DiffGeo

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1 Manifolds

Motivation: Some problems occur with our definition of curves and surfaces:

- The sphere is not a surface because no regular parametrisation (harry (Potter? WTF Jojo?) ball theorem)
- The hyperbola is no a surface, because it has two components thus it can not be parametrised by a single regular map on an open interval.

The notion of a submanifold resolves these problems at the expense of imposing other restrictions.

1.1 Submanifolds of $\mathcal{E}^n$

There are several equivalent characterisations of submanifolds in $\mathcal{E}^n$.

Definition. 1.1 (1. A submanifols can be locally flattened)**.** $M \subset \mathcal{E}^n$ is called a k-dimensional submanifold of $\mathcal{E}^n$ if for all $p \in M$ there exists a diffeomorphism $\phi : U \to \tilde{U}$, where $U \subset \mathcal{E}^n$ is an open neighbourhood of p and $\tilde{U} \subset \mathbb{R}^n$ is an open neighbourhood of 0 such that

$$\phi(M \cap U) = \tilde{U} \cap (\mathbb{R}^k \times \{0\}),$$

where $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^{n-k}$.

Definition. 1.2 (2. A submanifold is locally a level set)**.** $M \subset \mathcal{E}^n$ is a k-dimensional submanifold of $\mathcal{E}^n$ if for all $p \in M$ there exist open neighbourhood $U \subset \mathcal{E}^n$ of p and a submersion $F : U \to \mathbb{R}^{n-k}$ such that

$$M \cap U = F^{-1}\{0\}.$$

Where $dpF : \mathbb{R}^n \to \mathbb{R}^{n-k}$ surjects for all $p \in U$.

Bemerkung. In the definition above th is sufficient to require that $dpF : \mathbb{R}^n \to \mathbb{R}^{n-k}$ surjects: if dpF surjects then since $p \mapsto dpF$ is continuous, dpF surjects by the inertia principle on some open neighbourhood $\tilde{U} \subset U$ of p.

Definition. 1.3 (3. A submanifold can be locally parametrised)**.** $M \subset \mathcal{E}^n$ is a k-dimensional submanifold of $\mathcal{E}^n$ if for all $p \in M$ there exists an immersion $X : V \to U$ from an open neighbourhood $V \subset \mathbb{R}^k$ of 0 to an open neighbourhood $U \subset \mathcal{E}^n$ of p such that

$$M \cap U = X(V)$$

and $X : V \to M \cap U$ is a homeomorphism (with respect to the induced topology on $M \cap U$).

A homeomorphism is continuous and bijective.

Bemerkung. We get the following exlusions:

- X being an immersion excludes "kinks" such as the singularity of the nilparabola.
- X being injective excludes self intersections.
- Continuity of X^{-1} excludes "T-junctions".

Beweis. Proof of equivalence of these definitions:

For $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^{n-k}$ we define the submersions

$$\pi_1 : \mathbb{R}^k \oplus \mathbb{R}^{n-k} \to \mathbb{R}^k, (x, y) \mapsto x,$$

$$\pi_2 : \mathbb{R}^k \oplus \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}, (x, y) \mapsto y.$$

First we proof 1. implies 2.:

Let $F := \pi_2 \circ \phi : U \to \mathbb{R}^{n-k}$. F is a submersion.

Secondly we proof 1. implies 3.:

With $V = \pi_1(\tilde{U}) \subset \mathbb{R}^k$ we can have

$$X := \phi^{-1}\big|_v : V \to U$$

is the necessary map. If you are bored, you can check that this is an homeomorphism.

Now we proof 3. implies 1.:

Let $X : \mathbb{R}^k \supset V \to \mathcal{E}^n$ parametrisation of $M \cap U = X(V)$. Assume that $X(0) = p$. Let $(t_1, \ldots, t_{n-k})$ be an orthonormal basis of $d_0X(\mathbb{R}^k)^\perp \subset \mathbb{R}^n$. Define

$$C \times \mathbb{R}^{n-k} : (x, y) \mapsto \psi(x, y) = X(x) + \sum_{i=1}^{n-k} t_i y_i, \quad y = (y_1, \ldots, y_{n-k}).$$

Then

$$d_0\psi(v, w) = \underbrace{d_0X(v)}_{\in d_0X(\mathbb{R}^k)} + \underbrace{\sum_{i=1}^{n-k} t_i w_i}_{\in (d_0X(\mathbb{R}^k))^\perp} = 0$$

iff $w_i = 0$ for all i and $v = 0$ or $(v, w) = 0$.

Then we use the inverse mapping theorem, ψ has a smooth local inverse

$$\phi = (\psi|_{\tilde{U}})^{-1} : \psi(\tilde{U}) \to \tilde{U}$$

where $\tilde{U} \subset V \times \mathbb{R}^k$ open neighbourhood of 0. Without loss of generality, assume that $\psi(\tilde{U}) \subset U$ (otherwise take the intersection with U). Now, $q \in M \cap \psi(\tilde{U})$ implies there exists a $x \in V$ such that $q = X(x) = \psi(x, 0) \in \psi(\tilde{U})$. On the other hand

$$(x, 0) \in \tilde{U} \Rightarrow \psi(x, 0) = X(x) \in M$$

with means that $q = X(x) \in M \cap \psi(\tilde{U})$.

After replacing $\psi(\tilde{U})$ with U, then $\phi(U \cap M) = \tilde{U} \cap (\mathbb{R}^k \times \{0\})$.

2. $\Rightarrow$ 1. $F : U \to \mathbb{R}^{n-k}$ submersion. Let $t_1, \ldots, t_n$ be an orthonormal basis of $\mathbb{R}^n$ such that $t_1, \ldots, t_k$ is an orthonormal basis of $\ker d_pF$. Write $\mathbb{R}^n = \langle e_1, \ldots, e_k \rangle \oplus \mathbb{R}^{n-k}$ then $q \in U \Rightarrow q = p + \sum_{i=1}^n t_i q_i$ and

$$\phi : U \to \mathbb{R}^n \quad , \quad \phi(q) = \sum_{i=1}^k e_i q_i + F(q)$$

$$d_p\phi(v) = \sum_{i=1}^{k} e_i v_i + d_pF(v) = 0 \text{ for } v = \sum_{i=1}^{k} t_i v_i$$

is equivalent to $d_pF(v) = 0$ and thus $v \in \ker d_pF$ and also to

$$\sum_{i=1}^{k} e_i v_i = 0 \text{ thus } \forall i v_i = 0 \Leftrightarrow v = 0$$

Thus $d_p\phi$ is invertable and by the Inverse Mapping Theorem

$$\phi : U \to \phi(U) \text{ is a diffeomorphism (maybe after shrinking } U).$$

Now, $q \in M \cap U \Leftrightarrow F(q) = 0 \Leftrightarrow \phi(q) \in \phi(U) \cap (\mathbb{R}^k \times \{0\})$. Thus $\phi(M \cap U) = \phi(U) \cap (R^k \times \{0\})$ $\square$

Beispiel. The following are manifolds:

1. Plane: $\pi = \{p \in \mathcal{E}^3 : \langle p - O, n\rangle = d\}$, $F : \mathcal{E}^3 \to \mathbb{R}, F(q) = \langle p - O, n\rangle - d$ then $\pi = F^{-1}(\{0\})$. Also, $d_pF(v) = \langle v, n\rangle$ and hence $d_pF \not\equiv 0$. Thus d_pF surjects and F is a submersion.
2. Sphere: $S = \{p \in \mathcal{E}^3 \mid \langle p - O, p - O\rangle = r^2\}$ and S is implicitly given by the function $F : \mathcal{E}^3 \to \mathbb{R}, F(p) = \langle p - O, p - O\rangle - r^2$. Here $d_pF(v) = 2\langle v, p - O\rangle$ and d_pF surjects as long as $p \not\equiv 0$ which does not happen in S. $F\big|_{\mathcal{E}^3\setminus\{0\}}$ is a submersion.
3. Hyperboloids: $O + e_1x + e_2y + e_3z$ such that $F_\pm(O + e_1x + e_2y + e_3z) = 0$ where

$$F_\pm(O + e_1x + e_2y + e_3z) = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \pm 1$$

and

$$\nabla F = 2\left(e_1\frac{x}{a^2} + e_2\frac{y}{b^2} + e_3\frac{z}{c^2}\right).$$

Then $\nabla F = 0 \Leftrightarrow (x, y, z) = 0$. Therefore $F\big|_{\mathcal{E}^3\setminus\{0\}}$ is a submersion with $\langle\nabla F, v\rangle = dF(v)$.

Beispiel. A counterexample is the following *Lemniscate*. $O + e_1x + e_2y$, where $x^4 - x^2 + y^2 = 0$. A regular parametrization is given by $t \mapsto O + e_1x(t) + e_2y(t) = O + e_1\sin(t) + e_2\sin(t)\cos(t)$. The curve has a self-intersection at $(x(t), y(t)) = (0, 0)$ which is equivalent to $\forall k \in \mathbb{Z} t = k\pi$. Hence this is not a 1-Dimensional submanifold.

Definition. 1.4. The tangent space of a k-dimensional submanifold $M \subset \mathcal{E}^n$ of $p \in M$ is the k-dimensional subspace

$$T_pM = d_oX(\mathbb{R}^k) \subset \mathbb{R}^n$$

where $X : \mathbb{R}^k \supset V \to \mathcal{E}^n$ is a parametrisation of M around $p = X(o)$.

Bemerkung. T_pM is independent of the parametrisation X.

Let $\tilde{X} = X_o\mu$ around $p = \tilde{X}(\tilde{o})$ where $\mu : \tilde{V} \to V$ is a diffeomorphism with $\mu(\tilde{o}) = o$. Then

$$d_o\tilde{X} = d_{\tilde{o}}(X \circ \mu) = (d_oX) \circ d_{\tilde{o}}\mu.$$

Therefore

$$d_{\tilde{o}}\tilde{X}(\mathbb{R}^k) = (d_oX)(d_{\tilde{o}}\mu(\mathbb{R}^k)) = d_oX(\mathbb{R}^k).$$

Bemerkung. If we have $M = F^{-1}(\{0\})$ is defined as the level set of a submersion $F : U \to \mathbb{R}^{n-k}$ then

$$T_pM = \ker d_pF.$$

This follows from the following: $F \circ X = 0$ for any parametrisation $X : V \to \mathcal{E}^n$ around $p = X(o)$. Then the chain rule gives us

$$0 = d_o(F \circ X) = (d_oF) \circ d_oX.$$

$$0 = (d_pF) \circ (d_oX(\mathbb{R}^k)) = (d_pF)(T_pM).$$

Therefore $T_pM \subset \ker d_pF$. But since F is a submersion $\dim(\ker d_pF) = k$ and $\dim(T_pM) = k$.

Beispiel. Lets consider the set of orthogonal maps

$$O(3) = \{A \in \mathrm{GL}(3) : AA^T = id\} = \{A \in \mathrm{GL}(3) : F(A) = 0\}$$

where $F : \mathrm{GL}(3) \to \mathrm{Sym}(3), F(A) = AA^T - id$. $\mathrm{Sym}(3)$ is a 6-dimensional vector space.

$$d_AF : \mathrm{End}(\mathbb{R}^3) \to \mathrm{Sym}(3), d_AF(B) = BA^T + AB^T.$$

This surjects since any element of $\mathrm{Sym}(3)$ can be written as $Y + Y^T$ so let $B = YA$. Hence F is a submersion and $O(3)$ is a 3-dimensional submanifold of $\mathrm{End}(\mathbb{R}^3) \approx M_{3\times 3}$.

$$d_AF(B) = 0 \iff BA^T + AB^T = 0 \iff BA^T \in \square(3),$$

where $\square(3)$ is the skew-symmetric-endomorphisms. Therefore

$$T_AO(3) = \{B \in \mathrm{GL}(3) = \mathrm{End}(\mathbb{R}^3) | BA^T \in \square(3)\}$$

with is 3-dimensional and

$$T_ASO(3) = T_AO(3), \quad A \in SO(3).$$

This is an example of a Lie-group

Aufgabe. Think about $\mathrm{GL}(n)$ and $\mathrm{SL}(3)$.

Fun-fact:

$$\mathrm{End}(\mathbb{R}^n) = \mathrm{Sym}(n) \oplus \square(n).$$

1.2 Functions on submanifolds

Perviosly functions, vector fields etc. were defined on open sets of affine spaces, where notions of differentiability makes sense. Now we want to consider functions on domains that are submanifolds of $\mathcal{E}^n$. To do this we define derivatives so that the chain rule holds.

Definition. 1.5. A function $\phi : M \to \mathcal{E}$ on a submanifold $M \subset \mathbb{R}^n$ is said to be differentiable at $p \in M$ with derivative

$$d_p\phi := d_0(\phi \circ X) \circ (d_oX)^{-1} : T_pM \to \mathbb{R}$$

if $\phi_oX : \mathbb{R}^k \supset V \to \mathcal{E}$ is differentiable at o for some local parametrisation $X : V \to M$ of M around p with $X(o) = p$.

Bemerkung. This definition makes sense as it does not depend on our choice of parametrisation X.

Let $\tilde{X} = X \circ \psi$ is a reparametrisation at $p \in M$ for some diffeomorphism ψ then $\phi \circ \tilde{X} = \phi \circ X \circ \psi$ is differentiable as soon as $\phi \circ X$ is differentiable. Moreover, if we assume $\psi(o) = o$ then

$$d_o(\phi \circ \tilde{X}) \circ (d_o \tilde{X})^{-1} = d_o(\phi \circ X \circ \psi) \circ (d_o(X \circ \psi))^{-1}$$
$$= d_o(\phi \circ X) \circ d_o\psi \circ (d_o\psi)^{-1} \circ (d_o X)^{-1} = d_o(\phi \circ X) \circ (d_o X)^{-1}.$$

This definition can be easily generalised to $\mathcal{E}^n$-valued maps and thus to maps between submanifolds.

Bemerkung. Suppose that $\Phi : \mathcal{E}^n \to \mathcal{E}$ is differentiable and M is a submanifold of $\mathcal{E}^n$. Thus $\phi := \Phi\big|_M : M \to \mathcal{E}$ is differentiable with

$$d_p\phi = d_p\Phi\big|_{T_pM} : T_pM \to \mathbb{R}, \quad p \in M.$$

Let $X : V \to M$ be a parametrisation of M around $p = X(o)$, then $\phi \circ X = \Phi \circ X$ is differentiable and for $\xi = d_oX(x)$ then

$$d_p\phi(\xi) = d_o(\phi \circ X) \circ (d_oX)^{-1}(\xi) = d_0(\Phi \circ X)(x) = (d_{X(o)}\Phi) \circ d_oX(x) = d_p\Phi(\xi).$$

Definition. 1.6. Let $\phi : M \to \mathcal{E}$ be differentiable. Then the gradient of ϕ at $p \in M$ is the unique vector field $\operatorname{grad}\phi(p) \in T_pM$ with

$$d_p\phi(\xi) = \langle \xi, \operatorname{grad}\phi(p) \rangle\,, \quad \forall \xi \in T_pM.$$

Since $d_p\phi : T_pM \to \mathbb{R}$, $d_p\phi \in (T_pM)^*$, with Riesz-Fischer $\exists!$ vector $v \in T_pM$ such that $d_p\phi = \langle ., v \rangle$

Beispiel. Suppose $\mathbb{R}^2 \supset V \in (u, v) \mapsto X(u, v) \in \mathcal{E}^3$ is a parametrised surface with $I = Edu^2 + 2Fdudv + Gdv^2$. Let

$$X_u^* := \frac{1}{EG - F^2}(GX_u - FX_v), \quad X_v^* := \frac{1}{EG - F^2}(-FX_u + EX_v) \in T_{(u,v)}X.$$

Cause

$$\langle X_u^*, X_u \rangle = \langle X_v^*, X_v \rangle = 1, \quad \langle X_u^*, X_v \rangle = \langle X_v^*, X_u \rangle = 0$$

X_u^*, X_v^* is a dual basis.

Now let $\phi : M = X(V) \to \mathcal{E}$ be a differentiable function and define $\psi := \phi \circ X$. Then $\xi = d_{(u,v)}X(x) \in T_{X(u,v)}M$ and

$$\langle \operatorname{grad}\phi(X(u, v)), \xi \rangle = d_{X(u,v)}\phi(\xi) = (d_{(u,v)}\psi)(x).$$

Thus

$$\langle \operatorname{grad}\phi \circ X, X_u \rangle = \psi_u, \quad \langle \operatorname{grad}\phi \circ X, X_v \rangle = \psi_v.$$

Hence

$$\operatorname{grad}\phi \circ X = X_u^*\psi_u + X_v^*\psi_v = \frac{G\psi_u - F\psi_v}{EG - F^2}X_u + \frac{E\psi_v - F\psi_u}{EG - F^2}X_v,$$

where

$$\operatorname{grad}\phi \circ X = V \to TM.$$

Now we know how to differentiate functions on submanifolds : we can find analogues of notions such as I & II, shape operator, covariant derivative.

Definition. 1.7. Let ξ be a tangential verctor field, i.e., $\xi : M \to \mathbb{R}^n$ such that $\forall p \in M(\xi(p) \in T_pM)$. Let $\eta \in T_pM$. Then

$$\nabla_\eta \xi\big|_p = (d_p\xi(\eta))^T = (d_o(\xi \circ X)(y))^T$$

where $X : V \to \mathcal{E}^n$ is a local parametrisation of M at p with $p = X(o)$ and $d_oX(y) = \eta$. As usual $(\dots)^T$ denotes the tangential part, i.e., the orthogonal projection $\mathbb{R}^n \to T_pM$. ∇ is called the *Levi-Civita connection.*

Bemerkung. In case of parametrised surfaces $X : V \to \mathcal{E}^3$, $M = X(V)$. Then $Y := xi : V \to \mathcal{E}^3$ tangential vector field in the sense of chaper 2.

$$\nabla_\eta \xi = \nabla_y Y \quad d_{(u,v)}X(y) = \eta \quad X(u,v) = p$$

This yields a notion of second derivatives on surfaces:

Definition. 1.8. The *Hessian* of $\phi : M \to \mathcal{E}$ at $p \in M$

$$T_pM \times T_pM \ni (\xi,\eta) \mapsto (\text{hess}\phi)\big|_p(\xi,\eta) = \left\langle \eta, \nabla_\eta \text{grad}\phi\big|_p \right\rangle$$

is a symmetric tensor.

Beweis. Exercise/technical. □

Bemerkung. The Hessian is the covariant derivative of $d\phi : M \times TM \to \mathbb{R}$, $\xi, \eta : M \to TM$

$$\text{hess}\phi(\xi,\eta) = d(d\phi(\eta)(\xi) - d\phi(\nabla_\xi \eta) ='' (\nabla_\xi d\phi)(\eta)''$$

The Hessian depends on the covariant derivative, hence on the induced metric,not just on the differentiable structure on M.

Lemma. 1.9 (Poincaré Lemma). *A tangential vector field $\xi : M \to \mathbb{R}^n$ has a local potential, i.e., locally $\xi = \text{grad}\phi$, if and only if*

$$(\nu, \eta) \mapsto \langle \eta, \nabla_\nu \xi \rangle$$

is symmetric.

Beweis. We saw in the Lemma above that symmetry of $(\nu,\eta) \mapsto \langle \eta, \nabla : \nu\xi \rangle$ is a necessary condition for $\xi = \text{grad}\phi$. For sufficiency, let $X : V \to M \cap U$ be a local parametrisation. Let $\xi_1, \dots, \xi_k : M \to \mathbb{R}^k$ be a tangential v.f. such that $\xi_i \circ X = \dfrac{\partial X}{\partial x_i} = dX(\dfrac{\partial}{\partial x_i})$. Then given $\xi : M \to \mathbb{R}^n$ we seek a function $\psi : V \to \mathbb{R}$ such that $\dfrac{\partial \psi}{\partial x_i} = \langle \xi_i, \xi \rangle \circ X$. Then

$$\frac{\partial}{\partial x_i}(\langle \xi_i, \xi \rangle \circ X) = \left\langle \frac{\partial}{\partial x_i}(\xi : j \circ X), \xi \circ X \right\rangle + \left\langle \xi_j \circ X, \frac{\partial}{\partial x_i}(\xi \circ X) \right\rangle$$

then by the Leibniz rule:

$$= \left\langle \nabla_{\frac{\partial}{\partial x_i}}(\xi_j \circ X), \xi \circ X \right\rangle + \left\langle \xi_j \circ X, \nabla_{\frac{\partial}{\partial x_i}}(\xi \circ X) \right\rangle = \left[\left\langle \nabla_{\frac{\partial}{\partial \xi_i}} \xi_j, \xi \right\rangle + \left\langle \xi_j, \nabla_{\frac{\partial}{\partial \xi_i}} \xi \right\rangle \right] \circ X$$

LHS is symmetric as soon as $\langle \xi_j, \nabla_{\xi_i} \xi \rangle$ is. (N.B. $\nabla_{\xi_i} \xi_i$ because ∇ is "torsion free".) By Poincaré's Lemma $\exists \psi : V \to \mathbb{R}$ such that $\dfrac{\partial \psi}{\partial x_i} = \langle \xi_i, \xi \rangle \circ X$. Now define $\phi = \psi \circ X^{-1} : X(v) \to \mathbb{R}$. Check that $\xi = \mathrm{grad}\phi$. □

Lemma. 1.10. *Suppose that $X : \mathbb{R}^n \supset V \to \mathcal{E}^n$ is a local parametrisation of $M \subset \mathcal{E}^n$. Then*

$$\nabla_{\xi_i} \xi_j - \nabla_{\xi_j} \xi_i = 0,$$

where ξ_i, ξ_j are vector fields so that $\xi_i \circ X = \frac{\partial X}{\partial x_i}, \xi_j \circ X = \frac{\partial X}{\partial x_j} = dX\left(\frac{\partial}{\partial x_j}\right)$.

Beweis.

$$(\nabla_{\xi_i} \xi_j) \circ X = (d\xi_j(\xi_i) \circ X)^T = \left(d(\xi_j \circ X) dX^{-1}(\xi_i \circ X)\right)^T$$

$$= \left(d\left(\frac{\partial X}{\partial x_j}\right)\left(\frac{\partial}{\partial x_i}\right)\right)^T = \left(\frac{\partial^2 X}{\partial x_j \partial x_i}\right)^T.$$

similarly

$$\left(\nabla_{\xi_j} \xi_i\right) \circ X = \left(\frac{\partial^2 X}{\partial x_i \partial x_j}\right)^T.$$

Since Schwarz the mixed partial derivatives commute, nothing is left to proof. □

1.3 Vector fields and flows

Definition. 1.11. Let $\xi : M \to \mathbb{R}^n$ be a (tangential) vector field on a k-dimensional submanifold $M \subset \mathcal{E}^n$. A curve $C : I \to M$ on an open interval $I \subset \mathbb{R}$ is called an *integral curve of* ξ if

$$C' = \xi \circ C.$$

It is maximal if it cannot be extended as an integral curve.

Bemerkung. We do not require regularity. For example we could have $\xi = 0$. Then all integral curves are constant.

Lemma. 1.12. *Through any point $p \in M$ passes a unique maximal integral curve of a vector field ξ.*

Beweis. Let $X : V \to M$ be a local parametrisation of M around $X(o) = p \in M$ and write $\xi \circ X = dX(y)$, i.e.,

$$\xi \circ X(\tilde{o}) = d_{\tilde{o}} X(y(\tilde{o}))$$

for all $\tilde{o} \in V$ and $y : V \to \mathbb{R}^k$.

The ansatz $C = X \circ \gamma$ yields

$$C' = \xi \circ C$$

iff

$$dX \circ \gamma' = dX(y \circ \gamma).$$

Since $d_{\tilde{o}}X$ is an isomorphism for all $\tilde{o} \in V$, this holds for all $\gamma' = y \circ \gamma$. Then by applying Picard-Lindelöf then the initial-value-problem (IVP)

$$\gamma' = y \circ \gamma, \qquad \gamma(0) = o$$

has a solution $\gamma : J \to V$ on some open interval $J \subset \mathbb{R}$ with $0 \in J$. This is unique up to extension. $\square$

The max integral curves of a vector field ξ can be assembled into a single map.

Satz und Definition. 1.13. Give a smooth tangential vector field ξ on a submanifold M. There exists an unique smooth map called its *maximal flow*

$$\Phi : W \to M, \quad (t, p) \mapsto \Phi_t(p)$$

on an open neighbourhood W of $\{0\} \times M \subset \mathbb{R} \times M$ so that

1. $\Phi_0 = id$
2. $I_p := \{t | (t, p) \in W\}$ is an open interval containing 0 for all $p \in M$
3. $I_p \ni t \mapsto \Phi_i(p)$ is the maximal integral curve of ξ through p.

Beweis. 1),2) and 3) uniquely define Φ.

Check that $W = \bigcup_{p \in M} I_p \times \{p\}$ is an open subset of $\mathbb{R} \times M$ and that Φ is smooth. The smoothness dependence on the initial conditions. $\square$

Bemerkung und Definition. 1.14. If $M \subset \mathcal{E}^n$ is compact (closed and bounded) or more generally, ξ is compactly supported on M, i.e., there exists a compact $V \subset M$ such that $\xi|_{M \setminus V} = 0$, then

$$W = \mathbb{R} \times M, \text{ anf} \Phi_{s+t} = \Phi_s \circ \Phi_t.$$

Moreover $\Phi_t : M \to M$ is a diffeomorphism for any fixed $t \in \mathbb{R}$ and thus $\Phi : \mathbb{R} \times M \to M$ defines a 1-parameter group of diffeomorphisms. This is a subgroup of the group of diffeomorphisms from M to M and is often called *flow*.

Beispiel. Let $M = \{p \in \mathcal{E}^3 : \langle p - O, p - O \rangle = 1\}$ be the unit sphere centred around O. Define $\xi(p) = e_3 \times (p - O)$. This resembles the rotation of the earth.

If we write $p = O + e_1 x + e_2 y + e_3 z$ then

$$\xi(p) = -e_1 y + e_2 x$$

and

$$\Phi_t(p) = e_1(x \cos t - y \sin t) + e_2(x \sin t + y \cos t) + e_3 z.$$

Bemerkung. (c.f., $C_t(s)$ in Clairaut's theorem) resembles the flow $\Phi_t(C(s))$ on a surface of revolution.

WARNING: In the case ξ is not compactly supported there may not be an $\varepsilon > 0$ such that $W \supset (-\varepsilon, \varepsilon) \times M$. For example, with

$$M = \{O + e_1 u + e_2 v : |u| < 1\} \subset \mathcal{E}^2 \text{ and } \xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The flow for some fixed $p = O + e_1 u + e_2 v$ is

$$\Phi_t(p) : (-1-u, 1-u) \ni t \mapsto O + e_1(u+t) + e_2 v.$$

Therefore we cannot obtain a 1-parameter group of diffeomorphisms $M \to M$.

However, we obtain a *local flow*

Bemerkung. In general, the maximal flow $\Phi : W \to M$ is a local flow, i.e., There exists an open neighbourhood $(-\varepsilon, \varepsilon) \times U \subset M$ of $(0, p)$ for any point $p \in M$ such that

1. $\Phi_t|_U : U \to \Phi_t(U)$ is a diffeomorphism for all $t \in (-\varepsilon, \varepsilon)$
2. $\Phi_{s+t}(q) = (\Phi_s \circ \Phi_t)(q)$ wherever $q \in U$ and $s, t, a + t \in (-\varepsilon, \varepsilon)$

Beweis.

1. Fix $p \in M$. Since $W \subset \mathbb{R} \times M$ there exists an open neighbourhood U of p and an $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \times U \subset E$ (box topology). Since $\Phi_0|_U = id_U$ is a diffeomorphism, by inertia, $\Phi_t|_U$ is a diffeomorphism for all $t \in (-\varepsilon, \varepsilon)$. Perhaps after shrinking.
2. Suppose $s \mapsto C(s) = \Phi_{s+t}(q)$ for a fixed t. Then

$$C'(s) = \xi \circ C(s) \text{ and } C(0) = \Phi_t(q) \in M,$$

i.e., C is the integral curve of ξ through $\Phi_t(q)$. Hence by uniqueness $C(s) = \Phi_s \circ \Phi_t(q)$.

□

Das Logbuch ist unter *http://www.geometrie.tuwien.ac.at/hertrich-jeromin/tea/2018dgtm.txt* zu finde.