Logic

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Preface

The following notes, are to be regarded as such – notes. They should contain most of what is written down in the logic lecture at Imperial College London (2018) by professor Evans. More likely then not, there will be a considerable amount of spelling errors (–please report everything to email down below or in Github comments–) that hopefully do not alter any important meaning. These notes will be constantly reread (by you the readers as well as myself) so I hope that at the end of the term most errors will be corrected so that anybody reading this will find good lecture notes for the exam.

Anybody willing to help me, can write me an email at

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Help will only consist of being able to edit errors yourself. (So no need for any texing, except if you really want to.)

This is a project for my fellow students, so I hope it will be appreciated and used. I wish everbody reading this a lot of fun with the following content.

1 Propositional logic

1.1 Truth functions

Convention: In this course we write T for true and F for false.

Definition. 1.1. The alphabet of propositional logic consists of following symbols: *propositional variables* denoted (mostly) by $p, q, \ldots$ or $p_1, p_2, \ldots, q_1, q_2, \ldots$ and the *connectives* $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$.

Definition. 1.2. A *propositional formula* is a string of symbols obtained in the following way:

1. Any variable is a formula.

2. If ϕ and ψ are formulas then so are $(\phi \wedge \psi), (\phi \vee \psi), (\neg\phi), (\phi \rightarrow \psi), (\phi \leftrightarrow \psi)$.
3. Any formula is obtained in this way.

Definition. 1.3. A *truth function* of n variables is a function

$$f : \{T, F\}^n \rightarrow \{T, F\} \quad .$$

Exercise. How many functions are there for n variables?

Definition. 1.4. Suppose ϕ is a formula with variables $p_1, \ldots, p_n$ then we obtain a truth function $F_\phi : \{T, F\}^n \rightarrow \{T, F\}$ whose value at $(x_1, \ldots, x_n) \quad x_i \in \{T, F\}$ is the truth value of ϕ when p_i has value x_i. The function F_ϕ is the *truth function of* ϕ.

Example. What is the truth function of

$$(((p \rightarrow q) \wedge (q \rightarrow (\neg p))) \rightarrow (\neg p)) \quad ?$$

Definition. 1.5. A propositional formula ϕ whose truth function F_ϕ is always true is called *tautology*. Say that formulas ϕ, ψ are *logically equivalent* (l.e.) if they have the same truth function.

Remark. ϕ, ψ are l.e. iff $(\phi \leftrightarrow \psi)$ is a tautology. Also, suppose that we got some formula ϕ with variables $p_1, \ldots, p_n$ and $\phi_1, \ldots, \phi_n$ are formulas with variables $q_1, \ldots, q_r$. For each $i \leq n$ substitute ϕ in place of p_i in ϕ. Then the result is a formula ψ and if ϕ is a tautology, then so is ψ.

Proof. The first statement is easy. For the second remark that

$$F_\psi(q_1, \ldots, q_r) = F_\phi(F_{\phi_1}(q_1, \ldots, q_r), \ldots, F_{\phi_n}(q_1, \ldots, q_r))$$

by induction on the number of connectives in ϕ. $\square$

Example.
1. $(p_1 \wedge (p_2 \wedge p_3))$ is l.e. to $((p_1 \wedge p_2) \wedge p_3)$,
2. same with $\vee$,
3. $(p_1 \vee (p_2 \wedge p_3))$ is l.e. to $((p_1 \vee p_2) \wedge (p_1 \vee p_3))$

4. similar the other way around.
5. etc.

Remark. Note that by the remark above, we can boost these equivalences by substituting formulas for the variables.

Definition. 1.6. Say that a set of connectives is *adequate* if for evry $n \geq 1$, every truth function of n variables is the truth function of some formula which involves only connectives from the set and variables $p_1, \ldots, p_n$.

Theorem. 1.7. *The set* $\{\neg, \wedge, \vee\}$ *is adequate.*

Proof. Let $G : \{T, F\}^n \to \{T, F\}$

1. $G(v) = F$ for all $v \in \{T, F\}$. Take ϕ to be $(p_1 \wedge (\neg p_1))$ then $G = F_\phi$
2. (*Disjunctive Normal Form* List the $v \in \{T, F\}^n$ with $G(v) = T$ as $v_1, \ldots, v_r$. Write $v_i = (v_{i1}, \ldots, v_{in})$ Define
$$q_{ij} = \begin{cases} p_j & \text{if } v_{ij} = T \\ (\neg p_j) & \text{if } v_{ij} = F \end{cases}$$
So q_{ij} has value T iff p_j has value v_{ij}. Let ψ_i be
$$(q_{i1}, \ldots, q_{in})$$
Then $F_{\psi_i}(v) = T$ iff each q_{ij} has value T iff $v = v_i$.

   Let θ be $(\phi_1 \vee, \ldots, \vee \phi_r)$. Then $F_\theta(v) = T$ iff $F_{\psi_i}(v) = T$ for some i which is equivalent to $v = v_i$ for some $i \leq r$. Thus $F_\theta(v) = T$ iff $G(v) = T$ i.e. $F_\theta = G$. As θ was constructed using only $\neg, \vee, \wedge$ the statement follows.

□

Definition. 1.8. A formula in the form as θ in the proof above (1.7) is said to be in *disjunctive normal form (dnf).*

Remark. Apart from the very intuitive and useful dnf, having a small adequate set at our disposal is useful for the following reason. It shortens induction proofs over the structure of formulas by a considerable amount, as the reader will surely experience in due time.

Corollary. 1.9. *Suppose χ is a formula which truth function is not always false. Then χ is l.e. to a formula in dnf.*

Proof. Take $G = F_\chi$ and apply the second case from the proof above. □

Example. For
$$\chi : \quad ((p_1 \to p_2) \to (\neg p_2))$$
the truth function $F_\chi(v)$ is true precisely when $v = \{T, F\}$ or $v = \{F, F\}$. Hence the dnf is:
$$((p_1 \wedge (\neg p_2)) \vee ((\neg p_1) \wedge (\neg p_2))).$$

Corollary. 1.10. *The following sets of connectives are adequate:*

1. $\neg, \vee$
2. $\neg, \wedge$
3. $\neg, \rightarrow$.

Proof. 1. By 1.7 we just need to show, that $\wedge$ can be expressed using $\neg, \vee$. $(p \wedge q)$ is l.e. to $(\neg((\neg p) \vee (\neg q))$.

2. similar to the approach above. $(p \vee q)$ is l.e. to $(\neg((\neg p) \wedge (\neg q))$.
3. Due to the cases above, it suffices to express either $\wedge$ or $\vee$ using $\neg, \rightarrow$. $(p \vee q)$ is l.e. to $((\neg p) \rightarrow q)$.

$\square$

Example. Some sets of connectives that are not adequate are:

1. $\wedge, \vee$
2. $\neg, \leftrightarrow$

Proof. 1. If ϕ is build using $\wedge, \vee$ then $F_\phi(T, \ldots, T) = T$ as proven by induction over number of connectives.

2. exercise.

$\square$

Example. The *NOR* connective $\downarrow$ has truth table:

p	q	$(p \downarrow q)$
T	T	F
T	F	F
F	T	F
F	F	T

It is adequate on its own.

(exercise - express $\neg, \wedge$)

1.2 A formal system for propositional logic

Idea: Try to generate all tautologies from certain basic assumptions (axioms) using appropriate deduction rules.

Definition. 1.11. This is important!

A *formal deduction system* Σ has the following ingredients:

1. An *alphabet* A of symbols ($A \neq \emptyset$).
2. A non empty set $\mathcal{J}$ of the set of all finite sequences (‘strings‘) of the elements of A: the *formulas* of Σ.
3. A subset $\mathcal{A} \subseteq \mathcal{J}$ called the *axioms* of Σ.
4. A collection of *deduction rules.*

Definition. 1.12. A *proof* in Σ us a finite sequence of formulas in $\mathcal{J}$

$$\phi_1, \ldots, \phi_n$$

such that each ϕ_i is either an axiom *or* is obtained from $\phi_1, \ldots, \phi_{i-1}$ using one of the deduction rules. The last (or any) formula in a proof is a *theorem* of Σ. Write $\vdash_\Sigma \phi$ for 'ϕ is a theorem of Σ'.

Remark.

1. If $\phi \in \mathcal{A}$ then $\vdash_\Sigma \phi$.
2. We should have an algorithm to test whether a string of symbols really is a formula and whether it is an axiom. Then someone who is willing to follow an algorithm precisely (computer) should be able to generate all possible proofs in σ and check whether something is a proof. (We say Σ is *recursive* in this case.)

Definition. 1.13. The formal system L for propositional logic has:

- **Alphabet**: variables $p_1, p_2, p_3 \ldots$ connectives $\neg, \rightarrow$ punctuation),(.
- **Formulas**: as defined in 1.2 and will be called *L-formulas*.
- **Axioms**: Suppose ϕ, ψ, χ are L-formulas. The following are axioms of L:

  A1 $(\phi \rightarrow (\psi \rightarrow \phi))$

  A2 $((\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\psi \rightarrow \chi)))$

  A3 : $(((\neg\psi) \rightarrow (\neg\phi)) \rightarrow (\phi \rightarrow \psi))$
- **deduction rule**: *Modus Ponens (MP)* from $\phi, (\phi \rightarrow \psi)$ deduce ψ.

Example. Suppose ϕ is an L-formula. Then $\vdash_L (\phi \rightarrow \phi)$. A proof in L could be as follows:

1. $(\phi \rightarrow ((\phi \rightarrow \phi) \rightarrow \phi))$ use A1
2. $(\chi \rightarrow (\phi \rightarrow ((\phi \rightarrow \phi) \rightarrow \phi)))$ use A1 and MP
3. $((\phi \rightarrow (\phi \rightarrow \phi)) \rightarrow (\phi \rightarrow \phi))$
4. $(\phi \rightarrow (\phi \rightarrow \phi))$
5. $(\phi \rightarrow \phi)$.