# Logic

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# Contents

## 1 Propositional logic

Convention: In this course we write $T$ for true and $F$ for false.

**Definition. 1.1.** The language of propositional logic consists of following symbols: *propositional variables* denoted (mostly) by $p, q, \ldots$ or $p_1, p_2, \ldots, q_1, q_2, \ldots$ and the *connectives* $\wedge, \vee, \neg, \to, \leftrightarrow$.

**Definition. 1.2.** A *propositional formula* is a string of symbols obatained in the following way

1. Any variable is a formula

2. If $\phi$ and $\psi$ are formulas then so are $(\phi \wedge \psi), (\phi \vee \psi), (\neg\phi), (\phi \to \psi), (\phi \leftrightarrow \psi)$
3. Any formula is obtained in this way.

**Definition. 1.3.** A *truth function* of $n$ variables is a function

$$f : \{T, F\}^n \to \{T, F\} \quad .$$

***Exercise.*** How many functions are there for $n$ variables?

**Definition. 1.4.** Suppose $\phi$ is a formula with variables $p_1, \ldots, p_n$ then we obtain a truth function $F_\phi : \{T, F\}^n \to \{T, F\}$ whose value at $(x_1, \ldots, x_n) \quad x_i \in \{T, F\}$ is the truth value of $\phi$ when $p_i$ has value $x_i$. The function $F_\phi$ is the *truth function of $\phi$.*

***Example.*** What is the truth function of

$$(((p \to q) \wedge (q \to (\neg p))) \to (\neg p)) \quad ?$$

It is always true.

**Definition. 1.5.** A propositional formula $\phi$ whose truth function $F_\phi$ is always true is called *tautology.* Say that formulas $\phi, \psi$ are *logically equivalent* (l.e.) if they have the same truth function.

***Remark.*** $\phi, \psi$ are l.e. iff $(\phi \leftrightarrow \psi)$ is a tautology. Also, suppose that we got some formula $\phi$ with variables $p_1, \ldots, p_n$ and $\phi_1, \ldots, \phi_n$ are formulas with variables $q_1, \ldots, q_r$. For each $i \leq n$ substitute $\phi$ in place of $p_i$ in $\phi$. Then the result is a formula $\psi$ and if $\phi$ is a tautology, then so is $\psi$.

*Proof.* The first statement is easy. For the second remark that

$$F_\psi(q_1, \ldots, q_r) = F_\phi(F_{\phi_1}(q_1, \ldots, q_r), \ldots, F_{\phi_n}(q_1, \ldots, q_r))$$

by the induction on the number of connectives in $\phi$. □

***Example.*** 1. $(p_1 \wedge (p_2 \wedge p_3))$ is l.e. to $((p_1 \wedge p_2) \wedge p_3)$,

2. same with $\vee$,

3. $(p_1 \vee (p_2 \wedge p_3))$ is l.e. to $((p_1 \vee p_2) \wedge (p_1 \vee p_3))$

4. similar the other way around.

5. etc.

***Remark.*** Note that by the remark above, we can boost these equivalences by substituting formulas for the variables.

**Definition. 1.6.** Say that a set of connectives is *adequate* if for evry $n \geq 1$, every truth function of $n$ variables is the truth function of some formula which involves only connectives from the set and variables $p_1, \ldots, p_n$.

**Theorem. 1.7.** *The set* $\{\neg, \wedge, \vee\}$ *is adequate.*

*Proof.* Let $G : \{T, F\}^n \to \{T, F\}$

1. $G(v) = F$ for all $v \in \{T, F\}$. Take $\phi$ to be $(p_1 \wedge (\neg p_1))$ then $G = F_\phi$

2. (*Disjunctive Normal Form* List the $v \in \{T, F\}^n$ with $G(v) = T$ as $v_1, \ldots, v_r$. Write $v_i = (v_{i1}, \ldots, v_{in})$ Define
$$q_{ij} = \begin{cases} p_j \text{if} v_{ij} = T \\ (\neg p_j) \text{if} v_{ij} = F \end{cases}$$
So $q_{ij}$ has value $T$ iff $p_j$ has value $v_{ij}$. Let $\psi_i$ be
$$(q_{i1}, \ldots, q_{in})$$
Then $F_{\psi_i}(v) = T$ iff each $q_{ij}$ has value $T$ iff $v = v_i$.

   Let $\theta$ be $(\phi_1 \vee, \ldots, \vee \phi_r)$. Then $F_\theta(v) = T$ iff $F_{\psi_i}(v) = T$ for some $i$ which is equivalent to $v = v_i$ for some $i \leq r$. Thus $F_\theta(v) = T$ iff $G(v) = T$ i.e. $F_\theta = G$. As $\theta$ was constructed using only $\neg, \vee, \wedge$ the statement follows.

□

**Definition. 1.8.** A formula in the form as $\theta$ in the proof above (1.7) is said to be in *disjunctive normal form* (dnf).

**Corollary. 1.9.** *Suppose $\chi$ is a formula which truth function is not always false. Then $\chi$ is l.e. to a formula in dnf.*

*Proof.* Take $G = F_\chi$ and apply the second case from the proof above. □

***Example.*** For
$$\chi : \quad ((p_1 \to p_2) \to (\neg p_2))$$
the truth function $F_\chi(v)$ is true, precisely when $v = \{T, F\}$ or $v = \{F, F\}$. Hence the dnf is:
$$((p_1 \wedge (\neg p_2)) \vee ((\neg p_1) \vee (\neg p_2))).$$

**Corollary. 1.10.** *The following sets of connectives are adequate:*

*1.* $\neg, \vee$

*2.* $\neg, \wedge$

*3.* $\neg, \rightarrow$.

*Proof.* 1. By 1.7 we just need to show, that $\wedge$ can be expressed using $\neg, \vee$. $(p \wedge q)$ is l.e. to $(\neg((\neg p) \vee (\neg q))$.

2. similar to the approach above. $(p \vee q)$ is l.e. to $(\neg((\neg p) \wedge (\neg q))$.

3. Due to the cases above, it suffices to obtain either $\wedge$ or $\vee$ using $\neg, \rightarrow$. $(p \vee q)$ is l.e. to $((\neg p) \rightarrow q)$.

$\square$