One procedure for finding the effect on f^* of changes in the value of b (right-hand side of the constraint) would be to solve the problem all over with the new value of b. Another procedure would involve the use of the value of λ^*. When the original constraint is tightened by 1 unit (i.e., $db = -1$), Eq. (2.57) gives

$$df^* = \lambda^* db = 2(-1) = -2$$

Thus the new value of f^* is $f^* + df^* = 14.07$. On the other hand, if we relax the original constraint by 2 units (i.e., $db = 2$), we obtain

$$df^* = \lambda^* db = 2(+2) = 4$$

and hence the new value of f^* is $f^* + df^* = 20.07$.

2.5 MULTIVARIABLE OPTIMIZATION WITH INEQUALITY CONSTRAINTS

This section is concerned with the solution of the following problem:

$$\text{Minimize } f(\mathbf{X})$$

subject to

$$g_j(\mathbf{X}) \leq 0, \quad j = 1, 2, \ldots, m \tag{2.58}$$

The inequality constraints in Eq. (2.58) can be transformed to equality constraints by adding nonnegative slack variables, y_j^2, as

$$g_j(\mathbf{X}) + y_j^2 = 0, \quad j = 1, 2, \ldots, m \tag{2.59}$$

where the values of the slack variables are yet unknown. The problem now becomes

$$\text{Minimize } f(\mathbf{X})$$

subject to

$$G_j(\mathbf{X}, \mathbf{Y}) = g_j(\mathbf{X}) + y_j^2 = 0, \qquad j = 1, 2, \ldots, m \tag{2.60}$$

where $\mathbf{Y} = \{y_1, y_2, \ldots, y_m\}^{\mathrm{T}}$ is the vector of slack variables.

This problem can be solved conveniently by the method of Lagrange multipliers. For this, we construct the Lagrange function L as

$$L(\mathbf{X}, \mathbf{Y}, \boldsymbol{\lambda}) = f(\mathbf{X}) + \sum_{j=1}^{m} \lambda_j G_j(\mathbf{X}, \mathbf{Y}) \tag{2.61}$$

where $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}^{\mathrm{T}}$ is the vector of Lagrange multipliers. The stationary points of the Lagrange function can be found by solving the following equations (necessary conditions):

$$\frac{\partial L}{\partial x_i}(\mathbf{X}, \mathbf{Y}, \boldsymbol{\lambda}) = \frac{\partial f}{\partial x_i}(\mathbf{X}) + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i}(\mathbf{X}) = 0, \qquad i = 1, 2, \ldots, n \tag{2.62}$$

$$\frac{\partial L}{\partial \lambda_j}(\mathbf{X}, \mathbf{Y}, \boldsymbol{\lambda}) = G_j(\mathbf{X}, \mathbf{Y}) = g_j(\mathbf{X}) + y_j^2 = 0, \qquad j = 1, 2, \ldots, m \tag{2.63}$$

$$\frac{\partial L}{\partial y_j}(\mathbf{X}, \mathbf{Y}, \boldsymbol{\lambda}) = 2\lambda_j y_j = 0, \qquad j = 1, 2, \ldots, m \tag{2.64}$$

It can be seen that Eqs. (2.62) to (2.64) represent $(n + 2m)$ equations in the $(n + 2m)$ unknowns, $\mathbf{X}$, $\boldsymbol{\lambda}$, and $\mathbf{Y}$. The solution of Eqs. (2.62) to (2.64) thus gives the optimum solution vector, $\mathbf{X}^*$; the Lagrange multiplier vector, $\boldsymbol{\lambda}^*$; and the slack variable vector, $\mathbf{Y}^*$.

Equations (2.63) ensure that the constraints $g_j(\mathbf{X}) \le 0$, $j = 1, 2, \ldots, m$, are satisfied, while Eqs. (2.64) imply that either $\lambda_j = 0$ or $y_j = 0$. If $\lambda_j = 0$, it means that the jth constraint is inactive[†] and hence can be ignored. On the other hand, if $y_j = 0$, it means that the constraint is active ($g_j = 0$) at the optimum point. Consider the division of the constraints into two subsets, J_1 and J_2, where $J_1 + J_2$ represent the total set of constraints. Let the set J_1 indicate the indices of those constraints that are active at the optimum point and J_2 include the indices of all the inactive constraints.

Thus for $j \in J_1$,[‡] $y_j = 0$ (constraints are active), for $j \in J_2$, $\lambda_j = 0$ (constraints are inactive), and Eqs. (2.62) can be simplified as

$$\frac{\partial f}{\partial x_i} + \sum_{j \in J_1} \lambda_j \frac{\partial g_j}{\partial x_i} = 0, \qquad i = 1, 2, \ldots, n \tag{2.65}$$

Similarly, Eqs. (2.63) can be written as

$$g_j(\mathbf{X}) = 0, \qquad j \in J_1 \tag{2.66}$$

$$g_j(\mathbf{X}) + y_j^2 = 0, \qquad j \in J_2 \tag{2.67}$$

Equations (2.65) to (2.67) represent $n + p + (m - p) = n + m$ equations in the $n + m$ unknowns $x_i (i = 1, 2, \ldots, n)$, $\lambda_j (j \in J_1)$, and $y_j (j \in J_2)$, where p denotes the number of active constraints.

Assuming that the first p constraints are active, Eqs. (2.65) can be expressed as

$$-\frac{\partial f}{\partial x_i} = \lambda_1 \frac{\partial g_1}{\partial x_i} + \lambda_2 \frac{\partial g_2}{\partial x_i} + \ldots + \lambda_p \frac{\partial g_p}{\partial x_i}, \qquad i = 1, 2, \ldots, n \tag{2.68}$$

These equations can be written collectively as

$$-\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 + \cdots + \lambda_p \nabla g_p \tag{2.69}$$

[†]Those constraints that are satisfied with an equality sign, $g_j = 0$, at the optimum point are called the *active constraints*, while those that are satisfied with a strict inequality sign, $g_j < 0$, are termed *inactive constraints*.

[‡]The symbol $\in$ is used to denote the meaning “belongs to” or “element of”.

where ∇f and ∇g_j are the gradients of the objective function and the jth constraint, respectively:

$$\nabla f = \begin{Bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \\ \vdots \\ \partial f/\partial x_n \end{Bmatrix} \quad \text{and} \quad \nabla g_j = \begin{Bmatrix} \partial g_j/\partial x_1 \\ \partial g_j/\partial x_2 \\ \vdots \\ \partial g_j/\partial x_n \end{Bmatrix}$$

Equation (2.69) indicates that the negative of the gradient of the objective function can be expressed as a linear combination of the gradients of the active constraints at the optimum point.

Further, we can show that in the case of a minimization problem, the λ_j values ($j \in J_1$) have to be positive. For simplicity of illustration, suppose that only two constraints are active ($p = 2$) at the optimum point. Then Eq. (2.69) reduces to

$$-\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \tag{2.70}$$

Let $\mathbf{S}$ be a feasible direction[†] at the optimum point. By premultiplying both sides of Eq. (2.70) by $\mathbf{S}^T$, we obtain

$$-\mathbf{S}^T \nabla f = \lambda_1 \mathbf{S}^T \nabla g_1 + \quad \lambda_2 \mathbf{S}^T \nabla g_2 \tag{2.71}$$

where the superscript T denotes the transpose. Since $\mathbf{S}$ is a feasible direction, it should satisfy the relations

$$\begin{aligned} \mathbf{S}^{\mathrm{T}} \nabla g_1 &< 0 \\ \mathbf{S}^{\mathrm{T}} \nabla g_2 &< 0 \end{aligned} \tag{2.72}$$

Thus if $\lambda_1 > 0$ and $\lambda_2 > 0$, the quantity $\mathbf{S}^T \nabla f$ can be seen always to be positive. As ∇f indicates the gradient direction, along which the value of the function increases at the maximum rate,[‡] $\mathbf{S}^T \nabla f$ represents the component of the increment of f along the direction $\mathbf{S}$. If $\mathbf{S}^T \nabla f > 0$, the function value increases as we move along the direction $\mathbf{S}$. Hence if λ_1 and λ_2 are positive, we will not be able to find any direction in the feasible domain along which the function value can be decreased further. Since the point at which Eq. (2.72) is valid is assumed to be optimum, λ_1 and λ_2 have to be positive. This reasoning can be extended to cases where there are more than two constraints active. By proceeding in a similar manner, one can show that the λ_j values have to be negative for a maximization problem.

[†]A vector $\mathbf{S}$ is called a *feasible direction* from a point $\mathbf{X}$ if at least a small step can be taken along $\mathbf{S}$ that does not immediately leave the feasible region. Thus for problems with sufficiently smooth constraint surfaces, vector $\mathbf{S}$ satisfying the relation

$$\mathbf{S}^T \nabla g_j < 0$$

can be called a feasible direction. On the other hand, if the constraint is either linear or concave, as shown in Fig. 2.8b and *c*, any vector satisfying the relation

$$\mathbf{S}^T \nabla g_j \le 0$$

can be called a feasible direction. The geometric interpretation of a feasible direction is that the vector $\mathbf{S}$ makes an obtuse angle with all the constraint normals, except that for the linear or outward-curving (concave) constraints, the angle may go to as low as 90°.

[‡]See Section 6.10.2 for a proof of this statement.

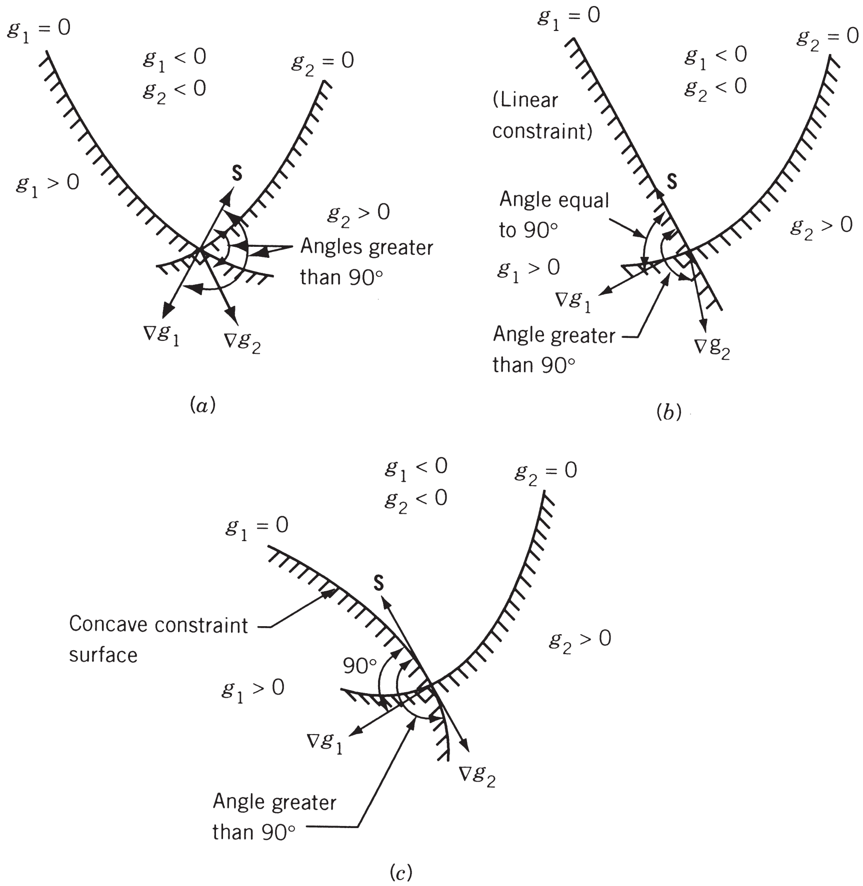

Figure 2.8 Feasible direction **S**.

Example 2.12 Consider the following optimization problem:

$$\text{Minimize} f(x_1, x_2) = x_1^2 + x_2^2$$

subject to

$$x_1 + 2x_2 \leq 15$$

$$1 \leq x_i \leq 10; \;\; i = 1, 2$$

Derive the conditions to be satisfied at the point $\mathbf{X}_1 = \{1, 7\}^{\mathrm{T}}$ by the search direction $\mathbf{S} = \{s_1, s_2\}^{\mathrm{T}}$ if it is a (a) usable direction, and (b) feasible direction.

SOLUTION The objective function and the constraints can be stated as

$$f(x_1, x_2) = x_1^2 + x_2^2$$

$$g_1(\mathbf{X}) = x_1 + 2x_2 \leq 15$$

$$g_2(\mathbf{X}) = 1 - x_1 \le 0$$

$$g_3(\mathbf{X}) = 1 - x_2 \le 0$$

$$g_4(\mathbf{X}) = x_1 - 10 \le 0$$

$$g_5(\mathbf{X}) = x_2 - 10 \le 0$$

At the given point $\mathbf{X}_1 = \{1, 7\}^{\mathrm{T}}$, all the constraints can be seen to be satisfied with g_1 and g_2 being active. The gradients of the objective and active constraint functions at point $\mathbf{X}_1 = \{1, 7\}^{\mathrm{T}}$ are given by

$$\nabla f = \begin{Bmatrix} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial f}{\partial x_2} \end{Bmatrix}_{\mathbf{X}_1} = \begin{Bmatrix} 2x_1 \\ 2x_2 \end{Bmatrix}_{\mathbf{X}_1} = \begin{Bmatrix} 2 \\ 14 \end{Bmatrix}$$

$$\nabla g_1 = \begin{Bmatrix} \dfrac{\partial g_1}{\partial x_1} \\ \dfrac{\partial g_1}{\partial x_2} \end{Bmatrix}_{\mathbf{X}_1} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

$$\nabla g_2 = \begin{Bmatrix} \dfrac{\partial g_2}{\partial x_1} \\ \dfrac{\partial g_2}{\partial x_2} \end{Bmatrix}_{\mathbf{X}_1} = \begin{Bmatrix} -1 \\ 0 \end{Bmatrix}$$

For the search direction $\mathbf{S} = \{s_1, s_2\}^{\mathrm{T}}$, the usability and feasibility conditions can be expressed as

(a) Usability condition:

$$\mathbf{S}^{\mathrm{T}} \nabla f \le 0 \quad \text{or} \quad (s_1 \quad s_2) \begin{Bmatrix} 2 \\ 14 \end{Bmatrix} \le 0 \quad \text{or} \quad 2s_1 + 14s_2 \le 0 \tag{E$_1$}$$

(b) Feasibility conditions:

$$\mathbf{S}^{\mathrm{T}} \nabla g_1 \le 0 \quad \text{or} \quad (s_1 \quad s_2) \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \le 0 \quad \text{or} \quad s_1 + 2s_2 \le 0 \tag{E$_2$}$$

$$\mathbf{S}^{\mathrm{T}} \nabla g_2 \le 0 \quad \text{or} \quad (s_1 \quad s_2) \begin{Bmatrix} -1 \\ 0 \end{Bmatrix} \le 0 \quad \text{or} \quad -s_1 \le 0 \tag{E$_3$}$$

Note: Any two numbers for s_1 and s_2 that satisfy the inequality (E$_1$) will constitute a usable direction $\mathbf{S}$. For example, $s_1 = 1$ and $s_2 = -1$ gives the usable direction $\mathbf{S} = \{1, -1\}^{\mathrm{T}}$. This direction can also be seen to be a feasible direction because it satisfies the inequalities (E$_2$) and (E$_3$).

2.5.1 Kuhn–Tucker Conditions

As shown above, the conditions to be satisfied at a constrained minimum point, $\mathbf{X}^*$, of the problem stated in Eq. (2.58) can be expressed as

$$\frac{\partial f}{\partial x_i} + \sum_{j \in J_1} \lambda_j \frac{\partial g_j}{\partial x_i} = 0, \qquad i = 1, 2, \ldots, n \tag{2.73}$$

$$\lambda_j > 0, \qquad j \in J_1 \tag{2.74}$$

These are called *Kuhn–Tucker conditions* after the mathematicians who derived them as the necessary conditions to be satisfied at a relative minimum of $f(\mathbf{X})$ [2.8]. These conditions are, in general, not sufficient to ensure a relative minimum. However, there is a class of problems, called *convex programming problems*,† for which the Kuhn–Tucker conditions are necessary and sufficient for a global minimum.

If the set of active constraints is not known, the Kuhn–Tucker conditions can be stated as follows:

$$\begin{aligned} \frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i} &= 0, \qquad i = 1, 2, \ldots, n \\ \lambda_j g_j &= 0,^{\ddagger} \quad j = 1, 2, \ldots, m \\ g_j &\le 0, \qquad j = 1, 2, \ldots, m \\ \lambda_j &\ge 0, \qquad j = 1, 2, \ldots, m \end{aligned} \tag{2.75}$$

Note that if the problem is one of maximization or if the constraints are of the type $g_j \ge 0$, the λ_j have to be nonpositive in Eqs. (2.75). On the other hand, if the problem is one of maximization with constraints in the form $g_j \ge 0$, the λ_j have to be nonnegative in Eqs. (2.75).

2.5.2 Constraint Qualification

When the optimization problem is stated as

$$\text{Minimize } f(\mathbf{X})$$

subject to

$$\begin{aligned} g_j(\mathbf{X}) &\le 0, \qquad j = 1, 2, \ldots, m \\ h_k(\mathbf{X}) &= 0 \qquad k = 1, 2, \ldots, p \end{aligned} \tag{2.76}$$

the Kuhn–Tucker conditions become

$$\nabla f + \sum_{j=1}^{m} \lambda_j \nabla g_j - \sum_{k=1}^{p} \beta_k \nabla h_k = \mathbf{0}$$

$$\lambda_j g_j = 0, \qquad j = 1, 2, \ldots, m$$

†See Sections 2.6 and 7.14 for a detailed discussion of convex programming problems.
‡This condition is the same as Eq. (2.64).

$$\begin{aligned} g_j &\leq 0, \qquad j = 1, 2, \ldots, m \\ h_k &= 0, \qquad k = 1, 2, \ldots, p \\ \lambda_j &\geq 0, \qquad j = 1, 2, \ldots, m \end{aligned} \tag{2.77}$$

where λ_j and β_k denote the Lagrange multipliers associated with the constraints $g_j \leq 0$ and $h_k = 0$, respectively. Although we found qualitatively that the Kuhn–Tucker conditions represent the necessary conditions of optimality, the following theorem gives the precise conditions of optimality.

Theorem 2.7 Let $\mathbf{X}^*$ be a feasible solution to the problem of Eqs. (2.76). If $\nabla g_j(\mathbf{X}^*)$, $j \in J_1$ and $\nabla h_k(\mathbf{X}^*)$, $k = 1, 2, \ldots, p$, are linearly independent, there exist $\boldsymbol{\lambda}^*$ and $\boldsymbol{\beta}^*$ such that $(\mathbf{X}^*, \boldsymbol{\lambda}^*, \boldsymbol{\beta}^*)$ satisfy Eqs. (2.77).

Proof: See Ref. [2.11].

The requirement that $\nabla g_j(\mathbf{X}^*)$, $j \in J_1$ and $\nabla h_k(\mathbf{X}^*)$, $k = 1, 2, \ldots, p$, be linearly independent is called the *constraint qualification*. If the constraint qualification is violated at the optimum point, Eqs. (2.77) may or may not have a solution. It is difficult to verify the constraint qualification without knowing $\mathbf{X}^*$ beforehand. However, the constraint qualification is always satisfied for problems having any of the following characteristics:

1. All the inequality and equality constraint functions are linear.
2. All the inequality constraint functions are convex, all the equality constraint functions are linear, and at least one feasible vector $\tilde{\mathbf{X}}$ exists that lies strictly inside the feasible region, so that

$$g_j(\tilde{\mathbf{X}}) < 0, \quad j = 1, 2, \ldots, m \quad \text{and} \quad h_k(\tilde{\mathbf{X}}) = 0, \ k = 1, 2, \ldots, p$$

Example 2.13 Consider the following problem:

$$\text{Minixize } f(x_1, x_2) = (x_1 - 1)^2 + x_2^2 \tag{E$_1$}$$

subject to

$$g_1(x_1, x_2) = x_1^3 - 2x_2 \leq 0 \tag{E$_2$}$$

$$g_2(x_1, x_2) = x_1^3 + 2x_2 \leq 0 \tag{E$_3$}$$

Determine whether the constraint qualification and the Kuhn–Tucker conditions are satisfied at the optimum point.

SOLUTION The feasible region and the contours of the objective function are shown in Fig. 2.9. It can be seen that the optimum solution is (0, 0). Since g_1 and g_2 are both active at the optimum point (0, 0), their gradients can be computed as

$$\nabla g_1(\mathbf{X}^*) = \begin{Bmatrix} 3x_1^2 \\ -2 \end{Bmatrix}_{(0,\,0)} = \begin{Bmatrix} 0 \\ -2 \end{Bmatrix} \quad \text{and} \quad \nabla g_2(\mathbf{X}^*) = \begin{Bmatrix} 3x_1^2 \\ 2 \end{Bmatrix}_{(0,\,0)} = \begin{Bmatrix} 0 \\ 2 \end{Bmatrix}$$

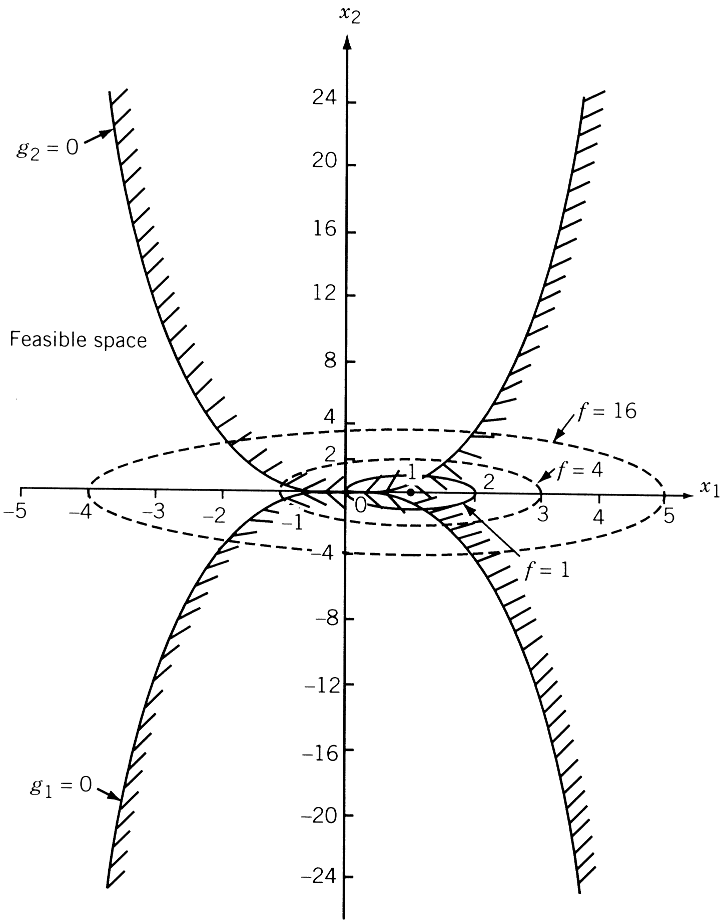

Figure 2.9 Feasible region and contours of the objective function.

It is clear that $\nabla g_1(\mathbf{X}^*)$ and $\nabla g_2(\mathbf{X}^*)$ are not linearly independent. Hence the constraint qualification is not satisfied at the optimum point. Noting that

$$\nabla f(\mathbf{X}^*) = \begin{Bmatrix} 2(x_1 - 1) \\ 2x_2 \end{Bmatrix}_{(0,\,0)} = \begin{Bmatrix} -2 \\ 0 \end{Bmatrix}$$

the Kuhn–Tucker conditions can be written, using Eqs. (2.73) and (2.74), as

$$-2 + \lambda_1(0) + \lambda_2(0) = 0 \tag{E$_4$}$$

$$0 + \lambda_1(-2) + \lambda_2(2) = 0 \tag{E$_5$}$$

$$\lambda_1 > 0 \tag{E$_6$}$$

$$\lambda_2 > 0 \tag{E$_7$}$$

Since Eq. (E$_4$) is not satisfied and Eq. (E$_5$) can be satisfied for negative values of $\lambda_1 = \lambda_2$ also, the Kuhn–Tucker conditions are not satisfied at the optimum point.

Example 2.14 A manufacturing firm producing small refrigerators has entered into a contract to supply 50 refrigerators at the end of the first month, 50 at the end of the second month, and 50 at the end of the third. The cost of producing x refrigerators in any month is given by $\$(x^2 + 1000)$. The firm can produce more refrigerators in any month and carry them to a subsequent month. However, it costs \$20 per unit for any refrigerator carried over from one month to the next. Assuming that there is no initial inventory, determine the number of refrigerators to be produced in each month to minimize the total cost.

SOLUTION Let x_1, x_2, and x_3 represent the number of refrigerators produced in the first, second, and third month, respectively. The total cost to be minimized is given by

$$\text{total cost} = \text{production cost} + \text{holding cost}$$

or

$$\begin{aligned} f(x_1, x_2, x_3) &= (x_1^2 + 1000) + (x_2^2 + 1000) + (x_3^2 + 1000) + 20(x_1 - 50) \\ &\quad + 20(x_1 + x_2 - 100) \\ &= x_1^2 + x_2^2 + x_3^2 + 40x_1 + 20x_2 \end{aligned}$$

The constraints can be stated as

$$\begin{aligned} g_1(x_1, x_2, x_3) &= x_1 - 50 \geq 0 \\ g_2(x_1, x_2, x_3) &= x_1 + x_2 - 100 \geq 0 \\ g_3(x_1, x_2, x_3) &= x_1 + x_2 + x_3 - 150 \geq 0 \end{aligned}$$

The Kuhn–Tucker conditions are given by

$$\frac{\partial f}{\partial x_i} + \lambda_1 \frac{\partial g_1}{\partial x_i} + \lambda_2 \frac{\partial g_2}{\partial x_i} + \lambda_3 \frac{\partial g_3}{\partial x_i} = 0, \qquad i = 1, 2, 3$$

that is,

$$2x_1 + 40 + \lambda_1 + \lambda_2 + \lambda_3 = 0 \tag{E$_1$}$$

$$2x_2 + 20 + \lambda_2 + \lambda_3 = 0 \tag{E$_2$}$$

$$2x_3 + \lambda_3 = 0 \tag{E$_3$}$$

$$\lambda_j g_j = 0, \qquad j = 1, 2, 3$$

that is,

$$\lambda_1(x_1 \quad - 50) = 0 \tag{E$_4$}$$

$$\lambda_2(x_1 + x_2 - 100) = 0 \tag{E$_5$}$$

$$\lambda_3(x_1 + x_2 + x_3 - 150) = 0 \tag{E$_6$}$$

$$g_j \geq 0, \qquad j = 1, 2, 3$$

that is,

$$x_1 - 50 \geq 0 \tag{E$_7$}$$

$$x_1 + x_2 - 100 \geq 0 \tag{E$_8$}$$

$$x_1 + x_2 + x_3 - 150 \geq 0 \tag{E$_9$}$$

$$\lambda_j \leq 0, \quad j = 1, 2, 3$$

that is,

$$\lambda_1 \leq 0 \tag{E$_{10}$}$$

$$\lambda_2 \leq 0 \tag{E$_{11}$}$$

$$\lambda_3 \leq 0 \tag{E$_{12}$}$$

The solution of Eqs. (E$_1$) to (E$_{12}$) can be found in several ways. We proceed to solve these equations by first nothing that either $\lambda_1 = 0$ or $x_1 = 50$ according to Eq. (E$_4$). Using this information, we investigate the following cases to identify the optimum solution of the problem.

***Case 1:* $\boldsymbol{\lambda_1 = 0}$.**

Equations (E$_1$) to (E$_3$) give

$$\begin{aligned} x_3 &= -\frac{\lambda_3}{2} \\ x_2 &= -10 - \frac{\lambda_2}{2} - \frac{\lambda_3}{2} \\ x_1 &= -20 - \frac{\lambda_2}{2} - \frac{\lambda_3}{2} \end{aligned} \tag{E$_{13}$}$$

Substituting Eqs. (E$_{13}$) in Eqs. (E$_5$) and (E$_6$), we obtain

$$\begin{aligned} \lambda_2(-130 - \lambda_2 - \lambda_3) &= 0 \\ \lambda_3(-180 - \lambda_2 - \tfrac{3}{2}\lambda_3) &= 0 \end{aligned} \tag{E$_{14}$}$$

The four possible solutions of Eqs. (E$_{14}$) are

1. $\lambda_2 = 0, -180 - \lambda_2 - \dfrac{3}{2}\lambda_3 = 0$. These equations, along with Eqs. (E$_{13}$), yield the solution

$$\lambda_2 = 0, \quad \lambda_3 = -120, \quad x_1 = 40, \quad x_2 = 50, \quad x_3 = 60$$

This solution satisfies Eqs. (E$_{10}$) to (E$_{12}$) but violates Eqs. (E$_7$) and (E$_8$) and hence cannot be optimum.

2. $\lambda_3 = 0,\ -130 - \lambda_2 - \lambda_3 = 0$. The solution of these equations leads to

$$\lambda_2 = -130, \quad \lambda_3 = 0, \quad x_1 = 45, \quad x_2 = 55,\ x_3 = 0$$

This solution can be seen to satisfy Eqs. (E_{10}) to (E_{12}) but violate Eqs. (E_7) and (E_9).

3. $\lambda_2 = 0$, $\lambda_3 = 0$. Equations (E_{13}) give

$$x_1 = -20, \quad x_2 = -10, \quad x_3 = 0$$

This solution satisfies Eqs. (E_{10}) to (E_{12}) but violates the constraints, Eqs. (E_7) to (E_9).

4. $-130 - \lambda_2 - \lambda_3 = 0, -180 - \lambda_2 - \frac{3}{2}\lambda_3 = 0$. The solution of these equations and Eqs. (E_{13}) yields

$$\lambda_2 = -30, \quad \lambda_3 = -100, \quad x_1 = 45, \quad x_2 = 55, \quad x_3 = 50$$

This solution satisfies Eqs. (E_{10}) to (E_{12}) but violates the constraint, Eq. (E_7).

Case 2: $x_1 = 50$.

In this case, Eqs. (E_1) to (E_3) give

$$\begin{aligned} \lambda_3 &= -2x_3 \\ \lambda_2 &= -20 - 2x_2 - \lambda_3 = -20 - 2x_2 + 2x_3 \\ \lambda_1 &= -40 - 2x_1 - \lambda_2 - \lambda_3 = -120 + 2x_2 \end{aligned} \tag{E$_{15}$}$$

Substitution of Eqs. (E_{15}) in Eqs. (E_5) and (E_6) leads to

$$\begin{aligned} (-20 - 2x_2 + 2x_3)(x_1 + x_2 - 100) &= 0 \\ (-2x_3)(x_1 + x_2 + x_3 - 150) &= 0 \end{aligned} \tag{E$_{16}$}$$

Once again, it can be seen that there are four possible solutions to Eqs. (E_{16}), as indicated below:

1. $-20 - 2x_2 + 2x_3 = 0$, $x_1 + x_2 + x_3 - 150 = 0$: The solution of these equations yields

$$x_1 = 50, \quad x_2 = 45, \quad x_3 = 55$$

This solution can be seen to violate Eq. (E_8).

2. $-20 - 2x_2 + 2x_3 = 0$, $-2x_3 = 0$: These equations lead to the solution

$$x_1 = 50, \quad x_2 = -10, \quad x_3 = 0$$

This solution can be seen to violate Eqs. (E_8) and (E_9).

3. $x_1 + x_2 - 100 = 0$, $-2x_3 = 0$: These equations give

$$x_1 = 50, \quad x_2 = 50, \quad x_3 = 0$$

This solution violates the constraint Eq. (E_9).

4. $x_1 + x_2 - 100 = 0,\ x_1 + x_2 + x_3 - 150 = 0$: The solution of these equations yields

$$x_1 = 50, \quad x_2 = 50, \quad x_3 = 50$$

This solution can be seen to satisfy all the constraint Eqs. (E_7) to (E_9). The values of λ_1, λ_2, and λ_3 corresponding to this solution can be obtained from Eqs. (E_{15}) as

$$\lambda_1 = -20, \quad \lambda_2 = -20, \quad \lambda_3 = -100$$

Since these values of λ_i satisfy the requirements [Eqs. (E_{10}) to (E_{12})], this solution can be identified as the optimum solution. Thus

$$x_1^* = 50, \quad x_2^* = 50, \quad x_3^* = 50$$

2.6 CONVEX PROGRAMMING PROBLEM

The optimization problem stated in Eq. (2.58) is called a *convex programming problem* if the objective function $f(\mathbf{X})$ and the constraint functions $g_j(\mathbf{X})$ are convex. The definition and properties of a convex function are given in Appendix A. Suppose that $f(\mathbf{X})$ and $g_j(\mathbf{X}),\ j = 1, 2, \ldots, m$, are convex functions. The Lagrange function of Eq. (2.61) can be written as

$$L(\mathbf{X}, \mathbf{Y}, \boldsymbol{\lambda}) = f(\mathbf{X}) + \sum_{j=1}^{m} \lambda_j [g_j(\mathbf{X}) + y_j^2] \tag{2.78}$$

If $\lambda_j \geq 0$, then $\lambda_j g_j(\mathbf{X})$ is convex, and since $\lambda_j y_j = 0$ from Eq. (2.64), $L(\mathbf{X}, \mathbf{Y}, \boldsymbol{\lambda})$ will be a convex function. As shown earlier, a necessary condition for $f(\mathbf{X})$ to be a relative minimum at $\mathbf{X}^*$ is that $L(\mathbf{X}, \mathbf{Y}, \boldsymbol{\lambda})$ have a stationary point at $\mathbf{X}^*$. However, if $\mathbf{L}(\mathbf{X}, \mathbf{Y}, \boldsymbol{\lambda})$ is a convex function, its derivative vanishes only at one point, which must be an absolute minimum of the function $f(\mathbf{X})$. Thus the Kuhn–Tucker conditions are both necessary and sufficient for an absolute minimum of $f(\mathbf{X})$ at $\mathbf{X}^*$.

Notes:

1. If the given optimization problem is known to be a convex programming problem, there will be no relative minima or saddle points, and hence the extreme point found by applying the Kuhn–Tucker conditions is guaranteed to be an absolute minimum of $f(\mathbf{X})$. However, it is often very difficult to ascertain whether the objective and constraint functions involved in a practical engineering problem are convex.
2. The derivation of the Kuhn–Tucker conditions was based on the development given for equality constraints in Section 2.4. One of the requirements for these conditions was that at least one of the Jacobians composed of the m constraints and m of the $n + m$ variables $(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_m)$ be nonzero. This requirement is implied in the derivation of the Kuhn–Tucker conditions.