DiffGeo

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1 Manifolds

Motivation: Some problems occur with our definition of curves and surfaces:

- The sphere is not a surface because no regular parametrisation (harry (Potter? WTF Jojo?) ball theorem)
- The hyperbola is no a surface, because it has two components thus it can not be parametrised by a single regular map on an open interval.

The notion of a submanifold resolves these problems at the expense of imposing other restrictions.

1.1 Submanifolds of $\mathcal{E}^n$

There are several equivalent characterisations of submanifolds in $\mathcal{E}^n$.

Definition. 1.1 (1. A submanifols can be locally flattened)**.** $M \subset \mathcal{E}^n$ is called a k-dimensional submanifold of $\mathcal{E}^n$ if for all $p \in M$ there exists a diffeomorphism $\phi : U \to \tilde{U}$, where $U \subset \mathcal{E}^n$ is an open neighbourhood of p and $\tilde{U} \subset \mathbb{R}^n$ is an open neighbourhood of 0 such that

$$\phi(M \cap U) = \tilde{U} \cap (\mathbb{R}^k \times \{0\}),$$

where $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^{n-k}$.

Definition. 1.2 (2. A submanifold is locally a level set)**.** $M \subset \mathcal{E}^n$ is a k-dimensional submanifold of $\mathcal{E}^n$ if for all $p \in M$ there exist open neighbourhood $U \subset \mathcal{E}^n$ of p and a submersion $F.U \to \mathbb{R}^{n-k}$ such that

$$M \cap U = F^{-1}\{0\}.$$

Where $dpF : \mathbb{R}^n \to \mathbb{R}^{n-k}$ surjects for all $p \in U$.

Bemerkung. In the definition above th is sufficient to require that $dpF : \mathbb{R}^n \to \mathbb{R}^{n-k}$ surjects: if dpF surjects then since $p \mapsto dpF$ is continuous, dpF surjects by the inertia principle on some open neighbourhood $\tilde{U} \subset U$ of p.

Definition. 1.3 (3. A submanifold can be locally parametrised)**.** $M \subset \mathcal{E}^n$ is a k-dimensional submanifold of $\mathcal{E}^n$ if for all $p \in M$ there exists an immersion $X : V \to U$ from an open neighbourhood $V \subset \mathbb{R}^k$ of 0 to an open neighbourhood $U \subset \mathcal{E}^n$ of p such that

$$M \cap U = X(V)$$

and $X : V \to M \cap U$ is a homeomorphism (with respect to the induced topology on $M \cap U$).

A homeomorphism is continuous and bijective.

Bemerkung. We get the following exlusions:

- X being an immersion excludes ”kinks” such as the singularity of the nilparabola.
- X being injective excludes self intersections.
- Continuity of X^{-1} excludes ”T-junctions”.

Beweis. Proof of equivalence of these definitions:

For $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^{n-k}$ we define the submersions

$$\pi_1 : \mathbb{R}^k \oplus \mathbb{R}^{n-k} \to \mathbb{R}^k, (x, y) \mapsto x,$$

$$\pi_2 : \mathbb{R}^k \oplus \mathbb{R}^{n-k} \to \mathbb{R}^k, (x, y) \mapsto y.$$

First we proof 1. implies 2.:

Let $F := \pi_2 \circ \phi : U \to \mathbb{R}^{n-k}$. F is a submersion.

Secondly we proof 1. implies 3.:

With $V = \pi_1(\tilde{U}) \subset \mathbb{R}^k$ we can have

$$X := \phi^{-1}\big|_v : V \to U$$

is the necessary map. If you are bored, you can check that this is an homeomorphism.

Now we proof 3. implies 1.:

Let $X : \mathbb{R}^k \supset V \to \mathcal{E}^n$ parametrisation of $M \cap U = X(V)$. Assume that $X(0) = p$. Let $(t_1, \dots, t_{n-k})$ be an orthonormal basis of $d_0X(\mathbb{R}^k)^\perp \subset \mathbb{R}^n$. Define

$$C \times \mathbb{R}^{n-k} : (x, y) \mapsto \psi(x, y) = X(x) + \sum_{i=1}^{n-k} t_i y_i, \quad y = (y_1, \dots, y_{n-k}).$$

Then

$$d_0\psi(v, w) = \underbrace{d_0X(v)}_{\in d_0X(\mathbb{R}^k)} + \underbrace{\sum_{i=1}^{n-k} t_i w_i}_{\in (d_0X(\mathbb{R}^k))^\perp} = 0$$

iff $w_i = 0$ for all i and $v = 0$ or $(v, w) = 0$.

Then we use the inverse mapping theorem, ψ has a smooth local inverse

$$\phi = (\psi|_{\tilde{U}})^{-1} : \psi(\tilde{U}) \to \tilde{U}$$

where $\tilde{U} \subset V \times \mathbb{R}^k$ open neighbourhood of 0. Without loss of generality, assume that $\psi(\tilde{U}) \subset U$ (otherwise take the intersection with U). Now, $q \in M \cap \psi(\tilde{U})$ implies there exists a $x \in V$ such that $q = X(x) = \psi(x, 0) \in \psi(\tilde{U})$. On the other hand

$$(x, 0) \in \tilde{U} \Rightarrow \psi(x, 0) = X(x) \in M$$

with means that $q = X(x) \in M \cap \psi(\tilde{U})$.

After replacing $\psi(\tilde{U})$ with U, then $\phi(U \cap M) = \tilde{U} \cap (\mathbb{R}^k \times \{0\})$.

2. $\Rightarrow$ 1. $F : U \to \mathbb{R}^{n-k}$ submersion. Let $t_1, \dots, t_n$ be an orthonormal basis of $\mathbb{R}^n$ such that $t_1, \dots, t_k$ is an orthonormal basis of $\ker d_pF$. Write $\mathbb{R}^n = \langle e_1, \dots, e_k \rangle \oplus \mathbb{R}^{n-k}$ then $q \in U \Rightarrow q = p + \sum_{i=1}^n t_i q_i$ and

$$\phi : U \to \mathbb{R}^n \quad , \quad \phi(q) = \sum_{i=1}^k e_i q_i + F(q)$$

$$d_p\phi(v) = \sum_{i=1}^{k} e_i v_i + d_pF(v) = 0 \text{ for } v = \sum_{i=1}^{k} t_i v_i$$

is equivalent to $d_pF(v) = 0$ and thus $v \in \ker d_pF$ and also to

$$\sum_{i=1}^{k} e_i v_i = 0 \text{ thus } \forall i v_i = 0 \Leftrightarrow v = 0$$

Thus $d_p\phi$ is invertable and by the Inverse Mapping Theorem

$$\phi : U \to \phi(U) \text{ is a diffeomorphism (maybe after shrinking } U).$$

Now, $q \in M \cap U \Leftrightarrow F(q) = 0 \Leftrightarrow \phi(q) \in \phi(U) \cap (\mathbb{R}^k \times \{0\})$. Thus $\phi(M \cap U) = \phi(U) \cap (R^k \times \{0\})$ $\square$

Beispiel. The following are manifolds:

1. Plane: $\pi = \{p \in \mathcal{E}^3 : \langle p - O, n\rangle = d\}$, $F : \mathcal{E}^3 \to \mathbb{R}, F(q) = \langle p - O, n\rangle - d$ then $\pi = F^{-1}(\{0\})$. Also, $d_pF(v) = \langle v, n\rangle$ and hence $d_pF \not\equiv 0$. Thus d_pF surjects and F is a submersion.
2. Sphere: $S = \{p \in \mathcal{E}^3 \mid \langle p - O, p - O\rangle = r^2\}$ and S is implicitly given by the function $F : \mathcal{E}^3 \to \mathbb{R}, F(p) = \langle p - O, p - O\rangle - r^2$. Here $d_pF(v) = 2\langle v, p - O\rangle$ and d_pF surjects as long as $p \not\equiv 0$ which does not happen in S. $F\big|_{\mathcal{E}^3 \setminus \{0\}}$ is a submersion.
3. Hyperboloids: $O + e_1x + e_2y + e_3z$ such that $F_\pm(O + e_1x + e_2y + e_3z) = 0$ where

$$F_\pm(O + e_1x + e_2y + e_3z) = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \pm 1$$

and

$$\nabla F = 2\left(e_1\frac{x}{a^2} + e_2\frac{y}{b^2} + e_3\frac{z}{c^2}\right).$$

Then $\nabla F = 0 \Leftrightarrow (x, y, z) = 0$. Therefore $F\big|_{\mathcal{E}^3 \setminus \{0\}}$ is a submersion with $\langle \nabla F, v\rangle = dF(v)$.

Beispiel. A counterexample is the following *Lemniscate*. $O + e_1x + e_2y$, where $x^4 - x^2 + y^2 = 0$. A regular parametrization is given by $t \mapsto O + e_1x(t) + e_2y(t) = O + e_1\sin(t) + e_2\sin(t)\cos(t)$. The curve has a self-intersection at $(x(t), y(t)) = (0, 0)$ which is equivalent to $\forall k \in \mathbb{Z} t = k\pi$. Hence this is not a 1-Dimensional submanifold.

Definition. 1.4. The tangent space of a k-dimensional submanifold $M \subset \mathcal{E}^n$ of $p \in M$ is the k-dimensional subspace

$$T_pM = d_oX(\mathbb{R}^k) \subset \mathbb{R}^n$$

where $X : \mathbb{R}^k \supset V \to \mathcal{E}^n$ is a parametrisation of M around $p = X(o)$.

Bemerkung. T_pM is independent of the parametrisation X.

Let $\tilde{X} = X_o\mu$ around $p = \tilde{X}(\tilde{o})$ where $\mu : \tilde{V} \to V$ is a diffeomorphism with $\mu(\tilde{o}) = o$. Then

$$d_o\tilde{X} = d_{\tilde{o}}(X \circ \mu) = (d_oX) \circ d_{\tilde{o}}\mu.$$

Therefore

$$d_{\tilde{o}}\tilde{X}(\mathbb{R}^k) = (d_oX)(d_{\tilde{o}}\mu(\mathbb{R}^k)) = d_oX(\mathbb{R}^k).$$

Bemerkung. If we have $M = F^{-1}(\{0\})$ is defined as the level set of a submersion $F : U \to \mathbb{R}^{n-k}$ then

$$T_pM = \ker d_pF.$$

This follows from the following: $F \circ X = 0$ for any parametrisation $X : V \to \mathcal{E}^n$ around $p = X(o)$. Then the chain rule gives us

$$0 = d_o(F \circ X) = (d_oF) \circ d_oX.$$

$$0 = (d_pF) \circ (d_oX(\mathbb{R}^k)) = (d_pF)(T_pM).$$

Therefore $T_pM \subset \ker d_pF$. But since F is a submersion $\dim(\ker d_pF) = k$ and $\dim(T_pM) = k$.

Beispiel. Lets consider a set of orthogonal maps

$$O(3) = \{A \in GL(3) : AA^T = id\} = \{A \in GL(3) : F(A) = 0\}$$

where $F : GL(3) \to Sym(3), F(A) = AA^T - id$. $Sym(3)$ is a 6-dimensional vector space.

$$d_AF : End(\mathbb{R}^3) \to Sym(3), d_AF(B) = BA^T + AB^T.$$

This surjects since any element of $Sym(3)$ can be written as $Y + Y^T$ so let $B = YA$. Hence F is a submersion and $O(3)$ is a 3-dimensional submanifold of $End(\mathbb{R}^3) \approx M_{3\times 3}$.

$$d_AF(B) = 0 \iff BA^T + AB^T = 0 \iff BA^T \in \square(3),$$

where $\square(3)$ is the skew-symmetric-endomorphisms. Therefore

$$T_AO(3) = \{B \in GL(3) = End(\mathbb{R}^3) | BA^T \in \square(3)\}$$

with is 3-dimensional and

$$T_ASO(3) = T_AO(3), \quad A \in SO(3).$$

This is an example of a Lie-group

Aufgabe. Think about GL(n) and $SL(3)$.

Fun-fact:

$$End(\mathbb{R}^n) = Sym(n) \oplus \square(n).$$

1.2 Functions on submanifolds

Perviosly functions, vector fields etc. were defined on open sets of affine spaces, where notions of differentiability makes sense. Now we want to consider functions on domains that are submanifolds of $\mathcal{E}^n$ To do this we define derivatives so that the chain rule holds.

Definition. 1.5. A function $\phi : M \to \mathcal{E}$ on a submanifold $M \subset \mathbb{R}^n$ is said to be differentiable at $p \in M$ with derivative

$$d_p\phi := d_0(\phi \circ X) \circ (d_oX)^{-1} : T_pM \to \mathbb{R}$$

uf $\phi_o X : \mathbb{R}^k \supset V \to \mathcal{E}$ is differentiable at o for some local parametrisation $X : V \to M$ of M around p with $X(o) = p$.

Bemerkung. This definition makes sense as it does not depend on our choice of parametrisation X.

Let $\tilde{X} = X \circ \psi$ is a reparametrisation at $p \in M$ for some diffeomorphism ψ then $\phi \circ \tilde{X} = \phi \circ X \circ \psi$ is differentiable as soon as $\phi \circ X$ is differentiable. Moreover, if we assume $\psi(o) = o$ then

$$d_o(\phi \circ \tilde{X}) \circ (d_o\tilde{X})^{-1} = d_o(\phi \circ X \circ \psi) \circ (d_o(X \circ \psi))^{-1}$$
$$= d_o(\phi \circ X) \circ d_o\psi \circ (d_o\psi)^{-1} \circ (d_oX)^{-1} = d_o(\phi \circ X) \circ (d_oX)^{-1}.$$

This definition can be easily generalised to $\mathcal{E}^n$-valued maps and thus to maps between submanifolds.

Bemerkung. Suppose that $\Phi : \mathcal{E}^n \to \mathcal{E}$ is differentiable and M is a submanifold of $\mathcal{E}^n$. Thus $\phi := \Phi\big|_M : M \to \mathcal{E}$ is differentiable with

$$d_p\phi = d_p\Phi\big|_{T_pM} : T_pM \to \mathbb{R}, \quad p \in M.$$

Let $X : V \to M$ be a parametrisation of M around $p = X(o)$, then $\phi \circ X = \Phi \circ X$ is differentiable and for $\xi = d_oX(x)$ then

$$d_p\phi(\xi) = d_o(\phi \circ X) \circ (d_oX)^{-1}(\xi) = d_0(\Phi \circ X)(x) = (d_{X(o)}\Phi) \circ d_oX(x) = d_p\Phi(\xi).$$

Definition. 1.6. Let $\phi : M \to \mathcal{E}$ be differentiable. Then the gradient of ϕ at $p \in M$ is the unique vector field $\operatorname{grad}\phi(p) \in T_pM$ with

$$d_p\phi(\xi) = \langle \xi, \operatorname{grad}\phi(p) \rangle, \quad \forall \xi \in T_pM.$$

Since $d_p\phi : T_pM \to \mathbb{R}$, $d_p\phi \in (T_pM)^*$, with Riesz-Fischer $\exists!$ vector $v \in T_pM$ such that $d_p\phi = \langle ., v \rangle$

Beispiel. Suppose $\mathbb{R}^2 \supset V \in (u, v) \mapsto X(u, v) \in \mathcal{E}^3$ is a parametrised surface with $I = Edu^2 + 2Fdudv + Gdv^2$. Let

$$X_u^* := \frac{1}{EG - F^2}(GX_u - FX_v), \quad X_v^* := \frac{1}{EG - F^2}(-FX_u + EX_v) \in T_{(u,v)}X.$$

Cause

$$\langle X_u^*, X_u \rangle = \langle X_v^*, X_v \rangle = 1, \quad \langle X_u^*, X_v \rangle = \langle X_v^*, X_u \rangle = 0$$

X_u^*, X_v^* is a dual basis.

Now let $\phi : M = X(V) \to \mathcal{E}$ be a differentiable function and define $\psi := \phi \circ X$. Then $\xi = d_{(u,v)}X(x) \in T_{X(u,v)}M$ and

$$\langle \operatorname{grad}\phi(X(u, v)), \xi \rangle = d_{X(u,v)}\phi(\xi) = (d_{(u,v)}\psi)(x).$$

Thus

$$\langle \operatorname{grad}\phi \circ X, X_u \rangle = \psi_u, \quad \langle \operatorname{grad}\phi \circ X, X_v \rangle = \psi_v.$$

Hence

$$\operatorname{grad}\phi \circ X = X_u^*\psi_u + X_v^*\psi_v = \frac{G\psi_u - F\psi_v}{EG - F^2}X_u + \frac{E\psi_v - F\psi_u}{EG - F^2}X_v,$$

where

$$\operatorname{grad}\phi \circ X = V \to TM.$$