_L (\phi \to \psi)$$

.

Corollary. 1.16. *Suppose ϕ, ψ, χ are L-formulas such that $\vdash_L (\phi \to \psi)$ and $\vdash_L (\psi \to \chi)$. Then $\vdash_L (\phi \to \chi)$*

Proof. Use 1.15 with $\Gamma = \emptyset$: Show $\{\phi\} \vdash_L \chi$. Here is a deduction of *chi* from *phi*:

1. $(phi \to \psi)$ (theorem of L)
2. $(\psi \to \chi)$ (theorem of L)
3. ϕ (assumption)
4. ψ (MP)
5. χ (MP).

Thus $\{phi\} \vdash_L \chi$. By 1.15: $\emptyset \vdash_L (\phi \to \chi)$ i.e. $\vdash_L (\phi \to \chi)$. □

Lemma. 1.17. *Suppose ϕ, ψ are L-formulas. Then*

1. $\vdash_L ((\neg\psi) \to (\psi \to \phi))$.
2. $\{(\neg\psi), \psi\} \vdash_L \phi$.
3. $\vdash_L (((\neg\phi) \to \phi) \to \phi)$

Proof.
1. problem sheet 1.
2. by 1. and MP (twice)

3. Suppose χ is any formula. Then $\{(\neg\phi), ((\neg\phi) \to \phi)\} \vdash_L \chi$ (by MP and 2.) Let α be any axiom and let χ be $(\neg\alpha)$. Apply 1.15 to 3 to get:

$$\{((\neg\phi) \to \phi)\} \vdash_L ((\neg\phi) \to (\neg\alpha))$$

A3: $(((\neg\psi) \to (\neg\phi)) \to (\phi \to \psi))$ and MP generate:

$$\{((\neg\phi) \to \phi)\} \vdash_L (\alpha \to \phi) \quad .$$

Since α is an axiom, by MP

$$\{((\neg\phi) \to \phi)\} \vdash_L \phi$$

and the application of 1.15 gives us:

$$\vdash_L (((\neg\phi) \to \phi) \to \phi) \quad .$$

□

Proof of 1.15: Suppose $\Gamma \cup \{\phi\} \vdash_L \psi$ using a deduction of length n. Show by induction on n that $\Gamma \vdash_L (\phi \to \psi)$.

Base step: $n = 1$. In this case ϕ is either an axiom or in Γ or is *phi*. In the first two cases $\Gamma \vdash_L \phi$ (one line deduction!) Using the A1 axiom $(\psi \to (\phi \to \psi))$ and MP we obtain $\Gamma \vdash_L (\phi \to \psi)$. In the last case – that $\phi = \psi$ – we already know

$$\Gamma \vdash (\phi \to \phi) \text{ by (1.13.)}$$

induction step: In our deduction of ψ from $\Gamma \cup \{\phi\}$ either ψ is an axiom or ψ is obtained from earlier steps using MP. In the last case these are formulas $\chi, (\chi \to \psi)$ earlier in the deduction. in the first case we argue as in the base case to get $\Gamma \vdash_L (\phi \to \psi)$. Otherwise we use the inductive hypothesis to get

$$\Gamma \vdash_L (\phi \to \chi)$$

and

$$\Gamma \vdash_L (\phi \to (\chi \to \psi)).$$

We have the A2 axiom

$$((\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\psi \to \chi)))$$

Using the two formulas we obtained, this axiom and MP twice we obtain $\Gamma \vdash_L (\phi \to \psi)$ as required, completing the induction step.

□

1.3 soundness and completeness of L

Theorem. 1.18 (soundness of L)**.** *Suppose ϕ is a theorem of L. Then ϕ is a tautology.*

Remark (notation). A (propositional) *valuation* v is an assignment of truth values to the propositional variables $p_1, p_2, \ldots$. So $v(p_i) \in \{T, F\}$ (for $i \in \mathbb{N}$). Note that, using the truth table rules, this assigns a truth value $v(\phi) \in \{T, F\}$ to every L-formula ϕ.

Proof of 1.18: By the induction on the length of a proof of ϕ it is enough to show:

1. every axiom is a tautology;
2. MP preserves tautologies. I.e. if $\psi, (\psi \to \chi)$ are tautologies, then so is χ.

For 1. use truth tables or argue as follows:

A2 Suppose for 1. if there is a valuation v with

$$v(((\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\psi \to \chi)))) = F$$

Then

$$v((\phi \to (\psi \to \chi))) = T$$

and

$$v(((\phi \to \psi) \to (\psi \to \chi))) = F$$

$v((\phi \to \psi) = T$ and $v((\phi \to \chi)) = F$ So by the last equality $v(\phi) = T, v(\chi) = F$ and also $(\psi) = T$ which contradicts the first equation.

A1 exercise

A3 exercise

For 2. if v is a valuation and $v(\psi) = T + v(\psi \to \chi) = T$ then $v(\chi) = T$.

□

Theorem. 1.19 (Generalization of soundness)**.** *Suppose Γ is a set of formulas and ϕ is a formula with $\Gamma \vdash_L \phi$. Suppose v is a valuation with $v(\phi) = T$ for all $\phi \in \Gamma$. Then $v(\phi) = T$.*

Proof. exercise □

Theorem. 1.20 (Completeness Theorem for L.)**.** *Suppose ϕ is a tautology. Then $\vdash_L \phi$.*

Remark (steps in the proof).

1. If $v(\phi) = T$ for all valuations v – we want to show $\vdash_L \phi$.
2. Try to prove a generalization: Suppose that for every v with $v(\Gamma) = T$ (i.e. $v(\phi) = T \forall \phi \in \Gamma$) we have $v(\phi) = T$. Then $\Gamma \vdash_L \phi$.
3. Equivalently, if $\Gamma \nvdash_L \phi$ show there is a valuation v with $v(\Gamma) = T$ and $v(\phi) = F$.

Definition. 1.21. A set Γ of L- formulas is *consistent* if there is no L-formula ϕ with

$$\Gamma \vdash_L \phi \text{ and } \Gamma \vdash_L (\neg\phi) \quad .$$

Proposition. 1.22. *Suppose Γ is a consistent set of L-formulas and $\Gamma \nvdash_L \phi$. Then $\Gamma \cup \{(\neg\phi)\}$ is consistent.*

Proof. Suppose not. So there is some formula ψ with

$$\Gamma \cup \{(\neg\phi)\} \vdash_L \psi$$

and

$$\Gamma \cup \{(\neg\phi)\} \vdash_L (\neg\psi)$$

Apply 1.15 to the second line above, then

$$\Gamma \vdash_L ((\neg\phi) \to (\neg\psi)).$$

By A3 and MP we obtain

$$\Gamma \vdash_L (\psi \to \psi).$$

Then $\Gamma \cup \{(\neg\phi)\} \vdash_L \phi$. By 1.15:

$$\Gamma \vdash_L ((\neg\phi) \to (\neg\phi)$$

and by a result from above

$$\vdash_L (((\neg\phi) \to \phi) \to \phi)$$

So by this, $\Gamma \vdash_L ((\neg\phi) \to (\neg\phi)$ and Modus Ponens we obtain

$$\Gamma \vdash_L \phi.$$

This contradicts $\Gamma \nvdash_L \phi$. ☐

Proposition. 1.23 (Lindenbaum Lemma)**.** *Suppose Γ is a consistent set of L-formulas. Then there is a consistent set of formulas $\Gamma^* \supseteq \Gamma$ such that for every ϕ either*

$$\Gamma^* \vdash_L \phi$$

or

$$\Gamma^* \vdash_L (\neg\phi) \quad .$$

(sometimes say Γ^* is complete*.)*

Proof. The set of all L-formulas is *countable*, so we can list the L-formulas as $\phi_0, \phi_1, \ldots$. (Why countable? Alphabet is countable: $\neg, \to,), (, p_1, p_2, \ldots$. Formulas are finite sequences from this alphabet, hence only countably many.) Define inductively sets of L-formulas $\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \ldots$, where

$$\Gamma_0 = \Gamma$$

and

$$\Gamma^* = \bigcup_{i \in \mathbb{N}} \Gamma_i \quad .$$

Suppose Γ_n has been defined. If $\Gamma_n \vdash_L \phi_n$ then let $\Gamma_{n+1} := \Gamma_n$. Otherwise, if $\Gamma_n \nvdash_L \phi_n$ then let $\Gamma_{n+1} := \Gamma_n \cup \{(\neg\phi_n)\}$. An easy induction using 1.22 shows that each Γ_i is consistent.

Claim:: Γ^* is consistent. If $\Gamma^* \vdash_L \phi$ and $\Gamma^* \vdash_L (\neg\phi)$ then as deductions are finite sequences of L -formulas

$$\Gamma_n \vdash_L \phi \text{ and } \Gamma_n \vdash_L (\neg\phi)$$

for some $n \in \mathbb{N}$: contradiction. Now to show that Γ^* is complete, let ϕ be any formula. So $\phi = \phi_n$ for some $n \in \mathbb{N}$. If $\Gamma^* \nvdash_L \phi$ then $\Gamma_n \nvdash_L \phi$. So by construction $\Gamma_{n+1} \vdash_L (\neg\phi)$. Therefore $\Gamma^* \vdash_L (\neg\phi)$ and since ϕ was arbitrary this proves the statement. □

Lemma. 1.24. *Let Γ^* be as above, then there is a valuation v such that for every L-formula ϕ*

$$v(\phi) = T \Leftrightarrow \Gamma^* \vdash_L \phi$$

Corollary. 1.25. *Suppose Δ is a set of L-formulas which is consistent and*

$$\Delta \nvdash_L \phi \quad .$$

Then there is a valuation v with

$$v(\Delta) = T \text{ and } v(\phi) = F \quad .$$

Proof. Let $\Gamma := \Delta \cup \{(\neg\phi)\}$. By 1.22, Γ is consistent. By the Lindenbaum lemma there is $\Gamma^* \supseteq \Gamma$ which is consistent and such that for every χ either $\Gamma^* \vdash_L \chi$ or $\Gamma^* \vdash_L \neg\chi$. By 1.24 there exists a valuation with $v(\Gamma^*) = T$. In particular $v(\Delta) = T$ and $v((\neg\phi)) = T$ thus $v(\phi) = F$. □

Theorem. 1.26 (completeness/Adequacy theorem for L)**.** *If $v(\phi) = T$ for every valuation v, then $\vdash_L \phi$.*

Proof. Suppose $\nvdash_L \phi$. then apply the corollary above with $\Delta = \emptyset$. (Why is this consistent? Soundness Theorem.) There is a valuation with $v(\phi) = F$. □

Proof of 1.24. Γ^* consistent set of L-formulas such that for every L-formula ϕ either $\Gamma^* \vdash_L \phi$ or $\Gamma^* \vdash_L \neg\phi$. Want valuation v with $v(\phi) = T$ for all $\phi \in \Gamma^*$. (i.e. $v(\phi) = T \Leftrightarrow \Gamma^* \vdash_L \phi$.) Note that for each variable p_i either $\Gamma^* \vdash_l p_i$ or $\Gamma^* \vdash_L (\neg p_i)$. So let v be the valuation with $v(p_i) = T \Leftrightarrow \Gamma^* \vdash_L p_i$. Now, prove by induction on the length of ϕ that $v(\phi) = T \Leftrightarrow \Gamma^* \vdash_L \phi$.

Base case: ϕ is just a propositional variable – this case holds by the definition of v.

inductive step: **Case 1:** ϕ is $(\neg\psi)$. "$\Rightarrow$": $v(\phi) = T$ then $v(\psi) = F$ and by the induction hypothesis $\Gamma^* \nvdash_L \psi$. Then by the completeness of Γ^*, $\Gamma^* \vdash_L (\neg\psi)$ i.e. $\Gamma^* \vdash_L \phi$. "$\Leftarrow$": Suppose $\Gamma^* \vdash_L \phi$. By consistency $\Gamma^* \nvdash_L \psi$. By the induction hypothesis $v(\psi) = F$. As v is a valuation we obtain $v(\phi) = T$ which concludes the first case.

Case 2: ϕ is $(\psi \to \chi)$. "$\Rightarrow$": Suppose $v(\phi) = F$. Then $v(psi) = T$ and $v(\chi) = F$. By the induction hypothesis $\Gamma^* \vdash_L \psi$ and $\Gamma^* \nvdash_L \chi$. If

$$\Gamma^* \vdash_L \phi$$

then using modus ponens and$\Gamma^* \vdash \psi$ we obtain

$$\Gamma^* \vdash_L \chi$$

which is a contradiction. So $\Gamma^* \nvdash_L \phi$. "$\Leftarrow$": Suppose $\Gamma^* \nvdash_L (\psi \to \chi)$. Then $\Gamma^* \nvdash \chi$ (due to A1). Also, due to a result from above, $\Gamma^* \nvdash_L (\neg\psi)$. By combining these results we obtain $v(\chi) = F$ and $v(\psi) = T$ therefore $v(\phi) = F$ which concludes the induction step.

$\square$

Corollary. 1.27. *Suppose Δ is a set of L-formulas and ϕ is an L-formula. Then*

1. *Δ is consistent if and only if there is a valuation v with $v(\Delta) = T$.*
2. *$\Delta \vdash_L \phi$ if and only if for every valuation v with $v(\Delta) = T$*

we have $v(\phi) = T$.

Proof. Exercise – deduce these from the preliminaries to 1.26. Warning: in the second statement Δ is not necessarily consistent. $\square$

Theorem. 1.28 (Compactness Theorem for L)**.** *Suppose Δ is a set of L-formulas. The following are equivalent:*

1. *There is a valuation v with $v(\Delta) = T$.*
2. *For every finite subset $\Delta_0 \subseteq \Delta$ there us a valuation w with $w(\Delta_0) = T$.*

Proof. By the above corollary, the first is true iff Δ is consistent. Similarly the second holds iff every finite subset is consistent. But if $\Delta \vdash_L \phi$ and $\Delta \vdash_L (\neg\phi)$ then as deductions are finite (+ therefore only involves finitely many L-formulas in Δ), for some finite $\Delta_0 \subseteq \Delta$, $\Delta_0 \vdash_L \phi$ and $\Delta_0 \vdash_L (\neg\phi)$. $\square$

Exercise. Let P be the set of sequences of