DiffGeo

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1 Curves on surfaces

1.1 Natural ribbon & special lines on surfaces

Definition. 1.1. Let $X : \mathbb{R}^2 \supset M \to \mathcal{E}^3$ a surface and $I \ni t \mapsto X(u(t), v(t))$ with a map $(u, v) : I \to M$ defines a curve on the surface X as soon as $X \circ (u, v)$ is regular:

$$\forall t \in I : (X_u u' + X_v v')(t) \neq 0 \quad \iff \quad \begin{pmatrix} u' \\ v' \end{pmatrix}(t) \neq 0$$

since $d_{(u,v)}X : \mathbb{R}^2 \to \mathbb{R}^3$ injects.

Beispiel. The parameter lines of a surface are the curves

$$t \mapsto X(u, v + t), t \mapsto X(u + t, v).$$

Bemerkung und Definition. 1.2. If $t \mapsto X(u(t), v(t))$ is a curve on a surface $X : M \to \mathcal{E}^3$ than $T_t(X \circ (u, v)) \subset T_{(u(t),v(t))}X$ or equivalently, the unit tangent field is always tangential to the surface

$$T = \frac{X_u u' + X_v v'}{\sqrt{Eu'^2 + 2Fu'v' + Gv'^2}}.$$

Thus the Gauss map N of X yields a unit normal vectorfield for $X_0(u, v)$

$$I \ni t \mapsto N(u(t), v(t)).$$

Hence this defines the *natural ribbon* of the curve. The corresponding frame is called the *Darboux frame.*

Definition. 1.3. A curve $t \mapsto X(u(t), v(t))$ on a surface $X : M \to \mathcal{E}^3$ is called

- a *curvature line* if its natural ribbon is a curvature ribbon, i.e., $\tau = 0$,
- an *asymptotic line* if its natural ribbon is an asymtotic ribbon, i.e., $\kappa_n = 0$.
- an *per-geodesic line* if its natural ribbon is an geodesic ribbon, i.e., $\kappa_g = 0$.

Bemerkung. A curve is a curvature line iff the Gauss map of X is parallel along the curve.

Satz 1.4 (Joachimsthal's theorem)**.** *Suppose two surfaces intersect along a curve that is a curvature line for one of the two surfaces. Then it is a curvature line for the other surface iff the two surfaces intersect at a constant angle.*

Beweis. Exercise. $\square$

Definition. 1.5. Rodriges' equation: The curve $t \mapsto X(u(t), v(t))$ is a curvature line iff

$$0 = (dN + \kappa dX) \begin{pmatrix} u' \\ v' \end{pmatrix}$$

where κ is a principle curvature of X at $(u, v) = (u(t), v(t))$ and $dX \begin{pmatrix} u' \\ v' \end{pmatrix}$ is the corresponding curve direction.

Beweis. The structure equation of the natural ribbon yield

$$\nabla^{\perp}(N\circ(u,v)) = (N\circ(u,v))' - \langle N\circ(u,v)', T\rangle\, T = N_u u' + N_v v' + \kappa_n \circ (u,v)(X_u u' + X_v v')$$
$$= (dN + \kappa_n dX)\begin{pmatrix} u' \\ v' \end{pmatrix}.$$

Therefore $t \mapsto (X,N)(u(t),v(t))$ is a curvature ribbon iff $(dN+\kappa_n dX)\begin{pmatrix} u' \\ v' \end{pmatrix} = 0$. On the other hand $dN = -\mathcal{S}\circ dX$. Therefore

$$(dN+\kappa_n dX)\begin{pmatrix} u' \\ v' \end{pmatrix} = (-\mathcal{S}+\kappa_n id)\circ dX\begin{pmatrix} u' \\ v' \end{pmatrix} = 0$$

iff κ_n is a principle curvature and $dX\begin{pmatrix} u' \\ v' \end{pmatrix}$ is the corresponding curve direction. □

Beispiel. Let X be a surface of revolution with Gauss map N (sec 2.2)

$$X(u,v) = O + e_1 r(u)\cos v + e_2 r(u)\sin v + e_3 h(u).$$

and

$$N(u,v) = -e_1 h'(u)\cos v - e_2 h'(u)\sin v + e_3 r'(u)$$

we deduce

$$N_u || X_u \text{ and } N_v || X_v$$

Hence the parameter line of X are curvature lines.

Satz und Definition. 1.6. $X : M \to \mathcal{E}^3$ is a *curvature line parametrisation* if all parameter lines are curvature lines. Any surface admits locally away form umbilics, a curvature line (re-)parametrisation.

Bemerkung. Suppose X is a curvature line parametrisation then (X_u, X_v) diagonalizes the shape operator, cause these are the Eigenvalues,

$$\mathcal{S}X_u || X_u \quad \mathcal{S}X_v || X_v.$$

Hence, as $\mathcal{S}$ is symmetric, $X_u \perp X_v$ and $N_u = -\mathcal{S}X_u \perp X_v$, or equivalently, $F = f = 0$ where

$$I = Edu^2 + 2Fdudv + Gdv^2$$

and

$$II = edu^2 + 2fdudv + gdv^2.$$

Conversely, if $f = F = 0$, then X is a curvature line parametrisation. Look at the matrix representation of $\mathcal{S}$.

Lemma. 1.7. *The normal curvature of a curve $t \mapsto X(u(t),v(t))$ on a surface is given by*

$$\kappa_n = \frac{II(\begin{pmatrix} u' \\ v' \end{pmatrix}, \begin{pmatrix} u' \\ v' \end{pmatrix})}{I(\begin{pmatrix} u' \\ v' \end{pmatrix}, \begin{pmatrix} u' \\ v' \end{pmatrix})}.$$

Beweis. The normal curvature of a ribbon (X, N) is given by

$$\kappa_n = \frac{1}{|X'|} \langle T', N \rangle = \frac{1}{|X'|^2} \langle X'', N \rangle = -\frac{1}{|X'|^2} \langle X', N' \rangle .$$

Applying the chain rule yields the result

$$X' = X_u u' + X_v v', \quad N' = N_u u' + N_v v'.$$

□

Bemerkung und Definition. 1.8. The normal curvature κ_n for a curve on a surface depends only on the tangent direction (and not on u'', v''). Thus we also call it the "normal curvature κ_n of a tangent direction".

Satz 1.9 (Euler's theorem)**.** *The normal curvatures κ_n at a point on a surface satisfy*

$$\min\{\kappa^+, \kappa^-\} \leq \kappa_n(\theta) = \kappa^+ \cos^2\theta + \kappa^- \sin^2\theta \leq \max\{\kappa^+, \kappa^-\},$$

where $\kappa^\pm$ are the principle curvatures and θ is the angle between the tangent direction of $\kappa_n(\theta)$ and the curvature direction of κ^+.

Beweis. Exercise. □

Korollar. 1.10. *The principle curvatures can be characterised as the extremal values of the normal curvature at a point on a surface.*

Korollar. 1.11. *If $t \mapsto (\begin{pmatrix} u' \\ v' \end{pmatrix}, \begin{pmatrix} u' \\ v' \end{pmatrix})$ is an asymptotic line, i.e. $\kappa_n = 0$, of X iff*

$$eu'^2 + 2fu'v' + gv'^2 = 0.$$

Beispiel. The helicoid

$$X(u, v) = O + e_1 \sinh u \cos v + e_2 \sinh u \sin v + e_3 v.$$

Then

$$II = -2dudv.$$

Hence the parameter lines of X are asymptotic lines

$$(II(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix})) = II(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}) = 0$$

$$t \mapsto X(u, t) = O + e_1 r \cos t + e_2 r \sin t + e_3 t,$$

where $r = \sinh u$.

Lemma. 1.12. *Fix a point $X(u, v)$ on a parametrised surface that an asymptotic line passes through $X(u, v)$ in two, one or no directions, depending on the sign of the Gauss curvature $K(u, v)$*

1.2 Geodesic and exponential map

Definition. 1.13. The *covariant derivative* of a tangent field $Y : I \to R^3$ along a curve $t \mapsto X(u(t), v(t))$ on a surface $X : M \to \mathcal{E}^3$ is the tangential part of its derivative

$$\frac{D}{dt}Y := Y' - N\langle Y', N\rangle .$$

A geodesic is an acceleration free curve $t \mapsto C(t) = X(u(t), v(t))$ on a surface ,i.e,

$$\frac{D}{dt}C' = 0$$

Beispiel. Circular helices as geodesies a circular cylinders

$$t \mapsto C(t) = O + e_1 r\cos t + e_2 r\sin t + e_3 ht = X(ht, t)$$

is a geodesic on the cylinder of radius $r > 0$, $h \in \mathbb{R}$ constant.

$$C'(t) = -e_1 r\sin t + e_2 r\cos t + e_3 h$$
$$C''(t) = -e_1 r\cos t - e_2 r\sin t \perp X_u(ht, t), X_v(ht, t)$$

Therefore

$$\frac{D}{dt}C' = 0$$

Satz 1.14. *Geodesics are the constant speed pre-geodesic lines ($\kappa_g \equiv 0$)*

Beweis. Firstly, every geodesic has constant speed by the Leibniz' rule.

$$\langle C', C'\rangle' = 2\langle C'', C'\rangle = 2\left\langle \frac{D}{dt}C', C'\right\rangle \equiv 0.$$

Secondly, assume $|C'| =$ const., then

$$\frac{C''}{|C'|^2} = \frac{T'}{|C'|} = \frac{1}{|C'|}(|C'|\kappa_n N - |C'|\kappa_g B)||N \Leftrightarrow \kappa_g = 0 \Leftrightarrow C \text{ is pre-geodesic line.}$$

□

Satz 1.15 (Clairaut's theorem)**.** *For a geodesic on a surface of revolution the quantity* $r\sin(\theta) \equiv$ const. *where* $r = r(s)$ *is the distance from the axis and* $\theta(s)$ *is the angle the geodesic makes with the profile curves.*

Beweis. Consider the surface of revolution:

$$X(u, v) = O + e_1 r(u)\cos(v) + e_2 r(u)\sin(v) + e_3 h(u);$$

$$N(u, v) = -e_1 g'(u)\cos(v) - e_2 h'(u)\sin(v) + e_3 r'(u)$$

$C(s) = X(u(s), v(s))$ be a geodesic on a surface of revolution , w.l.o.g., arc length parametrized. Set $C_t(s) = O + A(t)(C(s) - O) = X(u(s), v(s) + t)$ where $A(t)$ is given in matrix form by

$$A(t) = \begin{pmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SO}(3).$$

Note that $\forall t \in \mathbb{R}, C_t$ is an arclength parametrized geodesic and

$$\frac{\partial}{\partial t} C_t(s) = \frac{\partial}{\partial t} X(u(s), v(s) + t) = X_v(u(s), v(s) + t$$

and the normal of the normal ribbon for C_t is $s \mapsto N(u(s), v(s) + t$. Set $Y(s) := \frac{\partial}{\partial t}|_{t=0} C_t(s) = X_v(u(s), v(s))$.

$$|Y(s)| = \widetilde{r}(u(s)) = r(s).$$

Therefore the angle $\theta = \theta(s)$ between C and the profile curve satisfies

$$r \sin(\phi) = r \cos(\frac{\pi}{2} - \theta) = r \frac{\langle C', Y \rangle}{|C'||Y|} = \langle C', Y \rangle = \left\langle \frac{\partial}{\partial s} C_t, \frac{\partial}{\partial t} C_t \right\rangle \Big|_{t=0}.$$

We want to show $\frac{\partial}{\partial s} \left\langle \frac{\partial}{\partial s} C_t, \frac{\partial}{\partial t} C_t \right\rangle = 0$:

$$\frac{\partial}{\partial s} \left\langle \frac{\partial}{\partial s} C_t, \frac{\partial}{\partial t} C_t \right\rangle = \left\langle \frac{\partial^2}{\partial^2 s} C_t, \frac{\partial}{\partial t} C_t \right\rangle + \left\langle \frac{\partial}{\partial s} C_t, \frac{\partial}{\partial s} \frac{\partial}{\partial t} C_t \right\rangle$$

C_t is geodesic and thus $\frac{\partial^2}{\partial^2 s} C_t || N(u(s), v(s) + t)$ and $\frac{\partial}{\partial t} C_t(s) = X_v(u(s), v(s) + t)$. Hence

$$\left\langle \frac{\partial^2}{\partial^2 s} C_t, \frac{\partial}{\partial t} C_t \right\rangle = 0$$

and furthermore

$$\frac{\partial}{\partial s} \left\langle \frac{\partial}{\partial s} C_t, \frac{\partial}{\partial t} C_t \right\rangle = 0 \text{ fot all } t \in \mathbb{R}$$

and

$$\frac{\partial}{\partial s} (r \sin(\theta)) = 0.$$

□

Bemerkung. The proof can be generalized for surfaces invariant with respect to 1-parameter families of isometries.

Bemerkung und Beispiel. Clairaut's theorem only provides a necessary condition for a geodesic, not a sufficient one. For example: one sheeted hyperboloid

$$(u, v) \mapsto O + e_1 \cosh(u) \cos(v) + e_2 \cosh(u) \sin(v) + e_3 \sinh(u)$$

Straight line $C(t) = O + e_1 + (e_2 + e_3)t$ is a geodesic in X

$$r \sin(\theta) = \left\langle \frac{C'}{|C'|}, Y \right\rangle = \frac{\cosh(u) \cos(v)}{\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

$Y(s) = (-e_1 \sin(v(s)) + e_2 \cos(v(s))) \cosh(u)$ On the other hand, every circle of latitude in X satisfies $r \sin(\theta) \equiv \cosh(u) \equiv$ const. but in general these are not geodesic.

Differential equations of a geodesic:

Let $Y(t) = X_u(u(t), v(t))x(t) + X_v(u(t), v(t))y(t)$ be a tangent field along a curve $t \mapsto C(t) = X(u(t), v(t))$. Compute the covariant derivative

$$\begin{aligned}
\frac{D}{\partial t} &= X_u x' + (\nabla_{\frac{\partial}{\partial u}} X_u u' + \nabla_{\frac{\partial}{\partial v}} X_u v')x + X_v y' + (\nabla_{\frac{\partial}{\partial u}} X_v u' + \nabla_{\frac{\partial}{\partial v}} X_v v')y \\
&= X_u x' + (X_u \Gamma^1_{11} u' + X_v \Gamma^2_{11} u' + X_u \Gamma^1_{21} v' + X_v \Gamma^2_{21} v')x \\
&\quad + X_v y' + (X_u \Gamma^1_{12} u' + X_v \Gamma^2_{12} u' + X_u \Gamma^1_{22} v' + X_v \Gamma^1_{22} v')y \\
&= X_u(x' + (\Gamma^1_{11} u' + \Gamma^1_{21} v')x + (\Gamma^1_{12} u' + \Gamma^2_{22} v')y) + X_v(y' + (\Gamma^2_{11} u' + \Gamma^2_{21} v')x + (\Gamma^2_{12} u' + \Gamma^2_{22} v')y)
\end{aligned}$$

Now, let $Y = C' = X_u u' + X_v v'$, we get

$$\frac{D}{\partial t} C' = X_u(u'' + \Gamma^1_{11} u'^2 + 2\Gamma^1_{12} u'v' + \Gamma^1_{22} v'^2) + X_v(v'' + \Gamma^2_{11} u'^2 + 2\Gamma^2_{12} u'v' + \Gamma^2_{22} v'^2).$$

Then we learn that :

q: geodesic **Definition. 1.16.** *Geodesic Equation:* $t \mapsto C(t) = X(u(t), v(t))$ is a geodesic if and only if

$$0 = u'' + \Gamma^1_{11} u'^2 + 2\Gamma^1_{12} u'v' + \Gamma^1_{22} v'^2$$

and

$$0 = v'' + \Gamma^2_{11} u'^2 + 2\Gamma^2_{12} u'v' + \Gamma^2_{22} v'^2.$$

We can also write as

$$0 = u'' + (u', v')\Gamma^1 \begin{pmatrix} u' \\ v' \end{pmatrix}$$

$$0 = v'' + (u', v')\Gamma^2 \begin{pmatrix} u' \\ v' \end{pmatrix}$$

with

$$\Gamma^i = \begin{pmatrix} \Gamma^i_{11} & \Gamma^i_{12} \\ \Gamma^i_{21} & \Gamma^i_{22} \end{pmatrix}.$$

Bemerkung. For a geodesic curve $t \mapsto C(t) = X(u(t), v(t))$ we can compute the geodesic curvature of the Darboux frame

$$\frac{d}{\partial t} \frac{C'}{|C'|} = -B|C'|\kappa_g.$$

Take the cross product of this with $T = \dfrac{C'}{|C'|}$

$$-N|C'|\kappa_g = \frac{D}{\partial t}\left(\frac{C'}{|C'|}\right) \times \frac{C'}{|C'|} = \frac{1}{|C'|^2} \frac{D}{\partial t} C' \times C'$$

and thus

$$N\kappa_g = -\frac{1}{|C'|^3} \frac{D}{\partial t} C' \times C'.$$

Comparing $X_u \times X_v$ terms

$$\kappa_g = \frac{\sqrt{EG-F^2}}{\sqrt{Eu'^2+2Fu'v'+Gv'^2}} \det \begin{pmatrix} u' & u''+\Gamma^1_{11}u'^2+2\Gamma^1_{12}u'v'+\Gamma^1_{22}v'^2 \\ v' & v''+\Gamma^2_{11}u'^2+2\Gamma^2_{12}u'v'+\Gamma^2_{22}v'^2 \end{pmatrix}.$$

Korollar. 1.17. *Geodesics can be determined by the induced metric* I *alone.*

Beispiel. Geodesics on a circular cylinder are the straight lines after developing onto a plane: circular helices. I.e. 1.16 holds for exactly those curves.

Korollar. 1.18. *Given a point $(u_0, v_0) \in M$ and a direction $Y = d_{(u_0,v_0)}X(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix})$ There exists a unique (maximal) geodesic $t \mapsto C_Y(t) = X(u(t), v(t))$ on $X : M \to \mathcal{E}^3$ such that*

$$(u(0), v(0)) = (u_0, v_0) \text{ and } (u'(0), v'(0)) = (x_0, y_0).$$

Bemerkung. The initial conditions 1.18 say that an initial point and a tangential direction are given on the surface, if $X(u_0, v_0)$ is a double point 1.18 also specifies which leaf of the surface $C_Y(t)$ lives on.

Beweis. We are going to use Picard-Lindelöf. Let $(w_1, w_2, w_3, w_4) = (u, v, u', v')$. Thus we have the system

$$w_1' = w_3$$
$$w_2' = w_4$$

And because of 1.16

$$w_3' = -\Gamma^1_{11}w_3^2 - 2\Gamma^1_{12}w_3w_4 - \Gamma^1_{22}w_4^2$$
$$w_4' = -\Gamma^2_{11}w_3^2 - 2\Gamma^2_{12}w_3w_4 - \Gamma^2_{22}w_4^2.$$

So, the initial conditions 1.18 imply that $(w_1, w_2, w_3, w_4)(0) = (u_0, v_0, x_0, y_0)$ and we can use Picard-Lindelöf theorem by which the result follows.

□

lemma:I

Lemma. 1.19. *$C_{Ys}(t) = C_Y(st)$ for $s \in (0, 1)$*

Beweis. Suppose $C_Y : I \to \mathcal{E}^3$ is the geodesic satisfying 1.18, then (for an interval around 0)

$$\frac{D}{\partial t}(C_Y(st)' = \frac{D}{\partial t}C_Y'(st)s = (\frac{D}{\partial t}C')(st)s^2 = 0$$

and also

$$(C_Y(st))'(0) = C_Y'(s0)s = Ys$$

while

$$C_Y(S0) = C_Y(0).$$

By the uniqueness, $C_{Ys}(t) = C_Y(st)$ for $t \in I$ □

Bemerkung. By the smooth dependence of solutions C_Y of the initial value problem, we obtain a smooth map

$$R \times T_{(u_0,v_0)}X \ni (t, Y) \mapsto C_Y(t) \in \mathcal{E}^3,$$

which is defined on an open neighbourhood $I \times U$ of $(0,0) \in R \times T_{(u_0,v_0)}X$ with star shaped U and , w.l.o.g., $I \supset [0,1]$.

Consider all unit tangent vectors $Y \in T_{(u_0,v_0)}X$. Then by Picard-Lindelöf theorem there exists a $\varepsilon_Y > 0$ such that C_Y is defined on $(-\varepsilon_Y, \varepsilon_Y)$. Let Y_{min} be the direction for witch $\varepsilon_{Y_{min}}$ is the smallest possible ε_Y.

If $\varepsilon_{Y_{min}} < 1$, then $C_{Y\frac{\varepsilon_{min}}{2}} = C_Y(\frac{\varepsilon_{min}}{2}t)$ is defined on $[0,1]$, for all Y. let $U \subset B_{\frac{\varepsilon_{min}}{2}}(0)$.

Lemma und Definition. 1.20. Given a point $X(u_0, v_0)$ on a surface $X : M \to \mathcal{E}^3$

$$Y \mapsto \exp(Y) := C_Y(1)$$

defines a smooth map on an open neighbourhood U of $O \in T_{(u_0,v_0)}X$ with

$$d_0 \exp = id_{T_{(u_0,v_0)}X},$$

with $d_0 = \frac{d}{dt}|_{t=0}$. exp is called the exponential map of X at $X(u_0, v_0)$.

Beweis. exp is a smooth dependence of solutions of IVPs. Now we compute $d_0 \exp$ using directional derivatives. Let $Y \in T_{(u_0,v_0)}X$

$$d_0 \exp(Y) = \frac{d}{dt}|_{t=0} \exp(tY) = \frac{d}{dt}|_{t=0} C_{Yt}(1) = \frac{d}{dt}|_{t=0} C_Y(t) = Y.$$

Therefore $d_0 \exp = id_{T_{(u_0,v_0)}X}$. □

Bemerkung. Thus $\exp : T_{(u_0,v_0)}X \supset U \to X(M)$ yields a local diffeomorphism and in particular a reparametrisation of X around $X(u_0, v_0)$.

1.3 Geodesic polar coordinates and Minding's Theorem

Definition. 1.21. A reparametrisation of a surface by geodesic polar coordinates (r, θ) around a point $X(0,0)$ of a surface is given by the map

$$(r, \theta) \mapsto \exp(e_1 r \cos\theta + e_2 r \sin\theta) = C_{e_1 r \cos\theta + e_2 r \sin\theta}(1),$$

where (e_1, e_2) form an orthonormal basis of $T_{(0,0)}X$.

For fixed θ, $r \mapsto X(r,\theta) = C_{e_1 r \cos\theta + e_2 r \sin\theta}(1) = C_{e_1 \cos\theta + e_2 \sin\theta}(r)$ and $r \leq 1$. So parameter lines $\theta = const$ are geodesic.

We let $r \leq 1$ in contrast to the Lemma lemma:I 1.19 cause we expect $[0,1] \subset I$.

Bemerkung. This parametrisation is regular at $r = 0$, however it is regular on $(0, \varepsilon) \times \mathbb{R}$ for some $\varepsilon > 0$.

Lemma. 1.22. *In geodesic polar coordinates* (r, θ) *the induced metric is given by*

$$\mathrm{I} = dr^2 + Gd\theta^2$$

where

$$\sqrt{G}|_{r=0} = 0 \text{ and } \frac{\partial \sqrt{G}}{\partial r}|_{r=0} = 1.$$

Beweis. This proof is technical, see the notes. □

Beispiel. In geodesic polar coordinates (r, θ) the Gauss curvature is

$$K = -\frac{(\sqrt{G})_{rr}}{\sqrt{G}}.$$

Korollar. 1.23. *Geodesics are locally the shortest distance between two points.*

Beweis. Let $X : M \to \mathcal{E}^3$ parametrised by geodesic polar coordinates around $X(0,0)$. Let $c(t) = X(r(t), \theta(t))$ be a curve between $X(0,0)$ and $X(r(0), \theta(1))$. Then

$$\int_0^1 |C'(\tilde{t})| \; d\tilde{t} = \int_0^1 \sqrt{r'^2 + G(r, \theta)\theta'^2} \; d\tilde{t} \geq \int_0^1 r' \; d\tilde{t} = r(1).$$

Equality holds iff $\theta' = 0$ or $\theta = const.$

Therefore C is a parameter line of geodesic polar coordinates and thus geodesic (up to reparametrisation of the function r). □

Bemerkung. The surface $\mathbb{R}^2 \setminus \{(0,0)\}$ is an example of why we need locality in the corollary above. Cause there is no shortest distance on that surface between $(1,0)$ and $(-1,0)$.

Satz 1.24 (Minding's theorem)**.** *Any two surfaces with the same* constant *Gauss curvature are locally isometric, i.e., there exists local parametrisations* X_1 *and* X_2 *such that* $I_1 = I_2$.

Beweis. For surface $X : M \to \mathcal{E}^3$ parametrised by geodesic polar coordinates around $X(0,0)$ we have

$$I = dr^2 + Gd\theta^2 \text{ with } \sqrt{G}|_{r=0} \text{ and } \frac{\partial \sqrt{G}}{\partial r}|_{r=0} = 1$$

and $K = -\frac{(\sqrt{G})_{rr}}{\sqrt{G}}$. Therefore $\sqrt{G}_{rr} + K\sqrt{G} = 0$ and K is constant. Hence we have an initial value problem (for fixed θ)

$$(\sqrt{G})_{rr} + K\sqrt{G} = 0, \quad \sqrt{G}|_{r=0} = 0, \; \frac{\partial \sqrt{G}}{\partial r}|_{r=0} = 1.$$

The unique solution is

$$\sqrt{G} = \begin{cases} \frac{1}{\sqrt{K}} \sin(\sqrt{K}r), & K > 0 \\ r, & K = 0 \\ \frac{1}{\sqrt{-K}} \sinh(\sqrt{-K}r), & K < 0. \end{cases}$$

Thus G is determined by K and thus so is I. Thus any two surfaces with the same constant Gauss curvature have the same induced metric.

□

2 Manifolds

Motivation: Some problems occur with our definition of curves and surfaces:

- The sphere is not a surface because no regular parametrisation (harry (Potter? WTF Jojo?) ball theorem)
- The hyperbola is no a surface, because it has two components thus it can not be parametrised by a single regular map on an open interval.

The notion of a submanifold resolves these problems at the expense of imposing other restrictions.

3 Submanifolds of $\mathcal{E}^n$

There are several equivalent characterisations of submanifolds in $\mathcal{E}^n$.

Definition. 3.1 (1. A submanifols can be locally flattened)**.** $M \subset \mathcal{E}^n$ is called a k-dimensional submanifold of $\mathcal{E}^n$ if for all $p \in M$ there exists a diffeomorphism $\phi : U \to \tilde{U}$, where $U \subset \mathcal{E}^n$ is an open neighbourhood of p and $\tilde{U} \subset \mathbb{R}^n$ is an open neighbourhood of 0 such that

$$\phi(M \cap U) = \tilde{U} \cap (\mathbb{R}^k \times \{0\}),$$

where $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^{n-k}$.

Definition. 3.2 (2. A submanifold is locally a level set)**.** $M \subset \mathcal{E}^n$ is a k-dimensional submanifold of $\mathcal{E}^n$ if for all $p \in M$ there exist open neighbourhood $U \subset \mathcal{E}^n$ of p and a submersion $F.U \to \mathbb{R}^{n-k}$ such that

$$M \cap U = F^{-1}\{0\}.$$

Where $dpF : \mathbb{R}^n \to \mathbb{R}^{n-k}$ surjects for all $p \in U$.

Bemerkung. In the definition above th is sufficient to require that $dpF : \mathbb{R}^n \to \mathbb{R}^{n-k}$ surjects: if dpF surjects then since $p \mapsto dpF$ is continuous, dpF surjects by the inertia principle on some open neighbourhood $\tilde{U} \subset U$ of p.

Definition. 3.3 (3. A submanifold can be locally parametrised)**.** $M \subset \mathcal{E}^n$ is a k-dimensional submanifold of $\mathcal{E}^n$ if for all $p \in M$ there exists an immersion $X : V \to U$ from an open neighbourhood $V \subset \mathbb{R}^k$ of 0 to an open neighbourhood $U \subset \mathcal{E}^n$ of p such that

$$M \cap U = X(V)$$

and $X : V \to M \cap U$ is a homeomorphism (with respect to the induced topology on $M \cap U$).

A homeomorphism is continuous and bijective.

Bemerkung.

- X being an immersion excludes "kinks" such as the singularity of the nilparabola.
- X being injective excludes self intersections.
- Continuity of X^{-1} excludes "T-junctions".

Beweis. Proof of equivalence of these definitions:

For $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^{n-k}$ we define the submersions

$$\pi_1 : \mathbb{R}^k \oplus \mathbb{R}^{n-k} \to \mathbb{R}^k, (x, y) \mapsto x,$$

$$\pi_2 : \mathbb{R}^k \oplus \mathbb{R}^{n-k} \to \mathbb{R}^k, (x, y) \mapsto y.$$

First we proof 1. implies 2.:

Let $F := \pi_2 \circ \phi : U \to \mathbb{R}^{n-k}$. F is a submersion.

Secondly we proof 1. implies 3.:

With $V = \pi_1(\tilde{U}) \subset \mathbb{R}^k$ we can have

$$X := \phi^{-1}\big|_v : V \to U$$

is the necessary map. If you are bored, you can check that this is an homeomorphism.

Now we proof 3. implies 1.:

Let $X : \mathbb{R}^k \supset V \to \mathcal{E}^n$ parametrisation of $M \cap U = X(V)$. Assume that $X(0) = p$. Let $(t_1, \dots, t_{n-k})$ be an orthonormal basis of $d_0X(\mathbb{R}^k)^\perp \subset \mathbb{R}^n$. Define

$$C \times \mathbb{R}^{n-k} : (x, y) \mapsto \psi(x, y) = X(x) + \sum_{i=1}^{n-k} t_i y_i, \quad y = (y_1, \dots, y_{n-k}).$$

Then

$$d_0\psi(v, w) = \underbrace{d_0X(v)}_{\in d_0X(\mathbb{R}^k)} + \underbrace{\sum_{i=1}^{n-k} t_i w_i}_{\in (d_0X(\mathbb{R}^k))^\perp} = 0$$

iff $w_i = 0$ for all i and $v = 0$ or $(v, w) = 0$.

Then we use the inverse mapping theorem, ψ has a smooth local inverse

$$\phi = (\psi|_{\tilde{U}})^{-1} : \psi(\tilde{U}) \to \tilde{U}$$

where $\tilde{U} \subset V \times \mathbb{R}^k$ open neighbourhood of 0. Without loss of generality, assume that $\psi(\tilde{U}) \subset U$ (otherwise take the intersection with U). Now, $q \in M \cap \psi(\tilde{U})$ implies there exists a $x \in V$ such that $q = X(x) = \psi(x, 0) \in \psi(\tilde{U})$. On the other hand

$$(x, 0) \in \tilde{U} \Rightarrow \psi(x, 0) = X(x) \in M$$

with means that $q = X(x) \in M \cap \psi(\tilde{U})$.

After replacing $\psi(\tilde{U})$ with U, then $\phi(U \cap M) = \tilde{U} \cap (\mathbb{R}^k \times \{0\})$.

2. $\Rightarrow$ 1. $F : U \to \mathbb{R}^{n-k}$ submersion. Let $t_1, \dots, t_n$ be an orthonormal basis of $\mathbb{R}^n$ such that $t_1, \dots, t_k$ is an orthonormal basis of $\ker d_pF$. Write $\mathbb{R}^n = \langle e_1, \dots, e_k \rangle \oplus \mathbb{R}^{n-k}$ then $q \in U \Rightarrow q = p + \sum_{i=1}^n t_i q_i$ and

$$\phi : U \to \mathbb{R}^n \quad , \quad \phi(q) = \sum_{i=1}^k e_i q_i + F(q)$$

$$d_p\phi(v) = \sum_{i=1}^{k} e_i v_i + d_pF(v) = 0 \text{ for } v = \sum_{i=1}^{k} t_i v_i$$

is equivalent to $d_pF(v) = 0$ and thus $v \in \ker d_pF$ and also to

$$\sum_{i=1}^{k} e_i v_i = 0 \text{ thus } \forall i v_i = 0 \Leftrightarrow v = 0$$

Thus $d_p\phi$ is invertable and by the Inverse Mapping Theorem

$$\phi : U \to \phi(U) \text{ is a diffeomorphism (maybe after shrinking } U).$$

Now, $q \in M \cap U \Leftrightarrow F(q) = 0 \Leftrightarrow \phi(q) \in \phi(U) \cap (\mathbb{R}^k \times \{0\})$. Thus $\phi(M \cap U) = \phi(U) \cap (R^k \times \{0\})$ □

Beispiel. The following are manifolds:

1. Plane: $\pi = \{p \in \mathcal{E}^3 : \langle p - O, n\rangle = d\}$ $F : \mathcal{E}^3 \to \mathbb{R}, F(q) = \langle p - O, n\rangle - d$ then $\pi = f^{-1}(\{0\})$. Also, $d_pF(v) = \langle v, n\rangle$ and hence $d_pF \not\equiv 0$. Thus d_pF surjects and F is a submersion.
2. Sphere: $S = \{p \in \mathcal{E}^3 \mid \langle p - O, p - O\rangle = r^2\}$ and S is implicitly given by the function $F : \mathcal{E}^3 \to \mathbb{R}, F(p) = \langle p - O, p - O\rangle - r^2$. Here $d_pF(v) = 2\langle v, p - O\rangle$ and d_pF surjects as long as $p \not\equiv 0$ which does not happen in S. $F\big|_{\mathcal{E}^3\setminus\{0\}}$ is a submersion.
3. Hyperboloids: $O + e_1x + e_2y + e_3z$ such that $F_\pm(O + e_1x + e_2y + e_3z) = 0$ where $F_\pm(O + e_1x + e_2y + e_3z) = (\frac{x}{a})^2 + (\frac{y}{b})^2 + (\frac{z}{c})^2 \pm 1$ $\nabla F = 2(e_1\frac{x}{a^2} + e_2\frac{y}{b^2} + e_3\frac{z}{c^2})$ Then $\nabla F = 0 \Leftrightarrow (x, y, z) = 0$. Therefore $F\big|_{\mathcal{E}^3\setminus\{0\}}$ is a submersion. $\langle \nabla F, v\rangle = dF(v)$.

Beispiel. A counterexample is the following *Lemniscate*. $O + e_1x + e_2y$, where $x^4 - x^2 + y^2 = 0$. A regular parametrization is given by $t \mapsto O + e_1x(t) + e_2y(t) = O + e_1\sin(t) + e_2\sin(t)\cos(t)$. The curve has a self-intersection at $(x(t), y(t)) = (0, 0)$ which is equivalent to $\forall k \in \mathbb{Z} t = k\pi$. Hence this is not a 1-Dimensional submanifold.