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Introduction to Optimization

1.1 INTRODUCTION

Optimization is the act of obtaining the best result under given circumstances. In design, construction, and maintenance of any engineering system, engineers have to take many technological and managerial decisions at several stages. The ultimate goal of all such decisions is either to minimize the effort required or to maximize the desired benefit. Since the effort required or the benefit desired in any practical situation can be expressed as a function of certain decision variables, *optimization* can be defined as the process of finding the conditions that give the maximum or minimum value of a function. It can be seen from Fig. 1.1 that if a point x^* corresponds to the minimum value of function $f(x)$, the same point also corresponds to the maximum value of the negative of the function, $-f(x)$. Thus without loss of generality, optimization can be taken to mean minimization since the maximum of a function can be found by seeking the minimum of the negative of the same function.

In addition, the following operations on the objective function will not change the optimum solution x^* (see Fig. 1.2):

1. Multiplication (or division) of $f(x)$ by a positive constant c.
2. Addition (or subtraction) of a positive constant c to (or from) $f(x)$.

There is no single method available for solving all optimization problems efficiently. Hence a number of optimization methods have been developed for solving different types of optimization problems. The optimum seeking methods are also known as *mathematical programming techniques* and are generally studied as a part of operations research. *Operations research* is a branch of mathematics concerned with the application of scientific methods and techniques to decision making problems and with establishing the best or optimal solutions. The beginnings of the subject of operations research can be traced to the early period of World War II. During the war, the British military faced the problem of allocating very scarce and limited resources (such as fighter airplanes, radars, and submarines) to several activities (deployment to numerous targets and destinations). Because there were no systematic methods available to solve resource allocation problems, the military called upon a team of mathematicians to develop methods for solving the problem in a scientific manner. The methods developed by the team were instrumental in the winning of the Air Battle by Britain. These methods, such as linear programming, which were developed as a result of research on (military) operations, subsequently became known as the methods of operations research.

Engineering Optimization: Theory and Practice, Fourth Edition Singiresu S. Rao

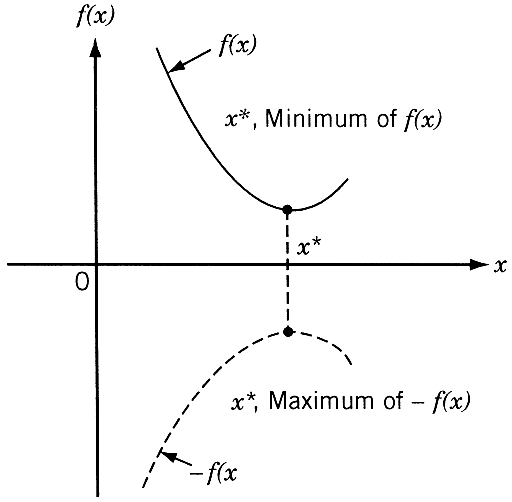

Figure 1.1 Minimum of $f(x)$ is same as maximum of $-f(x)$.

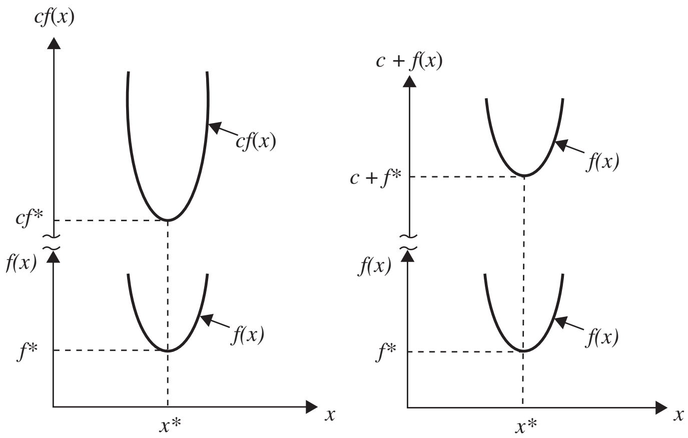

Figure 1.2 Optimum solution of $cf(x)$ or $c + f(x)$ same as that of $f(x)$.

Table 1.1 lists various mathematical programming techniques together with other well-defined areas of operations research. The classification given in Table 1.1 is not unique; it is given mainly for convenience.

Mathematical programming techniques are useful in finding the minimum of a function of several variables under a prescribed set of constraints. Stochastic process techniques can be used to analyze problems described by a set of random variables having known probability distributions. Statistical methods enable one to analyze the experimental data and build empirical models to obtain the most accurate representation of the physical situation. This book deals with the theory and application of mathematical programming techniques suitable for the solution of engineering design problems.

Table 1.1 Methods of Operations Research

Mathematical programming or optimization techniques	Stochastic process techniques	Statistical methods
Calculus methods	Statistical decision theory	Regression analysis
Calculus of variations	Markov processes	Cluster analysis, pattern recognition
Nonlinear programming	Queueing theory	Design of experiments
Geometric programming	Renewal theory	Discriminate analysis (factor analysis)
Quadratic programming	Simulation methods	
Linear programming	Reliability theory	
Dynamic programming		
Integer programming		
Stochastic programming		
Separable programming		
Multiobjective programming		
Network methods: CPM and PERT		
Game theory		
Modern or nontraditional optimization techniques		
Genetic algorithms		
Simulated annealing		
Ant colony optimization		
Particle swarm optimization		
Neural networks		
Fuzzy optimization		

1.2 HISTORICAL DEVELOPMENT

The existence of optimization methods can be traced to the days of Newton, Lagrange, and Cauchy. The development of differential calculus methods of optimization was possible because of the contributions of Newton and Leibnitz to calculus. The foundations of calculus of variations, which deals with the minimization of functionals, were laid by Bernoulli, Euler, Lagrange, and Weirstrass. The method of optimization for constrained problems, which involves the addition of unknown multipliers, became known by the name of its inventor, Lagrange. Cauchy made the first application of the steepest descent method to solve unconstrained minimization problems. Despite these early contributions, very little progress was made until the middle of the twentieth century, when high-speed digital computers made implementation of the optimization procedures possible and stimulated further research on new methods. Spectacular advances followed, producing a massive literature on optimization techniques. This advancement also resulted in the emergence of several well-defined new areas in optimization theory.

It is interesting to note that the major developments in the area of numerical methods of unconstrained optimization have been made in the United Kingdom only in the 1960s. The development of the simplex method by Dantzig in 1947 for linear programming problems and the annunciation of the principle of optimality in 1957 by Bellman for dynamic programming problems paved the way for development of the methods of constrained optimization. Work by Kuhn and Tucker in 1951 on the necessary and sufficiency conditions for the optimal solution of programming problems laid the foundations for a great deal of later research in nonlinear programming. The contributions of Zoutendijk and Rosen to nonlinear programming during the early 1960s have been significant. Although no single technique has been found to be universally applicable for nonlinear programming problems, work of Carroll and Fiacco and McCormick allowed many difficult problems to be solved by using the well-known techniques of unconstrained optimization. Geometric programming was developed in the 1960s by Duffin, Zener, and Peterson. Gomory did pioneering work in integer programming, one of the most exciting and rapidly developing areas of optimization. The reason for this is that most real-world applications fall under this category of problems. Dantzig and Charnes and Cooper developed stochastic programming techniques and solved problems by assuming design parameters to be independent and normally distributed.

The desire to optimize more than one objective or goal while satisfying the physical limitations led to the development of multiobjective programming methods. Goal programming is a well-known technique for solving specific types of multiobjective optimization problems. The goal programming was originally proposed for linear problems by Charnes and Cooper in 1961. The foundations of game theory were laid by von Neumann in 1928 and since then the technique has been applied to solve several mathematical economics and military problems. Only during the last few years has game theory been applied to solve engineering design problems.

Modern Methods of Optimization. The modern optimization methods, also sometimes called nontraditional optimization methods, have emerged as powerful and popular methods for solving complex engineering optimization problems in recent years. These methods include genetic algorithms, simulated annealing, particle swarm optimization, ant colony optimization, neural network-based optimization, and fuzzy optimization. The genetic algorithms are computerized search and optimization algorithms based on the mechanics of natural genetics and natural selection. The genetic algorithms were originally proposed by John Holland in 1975. The simulated annealing method is based on the mechanics of the cooling process of molten metals through annealing. The method was originally developed by Kirkpatrick, Gelatt, and Vecchi.

The particle swarm optimization algorithm mimics the behavior of social organisms such as a colony or swarm of insects (for example, ants, termites, bees, and wasps), a flock of birds, and a school of fish. The algorithm was originally proposed by Kennedy and Eberhart in 1995. The ant colony optimization is based on the cooperative behavior of ant colonies, which are able to find the shortest path from their nest to a food source. The method was first developed by Marco Dorigo in 1992. The neural network methods are based on the immense computational power of the nervous system to solve perceptional problems in the presence of massive amount of sensory data through its parallel processing capability. The method was originally used for optimization by Hopfield and Tank in 1985. The fuzzy optimization methods were developed to solve optimization problems involving design data, objective function, and constraints stated in imprecise form involving vague and linguistic descriptions. The fuzzy approaches for single and multiobjective optimization in engineering design were first presented by Rao in 1986.

1.3 ENGINEERING APPLICATIONS OF OPTIMIZATION

Optimization, in its broadest sense, can be applied to solve any engineering problem. Some typical applications from different engineering disciplines indicate the wide scope of the subject:

1. Design of aircraft and aerospace structures for minimum weight
2. Finding the optimal trajectories of space vehicles
3. Design of civil engineering structures such as frames, foundations, bridges, towers, chimneys, and dams for minimum cost
4. Minimum-weight design of structures for earthquake, wind, and other types of random loading
5. Design of water resources systems for maximum benefit
6. Optimal plastic design of structures
7. Optimum design of linkages, cams, gears, machine tools, and other mechanical components
8. Selection of machining conditions in metal-cutting processes for minimum production cost
9. Design of material handling equipment, such as conveyors, trucks, and cranes, for minimum cost
10. Design of pumps, turbines, and heat transfer equipment for maximum efficiency
11. Optimum design of electrical machinery such as motors, generators, and transformers
12. Optimum design of electrical networks
13. Shortest route taken by a salesperson visiting various cities during one tour
14. Optimal production planning, controlling, and scheduling
15. Analysis of statistical data and building empirical models from experimental results to obtain the most accurate representation of the physical phenomenon
16. Optimum design of chemical processing equipment and plants
17. Design of optimum pipeline networks for process industries
18. Selection of a site for an industry
19. Planning of maintenance and replacement of equipment to reduce operating costs
20. Inventory control
21. Allocation of resources or services among several activities to maximize the benefit
22. Controlling the waiting and idle times and queueing in production lines to reduce the costs
23. Planning the best strategy to obtain maximum profit in the presence of a competitor
24. Optimum design of control systems

1.4 STATEMENT OF AN OPTIMIZATION PROBLEM

An optimization or a mathematical programming problem can be stated as follows.

$$\text{Find } \mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \text{ which minimizes } f(\mathbf{X})$$

subject to the constraints

$$\begin{aligned} g_j(\mathbf{X}) &\le 0, \qquad j = 1, 2, \ldots, m \\ l_j(\mathbf{X}) &= 0, \qquad j = 1, 2, \ldots, p \end{aligned} \tag{1.1}$$

where $\mathbf{X}$ is an n-dimensional vector called the *design vector*, $f(\mathbf{X})$ is termed the *objective function*, and $g_j(\mathbf{X})$ and $l_j(\mathbf{X})$ are known as *inequality* and *equality* constraints, respectively. The number of variables n and the number of constraints m and/or p need not be related in any way. The problem stated in Eq. (1.1) is called a *constrained optimization problem*.† Some optimization problems do not involve any constraints and can be stated as

$$\text{Find } \mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \text{ which minimizes } f(\mathbf{X}) \tag{1.2}$$

Such problems are called *unconstrained optimization problems*.

1.4.1 Design Vector

Any engineering system or component is defined by a set of quantities some of which are viewed as variables during the design process. In general, certain quantities are usually fixed at the outset and these are called *preassigned parameters*. All the other quantities are treated as variables in the design process and are called *design* or *decision variables* x_i, $i = 1, 2, \ldots, n$. The design variables are collectively represented as a design vector $\mathbf{X} = \{x_1, x_2, \ldots, x_n\}^{\mathrm{T}}$. As an example, consider the design of the gear pair shown in Fig. 1.3, characterized by its face width b, number of teeth T_1 and T_2, center distance d, pressure angle ψ, tooth profile, and material. If center distance d, pressure angle ψ, tooth profile, and material of the gears are fixed in advance, these quantities can be called *preassigned parameters*. The remaining quantities can be collectively represented by a design vector $\mathbf{X} = \{x_1, x_2, x_3\}^{\mathrm{T}} = \{b, T_1, T_2\}^{\mathrm{T}}$. If there are no restrictions on the choice of b, T_1, and T_2, any set of three numbers will constitute a design for the gear pair. If an n-dimensional Cartesian space with each coordinate axis representing a design variable x_i $(i = 1, 2, \ldots, n)$ is considered, the space is called

†In the mathematical programming literature, the equality constraints $l_j(\mathbf{X}) = 0$, $j = 1, 2, \ldots, p$ are often neglected, for simplicity, in the statement of a constrained optimization problem, although several methods are available for handling problems with equality constraints.

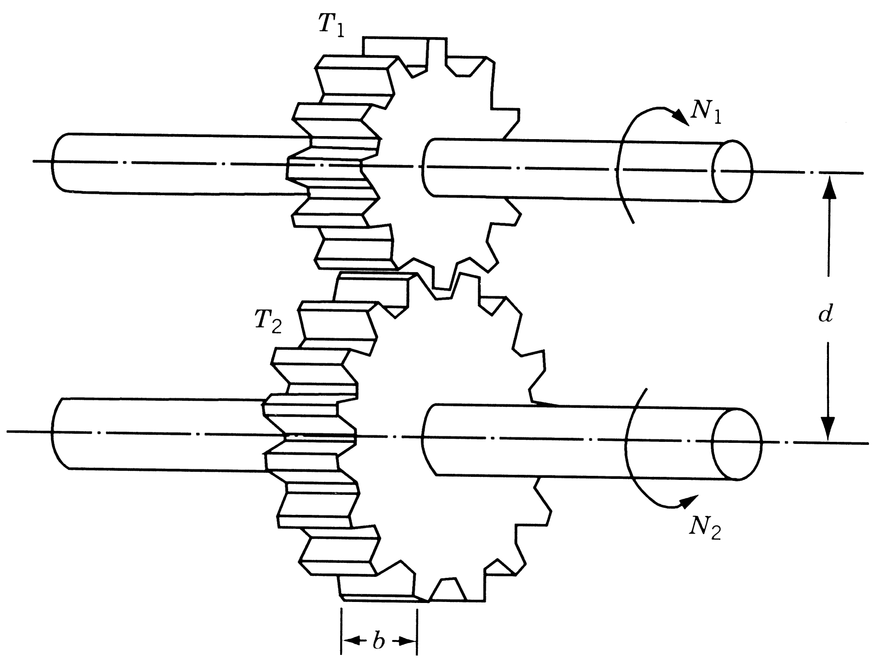

Figure 1.3 Gear pair in mesh.

the *design variable space* or simply *design space*. Each point in the n-dimensional design space is called a *design point* and represents either a possible or an impossible solution to the design problem. In the case of the design of a gear pair, the design point $\{1.0, 20, 40\}^{\mathrm{T}}$, for example, represents a possible solution, whereas the design point $\{1.0, -20, 40.5\}^{\mathrm{T}}$ represents an impossible solution since it is not possible to have either a negative value or a fractional value for the number of teeth.

1.4.2 Design Constraints

In many practical problems, the design variables cannot be chosen arbitrarily; rather, they have to satisfy certain specified functional and other requirements. The restrictions that must be satisfied to produce an acceptable design are collectively called *design constraints*. Constraints that represent limitations on the behavior or performance of the system are termed *behavior* or *functional constraints*. Constraints that represent physical limitations on design variables, such as availability, fabricability, and transportability, are known as *geometric* or *side constraints*. For example, for the gear pair shown in Fig. 1.3, the face width b cannot be taken smaller than a certain value, due to strength requirements. Similarly, the ratio of the numbers of teeth, T_1/T_2, is dictated by the speeds of the input and output shafts, N_1 and N_2. Since these constraints depend on the performance of the gear pair, they are called behavior constraints. The values of T_1 and T_2 cannot be any real numbers but can only be integers. Further, there can be upper and lower bounds on T_1 and T_2 due to manufacturing limitations. Since these constraints depend on the physical limitations, they are called side *constraints*.

1.4.3 Constraint Surface

For illustration, consider an optimization problem with only inequality constraints $g_j(\mathbf{X}) \leq 0$. The set of values of $\mathbf{X}$ that satisfy the equation $g_j(\mathbf{X}) = 0$ forms a hypersurface in the design space and is called a *constraint surface*. Note that this is an $(n-1)$-dimensional subspace, where n is the number of design variables. The constraint surface divides the design space into two regions: one in which $g_j(\mathbf{X}) < 0$ and the other in which $g_j(\mathbf{X}) > 0$. Thus the points lying on the hypersurface will satisfy the constraint $g_j(\mathbf{X})$ critically, whereas the points lying in the region where $g_j(\mathbf{X}) > 0$ are infeasible or unacceptable, and the points lying in the region where $g_j(\mathbf{X}) < 0$ are feasible or acceptable. The collection of all the constraint surfaces $g_j(\mathbf{X}) = 0$, $j = 1, 2, \ldots, m$, which separates the acceptable region is called the *composite constraint surface*.

Figure 1.4 shows a hypothetical two-dimensional design space where the infeasible region is indicated by hatched lines. A design point that lies on one or more than one constraint surface is called a *bound point*, and the associated constraint is called an *active constraint*. Design points that do not lie on any constraint surface are known as *free points*. Depending on whether a particular design point belongs to the acceptable or unacceptable region, it can be identified as one of the following four types:

1. Free and acceptable point
2. Free and unacceptable point
3. Bound and acceptable point
4. Bound and unacceptable point

All four types of points are shown in Fig. 1.4.

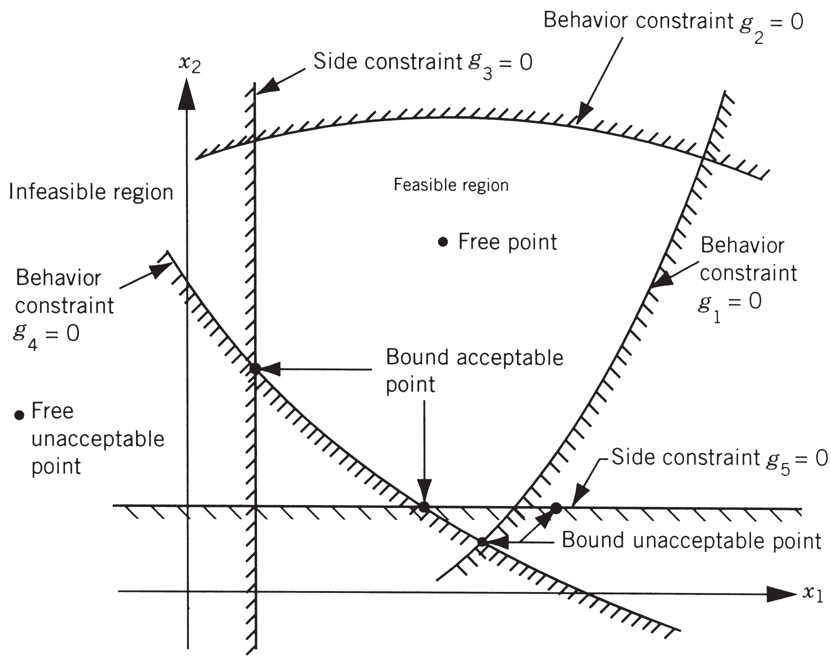

Figure 1.4 Constraint surfaces in a hypothetical two-dimensional design space.

1.4.4 Objective Function

The conventional design procedures aim at finding an acceptable or adequate design that merely satisfies the functional and other requirements of the problem. In general, there will be more than one acceptable design, and the purpose of optimization is to choose the best one of the many acceptable designs available. Thus a criterion has to be chosen for comparing the different alternative acceptable designs and for selecting the best one. The criterion with respect to which the design is optimized, when expressed as a function of the design variables, is known as the *criterion* or *merit* or *objective function*. The choice of objective function is governed by the nature of problem. The objective function for minimization is generally taken as weight in aircraft and aerospace structural design problems. In civil engineering structural designs, the objective is usually taken as the minimization of cost. The maximization of mechanical efficiency is the obvious choice of an objective in mechanical engineering systems design. Thus the choice of the objective function appears to be straightforward in most design problems. However, there may be cases where the optimization with respect to a particular criterion may lead to results that may not be satisfactory with respect to another criterion. For example, in mechanical design, a gearbox transmitting the maximum power may not have the minimum weight. Similarly, in structural design, the minimum weight design may not correspond to minimum stress design, and the minimum stress design, again, may not correspond to maximum frequency design. Thus the selection of the objective function can be one of the most important decisions in the whole optimum design process.

In some situations, there may be more than one criterion to be satisfied simultaneously. For example, a gear pair may have to be designed for minimum weight and maximum efficiency while transmitting a specified horsepower. An optimization problem involving multiple objective functions is known as a *multiobjective programming problem*. With multiple objectives there arises a possibility of conflict, and one simple way to handle the problem is to construct an overall objective function as a linear combination of the conflicting multiple objective functions. Thus if $f_1(\mathbf{X})$ and $f_2(\mathbf{X})$ denote two objective functions, construct a new (overall) objective function for optimization as

$$f(\mathbf{X}) = \alpha_1 f_1(\mathbf{X}) + \alpha_2 f_2(\mathbf{X}) \tag{1.3}$$

where α_1 and α_2 are constants whose values indicate the relative importance of one objective function relative to the other.

1.4.5 Objective Function Surfaces

The locus of all points satisfying $f(\mathbf{X}) = C =$ constant forms a hypersurface in the design space, and each value of C corresponds to a different member of a family of surfaces. These surfaces, called *objective function surfaces*, are shown in a hypothetical two-dimensional design space in Fig. 1.5.

Once the objective function surfaces are drawn along with the constraint surfaces, the optimum point can be determined without much difficulty. But the main problem is that as the number of design variables exceeds two or three, the constraint and objective function surfaces become complex even for visualization and the problem

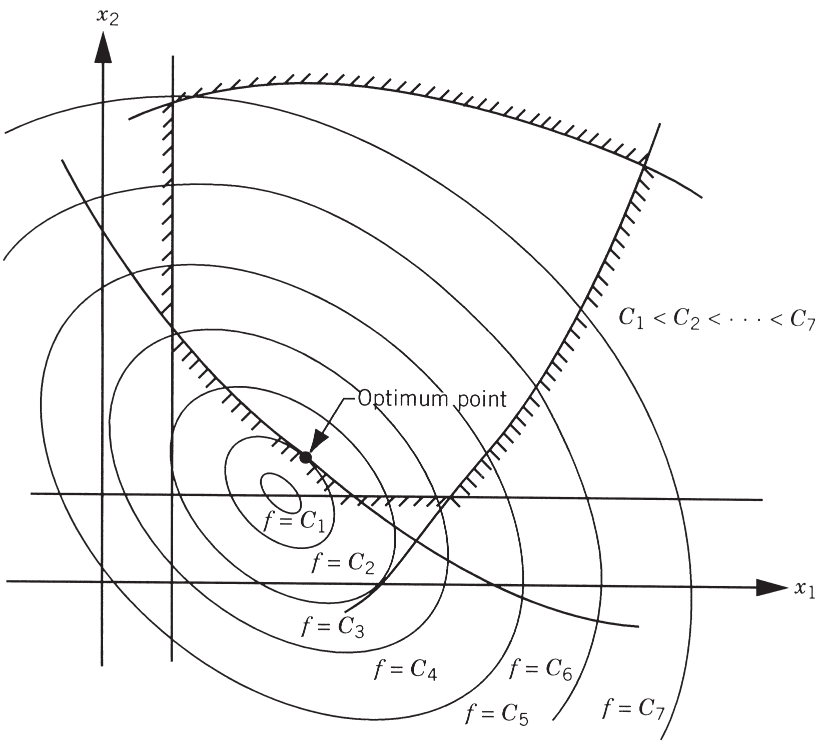

Figure 1.5 Contours of the objective function.

has to be solved purely as a mathematical problem. The following example illustrates the graphical optimization procedure.

Example 1.1 Design a uniform column of tubular section, with hinge joints at both ends, (Fig. 1.6) to carry a compressive load $P = 2500\,\text{kg}_\text{f}$ for minimum cost. The column is made up of a material that has a yield stress (σ_y) of $500\,\text{kg}_\text{f}/\text{cm}^2$, modulus of elasticity (E) of $0.85 \times 10^6\,\text{kg}_\text{f}/\text{cm}^2$, and weight density ($\rho$) of $0.0025\,\text{kg}_\text{f}/\text{cm}^3$. The length of the column is 250 cm. The stress induced in the column should be less than the buckling stress as well as the yield stress. The mean diameter of the column is restricted to lie between 2 and 14 cm, and columns with thicknesses outside the range 0.2 to 0.8 cm are not available in the market. The cost of the column includes material and construction costs and can be taken as $5W + 2d$, where W is the weight in kilograms force and d is the mean diameter of the column in centimeters.

SOLUTION The design variables are the mean diameter (d) and tube thickness (t):

$$\mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} d \\ t \end{Bmatrix} \tag{E$_1$}$$

The objective function to be minimized is given by

$$f(\mathbf{X}) = 5W + 2d = 5\rho l \pi\, dt + 2d = 9.82 x_1 x_2 + 2x_1 \tag{E$_2$}$$

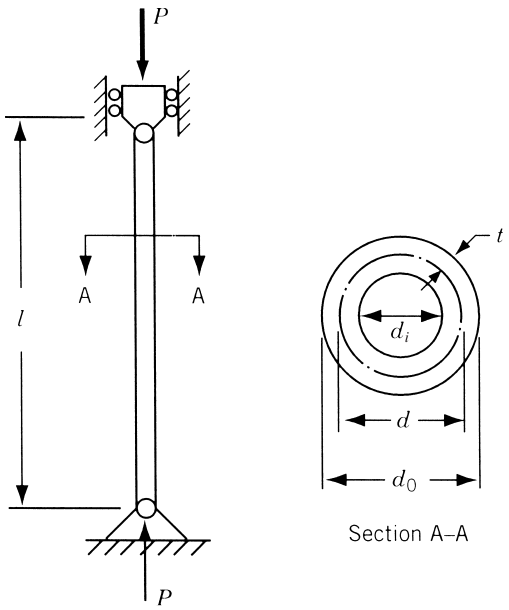

Figure 1.6 Tubular column under compression.

The behavior constraints can be expressed as

$$\text{stress induced} \leq \text{yield stress}$$
$$\text{stress induced} \leq \text{buckling stress}$$

The induced stress is given by

$$\text{induced stress} = \sigma_i = \frac{P}{\pi\, dt} = \frac{2500}{\pi x_1 x_2} \tag{E$_3$}$$

The buckling stress for a pin-connected column is given by

$$\text{buckling stress} = \sigma_b = \frac{\text{Euler buckling load}}{\text{cross-sectional area}} = \frac{\pi^2 EI}{l^2}\frac{1}{\pi\, dt} \tag{E$_4$}$$

where

$$\begin{aligned}
I &= \text{second moment of area of the cross section of the column} \\
&= \frac{\pi}{64}(d_o^4 - d_i^4) \\
&= \frac{\pi}{64}(d_o^2 + d_i^2)(d_o + d_i)(d_o - d_i) = \frac{\pi}{64}[(d+t)^2 + (d-t)^2] \\
&\quad \times [(d+t) + (d-t)][(d+t) - (d-t)] \\
&= \frac{\pi}{8}\, dt(d^2 + t^2) = \frac{\pi}{8} x_1 x_2 (x_1^2 + x_2^2)
\end{aligned} \tag{E$_5$}$$

Thus the behavior constraints can be restated as

$$g_1(\mathbf{X}) = \frac{2500}{\pi x_1 x_2} - 500 \le 0 \tag{E$_6$}$$

$$g_2(\mathbf{X}) = \frac{2500}{\pi x_1 x_2} - \frac{\pi^2(0.85 \times 10^6)(x_1^2 + x_2^2)}{8(250)^2} \le 0 \tag{E$_7$}$$

The side constraints are given by

$$2 \le d \le 14$$

$$0.2 \le t \le 0.8$$

which can be expressed in standard form as

$$g_3(\mathbf{X}) = -x_1 + 2.0 \le 0 \tag{E$_8$}$$

$$g_4(\mathbf{X}) = x_1 - 14.0 \le 0 \tag{E$_9$}$$

$$g_5(\mathbf{X}) = -x_2 + 0.2 \le 0 \tag{E$_{10}$}$$

$$g_6(\mathbf{X}) = x_2 - 0.8 \le 0 \tag{E$_{11}$}$$

Since there are only two design variables, the problem can be solved graphically as shown below.

First, the constraint surfaces are to be plotted in a two-dimensional design space where the two axes represent the two design variables x_1 and x_2. To plot the first constraint surface, we have

$$g_1(\mathbf{X}) = \frac{2500}{\pi x_1 x_2} - 500 \le 0$$

that is,

$$x_1 x_2 \ge 1.593$$

Thus the curve $x_1 x_2 = 1.593$ represents the constraint surface $g_1(\mathbf{X}) = 0$. This curve can be plotted by finding several points on the curve. The points on the curve can be found by giving a series of values to x_1 and finding the corresponding values of x_2 that satisfy the relation $x_1 x_2 = 1.593$:

x_1	2.0	4.0	6.0	8.0	10.0	12.0	14.0
x_2	0.7965	0.3983	0.2655	0.1990	0.1593	0.1328	0.1140

These points are plotted and a curve $P_1 Q_1$ passing through all these points is drawn as shown in Fig. 1.7, and the infeasible region, represented by $g_1(\mathbf{X}) > 0$ or $x_1 x_2 < 1.593$, is shown by hatched lines.[†] Similarly, the second constraint $g_2(\mathbf{X}) \le 0$ can be expressed as $x_1 x_2 (x_1^2 + x_2^2) \ge 47.3$ and the points lying on the constraint surface $g_2(\mathbf{X}) = 0$ can be obtained as follows for $x_1 x_2 (x_1^2 + x_2^2) = 47.3$:

[†]The infeasible region can be identified by testing whether the origin lies in the feasible or infeasible region.

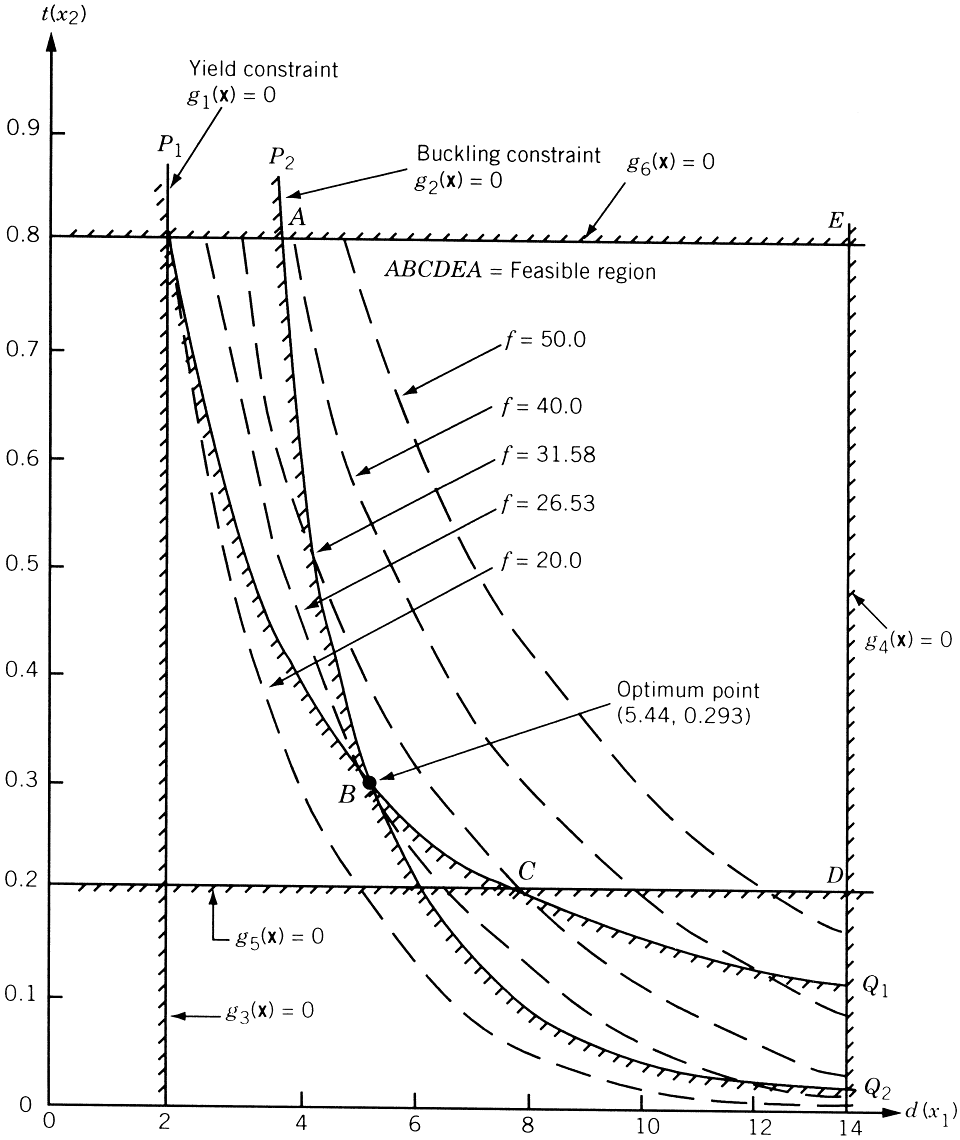

Figure 1.7 Graphical optimization of Example 1.1.

x_1	2	4	6	8	10	12	14
x_2	2.41	0.716	0.219	0.0926	0.0473	0.0274	0.0172

These points are plotted as curve P_2Q_2, the feasible region is identified, and the infeasible region is shown by hatched lines as in Fig. 1.7. The plotting of side constraints is very simple since they represent straight lines. After plotting all the six constraints, the feasible region can be seen to be given by the bounded area *ABCDEA*.

Next, the contours of the objective function are to be plotted before finding the optimum point. For this, we plot the curves given by

$$f(\mathbf{X}) = 9.82x_1x_2 + 2x_1 = c = \text{constant}$$

for a series of values of c. By giving different values to c, the contours of f can be plotted with the help of the following points.

For $9.82x_1x_2 + 2x_1 = 50.0$:

x_2	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
x_1	16.77	12.62	10.10	8.44	7.24	6.33	5.64	5.07

For $9.82x_1x_2 + 2x_1 = 40.0$:

x_2	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
x_1	13.40	10.10	8.08	6.75	5.79	5.06	4.51	4.05

For $9.82x_1x_2 + 2x_1 = 31.58$ (passing through the corner point C):

x_2	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
x_1	10.57	7.96	6.38	5.33	4.57	4.00	3.56	3.20

For $9.82x_1x_2 + 2x_1 = 26.53$ (passing through the corner point B):

x_2	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
x_1	8.88	6.69	5.36	4.48	3.84	3.36	2.99	2.69

For $9.82x_1x_2 + 2x_1 = 20.0$:

x_2	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
x_1	6.70	5.05	4.04	3.38	2.90	2.53	2.26	2.02

These contours are shown in Fig. 1.7 and it can be seen that the objective function cannot be reduced below a value of 26.53 (corresponding to point B) without violating some of the constraints. Thus the optimum solution is given by point B with $d^* = x_1^* = 5.44$ cm and $t^* = x_2^* = 0.293$ cm with $f_{\min} = 26.53$.

1.5 CLASSIFICATION OF OPTIMIZATION PROBLEMS

Optimization problems can be classified in several ways, as described below.

1.5.1 Classification Based on the Existence of Constraints

As indicated earlier, any optimization problem can be classified as constrained or unconstrained, depending on whether constraints exist in the problem.

1.6 OPTIMIZATION TECHNIQUES

The various techniques available for the solution of different types of optimization problems are given under the heading of mathematical programming techniques in Table 1.1. The classical methods of differential calculus can be used to find the unconstrained maxima and minima of a function of several variables. These methods assume that the function is differentiable twice with respect to the design variables and the derivatives are continuous. For problems with equality constraints, the Lagrange multiplier method can be used. If the problem has inequality constraints, the Kuhn–Tucker conditions can be used to identify the optimum point. But these methods lead to a set of nonlinear simultaneous equations that may be difficult to solve. The classical methods of optimization are discussed in Chapter 2.

The techniques of nonlinear, linear, geometric, quadratic, or integer programming can be used for the solution of the particular class of problems indicated by the name of the technique. Most of these methods are numerical techniques wherein an approximate solution is sought by proceeding in an iterative manner by starting from an initial solution. Linear programming techniques are described in Chapters 3 and 4. The quadratic programming technique, as an extension of the linear programming approach, is discussed in Chapter 4. Since nonlinear programming is the most general method of optimization that can be used to solve any optimization problem, it is dealt with in detail in Chapters 5–7. The geometric and integer programming methods are discussed in Chapters 8 and 10, respectively. The dynamic programming technique, presented in Chapter 9, is also a numerical procedure that is useful primarily for the solution of optimal control problems. Stochastic programming deals with the solution of optimization problems in which some of the variables are described by probability distributions. This topic is discussed in Chapter 11.

In Chapter 12 we discuss calculus of variations, optimal control theory, and optimality criteria methods. The modern methods of optimization, including genetic algorithms, simulated annealing, particle swarm optimization, ant colony optimization, neural network-based optimization, and fuzzy optimization, are presented in Chapter 13. Several practical aspects of optimization are outlined in Chapter 14. The reduction of size of optimization problems, fast reanalysis techniques, the efficient computation of the derivatives of static displacements and stresses, eigenvalues and eigenvectors, and transient response are outlined. The aspects of sensitivity of optimum solution to problem parameters, multilevel optimization, parallel processing, and multiobjective optimization are also presented in this chapter.

1.7 ENGINEERING OPTIMIZATION LITERATURE

The literature on engineering optimization is large and diverse. Several text-books are available and dozens of technical periodicals regularly publish papers related to engineering optimization. This is primarily because optimization is applicable to all areas of engineering. Researchers in many fields must be attentive to the developments in the theory and applications of optimization.

The most widely circulated journals that publish papers related to engineering optimization are *Engineering Optimization, ASME Journal of Mechanical Design, AIAA Journal, ASCE Journal of Structural Engineering, Computers and Structures, International Journal for Numerical Methods in Engineering, Structural Optimization, Journal of Optimization Theory and Applications, Computers and Operations Research, Operations Research*, *Management Science, Evolutionary Computation, IEEE Transactions on Evolutionary Computation, European Journal of Operations Research, IEEE Transactions on Systems, Man and Cybernetics*, and *Journal of Heuristics*. Many of these journals are cited in the chapter references.

1.8 SOLUTION OF OPTIMIZATION PROBLEMS USING MATLAB

The solution of most practical optimization problems requires the use of computers. Several commercial software systems are available to solve optimization problems that arise in different engineering areas. MATLAB is a popular software that is used for the solution of a variety of scientific and engineering problems.[†] MATLAB has several toolboxes each developed for the solution of problems from a specific scientific area. The specific toolbox of interest for solving optimization and related problems is called the *optimization toolbox*. It contains a library of programs or m-files, which can be used for the solution of minimization, equations, least squares curve fitting, and related problems. The basic information necessary for using the various programs can be found in the user's guide for the optimization toolbox [1.124]. The programs or m-files, also called functions, available in the minimization section of the optimization toolbox are given in Table 1.2. The use of the programs listed in Table 1.2 is demonstrated at the end of different chapters of the book. Basically, the solution procedure involves three steps after formulating the optimization problem in the format required by the MATLAB program (or function) to be used. In most cases, this involves stating the objective function for minimization and the constraints in "$\leq$" form with zero or constant value on the righthand side of the inequalities. After this, step 1 involves writing an m-file for the objective function. Step 2 involves writing an m-file for the constraints. Step 3 involves setting the various parameters at proper values depending on the characteristics of the problem and the desired output and creating an appropriate file to invoke the desired MATLAB program (and coupling the m-files created to define the objective and constraints functions of the problem). As an example, the use of the program, `fmincon`, for the solution of a constrained nonlinear programming problem is demonstrated in Example 1.11.

Example 1.11 Find the solution of the following nonlinear optimization problem (same as the problem in Example 1.1) using the MATLAB function `fmincon`:

$$\text{Minimize } f(x_1, x_2) = 9.82x_1x_2 + 2x_1$$

subject to

$$g_1(x_1, x_2) = \frac{2500}{\pi x_1 x_2} - 500 \leq 0$$

[†]The basic concepts and procedures of MATLAB are summarized in Appendix C.

Table 1.2 MATLAB Programs or Functions for Solving Optimization Problems

Type of optimization problem	Standard form for solution by MATLAB	Name of MATLAB program or function to solve the problem
Function of one variable or scalar minimization	Find x to minimize $f(x)$ with $x_1 < x < x_2$	`fminbnd`
Unconstrained minimization of function of several variables	Find $\mathbf{x}$ to minimize $f(\mathbf{x})$	`fminunc` or `fminsearch`
Linear programming problem	Find $\mathbf{x}$ to minimize $\mathbf{f}^T\mathbf{x}$ subject to $[A]\mathbf{x} \le \mathbf{b}$, $[A_{\text{eq}}]\mathbf{x} = \mathbf{b}_{\text{eq}}$, $\mathbf{l} \le \mathbf{x} \le \mathbf{u}$	`linprog`
Quadratic programming problem	Find $\mathbf{x}$ to minimize $\frac{1}{2}\mathbf{x}^T[H]\mathbf{x} + \mathbf{f}^T\mathbf{x}$ subject to $[A]\mathbf{x} \le \mathbf{b}$, $[A_{\text{eq}}]\mathbf{x} = \mathbf{b}_{\text{eq}}$, $\mathbf{l} \le \mathbf{x} \le \mathbf{u}$	`quadprog`
Minimization of function of several variables subject to constraints	Find $\mathbf{x}$ to minimize $f(\mathbf{x})$ subject to $\mathbf{c}(\mathbf{x}) \le \mathbf{0}$, $\mathbf{c}_{\text{eq}} = \mathbf{0}$ $[A]\mathbf{x} \le \mathbf{b}$, $[A_{\text{eq}}]\mathbf{x} = \mathbf{b}_{\text{eq}}$, $\mathbf{l} \le \mathbf{x} \le \mathbf{u}$	`fmincon`
Goal attainment problem	Find $\mathbf{x}$ and γ to minimize γ such that $F(\mathbf{x}) - \mathbf{w}\gamma \le \mathbf{goal}$, $\mathbf{c}(\mathbf{x}) \le \mathbf{0}$, $\mathbf{c}_{\text{eq}} = \mathbf{0}$ $[A]\mathbf{x} \le \mathbf{b}$, $[A_{\text{eq}}]\mathbf{x} = \mathbf{b}_{\text{eq}}$, $\mathbf{l} \le \mathbf{x} \le \mathbf{u}$	`fgoalattain`
Minimax problem	$\underset{\mathbf{x}}{\text{Minimize}} \; \underset{[F_i]}{\text{Max}} \; [F_i(\mathbf{x})\}$ such that $\mathbf{c}(\mathbf{x}) \le \mathbf{0}$, $\mathbf{c}_{\text{eq}} = \mathbf{0}$ $[A]\mathbf{x} \le \mathbf{b}$, $[A_{\text{eq}}]\mathbf{x} = \mathbf{b}_{\text{eq}}$, $\mathbf{l} \le \mathbf{x} \le \mathbf{u}$	`fminimax`
Binary integer programming problem	Find $\mathbf{x}$ to minimize $\mathbf{f}^T\mathbf{x}$ subject to $[A]\mathbf{x} \le \mathbf{b}$, $[A_{\text{eq}}]\mathbf{x} = \mathbf{b}_{\text{eq}}$, each component of $\mathbf{x}$ is binary	`bintprog`

$$g_2(x_1, x_2) = \frac{2500}{\pi x_1 x_2} - \frac{\pi^2(x_1^2 + x_2^2)}{0.5882} \le 0$$

$$g_3(x_1, x_2) = -x_1 + 2 \le 0$$

$$g_4(x_1, x_2) = x_1 - 14 \le 0$$

$$g_5(x_1, x_2) = -x_2 + 0.2 \le 0$$

$$g_6(x_1, x_2) = x_2 - 0.8 \le 0$$

SOLUTION

Step 1: Write an M-file `probofminobj.m` for the objective function.

```
function f= probofminobj (x)
f= 9.82*x(1)*x(2)+2*x(1);
```

Step 2: Write an M-file `conprobformin.m` for the constraints.

```
function [c, ceq] = conprobformin(x)
% Nonlinear inequality constraints
c = [2500/(pi*x(1)*x(2))-500;2500/(pi*x(1)*x(2))-
(pi^2*(x(1)^2+x(2)^2))/0.5882;-x(1)+2;x(1)-14;-x(2)+0.2;
x(2)-0.8];
% Nonlinear equality constraints
ceq = [];
```

Step 3: Invoke constrained optimization program (write this in new matlab file).

```
clc
clear all
warning off
x0 = [7 0.4]; % Starting guess\
fprintf ('The values of function value and constraints
at starting point\n');
f=probofminobj (x0)
[c, ceq] = conprobformin (x0)
options = optimset ('LargeScale', 'off');
[x, fval]=fmincon (@probofminobj, x0, [], [], [], [], [],
[], @conprobformin, options)
fprintf('The values of constraints at optimum solution\n');
[c, ceq] = conprobformin(x) % Check the constraint values at x
```

This produces the solution or output as follows:

```
The values of function value and constraints at starting point
f=
 41.4960
c =
-215.7947
-540.6668
-5.0000
-7.0000
-0.2000
-0.4000
ceq =
  []
Optimization terminated: first-order optimality
measure less
```

```
than options. TolFun and maximum constraint violation
is less
than options.TolCon.
Active inequalities (to within options.TolCon = 1e-006):
lower upper ineqlin ineqnonlin
              1
              2
x=
 5.4510 0.2920
fval =
 26.5310
The values of constraints at optimum solution
c=
 -0.0000
 -0.0000
 -3.4510
 -8.5490
 -0.0920
 -0.5080
ceq =
  []
```

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2

Classical Optimization Techniques

2.1 INTRODUCTION

The classical methods of optimization are useful in finding the optimum solution of continuous and differentiable functions. These methods are analytical and make use of the techniques of differential calculus in locating the optimum points. Since some of the practical problems involve objective functions that are not continuous and/or differentiable, the classical optimization techniques have limited scope in practical applications. However, a study of the calculus methods of optimization forms a basis for developing most of the numerical techniques of optimization presented in subsequent chapters. In this chapter we present the necessary and sufficient conditions in locating the optimum solution of a single-variable function, a multivariable function with no constraints, and a multivariable function with equality and inequality constraints.

2.2 SINGLE-VARIABLE OPTIMIZATION

A function of one variable $f(x)$ is said to have a *relative* or *local minimum* at $x = x^*$ if $f(x^*) \leq f(x^* + h)$ for all sufficiently small positive and negative values of h. Similarly, a point x^* is called a *relative* or *local maximum* if $f(x^*) \geq f(x^* + h)$ for all values of h sufficiently close to zero. A function $f(x)$ is said to have a *global* or *absolute minimum* at x^* if $f(x^*) \leq f(x)$ for all x, and not just for all x close to x^*, in the domain over which $f(x)$ is defined. Similarly, a point x^* will be a global maximum of $f(x)$ if $f(x^*) \geq f(x)$ for all x in the domain. Figure 2.1 shows the difference between the local and global optimum points.

A *single-variable optimization problem* is one in which the value of $x = x^*$ is to be found in the interval $[a, b]$ such that x^* minimizes $f(x)$. The following two theorems provide the necessary and sufficient conditions for the relative minimum of a function of a single variable.

Theorem 2.1 Necessary Condition If a function $f(x)$ is defined in the interval $a \leq x \leq b$ and has a relative minimum at $x = x^*$, where $a < x^* < b$, and if the derivative $df(x)/dx = f'(x)$ exists as a finite number at $x = x^*$, then $f'(x^*) = 0$.

Proof: It is given that

$$f'(x^*) = \lim_{h \to 0} \frac{f(x^* + h) - f(x^*)}{h} \tag{2.1}$$

Engineering Optimization: Theory and Practice, Fourth Edition Singiresu S. Rao

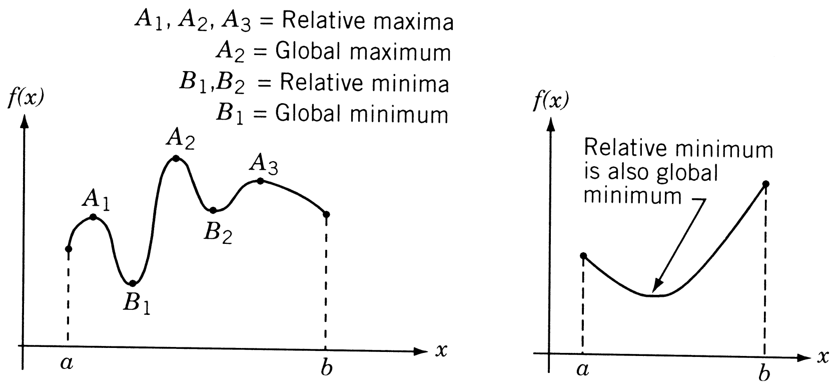

Figure 2.1 Relative and global minima.

exists as a definite number, which we want to prove to be zero. Since x^* is a relative minimum, we have

$$f(x^*) \leq f(x^* + h)$$

for all values of h sufficiently close to zero. Hence

$$\frac{f(x^* + h) - f(x^*)}{h} \geq 0 \quad \text{if } h > 0$$

$$\frac{f(x^* + h) - f(x^*)}{h} \leq 0 \quad \text{if } h < 0$$

Thus Eq. (2.1) gives the limit as h tends to zero through positive values as

$$f'(x^*) \geq 0 \tag{2.2}$$

while it gives the limit as h tends to zero through negative values as

$$f'(x^*) \leq 0 \tag{2.3}$$

The only way to satisfy both Eqs. (2.2) and (2.3) is to have

$$f'(x^*) = 0 \tag{2.4}$$

This proves the theorem.

Notes:

1. This theorem can be proved even if x^* is a relative maximum.
2. The theorem does not say what happens if a minimum or maximum occurs at a point x^* where the derivative fails to exist. For example, in Fig. 2.2,

$$\lim_{h \to 0} \frac{f(x^* + h) - f(x^*)}{h} = m^+ \text{(positive) or } m^- \text{(negative)}$$

depending on whether h approaches zero through positive or negative values, respectively. Unless the numbers m^+ and m^- are equal, the derivative $f'(x^*)$ does not exist. If $f'(x^*)$ does not exist, the theorem is not applicable.

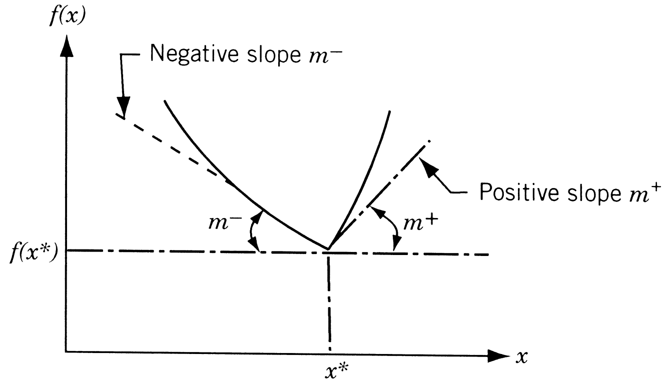

Figure 2.2 Derivative undefined at x^*.

3. The theorem does not say what happens if a minimum or maximum occurs at an endpoint of the interval of definition of the function. In this case

$$\lim_{h \to 0} \frac{f(x^* + h) - f(x^*)}{h}$$

exists for positive values of h only or for negative values of h only, and hence the derivative is not defined at the endpoints.
4. The theorem does not say that the function necessarily will have a minimum or maximum at every point where the derivative is zero. For example, the derivative $f'(x) = 0$ at $x = 0$ for the function shown in Fig. 2.3. However, this point is neither a minimum nor a maximum. In general, a point x^* at which $f'(x^*) = 0$ is called a *stationary point*.

If the function $f(x)$ possesses continuous derivatives of every order that come in question, in the neighborhood of $x = x^*$, the following theorem provides the sufficient condition for the minimum or maximum value of the function.

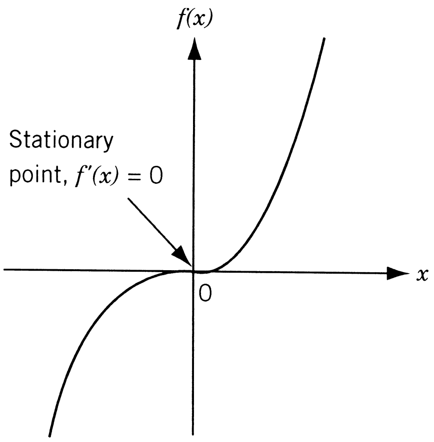

Figure 2.3 Stationary (inflection) point.

Theorem 2.2 Sufficient Condition Let $f'(x^*) = f''(x^*) = \cdots = f^{(n-1)}(x^*) = 0$, but $f^{(n)}(x^*) \neq 0$. Then $f(x^*)$ is (i) a minimum value of $f(x)$ if $f^{(n)}(x^*) > 0$ and n is even; (ii) a maximum value of $f(x)$ if $f^{(n)}(x^*) < 0$ and n is even; (iii) neither a maximum nor a minimum if n is odd.

Proof: Applying Taylor's theorem with remainder after n terms, we have

$$\begin{aligned} f(x^* + h) =& f(x^*) + hf'(x^*) + \frac{h^2}{2!} f''(x^*) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(x^*) \\ &+ \frac{h^n}{n!} f^{(n)}(x^* + \theta h) \qquad \text{for} \quad 0 < \theta < 1 \end{aligned} \tag{2.5}$$

Since $f'(x^*) = f''(x^*) = \cdots = f^{(n-1)}(x^*) = 0$, Eq. (2.5) becomes

$$f(x^* + h) - f(x^*) = \frac{h^n}{n!} f^{(n)}(x^* + \theta h)$$

As $f^{(n)}(x^*) \neq 0$, there exists an interval around x^* for every point x of which the nth derivative $f^{(n)}(x)$ has the same sign, namely, that of $f^{(n)}(x^*)$. Thus for every point $x^* + h$ of this interval, $f^{(n)}(x^* + \theta h)$ has the sign of $f^{(n)}(x^*)$. When n is even, $h^n/n!$ is positive irrespective of whether h is positive or negative, and hence $f(x^* + h) - f(x^*)$ will have the same sign as that of $f^{(n)}(x^*)$. Thus x^* will be a relative minimum if $f^{(n)}(x^*)$ is positive and a relative maximum if $f^{(n)}(x^*)$ is negative. When n is odd, $h^n/n!$ changes sign with the change in the sign of h and hence the point x^* is neither a maximum nor a minimum. In this case the point x^* is called a *point of inflection*.

Example 2.1 Determine the maximum and minimum values of the function

$$f(x) = 12x^5 - 45x^4 + 40x^3 + 5$$

SOLUTION Since $f'(x) = 60(x^4 - 3x^3 + 2x^2) = 60x^2(x-1)(x-2)$, $f'(x) = 0$ at $x = 0$, $x = 1$, and $x = 2$. The second derivative is

$$f''(x) = 60(4x^3 - 9x^2 + 4x)$$

At $x = 1$, $f''(x) = -60$ and hence $x = 1$ is a relative maximum. Therefore,

$$f_{\max} = f(x = 1) = 12$$

At $x = 2$, $f''(x) = 240$ and hence $x = 2$ is a relative minimum. Therefore,

$$f_{\min} = f(x = 2) = -11$$

At $x = 0$, $f''(x) = 0$ and hence we must investigate the next derivative:

$$f'''(x) = 60(12x^2 - 18x + 4) = 240 \quad \text{at} \quad x = 0$$

Since $f'''(x) \neq 0$ at $x = 0$, $x = 0$ is neither a maximum nor a minimum, and it is an inflection point.

Example 2.2 In a two-stage compressor, the working gas leaving the first stage of compression is cooled (by passing it through a heat exchanger) before it enters the second stage of compression to increase the efficiency [2.13]. The total work input to a compressor (W) for an ideal gas, for isentropic compression, is given by

$$W = c_p T_1 \left[\left(\frac{p_2}{p_1} \right)^{(k-1)/k} + \left(\frac{p_3}{p_2} \right)^{(k-1)/k} - 2 \right]$$

where c_p is the specific heat of the gas at constant pressure, k is the ratio of specific heat at constant pressure to that at constant volume of the gas, and T_1 is the temperature at which the gas enters the compressor. Find the pressure, p_2, at which intercooling should be done to minimize the work input to the compressor. Also determine the minimum work done on the compressor.

SOLUTION The necessary condition for minimizing the work done on the compressor is

$$\frac{dW}{dp_2} = c_p T_1 \frac{k}{k-1} \left[\left(\frac{1}{p_1} \right)^{(k-1)/k} \frac{k-1}{k} (p_2)^{-1/k} \right.$$
$$\left. + (p_3)^{(k-1)/k} \frac{-k+1}{k} (p_2)^{(1-2k)/k} \right] = 0$$

which yields

$$p_2 = (p_1 p_3)^{1/2}$$

The second derivative of W with respect to p_2 gives

$$\frac{d^2W}{dp_2^2} = c_p T_1 \left[- \left(\frac{1}{P_1} \right)^{(k-1)/k} \frac{1}{k} (p_2)^{-(1+k)/k} \right.$$
$$\left. - (p_3)^{(k-1)/k} \frac{1-2k}{k} (p_2)^{(1-3k)/k} \right]$$

$$\left(\frac{d^2W}{dp_2^2} \right)_{p2=(p1\,p2)^{1/2}} = \frac{2c_p T_1 \dfrac{k-1}{k}}{p_1^{(3k-1)/2k} p_3^{(k+1)/2k}}$$

Since the ratio of specific heats k is greater than 1, we get

$$\frac{d^2W}{dp_2^2} > 0 \qquad \text{at} \quad p_2 = (p_1 p_3)^{1/2}$$

and hence the solution corresponds to a relative minimum. The minimum work done is given by

$$W_{\min} = 2c_p T_1 \frac{k}{k-1} \left[\left(\frac{p_3}{p_1} \right)^{(k-1)/2k} - 1 \right]$$

2.3 MULTIVARIABLE OPTIMIZATION WITH NO CONSTRAINTS

In this section we consider the necessary and sufficient conditions for the minimum or maximum of an unconstrained function of several variables. Before seeing these conditions, we consider the Taylor's series expansion of a multivariable function.

Definition: rth Differential of f. If all partial derivatives of the function f through order $r \geq 1$ exist and are continuous at a point $\mathbf{X}^*$, the polynomial

$$d^r f(\mathbf{X}^*) = \underbrace{\sum_{i=1}^{n} \sum_{j=1}^{n} \cdots \sum_{k=1}^{n}}_{r \text{ summations}} h_i h_j \cdots h_k \frac{\partial^r f(\mathbf{X}^*)}{\partial x_i \partial x_j \cdots \partial x_k} \tag{2.6}$$

is called the rth differential of f at $\mathbf{X}^*$. Notice that there are r summations and one h_i is associated with each summation in Eq. (2.6).

For example, when $r = 2$ and $n = 3$, we have

$$d^2 f(\mathbf{X}^*) = d^2 f(x_1^*, x_2^*, x_3^*) = \sum_{i=1}^{3} \sum_{j=1}^{3} h_i h_j \frac{\partial^2 f(\mathbf{X}^*)}{\partial x_i\, \partial x_j}$$

$$= h_1^2 \frac{\partial^2 f}{\partial x_1^2}(\mathbf{X}^*) + h_2^2 \frac{\partial^2 f}{\partial x_2^2}(\mathbf{X}^*) + h_3^2 \frac{\partial^2 f}{\partial x_3^2}(\mathbf{X}^*)$$

$$+ 2h_1 h_2 \frac{\partial^2 f}{\partial x_1 \partial x_2}(\mathbf{X}^*) + 2h_2 h_3 \frac{\partial^2 f}{\partial x_2 \partial x_3}(\mathbf{X}^*) + 2h_1 h_3 \frac{\partial^2 f}{\partial x_1 \partial x_3}(\mathbf{X}^*)$$

The Taylor's series expansion of a function $f(\mathbf{X})$ about a point $\mathbf{X}^*$ is given by

$$f(\mathbf{X}) = f(\mathbf{X}^*) + df(\mathbf{X}^*) + \frac{1}{2!} d^2 f(\mathbf{X}^*) + \frac{1}{3!} d^3 f(\mathbf{X}^*)$$

$$+ \cdots + \frac{1}{N!} d^N f(\mathbf{X}^*) + R_N(\mathbf{X}^*, \mathbf{h}) \tag{2.7}$$

where the last term, called the *remainder*, is given by

$$R_N(\mathbf{X}^*, \mathbf{h}) = \frac{1}{(N+1)!} d^{N+1} f(\mathbf{X}^* + \theta \mathbf{h}) \tag{2.8}$$

where $0 < \theta < 1$ and $\mathbf{h} = \mathbf{X} - \mathbf{X}^*$.

Example 2.3 Find the second-order Taylor's series approximation of the function

$$f(x_1, x_2, x_3) = x_2^2 x_3 + x_1 e^{x3}$$

about the point $\mathbf{X}^* = \{1, 0, -2\}^{\mathrm{T}}$.

SOLUTION The second-order Taylor's series approximation of the function f about point $\mathbf{X}^*$ is given by

$$f(\mathbf{X}) = f\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + df\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + \frac{1}{2!} d^2 f\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

where

$$f\begin{pmatrix}1\\0\\-2\end{pmatrix} = e^{-2}$$

$$df\begin{pmatrix}1\\0\\-2\end{pmatrix} = h_1\frac{\partial f}{\partial x_1}\begin{pmatrix}1\\0\\-2\end{pmatrix} + h_2\frac{\partial f}{\partial x_2}\begin{pmatrix}1\\0\\-2\end{pmatrix} + h_3\frac{\partial f}{\partial x_3}\begin{pmatrix}1\\0\\-2\end{pmatrix}$$

$$= [h_1e^{x3} + h_2(2x_2x_3) + h_3x_2^2 + h_3x_1e^{x3}]\begin{pmatrix}1\\0\\-2\end{pmatrix} = h_1e^{-2} + h_3e^{-2}$$

$$d^2f\begin{pmatrix}1\\0\\-2\end{pmatrix} = \sum_{i=1}^{3}\sum_{j=1}^{3} h_ih_j\frac{\partial^2 f}{\partial x_i\partial x_j}\begin{pmatrix}1\\0\\-2\end{pmatrix} = \left(h_1^2\frac{\partial^2 f}{\partial x_1^2} + h_2^2\frac{\partial^2 f}{\partial x_2^2} + h_3^2\frac{\partial^2 f}{\partial x_3^2}\right.$$

$$\left. + 2h_1h_2\frac{\partial^2 f}{\partial x_1\partial x_2} + 2h_2h_3\frac{\partial^2 f}{\partial x_2\partial x_3} + 2h_1h_3\frac{\partial^2 f}{\partial x_1\partial x_3}\right)\begin{pmatrix}1\\0\\-2\end{pmatrix}$$

$$= [h_1^2(0) + h_2^2(2x_3) + h_3^2(x_1e^{x3}) + 2h_1h_2(0) + 2h_2h_3(2x_2)$$

$$+ 2h_1h_3(e^{x3})]\begin{pmatrix}1\\0\\-2\end{pmatrix} = -4h_2^2 + e^{-2}h_3^2 + 2h_1\,h_3e^{-2}$$

Thus the Taylor's series approximation is given by

$$f(\mathbf{X}) \simeq e^{-2} + e^{-2}(h_1 + h_3) + \frac{1}{2!}(-4h_2^2 + e^{-2}h_3^2 + 2h_1\,h_3e^{-2})$$

where $h_1 = x_1 - 1$, $h_2 = x_2$, and $h_3 = x_3 + 2$.

Theorem 2.3 Necessary Condition If $f(\mathbf{X})$ has an extreme point (maximum or minimum) at $\mathbf{X} = \mathbf{X}^*$ and if the first partial derivatives of $f(\mathbf{X})$ exist at $\mathbf{X}^*$, then

$$\frac{\partial f}{\partial x_1}(\mathbf{X}^*) = \frac{\partial f}{\partial x_2}(\mathbf{X}^*) = \cdots = \frac{\partial f}{\partial x_n}(\mathbf{X}^*) = 0 \tag{2.9}$$

Proof: The proof given for Theorem 2.1 can easily be extended to prove the present theorem. However, we present a different approach to prove this theorem. Suppose that one of the first partial derivatives, say the kth one, does not vanish at $\mathbf{X}^*$. Then, by Taylor's theorem,

$$f(\mathbf{X}^* + \mathbf{h}) = f(\mathbf{X}^*) + \sum_{i=1}^{n} h_i\frac{\partial f}{\partial x_i}(\mathbf{X}^*) + R_1(\mathbf{X}^*, \mathbf{h})$$

that is,

$$f(\mathbf{X}^* + \mathbf{h}) - f(\mathbf{X}^*) = h_k \frac{\partial f}{\partial x_k}(\mathbf{X}^*) + \frac{1}{2!} d^2 f(\mathbf{X}^* + \theta \mathbf{h}), \qquad 0 < \theta < 1$$

Since $d^2 f(\mathbf{X}^* + \theta\mathbf{h})$ is of order h_i^2, the terms of order $\mathbf{h}$ will dominate the higher-order terms for small $\mathbf{h}$. Thus the sign of $f(\mathbf{X}^* + \mathbf{h}) - f(\mathbf{X}^*)$ is decided by the sign of $h_k\, \partial f(\mathbf{X}^*)/\partial x_k$. Suppose that $\partial f(\mathbf{X}^*)/\partial x_k > 0$. Then the sign of $f(\mathbf{X}^* + \mathbf{h}) - f(\mathbf{X}^*)$ will be positive for $h_k > 0$ and negative for $h_k < 0$. This means that $\mathbf{X}^*$ cannot be an extreme point. The same conclusion can be obtained even if we assume that $\partial f(\mathbf{X}^*)/\partial x_k < 0$. Since this conclusion is in contradiction with the original statement that $\mathbf{X}^*$ is an extreme point, we may say that $\partial f/\partial x_k = 0$ at $\mathbf{X} = \mathbf{X}^*$. Hence the theorem is proved.

Theorem 2.4 Sufficient Condition A sufficient condition for a stationary point $\mathbf{X}^*$ to be an extreme point is that the matrix of second partial derivatives (Hessian matrix) of $f(\mathbf{X})$ evaluated at $\mathbf{X}^*$ is (i) positive definite when $\mathbf{X}^*$ is a relative minimum point, and (ii) negative definite when $\mathbf{X}^*$ is a relative maximum point.

Proof: From Taylor's theorem we can write

$$f(\mathbf{X}^* + \mathbf{h}) = f(\mathbf{X}^*) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(\mathbf{X}^*) + \frac{1}{2!} \sum_{i=1}^{n} \sum_{j=1}^{n} h_i h_j \left. \frac{\partial^2 f}{\partial x_i \partial x_j} \right|_{\mathbf{X}=\mathbf{X}^*+\theta\mathbf{h}}, \qquad 0 < \theta < 1 \tag{2.10}$$

Since $\mathbf{X}^*$ is a stationary point, the necessary conditions give (Theorem 2.3)

$$\frac{\partial f}{\partial x_i} = 0, \qquad i = 1, 2, \ldots, n$$

Thus Eq. (2.10) reduces to

$$f(\mathbf{X}^* + \mathbf{h}) - f(\mathbf{X}^*) = \frac{1}{2!} \sum_{i=1}^{n} \sum_{j=1}^{n} h_i h_j \left. \frac{\partial^2 f}{\partial x_i \partial x_j} \right|_{\mathbf{X}=\mathbf{X}^*+\theta\mathbf{h}}, \qquad 0 < \theta < 1$$

Therefore, the sign of

$$f(\mathbf{X}^* + \mathbf{h}) - f(\mathbf{X}^*)$$

will be same as that of

$$\sum_{i=1}^{n} \sum_{j=1}^{n} h_i h_j \left. \frac{\partial^2 f}{\partial x_i \partial x_j} \right|_{\mathbf{X}=\mathbf{X}^*+\theta\mathbf{h}}$$

Since the second partial derivative of $\partial^2 f(\mathbf{X})/\partial x_i \partial x_j$ is continuous in the neighborhood of $\mathbf{X}^*$,

$$\left. \frac{\partial^2 f}{\partial x_i \partial x_j} \right|_{\mathbf{X}=\mathbf{X}^*+\theta\mathbf{h}}$$

will have the same sign as $(\partial^2 f/\partial x_i \partial x_j)|\ \mathbf{X} = \mathbf{X}^*$ for all sufficiently small $\mathbf{h}$. Thus $f(\mathbf{X}^* + \mathbf{h}) - f(\mathbf{X}^*)$ will be positive, and hence X^* will be a relative minimum, if

$$Q = \sum_{i=1}^{n} \sum_{j=1}^{n} h_i h_j \left. \frac{\partial^2 f}{\partial x_i \partial x_j} \right|_{\mathbf{X}=\mathbf{X}^*} \tag{2.11}$$

is positive. This quantity Q is a quadratic form and can be written in matrix form as

$$Q = \mathbf{h}^{\mathrm{T}} \mathbf{J} \mathbf{h}|_{\mathbf{X}=\mathbf{X}^*} \tag{2.12}$$

where

$$\mathbf{J}|_{\mathbf{X}=\mathbf{X}^*} = \left[\left. \frac{\partial^2 f}{\partial x_i \partial x_j} \right|_{\mathbf{X}=\mathbf{X}^*} \right] \tag{2.13}$$

is the matrix of second partial derivatives and is called the *Hessian matrix* of $f(\mathbf{X})$.

It is known from matrix algebra that the quadratic form of Eq. (2.11) or (2.12) will be positive for all $\mathbf{h}$ if and only if $[\mathbf{J}]$ is positive definite at $\mathbf{X} = \mathbf{X}^*$. This means that a sufficient condition for the stationary point $\mathbf{X}^*$ to be a relative minimum is that the Hessian matrix evaluated at the same point be positive definite. This completes the proof for the minimization case. By proceeding in a similar manner, it can be proved that the Hessian matrix will be negative definite if $\mathbf{X}^*$ is a relative maximum point.

Note: A matrix $\mathbf{A}$ will be positive definite if all its eigenvalues are positive; that is, all the values of λ that satisfy the determinantal equation

$$|\mathbf{A} - \lambda \mathbf{I}| = 0 \tag{2.14}$$

should be positive. Similarly, the matrix $[\mathbf{A}]$ will be negative definite if its eigenvalues are negative.

Another test that can be used to find the positive definiteness of a matrix $\mathbf{A}$ of order n involves evaluation of the determinants

$$A = |a_{11}|, \quad A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad A_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{32} \end{vmatrix}, \ldots, \quad A_n = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} \tag{2.15}$$

The matrix $\mathbf{A}$ will be positive definite if and only if all the values $A_1, A_2, A_3, \ldots, A_n$ are positive. The matrix $\mathbf{A}$ will be negative definite if and only if the sign of A_j is $(-1)^j$ for $j = 1, 2, \ldots, n$. If some of the A_j are positive and the remaining A_j are zero, the matrix $\mathbf{A}$ will be positive semidefinite.

Example 2.4 Figure 2.4 shows two frictionless rigid bodies (carts) A and B connected by three linear elastic springs having spring constants k_1, k_2, and k_3. The springs are at their natural positions when the applied force P is zero. Find the displacements x_1 and x_2 under the force P by using the principle of minimum potential energy.

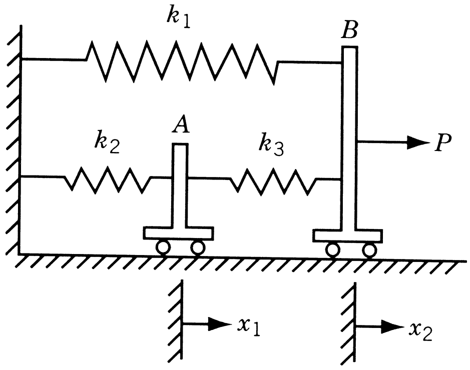

Figure 2.4 Spring–cart system.

SOLUTION According to the principle of minimum potential energy, the system will be in equilibrium under the load P if the potential energy is a minimum. The potential energy of the system is given by

$$\begin{aligned}&\text{potential energy } (U)\\ &\quad = \text{strain energy of springs} - \text{work done by external forces}\\ &\quad = [\tfrac{1}{2}k_2 x_1^2 + \tfrac{1}{2}k_3(x_2 - x_1)^2 + \tfrac{1}{2}k_1 x_2^2] - Px_2\end{aligned}$$

The necessary conditions for the minimum of U are

$$\frac{\partial U}{\partial x_1} = k_2 x_1 - k_3(x_2 - x_1) = 0 \tag{E$_1$}$$

$$\frac{\partial U}{\partial x_2} = k_3(x_2 - x_1) + k_1 x_2 - P = 0 \tag{E$_2$}$$

The values of x_1 and x_2 corresponding to the equilibrium state, obtained by solving Eqs. (E_1) and (E_2), are given by

$$x_1^* = \frac{Pk_3}{k_1 k_2 + k_1 k_3 + k_2 k_3}$$

$$x_2^* = \frac{P(k_2 + k_3)}{k_1 k_2 + k_1 k_3 + k_2 k_3}$$

The sufficiency conditions for the minimum at (x_1^*, x_2^*) can also be verified by testing the positive definiteness of the Hessian matrix of U. The Hessian matrix of U evaluated at (x_1^*, x_2^*) is

$$\mathbf{J}\Big|_{(x_1^*,x_2^*)} = \begin{bmatrix} \dfrac{\partial^2 U}{\partial x_1^2} & \dfrac{\partial^2 U}{\partial x_1 \partial x_2} \\[2ex] \dfrac{\partial^2 U}{\partial x_1 \partial x_2} & \dfrac{\partial^2 U}{\partial x_2^2} \end{bmatrix}_{(x_1^*,x_2^*)} = \begin{bmatrix} k_2 + k_3 & -k_3 \\ -k_3 & k_1 + k_3 \end{bmatrix}$$

The determinants of the square submatrices of **J** are

$$J_1 = \left| k_2 + k_3 \right| = k_2 + k_3 > 0$$

$$J_2 = \begin{vmatrix} k_2 + k_3 & -k_3 \\ -k_3 & k_1 + k_3 \end{vmatrix} = k_1 k_2 + k_1 k_3 + k_2 k_3 > 0$$

since the spring constants are always positive. Thus the matrix **J** is positive definite and hence (x_1^*, x_2^*) corresponds to the minimum of potential energy.

2.3.1 Semidefinite Case

We now consider the problem of determining the sufficient conditions for the case when the Hessian matrix of the given function is semidefinite. In the case of a function of a single variable, the problem of determining the sufficient conditions for the case when the second derivative is zero was resolved quite easily. We simply investigated the higher-order derivatives in the Taylor's series expansion. A similar procedure can be followed for functions of n variables. However, the algebra becomes quite involved, and hence we rarely investigate the stationary points for sufficiency in actual practice. The following theorem, analogous to Theorem 2.2, gives the sufficiency conditions for the extreme points of a function of several variables.

Theorem 2.5 Let the partial derivatives of f of all orders up to the order $k \geq 2$ be continuous in the neighborhood of a stationary point $\mathbf{X}^*$, and

$$d^r f|_{\mathbf{X}=\mathbf{X}^*} = 0, \qquad 1 \leq r \leq k - 1$$

$$d^k f|_{\mathbf{X}=\mathbf{X}^*} \neq 0$$

so that $d^k f|_{\mathbf{X}=\mathbf{X}^*}$ is the first nonvanishing higher-order differential of f at $\mathbf{X}^*$. If k is even, then (i) $\mathbf{X}^*$ is a relative minimum if $d^k f|_{\mathbf{X}=\mathbf{X}^*}$ is positive definite, (ii) $\mathbf{X}^*$ is a relative maximum if $d^k f|_{\mathbf{X}=\mathbf{X}^*}$ is negative definite, and (iii) if $d^k f|_{\mathbf{X}=\mathbf{X}^*}$ is semidefinite (but not definite), no general conclusion can be drawn. On the other hand, if k is odd, $\mathbf{X}^*$ is not an extreme point of $f(\mathbf{X})$.

Proof: A proof similar to that of Theorem 2.2 can be found in Ref. [2.5].

2.3.2 Saddle Point

In the case of a function of two variables, $f(x, y)$, the Hessian matrix may be neither positive nor negative definite at a point (x^*, y^*) at which

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$$

In such a case, the point (x^*, y^*) is called a *saddle point*. The characteristic of a saddle point is that it corresponds to a relative minimum or maximum of $f(x, y)$ with respect to one variable, say, x (the other variable being fixed at $y = y^*$) and a relative maximum or minimum of $f(x, y)$ with respect to the second variable y (the other variable being fixed at x^*).

As an example, consider the function $f(x, y) = x^2 - y^2$. For this function,

$$\frac{\partial f}{\partial x} = 2x \quad \text{and} \quad \frac{\partial f}{\partial y} = -2y$$

These first derivatives are zero at $x^* = 0$ and $y^* = 0$. The Hessian matrix of f at (x^*, y^*) is given by

$$\mathbf{J} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

Since this matrix is neither positive definite nor negative definite, the point ($x^* = 0$, $y^* = 0$) is a saddle point. The function is shown graphically in Fig. 2.5. It can be seen that $f(x, y^*) = f(x, 0)$ has a relative minimum and $f(x^*, y) = f(0, y)$ has a relative maximum at the saddle point (x^*, y^*). Saddle points may exist for functions of more than two variables also. The characteristic of the saddle point stated above still holds provided that x and y are interpreted as vectors in multidimensional cases.

Example 2.5 Find the extreme points of the function

$$f(x_1, x_2) = x_1^3 + x_2^3 + 2x_1^2 + 4x_2^2 + 6$$

SOLUTION The necessary conditions for the existence of an extreme point are

$$\frac{\partial f}{\partial x_1} = 3x_1^2 + 4x_1 = x_1(3x_1 + 4) = 0$$

$$\frac{\partial f}{\partial x_2} = 3x_2^2 + 8x_2 = x_2(3x_2 + 8) = 0$$

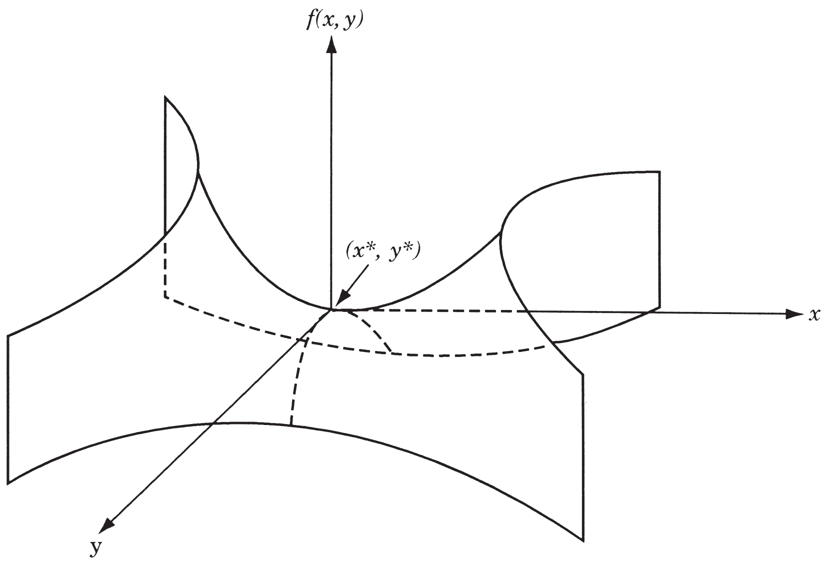

Figure 2.5 Saddle point of the function $f(x, y) = x^2 - y^2$.

These equations are satisfied at the points

$$(0, 0), \quad (0, -\tfrac{8}{3}), \quad (-\tfrac{4}{3}, 0), \quad \text{and} \quad (-\tfrac{4}{3}, -\tfrac{8}{3})$$

To find the nature of these extreme points, we have to use the sufficiency conditions. The second-order partial derivatives of f are given by

$$\frac{\partial^2 f}{\partial x_1^2} = 6x_1 + 4$$

$$\frac{\partial^2 f}{\partial x_2^2} = 6x_2 + 8$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = 0$$

The Hessian matrix of f is given by

$$\mathbf{J} = \begin{bmatrix} 6x_1 + 4 & 0 \\ 0 & 6x_2 + 8 \end{bmatrix}$$

If $J_1 = |6x_1 + 4|$ and $J_2 = \begin{vmatrix} 6x_1 + 4 & 0 \\ 0 & 6x_2 + 8 \end{vmatrix}$, the values of J_1 and J_2 and the nature of the extreme point are as given below:

Point $\mathbf{X}$	Value of J_1	Value of J_2	Nature of $\mathbf{J}$	Nature of $\mathbf{X}$	$f(\mathbf{X})$
(0, 0)	+4	+32	Positive definite	Relative minimum	6
$(0, -\frac{8}{3})$	+4	−32	Indefinite	Saddle point	418/27
$(-\frac{4}{3}, 0)$	−4	−32	Indefinite	Saddle point	194/27
$(-\frac{4}{3}, -\frac{8}{3})$	−4	+32	Negative definite	Relative maximum	50/3

2.4 MULTIVARIABLE OPTIMIZATION WITH EQUALITY CONSTRAINTS

In this section we consider the optimization of continuous functions subjected to equality constraints:

$$\begin{aligned} &\text{Minimize } f = f(\mathbf{X}) \\ &\text{subject to} \\ &g_j(\mathbf{X}) = 0, \quad j = 1, 2, \ldots, m \end{aligned} \tag{2.16}$$

where

$$\mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$$

Here m is less than or equal to n; otherwise (if $m > n$), the problem becomes overdefined and, in general, there will be no solution. There are several methods available for the solution of this problem. The methods of direct substitution, constrained variation, and Lagrange multipliers are discussed in the following sections.

2.4.1 Solution by Direct Substitution

For a problem with n variables and m equality constraints, it is theoretically possible to solve simultaneously the m equality constraints and express any set of m variables in terms of the remaining $n - m$ variables. When these expressions are substituted into the original objective function, there results a new objective function involving only $n - m$ variables. The new objective function is not subjected to any constraint, and hence its optimum can be found by using the unconstrained optimization techniques discussed in Section 2.3.

This method of direct substitution, although it appears to be simple in theory, is not convenient from a practical point of view. The reason for this is that the constraint equations will be nonlinear for most of practical problems, and often it becomes impossible to solve them and express any m variables in terms of the remaining $n - m$ variables. However, the method of direct substitution might prove to be very simple and direct for solving simpler problems, as shown by the following example.

Example 2.6 Find the dimensions of a box of largest volume that can be inscribed in a sphere of unit radius.

SOLUTION Let the origin of the Cartesian coordinate system x_1, x_2, x_3 be at the center of the sphere and the sides of the box be $2x_1$, $2x_2$, and $2x_3$. The volume of the box is given by

$$f(x_1, x_2, x_3) = 8x_1x_2x_3 \tag{E$_1$}$$

Since the corners of the box lie on the surface of the sphere of unit radius, x_1, x_2, and x_3 have to satisfy the constraint

$$x_1^2 + x_2^2 + x_3^2 = 1 \tag{E$_2$}$$

This problem has three design variables and one equality constraint. Hence the equality constraint can be used to eliminate any one of the design variables from the objective function. If we choose to eliminate x_3, Eq. (E$_2$) gives

$$x_3 = (1 - x_1^2 - x_2^2)^{1/2} \tag{E$_3$}$$

Thus the objective function becomes

$$f(x_1, x_2) = 8x_1x_2(1 - x_1^2 - x_2^2)^{1/2} \tag{E$_4$}$$

which can be maximized as an unconstrained function in two variables.

The necessary conditions for the maximum of f give

$$\frac{\partial f}{\partial x_1} = 8x_2 \left[(1 - x_1^2 - x_2^2)^{1/2} - \frac{x_1^2}{(1 - x_1^2 - x_2^2)^{1/2}} \right] = 0 \qquad (\text{E}_5)$$

$$\frac{\partial f}{\partial x_2} = 8x_1 \left[(1 - x_1^2 - x_2^2)^{1/2} - \frac{x_2^2}{(1 - x_1^2 - x_2^2)^{1/2}} \right] = 0 \qquad (\text{E}_6)$$

Equations (E_5) and (E_6) can be simplified to obtain

$$1 - 2x_1^2 - x_2^2 = 0$$

$$1 - x_1^2 - 2x_2^2 = 0$$

from which it follows that $x_1^* = x_2^* = 1/\sqrt{3}$ and hence $x_3^* = 1/\sqrt{3}$. This solution gives the maximum volume of the box as

$$f_{\max} = \frac{8}{3\sqrt{3}}$$

To find whether the solution found corresponds to a maximum or a minimum, we apply the sufficiency conditions to $f(x_1, x_2)$ of Eq. (E_4). The second-order partial derivatives of f at (x_1^*, x_2^*) are given by

$$\frac{\partial^2 f}{\partial x_1^2} = -\frac{32}{\sqrt{3}} \text{ at } (x_1^*,\ x_2^*)$$

$$\frac{\partial^2 f}{\partial x_2^2} = -\frac{32}{\sqrt{3}} \text{ at } (x_1^*,\ x_2^*)$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = -\frac{16}{\sqrt{3}} \text{ at } (x_1^*,\ x_2^*)$$

Since

$$\frac{\partial^2 f}{\partial x_1^2} < 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left(\frac{\partial^2 f}{\partial x_1 \partial x_2}\right)^2 > 0$$

the Hessian matrix of f is negative definite at $(x_1^*,\ x_2^*)$. Hence the point $(x_1^*,\ x_2^*)$ corresponds to the maximum of f.

2.4.2 Solution by the Method of Constrained Variation

The basic idea used in the method of constrained variation is to find a closed-form expression for the first-order differential of f (*df*) at all points at which the constraints $g_j(\mathbf{X}) = 0$, $j = 1, 2, \ldots, m$, are satisfied. The desired optimum points are then obtained by setting the differential *df* equal to zero. Before presenting the general method, we indicate its salient features through the following simple problem with $n = 2$ and $m = 1$:

$$\text{Minimize } f(x_1, x_2) \tag{2.17}$$

subject to

$$g(x_1, x_2) = 0 \tag{2.18}$$

A necessary condition for f to have a minimum at some point (x_1^*, x_2^*) is that the total derivative of $f(x_1, x_2)$ with respect to x_1 must be zero at (x_1^*, x_2^*). By setting the total differential of $f(x_1, x_2)$ equal to zero, we obtain

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 = 0 \tag{2.19}$$

Since $g(x_1^*, x_2^*) = 0$ at the minimum point, any variations dx_1 and dx_2 taken about the point (x_1^*, x_2^*) are called *admissible variations* provided that the new point lies on the constraint:

$$g(x_1^* + dx_1, x_2^* + dx_2) = 0 \tag{2.20}$$

The Taylor's series expansion of the function in Eq. (2.20) about the point (x_1^*, x_2^*) gives

$$\begin{aligned} &g(x_1^* + dx_1, x_2^* + dx_2) \\ &\simeq g(x_1^*, x_2^*) + \frac{\partial g}{\partial x_1}(x_1^*, x_2^*)\, dx_1 + \frac{\partial g}{\partial x_2}(x_1^*, x_2^*)\, dx_2 = 0 \end{aligned} \tag{2.21}$$

where dx_1 and dx_2 are assumed to be small. Since $g(x_1^*, x_2^*) = 0$, Eq. (2.21) reduces to

$$dg = \frac{\partial g}{\partial x_1} dx_1 + \frac{\partial g}{\partial x_2} dx_2 = 0 \qquad \text{at} \quad (x_1^*, x_2^*) \tag{2.22}$$

Thus Eq. (2.22) has to be satisfied by all admissible variations. This is illustrated in Fig. 2.6, where PQ indicates the curve at each point of which Eq. (2.18) is satisfied. If A is taken as the base point (x_1^*, x_2^*), the variations in x_1 and x_2 leading to points B and C are called *admissible variations*. On the other hand, the variations in x_1 and x_2 representing point D are not admissible since point D does not

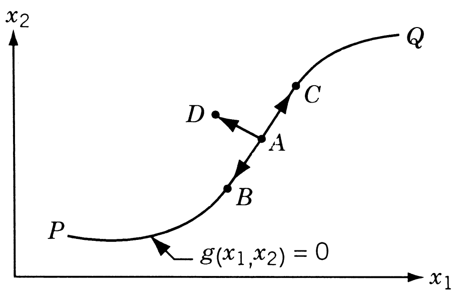

Figure 2.6 Variations about A.

lie on the constraint curve, $g(x_1, x_2) = 0$. Thus any set of variations (dx_1, dx_2) that does not satisfy Eq. (2.22) leads to points such as D, which do not satisfy constraint Eq. (2.18).

Assuming that $\partial g/\partial x_2 \neq 0$, Eq. (2.22) can be rewritten as

$$dx_2 = -\frac{\partial g/\partial x_1}{\partial g/\partial x_2}(x_1^*, x_2^*)dx_1 \tag{2.23}$$

This relation indicates that once the variation in $x_1 (dx_1)$ is chosen arbitrarily, the variation in x_2 (dx_2) is decided automatically in order to have dx_1 and dx_2 as a set of admissible variations. By substituting Eq. (2.23) in Eq. (2.19), we obtain

$$df = \left(\frac{\partial f}{\partial x_1} - \frac{\partial g/\partial x_1}{\partial g/\partial x_2}\frac{\partial f}{\partial x_2}\right)\bigg|_{(x_1^*, x_2^*)} dx_1 = 0 \tag{2.24}$$

The expression on the left-hand side is called the *constrained variation* of f. Note that Eq. (2.24) has to be satisfied for all values of dx_1. Since dx_1 can be chosen arbitrarily, Eq. (2.24) leads to

$$\left(\frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_1}\right)\bigg|_{(x_1^*, x_2^*)} = 0 \tag{2.25}$$

Equation (2.25) represents a necessary condition in order to have (x_1^*, x_2^*) as an extreme point (minimum or maximum).

Example 2.7 A beam of uniform rectangular cross section is to be cut from a log having a circular cross section of diameter $2a$. The beam has to be used as a cantilever beam (the length is fixed) to carry a concentrated load at the free end. Find the dimensions of the beam that correspond to the maximum tensile (bending) stress carrying capacity.

SOLUTION From elementary strength of materials, we know that the tensile stress induced in a rectangular beam (σ) at any fiber located a distance y from the neutral axis is given by

$$\frac{\sigma}{y} = \frac{M}{I}$$

where M is the bending moment acting and I is the moment of inertia of the cross section about the x axis. If the width and depth of the rectangular beam shown in Fig. 2.7 are $2x$ and $2y$, respectively, the maximum tensile stress induced is given by

$$\sigma_{\max} = \frac{M}{I}y = \frac{My}{\frac{1}{12}(2x)(2y)^3} = \frac{3}{4}\frac{M}{xy^2}$$

Thus for any specified bending moment, the beam is said to have maximum tensile stress carrying capacity if the maximum induced stress $(\sigma_{\max})$ is a minimum. Hence we need to minimize k/xy^2 or maximize Kxy^2, where $k = 3M/4$ and $K = 1/k$, subject to the constraint

$$x^2 + y^2 = a^2$$

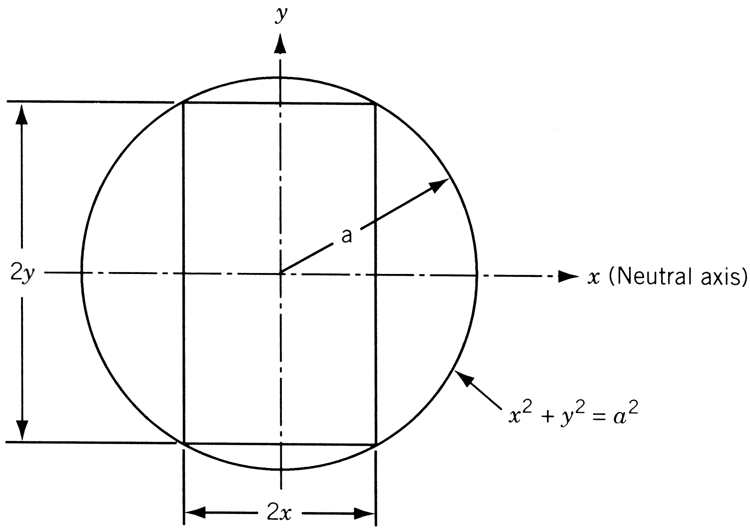

Figure 2.7 Cross section of the log.

This problem has two variables and one constraint; hence Eq. (2.25) can be applied for finding the optimum solution. Since

$$f = kx^{-1}y^{-2} \tag{E$_1$}$$

$$g = x^2 + y^2 - a^2 \tag{E$_2$}$$

we have

$$\frac{\partial f}{\partial x} = -kx^{-2}y^{-2}$$

$$\frac{\partial f}{\partial y} = -2kx^{-1}y^{-3}$$

$$\frac{\partial g}{\partial x} = 2x$$

$$\frac{\partial g}{\partial y} = 2y$$

Equation (2.25) gives

$$-kx^{-2}y^{-2}(2y) + 2kx^{-1}y^{-3}(2x) = 0 \quad \text{at} \quad (x^*, y^*)$$

that is,

$$y^* = \sqrt{2}x^* \tag{E$_3$}$$

Thus the beam of maximum tensile stress carrying capacity has a depth of $\sqrt{2}$ times its breadth. The optimum values of x and y can be obtained from Eqs. (E$_3$) and (E$_2$) as

$$x^* = \frac{a}{\sqrt{3}} \quad \text{and} \quad y^* = \sqrt{2}\frac{a}{\sqrt{3}}$$

Necessary Conditions for a General Problem. The procedure indicated above can be generalized to the case of a problem in n variables with m constraints. In this case, each constraint equation $g_j(\mathbf{X}) = 0$, $j = 1, 2, \ldots, m$, gives rise to a linear equation in the variations dx_i, $i = 1, 2, \ldots, n$. Thus there will be in all m linear equations in n variations. Hence any m variations can be expressed in terms of the remaining $n - m$ variations. These expressions can be used to express the differential of the objective function, df, in terms of the $n - m$ independent variations. By letting the coefficients of the independent variations vanish in the equation $df = 0$, one obtains the necessary conditions for the constrained optimum of the given function. These conditions can be expressed as [2.6]

$$J\left(\frac{f, g_1, g_2, \ldots, g_m}{x_k, x_1, x_2, x_3, \ldots, x_m}\right) = \begin{vmatrix} \dfrac{\partial f}{\partial x_k} & \dfrac{\partial f}{\partial x_1} & \dfrac{\partial f}{\partial x_2} & \cdots & \dfrac{\partial f}{\partial x_m} \\ \dfrac{\partial g_1}{\partial x_k} & \dfrac{\partial g_1}{\partial x_1} & \dfrac{\partial g_1}{\partial x_2} & \cdots & \dfrac{\partial g_1}{\partial x_m} \\ \dfrac{\partial g_2}{\partial x_k} & \dfrac{\partial g_2}{\partial x_1} & \dfrac{\partial g_2}{\partial x_2} & \cdots & \dfrac{\partial g_2}{\partial x_m} \\ \vdots & & & & \\ \dfrac{\partial g_m}{\partial x_k} & \dfrac{\partial g_m}{\partial x_1} & \dfrac{\partial g_m}{\partial x_2} & \cdots & \dfrac{\partial g_m}{\partial x_m} \end{vmatrix} = 0 \tag{2.26}$$

where $k = m + 1, m + 2, \ldots, n$. It is to be noted that the variations of the first m variables $(dx_1, dx_2, \ldots, dx_m)$ have been expressed in terms of the variations of the remaining $n - m$ variables $(dx_{m+1}, dx_{m+2}, \ldots, dx_n)$ in deriving Eqs. (2.26). This implies that the following relation is satisfied:

$$J\left(\frac{g_1, g_2, \ldots, g_m}{x_1, x_2, \ldots, x_m}\right) \neq 0 \tag{2.27}$$

The $n - m$ equations given by Eqs. (2.26) represent the necessary conditions for the extremum of $f(\mathbf{X})$ under the m equality constraints, $g_j(\mathbf{X}) = 0$, $j = 1, 2, \ldots, m$.

Example 2.8

$$\text{Minimize } f(\mathbf{Y}) = \tfrac{1}{2}(y_1^2 + y_2^2 + y_3^2 + y_4^2) \tag{E$_1$}$$

subject to

$$g_1(\mathbf{Y}) = y_1 + 2y_2 + 3y_3 + 5y_4 - 10 = 0 \tag{E$_2$}$$

$$g_2(\mathbf{Y}) = y_1 + 2y_2 + 5y_3 + 6y_4 - 15 = 0 \tag{E$_3$}$$

SOLUTION This problem can be solved by applying the necessary conditions given by Eqs. (2.26). Since $n = 4$ and $m = 2$, we have to select two variables as independent variables. First we show that any arbitrary set of variables cannot be chosen as independent variables since the remaining (dependent) variables have to satisfy the condition of Eq. (2.27).

In terms of the notation of our equations, let us take the independent variables as

$$x_3 = y_3 \quad \text{and} \quad x_4 = y_4 \qquad \text{so that} \quad x_1 = y_1 \quad \text{and} \quad x_2 = y_2$$

Then the Jacobian of Eq. (2.27) becomes

$$J\left(\frac{g_1, g_2}{x_1, x_2}\right) = \begin{vmatrix} \dfrac{\partial g_1}{\partial y_1} & \dfrac{\partial g_1}{\partial y_2} \\ \dfrac{\partial g_2}{\partial y_1} & \dfrac{\partial g_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0$$

and hence the necessary conditions of Eqs. (2.26) cannot be applied.

Next, let us take the independent variables as $x_3 = y_2$ and $x_4 = y_4$ so that $x_1 = y_1$ and $x_2 = y_3$. Then the Jacobian of Eq. (2.27) becomes

$$J\left(\frac{g_1, g_2}{x_1, x_2}\right) = \begin{vmatrix} \dfrac{\partial g_1}{\partial y_1} & \dfrac{\partial g_1}{\partial y_3} \\ \dfrac{\partial g_2}{\partial y_1} & \dfrac{\partial g_2}{\partial y_3} \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} = 2 \neq 0$$

and hence the necessary conditions of Eqs. (2.26) can be applied. Equations (2.26) give for $k = m + 1 \ = 3$

$$\begin{aligned} \begin{vmatrix} \dfrac{\partial f}{\partial x_3} & \dfrac{\partial f}{\partial x_1} & \dfrac{\partial f}{\partial x_2} \\ \dfrac{\partial g_1}{\partial x_3} & \dfrac{\partial g_1}{\partial x_1} & \dfrac{\partial g_1}{\partial x_2} \\ \dfrac{\partial g_2}{\partial x_3} & \dfrac{\partial g_2}{\partial x_1} & \dfrac{\partial g_2}{\partial x_2} \end{vmatrix} &= \begin{vmatrix} \dfrac{\partial f}{\partial y_2} & \dfrac{\partial f}{\partial y_1} & \dfrac{\partial f}{\partial y_3} \\ \dfrac{\partial g_1}{\partial y_2} & \dfrac{\partial g_1}{\partial y_1} & \dfrac{\partial g_1}{\partial y_3} \\ \dfrac{\partial g_2}{\partial y_2} & \dfrac{\partial g_2}{\partial y_1} & \dfrac{\partial g_2}{\partial y_3} \end{vmatrix} \\ &= \begin{vmatrix} y_2 & y_1 & y_3 \\ 2 & 1 & 3 \\ 2 & 1 & 5 \end{vmatrix} \\ &= y_2(5-3) - y_1(10-6) + y_3(2-2) \\ &= 2y_2 - 4y_1 = 0 \end{aligned} \tag{E$_4$}$$

and for $k = m + 2 = n = 4$,

$$\begin{vmatrix} \dfrac{\partial f}{\partial x_4} & \dfrac{\partial f}{\partial x_1} & \dfrac{\partial f}{\partial x_2} \\ \dfrac{\partial g_1}{\partial x_4} & \dfrac{\partial g_1}{\partial x_1} & \dfrac{\partial g_1}{\partial x_2} \\ \dfrac{\partial g_2}{\partial x_4} & \dfrac{\partial g_2}{\partial x_1} & \dfrac{\partial g_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial f}{\partial y_4} & \dfrac{\partial f}{\partial y_1} & \dfrac{\partial f}{\partial y_3} \\ \dfrac{\partial g_1}{\partial y_4} & \dfrac{\partial g_1}{\partial y_1} & \dfrac{\partial g_1}{\partial y_3} \\ \dfrac{\partial g_2}{\partial y_4} & \dfrac{\partial g_2}{\partial y_1} & \dfrac{\partial g_2}{\partial y_3} \end{vmatrix}$$

$$= \begin{vmatrix} y_4 & y_1 & y_3 \\ 5 & 1 & 3 \\ 6 & 1 & 5 \end{vmatrix}$$

$$= y_4(5-3) - y_1(25-18) + y_3(5-6)$$

$$= 2y_4 - 7y_1 - y_3 = 0 \tag{E$_5$}$$

Equations (E_4) and (E_5) give the necessary conditions for the minimum or the maximum of f as

$$\begin{aligned} y_1 &= \tfrac{1}{2} y_2 \\ y_3 &= 2y_4 - 7y_1 = 2y_4 - \tfrac{7}{2} y_2 \end{aligned} \tag{E$_6$}$$

When Eqs. (E_6) are substituted, Eqs. (E_2) and (E_3) take the form

$$-8y_2 + 11y_4 = 10$$

$$-15y_2 + 16y_4 = 15$$

from which the desired optimum solution can be obtained as

$$y_1^* = -\tfrac{5}{74}$$

$$y_2^* = -\tfrac{5}{37}$$

$$y_3^* = \tfrac{155}{74}$$

$$y_4^* = \tfrac{30}{37}$$

Sufficiency Conditions for a General Problem. By eliminating the first m variables, using the m equality constraints (this is possible, at least in theory), the objective function f can be made to depend only on the remaining variables, $x_{m+1}, x_{m+2}, \ldots, x_n$. Then the Taylor's series expansion of f, in terms of these variables, about the extreme point $\mathbf{X}^*$ gives

$$f(\mathbf{X}^* + d\mathbf{X}) \simeq f(\mathbf{X}^*) + \sum_{i=m+1}^{n} \left(\frac{\partial f}{\partial x_i}\right)_g dx_i + \frac{1}{2!} \sum_{i=m+1}^{n} \sum_{j=m+1}^{n} \left(\frac{\partial^2 f}{\partial x_i\, \partial x_j}\right)_g dx_i\, dx_j \tag{2.28}$$

where $(\partial f/\partial x_i)_g$ is used to denote the partial derivative of f with respect to x_i (holding all the other variables $x_{m+1}, x_{m+2}, \ldots, x_{i-1}, x_{i+1}, x_{i+2}, \ldots, x_n$ constant) when $x_1, x_2, \ldots, x_m$ are allowed to change so that the constraints $g_j(\mathbf{X}^* + d\mathbf{X}) = 0$, $j = 1, 2, \ldots, m$, are satisfied; the second derivative, $(\partial^2 f/\partial x_i \partial x_j)_g$, is used to denote a similar meaning.

As an example, consider the problem of minimizing

$$f(\mathbf{X}) = f(x_1, x_2, x_3)$$

subject to the only constraint

$$g_1(\mathbf{X}) = x_1^2 + x_2^2 + x_3^2 - 8 = 0$$

Since $n = 3$ and $m = 1$ in this problem, one can think of any of the m variables, say x_1, to be dependent and the remaining $n - m$ variables, namely x_2 and x_3, to be independent. Here the constrained partial derivative $(\partial f/\partial x_2)_g$, for example, means the rate of change of f with respect to x_2 (holding the other independent variable x_3 constant) and at the same time allowing x_1 to change about $\mathbf{X}^*$ so as to satisfy the constraint $g_1(\mathbf{X}) = 0$. In the present case, this means that dx_1 has to be chosen to satisfy the relation

$$g_1(\mathbf{X}^* + d\mathbf{X}) \simeq g_1(\mathbf{X}^*) + \frac{\partial g_1}{\partial x_1}(\mathbf{X}^*)dx_1 + \frac{\partial g_1}{\partial x_2}(\mathbf{X}^*)dx_2 + \frac{\partial g_1}{\partial x_3}(\mathbf{X}^*)dx_3 = 0$$

that is,

$$2x_1^*\, dx_1 + 2x_2^*\, dx_2 = 0$$

since $g_1(\mathbf{X}^*) = 0$ at the optimum point and $dx_3 = 0$ (x_3 is held constant).

Notice that $(\partial f/\partial x_i)_g$ has to be zero for $i = m + 1, m + 2, \ldots, n$ since the dx_i appearing in Eq. (2.28) are all independent. Thus the necessary conditions for the existence of constrained optimum at $\mathbf{X}^*$ can also be expressed as

$$\left(\frac{\partial f}{\partial x_i}\right)_g = 0, \qquad i = m + 1,\ m + 2, \ldots, n \tag{2.29}$$

Of course, with little manipulation, one can show that Eqs. (2.29) are nothing but Eqs. (2.26). Further, as in the case of optimization of a multivariable function with no constraints, one can see that a sufficient condition for $\mathbf{X}^*$ to be a constrained relative minimum (maximum) is that the quadratic form Q defined by

$$Q = \sum_{i=m+1}^{n} \sum_{j=m+1}^{n} \left(\frac{\partial^2 f}{\partial x_i\, \partial x_j}\right)_g dx_i\; dx_j \tag{2.30}$$

is positive (negative) for all nonvanishing variations dx_i. As in Theorem 2.4, the matrix

$$\begin{bmatrix} \left(\dfrac{\partial^2 f}{\partial x_{m+1}^2}\right)_g & \left(\dfrac{\partial^2 f}{\partial x_{m+1}\, \partial x_{m+2}}\right)_g & \cdots & \left(\dfrac{\partial^2 f}{\partial x_{m+1}\, \partial x_n}\right)_g \\ \vdots & & & \\ \left(\dfrac{\partial^2 f}{\partial x_n\, \partial x_{m+1}}\right)_g & \left(\dfrac{\partial^2 f}{\partial x_n\, \partial x_{m+2}}\right)_g & \cdots & \left(\dfrac{\partial^2 f}{\partial x_n^2}\right)_g \end{bmatrix}$$

has to be positive (negative) definite to have Q positive (negative) for all choices of dx_i. It is evident that computation of the constrained derivatives $(\partial^2 f/\partial x_i\, \partial x_j)_g$ is a difficult task and may be prohibitive for problems with more than three constraints. Thus the method of constrained variation, although it appears to be simple in theory, is very difficult to apply since the necessary conditions themselves involve evaluation of determinants of order $m + 1$. This is the reason that the method of Lagrange multipliers, discussed in the following section, is more commonly used to solve a multivariable optimization problem with equality constraints.

2.4.3 Solution by the Method of Lagrange Multipliers

The basic features of the Lagrange multiplier method is given initially for a simple problem of two variables with one constraint. The extension of the method to a general problem of n variables with m constraints is given later.

Problem with Two Variables and One Constraint. Consider the problem

$$\text{Minimize } f(x_1, x_2) \tag{2.31}$$

subject to

$$g(x_1, x_2) = 0$$

For this problem, the necessary condition for the existence of an extreme point at $\mathbf{X} = \mathbf{X}^*$ was found in Section 2.4.2 to be

$$\left.\left(\frac{\partial f}{\partial x_1} - \frac{\partial f/\partial x_2}{\partial g/\partial x_2}\frac{\partial g}{\partial x_1}\right)\right|_{(x_1^*,\, x_2^*)} = 0 \tag{2.32}$$

By defining a quantity λ, called the *Lagrange multiplier*, as

$$\lambda = -\left.\left(\frac{\partial f/\partial x_2}{\partial g/\partial x_2}\right)\right|_{(x_1^*,\, x_2^*)} \tag{2.33}$$

Equation (2.32) can be expressed as

$$\left.\left(\frac{\partial f}{\partial x_1} + \lambda\frac{\partial g}{\partial x_1}\right)\right|_{(x_1^*,\, x_2^*)} = 0 \tag{2.34}$$

and Eq. (2.33) can be written as

$$\left.\left(\frac{\partial f}{\partial x_2} + \lambda\frac{\partial g}{\partial x_2}\right)\right|_{(x_1^*,\, x_2^*)} = 0 \tag{2.35}$$

In addition, the constraint equation has to be satisfied at the extreme point, that is,

$$g(x_1, x_2)|_{(x_1^*, x_2^*)} = 0 \tag{2.36}$$

Thus Eqs. (2.34) to (2.36) represent the necessary conditions for the point (x_1^*, x_2^*) to be an extreme point.

Notice that the partial derivative $(\partial g/\partial x_2)|_{(x_1^*,\, x_2^*)}$ has to be nonzero to be able to define λ by Eq. (2.33). This is because the variation dx_2 was expressed in terms of dx_1 in the derivation of Eq. (2.32) [see Eq. (2.23)]. On the other hand, if we choose to express dx_1 in terms of dx_2, we would have obtained the requirement that $(\partial g/\partial x_1)|_{(x_1^*,x_2^*)}$ be nonzero to define λ. Thus the derivation of the necessary conditions by the method of Lagrange multipliers requires that at least one of the partial derivatives of $g(x_1, x_2)$ be nonzero at an extreme point.

The necessary conditions given by Eqs. (2.34) to (2.36) are more commonly generated by constructing a function L, known as the Lagrange function, as

$$L(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda g(x_1, x_2) \tag{2.37}$$

By treating L as a function of the three variables x_1, x_2, and λ, the necessary conditions for its extremum are given by

$$\begin{aligned}
\frac{\partial L}{\partial x_1}(x_1, x_2, \lambda) &= \frac{\partial f}{\partial x_1}(x_1, x_2) + \lambda \frac{\partial g}{\partial x_1}(x_1, x_2) = 0 \\
\frac{\partial L}{\partial x_2}(x_1, x_2, \lambda) &= \frac{\partial f}{\partial x_2}(x_1, x_2) + \lambda \frac{\partial g}{\partial x_2}(x_1, x_2) = 0 \\
\frac{\partial L}{\partial \lambda}(x_1, x_2, \lambda) &= g(x_1, x_2) = 0
\end{aligned} \tag{2.38}$$

Equations (2.38) can be seen to be same as Eqs. (2.34) to (2.36). The sufficiency conditions are given later.

Example 2.9 Find the solution of Example 2.7 using the Lagrange multiplier method:

$$\text{Minimize } f(x, y) = kx^{-1}y^{-2}$$

subject to

$$g(x, y) = x^2 + y^2 - a^2 = 0$$

SOLUTION The Lagrange function is

$$L(x, y, \lambda) = f(x, y) + \lambda g(x, y) = kx^{-1}y^{-2} + \lambda(x^2 + y^2 - a^2)$$

The necessary conditions for the minimum of $f(x, y)$ [Eqs. (2.38)] give

$$\frac{\partial L}{\partial x} = -kx^{-2}y^{-2} + 2x\lambda = 0 \tag{E$_1$}$$

$$\frac{\partial L}{\partial y} = -2kx^{-1}y^{-3} + 2y\lambda = 0 \tag{E$_2$}$$

$$\frac{\partial L}{\partial \lambda} = x^2 + y^2 - a^2 = 0 \tag{E$_3$}$$

Equations (E_1) and (E_2) yield

$$2\lambda = \frac{k}{x^3y^2} = \frac{2k}{xy^4}$$

from which the relation $x^* = (1/\sqrt{2})y^*$ can be obtained. This relation, along with Eq. (E_3), gives the optimum solution as

$$x^* = \frac{a}{\sqrt{3}} \quad \text{and} \quad y^* = \sqrt{2}\frac{a}{\sqrt{3}}$$

Necessary Conditions for a General Problem. The equations derived above can be extended to the case of a general problem with n variables and m equality constraints:

$$\text{Minimize } f(\mathbf{X}) \tag{2.39}$$

subject to

$$g_j(\mathbf{X}) = 0, \quad j = 1, 2, \ldots, m$$

The Lagrange function, L, in this case is defined by introducing one Lagrange multiplier λ_j for each constraint $g_j(\mathbf{X})$ as

$$\begin{aligned} L(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_m) \\ = f(\mathbf{X}) + \lambda_1 g_1(\mathbf{X}) + \lambda_2 g_2(\mathbf{X}) + \cdots + \lambda_m g_m(\mathbf{X}) \end{aligned} \tag{2.40}$$

By treating L as a function of the $n + m$ unknowns, $x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_m$, the necessary conditions for the extremum of L, which also correspond to the solution of the original problem stated in Eq. (2.39), are given by

$$\frac{\partial L}{\partial x_i} = \frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i} = 0, \qquad i = 1, 2, \ldots, n \tag{2.41}$$

$$\frac{\partial L}{\partial \lambda_j} = g_j(\mathbf{X}) = 0, \qquad j = 1, 2, \ldots, m \tag{2.42}$$

Equations (2.41) and (2.42) represent $n + m$ equations in terms of the $n + m$ unknowns, x_i and λ_j. The solution of Eqs. (2.41) and (2.42) gives

$$\mathbf{X}^* = \begin{Bmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{Bmatrix} \quad \text{and} \quad \lambda^* = \begin{Bmatrix} \lambda_1^* \\ \lambda_2^* \\ \vdots \\ \lambda_m^* \end{Bmatrix}$$

The vector $\mathbf{X}^*$ corresponds to the relative constrained minimum of $f(\mathbf{X})$ (sufficient conditions are to be verified) while the vector λ^* provides the sensitivity information, as discussed in the next subsection.

Sufficiency Conditions for a General Problem. A sufficient condition for $f(\mathbf{X})$ to have a constrained relative minimum at $\mathbf{X}^*$ is given by the following theorem.

Theorem 2.6 Sufficient Condition A sufficient condition for $f(\mathbf{X})$ to have a relative minimum at $\mathbf{X}^*$ is that the quadratic, Q, defined by

$$Q = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 L}{\partial x_i \, \partial x_j} dx_i \, dx_j \tag{2.43}$$

evaluated at $\mathbf{X} = \mathbf{X}^*$ must be positive definite for all values of $d\mathbf{X}$ for which the constraints are satisfied.

Proof: The proof is similar to that of Theorem 2.4.

Notes:

1. If

$$Q = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 L}{\partial x_i \, \partial x_j}(\mathbf{X}^*, \boldsymbol{\lambda}^*) dx_i \, dx_j$$

is negative for all choices of the admissible variations dx_i, $\mathbf{X}^*$ will be a constrained maximum of $f(\mathbf{X})$.

2. It has been shown by Hancock [2.1] that a necessary condition for the quadratic form Q, defined by Eq. (2.43), to be positive (negative) definite for all admissible variations $d\mathbf{X}$ is that each root of the polynomial z_i, defined by the following determinantal equation, be positive (negative):

$$\begin{vmatrix} L_{11} - z & L_{12} & L_{13} & \ldots & L_{1n} & g_{11} & g_{21} & \ldots & g_{m1} \\ L_{21} & L_{22} - z & L_{23} & \ldots & L_{2n} & g_{12} & g_{22} & \ldots & g_{m2} \\ \vdots & & & & & & & & \\ L_{n1} & L_{n2} & L_{n3} & \ldots & L_{nn} - z & g_{1n} & g_{2n} & \ldots & g_{mn} \\ g_{11} & g_{12} & g_{13} & \ldots & g_{1n} & 0 & 0 & \ldots & 0 \\ g_{21} & g_{22} & g_{23} & \ldots & g_{2n} & 0 & 0 & \ldots & 0 \\ \vdots & & & & & & & & \\ g_{m1} & g_{m2} & g_{m3} & \ldots & g_{mn} & 0 & 0 & \ldots & 0 \end{vmatrix} = 0 \tag{2.44}$$

where

$$L_{ij} = \frac{\partial^2 L}{\partial x_i \, \partial x_j}(\mathbf{X}^*, \boldsymbol{\lambda}^*) \tag{2.45}$$

$$g_{ij} = \frac{\partial g_i}{\partial x_j}(\mathbf{X}^*) \tag{2.46}$$

3. Equation (2.44), on expansion, leads to an $(n - m)$th-order polynomial in z. If some of the roots of this polynomial are positive while the others are negative, the point $\mathbf{X}^*$ is not an extreme point.

The application of the necessary and sufficient conditions in the Lagrange multiplier method is illustrated with the help of the following example.

Example 2.10 Find the dimensions of a cylindrical tin (with top and bottom) made up of sheet metal to maximize its volume such that the total surface area is equal to $A_0 = 24\pi$.

SOLUTION If x_1 and x_2 denote the radius of the base and length of the tin, respectively, the problem can be stated as

$$\text{Maximize } f(x_1, x_2) = \pi x_1^2 x_2$$

subject to

$$2\pi x_1^2 + 2\pi x_1 x_2 = A_0 = 24\pi$$

The Lagrange function is

$$L(x_1, x_2, \lambda) = \pi x_1^2 x_2 + \lambda(2\pi x_1^2 + 2\pi x_1 x_2 - A_0)$$

and the necessary conditions for the maximum of f give

$$\frac{\partial L}{\partial x_1} = 2\pi x_1 x_2 + 4\pi \lambda x_1 + 2\pi \lambda x_2 = 0 \quad (\mathrm{E}_1)$$

$$\frac{\partial L}{\partial x_2} = \pi x_1^2 + 2\pi \lambda x_1 = 0 \quad (\mathrm{E}_2)$$

$$\frac{\partial L}{\partial \lambda} = 2\pi x_1^2 + 2\pi x_1 x_2 - A_0 = 0 \quad (\mathrm{E}_3)$$

Equations (E_1) and (E_2) lead to

$$\lambda = -\frac{x_1 x_2}{2x_1 + x_2} = -\frac{1}{2} x_1$$

that is,

$$x_1 = \tfrac{1}{2} x_2 \quad (\mathrm{E}_4)$$

and Eqs. (E_3) and (E_4) give the desired solution as

$$x_1^* = \left(\frac{A_0}{6\pi}\right)^{1/2}, \; x_2^* = \left(\frac{2A_0}{3\pi}\right)^{1/2}, \; \text{and}\, \lambda^* = -\left(\frac{A_0}{24\pi}\right)^{1/2}$$

This gives the maximum value of f as

$$f^* = \left(\frac{A_0^3}{54\pi}\right)^{1/2}$$

If $A_0 = 24\pi$, the optimum solution becomes

$$x_1^* = 2, \quad x_2^* = 4, \quad \lambda^* = -1, \quad \text{and} \quad f^* = 16\pi$$

To see that this solution really corresponds to the maximum of f, we apply the sufficiency condition of Eq. (2.44). In this case

$$L_{11} = \left.\frac{\partial^2 L}{\partial x_1^2}\right|_{(\mathbf{X}^*, \lambda^*)} = 2\pi x_2^* + 4\pi \lambda^* = 4\pi$$

$$L_{12} = \left.\frac{\partial^2 L}{\partial x_1 \partial x_2}\right|_{(\mathbf{X}^*, \lambda^*)} = L_{21} = 2\pi x_1^* + 2\pi \lambda^* = 2\pi$$

$$L_{22} = \left.\frac{\partial^2 L}{\partial x_2^2}\right|_{(\mathbf{X}^*,\lambda^*)} = 0$$

$$g_{11} = \left.\frac{\partial g_1}{\partial x_1}\right|_{(\mathbf{X}^*,\lambda^*)} = 4\pi x_1^* + 2\pi x_2^* = 16\pi$$

$$g_{12} = \left.\frac{\partial g_1}{\partial x_2}\right|_{(\mathbf{X}^*,\lambda^*)} = 2\pi x_1^* = 4\pi$$

Thus Eq. (2.44) becomes

$$\begin{vmatrix} 4\pi - z & 2\pi & 16\pi \\ 2\pi & 0 - z & 4\pi \\ 16\pi & 4\pi & 0 \end{vmatrix} = 0$$

that is,

$$272\pi^2 z + 192\pi^3 = 0$$

This gives

$$z = -\tfrac{12}{17}\pi$$

Since the value of z is negative, the point (x_1^*, x_2^*) corresponds to the maximum of f.

Interpretation of the Lagrange Multipliers. To find the physical meaning of the Lagrange multipliers, consider the following optimization problem involving only a single equality constraint:

$$\text{Minimize } f(\mathbf{X}) \tag{2.47}$$

subject to

$$\underset{\sim}{g}(\mathbf{X}) = b \quad \text{or} \quad g(\mathbf{X}) = b - \underset{\sim}{g}(\mathbf{X}) = 0 \tag{2.48}$$

where b is a constant. The necessary conditions to be satisfied for the solution of the problem are

$$\frac{\partial f}{\partial x_i} + \lambda \frac{\partial g}{\partial x_i} = 0, \qquad i = 1, 2, \ldots, n \tag{2.49}$$

$$g = 0 \tag{2.50}$$

Let the solution of Eqs. (2.49) and (2.50) be given by $\mathbf{X}^*$, λ^*, and $f^* = f(\mathbf{X}^*)$. Suppose that we want to find the effect of a small relaxation or tightening of the constraint on the optimum value of the objective function (i.e., we want to find the effect of a small change in b on f^*). For this we differentiate Eq. (2.48) to obtain

$$db - d\underset{\sim}{g} = 0$$

or

$$db = d\underset{\sim}{g} = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} dx_i \tag{2.51}$$

Equation (2.49) can be rewritten as

$$\frac{\partial f}{\partial x_i} + \lambda \frac{\partial g}{\partial x_i} = \frac{\partial f}{\partial x_i} - \lambda \frac{\partial \underset{\sim}{g}}{\partial x_i} = 0 \tag{2.52}$$

or

$$\frac{\partial \underset{\sim}{g}}{\partial x_i} = \frac{\partial f / \partial x_i}{\lambda}, \qquad i = 1, 2, \ldots, n \tag{2.53}$$

Substituting Eq. (2.53) into Eq. (2.51), we obtain

$$db = \sum_{i=1}^{n} \frac{1}{\lambda} \frac{\partial f}{\partial x_i} dx_i = \frac{df}{\lambda} \tag{2.54}$$

since

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \tag{2.55}$$

Equation (2.54) gives

$$\lambda = \frac{df}{db} \quad \text{or} \quad \lambda^* = \frac{df^*}{db} \tag{2.56}$$

or

$$df^* = \lambda^* db \tag{2.57}$$

Thus λ^* denotes the sensitivity (or rate of change) of f with respect to b or the marginal or incremental change in f^* with respect to b at x^*. In other words, λ^* indicates how tightly the constraint is binding at the optimum point. Depending on the value of λ^* (positive, negative, or zero), the following physical meaning can be attributed to λ^*:

1. $\lambda^* > 0$. In this case, a unit decrease in b is positively valued since one gets a smaller minimum value of the objective function f. In fact, the decrease in f^* will be exactly equal to λ^* since $df = \lambda^*(-1) = -\lambda^* < 0$. Hence λ^* may be interpreted as the marginal gain (further reduction) in f^* due to the tightening of the constraint. On the other hand, if b is increased by 1 unit, f will also increase to a new optimum level, with the amount of increase in f^* being determined by the magnitude of λ^* since $df = \lambda^*(+1) > 0$. In this case, λ^* may be thought of as the marginal cost (increase) in f^* due to the relaxation of the constraint.
2. $\lambda^* < 0$. Here a unit increase in b is positively valued. This means that it decreases the optimum value of f. In this case the marginal gain (reduction) in f^* due to a relaxation of the constraint by 1 unit is determined by the value of λ^* as $df^* = \lambda^*(+1) < 0$. If b is decreased by 1 unit, the marginal cost (increase) in f^* by the tightening of the constraint is $df^* = \lambda^*(-1) > 0$ since, in this case, the minimum value of the objective function increases.

3. $\lambda^* = 0$. In this case, any incremental change in b has absolutely no effect on the optimum value of f and hence the constraint will not be binding. This means that the optimization of f subject to $g = 0$ leads to the same optimum point $\mathbf{X}^*$ as with the unconstrained optimization of f.

In economics and operations research, Lagrange multipliers are known as *shadow prices* of the constraints since they indicate the changes in optimal value of the objective function per unit change in the right-hand side of the equality constraints.

Example 2.11 Find the maximum of the function $f(\mathbf{X}) = 2x_1 + x_2 + 10$ subject to $g(\mathbf{X}) = x_1 + 2x_2^2 = 3$ using the Lagrange multiplier method. Also find the effect of changing the right-hand side of the constraint on the optimum value of f.

SOLUTION The Lagrange function is given by

$$L(X, \lambda) = 2x_1 + x_2 + 10 + \lambda(3 - x_1 - 2x_2^2) \qquad (\text{E}_1)$$

The necessary conditions for the solution of the problem are

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= 2 - \lambda = 0 \\ \frac{\partial L}{\partial x_2} &= 1 - 4\lambda x_2 = 0 \\ \frac{\partial L}{\partial \lambda} &= 3 - x_1 - 2x_2^2 = 0 \end{aligned} \qquad (\text{E}_2)$$

The solution of Eqs. (E_2) is

$$\begin{aligned} \mathbf{X}^* &= \begin{Bmatrix} x_1^* \\ x_2^* \end{Bmatrix} = \begin{Bmatrix} 2.97 \\ 0.13 \end{Bmatrix} \\ \lambda^* &= 2.0 \end{aligned} \qquad (\text{E}_3)$$

The application of the sufficiency condition of Eq. (2.44) yields

$$\begin{vmatrix} L_{11} - z & L_{12} & g_{11} \\ L_{21} & L_{22} - z & g_{12} \\ g_{11} & g_{12} & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} -z & 0 & -1 \\ 0 & -4\lambda - z & -4x_2 \\ -1 & -4x_2 & 0 \end{vmatrix} = \begin{vmatrix} -z & 0 & -1 \\ 0 & -8 - z & -0.52 \\ -1 & -0.52 & 0 \end{vmatrix} = 0$$

$$0.2704z + 8 + z = 0$$

$$z = -6.2972$$

Hence $\mathbf{X}^*$ will be a maximum of f with $f^* = f(\mathbf{X}^*) = 16.07$.

One procedure for finding the effect on f^* of changes in the value of b (right-hand side of the constraint) would be to solve the problem all over with the new value of b. Another procedure would involve the use of the value of λ^*. When the original constraint is tightened by 1 unit (i.e., $db = -1$), Eq. (2.57) gives

$$df^* = \lambda^* db = 2(-1) = -2$$

Thus the new value of f^* is $f^* + df^* = 14.07$. On the other hand, if we relax the original constraint by 2 units (i.e., $db = 2$), we obtain

$$df^* = \lambda^* db = 2(+2) = 4$$

and hence the new value of f^* is $f^* + df^* = 20.07$.

2.5 MULTIVARIABLE OPTIMIZATION WITH INEQUALITY CONSTRAINTS

This section is concerned with the solution of the following problem:

$$\text{Minimize } f(\mathbf{X})$$

subject to

$$g_j(\mathbf{X}) \le 0, \quad j = 1, 2, \ldots, m \tag{2.58}$$

The inequality constraints in Eq. (2.58) can be transformed to equality constraints by adding nonnegative slack variables, y_j^2, as

$$g_j(\mathbf{X}) + y_j^2 = 0, \quad j = 1, 2, \ldots, m \tag{2.59}$$

where the values of the slack variables are yet unknown. The problem now becomes

$$\text{Minimize } f(\mathbf{X})$$

subject to

$$G_j(\mathbf{X}, \mathbf{Y}) = g_j(\mathbf{X}) + y_j^2 = 0, \qquad j = 1, 2, \ldots, m \tag{2.60}$$

where $\mathbf{Y} = \{y_1, y_2, \ldots, y_m\}^{\mathrm{T}}$ is the vector of slack variables.

This problem can be solved conveniently by the method of Lagrange multipliers. For this, we construct the Lagrange function L as

$$L(\mathbf{X}, \mathbf{Y}, \boldsymbol{\lambda}) = f(\mathbf{X}) + \sum_{j=1}^{m} \lambda_j G_j(\mathbf{X}, \mathbf{Y}) \tag{2.61}$$

where $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}^{\mathrm{T}}$ is the vector of Lagrange multipliers. The stationary points of the Lagrange function can be found by solving the following equations