$$
\dfrac{
\text{(перест)}\dfrac{\text{(утнч)}\dfrac{\text{(mod pon)}\dfrac{\text{(перест)}\dfrac{\text{(утнч)}\dfrac{A \vdash A}{A,(A\to B) \vdash A}}{(A\to B),A \vdash A} \quad \text{(утнч)}\dfrac{A\to B \vdash A\to B}{(A\to B),A \vdash A\to B}}{(A\to B),A \vdash B}}{(A\to B),A,(B\to C) \vdash B}}{(A\to B),(B\to C),A \vdash B}
\quad
\dfrac{\dfrac{\dfrac{B\to C \vdash B\to C}{(B\to C),(A\to B) \vdash B\to C}\text{(утнч)}}{(B\to C),(A\to B),A \vdash B\to C}\text{(утнч)}}{(A\to B),(B\to C),A \vdash B\to C}\text{(перест)}
}{
\dfrac{(A\to B),(B\to C),A \vdash C}{(A\to B),(B\to C) \vdash (A\to C)}\text{(}\to\text{ введ)}
}\text{(mod pon)}
$$

$$
\dfrac{\dfrac{A\wedge B \vdash A\wedge B}{A\wedge B \vdash A}\text{(}\wedge\text{ удал прав)}}{\vdash (A\wedge B)\to A}\text{(}\to\text{ введ)}
$$

$\boxed{\neg A\vee\neg B \vdash \neg(A\wedge B)}$:

$$
\dfrac{\neg A\vee\neg B \vdash \neg A\vee\neg B \quad \dfrac{(1)}{\neg A\vee\neg B,\neg A \vdash \neg(A\wedge B)} \quad \dfrac{(2)}{\neg A\vee\neg B,\neg B \vdash \neg(A\wedge B)}}{\neg A\vee\neg B \vdash \neg(A\wedge B)}\text{(уд }\vee\text{)}
$$

(1):

$$
\dfrac{\text{(ут, ут, пер)}\dfrac{\neg A \vdash \neg A}{\neg A\vee\neg B,\neg A,\neg\neg(A\wedge B) \vdash \neg A} \quad \dfrac{\dfrac{\neg\neg(A\wedge B) \vdash A\wedge B\text{ - д.а.}}{\neg\neg(A\wedge B) \vdash A}\text{(уд }\wedge\text{ прав)}}{\neg A\vee\neg B,\neg A,\neg\neg(A\wedge B) \vdash A}\text{(ут, ут, пер, пер)}}{\neg A\vee\neg B,\neg A \vdash \neg(A\wedge B)}\text{(не)}
$$

(2):

$$
\dfrac{\text{(ут, ут, пер)}\dfrac{\neg B \vdash \neg B}{\neg A\vee\neg B,\neg B,\neg\neg(A\wedge B) \vdash \neg B} \quad \dfrac{\dfrac{\neg\neg(A\wedge B) \vdash A\wedge B\text{ - д.а.}}{\neg\neg(A\wedge B) \vdash B}\text{(уд }\wedge\text{ слев)}}{\neg A\vee\neg B,\neg B,\neg\neg(A\wedge B) \vdash B}\text{(ут, ут, пер, пер)}}{\neg A\vee\neg B,\neg A \vdash \neg(A\wedge B)}\text{(не)}
$$

$\boxed{\neg A\wedge\neg B \vdash \neg(A\vee B)}$:

$$
\dfrac{\text{(ут)}\dfrac{\neg A\wedge\neg B \vdash \neg A\wedge\neg B}{\neg A\wedge\neg B,\neg\neg(A\vee B) \vdash \neg A\wedge\neg B} \quad \dfrac{\dfrac{(1)}{\neg\neg(A\vee B) \vdash \neg(\neg A\wedge\neg B)}}{\neg A\wedge\neg B,\neg\neg(A\vee B) \vdash \neg(\neg A\wedge\neg B)}\text{(ут, пер)}}{\neg A\wedge\neg B \vdash \neg(A\vee B)}\text{(не)}
$$

(1):

$$
\dfrac{\neg\neg(A\vee B) \vdash A\vee B\text{ д.а.} \quad \dfrac{(1.1)}{\neg\neg(A\vee B),A \vdash \neg(\neg A\wedge\neg B)} \quad \dfrac{(1.2)}{\neg\neg(A\vee B),B \vdash \neg(\neg A\wedge\neg B)}}{\neg\neg(A\vee B) \vdash \neg(\neg A\wedge\neg B)}\text{(уд }\vee\text{)}
$$

(1.1):

$$\dfrac{\text{(ут, ут, пер)}\;\dfrac{A \;\vdash\; A}{\neg\neg(A\vee B), A, \neg\neg(\neg A\wedge\neg B) \;\vdash\; A} \qquad \dfrac{\dfrac{\neg\neg(\neg A\wedge\neg B) \;\vdash\; \neg A\wedge\neg B \text{ д.а}}{\neg\neg(\neg A\wedge \neg B) \;\vdash\; \neg A}\;\text{(уд } \wedge \text{ справ)}}{\neg\neg(A\vee B),A,\neg\neg(\neg A\wedge\neg B) \;\vdash\; \neg A}\;\text{(ут, ут,пер, пер)}}{\neg\neg(A\vee B), A \;\vdash\; \neg(\neg A\wedge\neg B)}\;\text{(не)}$$

(1.2):

$$\dfrac{\text{(ут, ут, пер)}\;\dfrac{B \;\vdash\; B}{\neg\neg(A\vee B), B, \neg\neg(\neg A\wedge\neg B) \;\vdash\; B} \qquad \dfrac{\dfrac{\neg\neg(\neg A\wedge\neg B) \;\vdash\; \neg A\wedge\neg B \text{ д.а.}}{\neg\neg(\neg A\wedge \neg B) \;\vdash\; \neg B}\;\text{(уд } \wedge \text{ справ)}}{\neg\neg(A\vee B),B,\neg\neg(\neg A\wedge\neg B) \;\vdash\; \neg B}\;\text{(ут, ут,пер, пер)}}{\neg\neg(A\vee B), B \;\vdash\; \neg(\neg A\wedge\neg B)}\;\text{(не)}$$

$\boxed{(A\wedge B)\vee C \;\vdash\; ((A\vee C)\wedge(B\vee C))}$:

$$\dfrac{(A\wedge B)\vee C \;\vdash\; (A\wedge B)\vee C \qquad \dfrac{(1)}{(A\wedge B)\vee C, A\wedge B \;\vdash\; \ldots} \qquad \dfrac{(2)}{(A\wedge B)\vee C, C \;\vdash\; \ldots}}{(A\wedge B)\vee C \;\vdash\; ((A\vee C)\wedge(B\vee C))}\;\text{(уд } \vee\text{)}$$

(1):

$$\dfrac{\text{(введ } \vee \text{ справ)}\;\dfrac{\text{(уд } \wedge \text{ слев)}\;\dfrac{\text{(ут, пер)}\;\dfrac{A\wedge B \;\vdash\; A\wedge B}{(A\wedge B)\vee C, A\wedge B \;\vdash\; A\wedge B}}{(A\wedge B)\vee C, A\wedge B \;\vdash\; B}}{(A\wedge B)\vee C, A\wedge B \;\vdash\; B\vee C} \qquad \dfrac{\dfrac{\dfrac{A\wedge B \;\vdash\; A\wedge B}{(A\wedge B)\vee C, A\wedge B \;\vdash\; A\wedge B}\;\text{(ут, пер)}}{(A\wedge B)\vee C, A\wedge B \;\vdash\; A}\;\text{(уд } \wedge \text{ справ)}}{(A\wedge B)\vee C, A\wedge B \;\vdash\; A\vee C}\;\text{(введ } \vee \text{ справ)}}{(A\wedge B)\vee C, A\wedge B \;\vdash\; ((A\vee C)\wedge(B\vee C)}\;\text{(введ } \wedge\text{)}$$

(2):

$$\dfrac{\text{(введ } \vee \text{ слев)}\;\dfrac{\text{(ут, пер)}\;\dfrac{C \;\vdash\; C}{(A\wedge B)\vee C, C \;\vdash\; C}}{(A\wedge B)\vee C, C \;\vdash\; B\vee C} \qquad \dfrac{\dfrac{C \;\vdash\; C}{(A\wedge B)\vee C, C \;\vdash\; C}\;\text{(ут, пер)}}{(A\wedge B)\vee C, C \;\vdash\; A\vee C}\;\text{(введ } \vee \text{ слев)}}{(A\wedge B)\vee C, C \;\vdash\; ((A\vee C)\wedge(B\vee C)}\;\text{(введ } \wedge\text{)}$$

$\boxed{(A\vee B)\wedge C \;\vdash\; ((A\wedge C)\vee(B\wedge C))}$:

$\phi = ((A\wedge C)\vee(B\wedge C))$

$\psi = (A\wedge B)\vee C)$

$$\dfrac{\text{(уд } \wedge \text{ пр)}\;\dfrac{\psi \;\vdash\; \psi}{\psi \;\vdash\; A\vee B} \qquad \dfrac{\dfrac{(1)}{(A\vee B)\wedge C, A \;\vdash\; (A\wedge C)}}{(A\vee B)\wedge C, A \;\vdash\; \phi}\;\text{(в } \vee \text{ пр)} \qquad \dfrac{\dfrac{(2)}{(A\vee B)\wedge C, B \;\vdash\; (B\wedge C)}}{(A\vee B)\wedge C, B \;\vdash\; \phi}\;\text{(в } \vee \text{ лев)}}{(A\vee B)\wedge C \;\vdash\; ((A\wedge C)\vee(B\wedge C))}\;\text{(уд } \vee\text{)}$$

(1):

$$\dfrac{\text{(введ } \wedge\text{)}\;\dfrac{A \;\vdash\; A}{(A\vee B)\wedge C, A \;\vdash\; A} \qquad \dfrac{\dfrac{(A\vee B)\wedge C \;\vdash\; (A\vee B)\wedge C}{(A\vee B)\wedge C, A \;\vdash\; (A\vee B)\wedge C}\;\text{(ут)}}{(A\vee B)\wedge C, A \;\vdash\; C}\;\text{(уд } \wedge \text{ лев)}}{(A\vee B)\wedge C, A \;\vdash\; (A\wedge C)}\;\text{(введ } \wedge\text{)}$$

(2):

$$\dfrac{\text{(введ }\wedge\text{)}\ \dfrac{B \ \vdash\ B}{(A \vee B) \wedge C, B \ \vdash\ B} \qquad \dfrac{\dfrac{(A \vee B) \wedge C \ \vdash\ (A \vee B) \wedge C}{(A \vee B) \wedge C, B \ \vdash\ (A \vee B) \wedge C}\text{(ут)}}{(A \vee B) \wedge C, B \ \vdash\ C}\text{(уд }\wedge\text{ лев)}}{(A \vee B) \wedge C, B \ \vdash\ (B \wedge C)}\text{(введ }\wedge\text{)}$$

$\boxed{((A \wedge C) \vee (B \wedge C)) \ \vdash\ (A \vee B) \wedge C}$:
$\phi = ((A \wedge C) \vee (B \wedge C))$

$$\dfrac{\phi \ \vdash\ \phi \qquad \dfrac{(1)}{\phi, A \wedge C \ \vdash\ (A \vee B) \wedge C} \qquad \dfrac{(2)}{\phi, B \wedge C \ \vdash\ (A \vee B) \wedge C}}{\phi \ \vdash\ (A \vee B) \wedge C}\text{(уд }\vee\text{)}$$

(1):

$$\dfrac{\text{(введ }\vee\text{ лев)}\ \dfrac{\text{(уд }\wedge\text{ прав)}\ \dfrac{\text{(ут, пер)}\ \dfrac{A \wedge C \ \vdash\ A \wedge C}{((A \wedge C) \vee (B \wedge C)), A \wedge C \ \vdash\ A \wedge C}}{((A \wedge C) \vee (B \wedge C)), A \wedge C \ \vdash\ A}}{((A \wedge C) \vee (B \wedge C)), A \wedge C \ \vdash\ A \vee B} \qquad \dfrac{\dfrac{A \wedge C \ \vdash\ A \wedge C}{((A \wedge C) \vee (B \wedge C)), A \wedge C \ \vdash\ A \wedge C}\text{(ут, пер)}}{((A \wedge C) \vee (B \wedge C)), A \wedge C \ \vdash\ C}\text{(уд }\wedge\text{ лев)}}{((A \wedge C) \vee (B \wedge C)), A \wedge C \ \vdash\ (A \vee B) \wedge C}\text{(введ }\wedge\text{)}$$

(2):

$$\dfrac{\text{(введ }\vee\text{ лев)}\ \dfrac{\text{(уд }\wedge\text{ прав)}\ \dfrac{\text{(ут, пер)}\ \dfrac{B \wedge C \ \vdash\ B \wedge C}{((A \wedge C) \vee (B \wedge C)), B \wedge C \ \vdash\ B \wedge C}}{((A \wedge C) \vee (B \wedge C)), B \wedge C \ \vdash\ B}}{((A \wedge C) \vee (B \wedge C)), B \wedge C \ \vdash\ A \vee B} \qquad \dfrac{\dfrac{B \wedge C \ \vdash\ B \wedge C}{((A \wedge C) \vee (B \wedge C)), B \wedge C \ \vdash\ B \wedge C}\text{(ут, пер)}}{((A \wedge C) \vee (B \wedge C)), B \wedge C \ \vdash\ C}\text{(уд }\wedge\text{ лев)}}{((A \wedge C) \vee (B \wedge C)), B \wedge C \ \vdash\ (A \vee B) \wedge C}\text{(введ }\wedge\text{)}$$

Лемма 1. *Если $\phi \ \vdash\ \psi$ выводима, то из $\Gamma \ \vdash\ \phi$ можно вывести $\Gamma \ \vdash\ \psi$*

Доказательство.
$$\dfrac{\dfrac{\dfrac{\dfrac{\phi \ \vdash\ \psi}{\vdash\ \phi \to \psi}}{\cdots}}{\Gamma \ \vdash\ \phi \to \psi} \quad \Gamma \ \vdash\ \phi}{\Gamma \ \vdash\ \psi}$$

□

Лемма 2. *Если $\psi \ \vdash\ \phi$ выводима, то из $\Gamma, \phi \ \vdash\ \theta$ можно вывести $\Gamma, \psi \ \vdash\ \theta$*

Доказательство.
$$\dfrac{\dfrac{\dfrac{\psi \ \vdash\ \phi}{\cdots}}{\Gamma, \psi \ \vdash\ \phi} \quad \dfrac{\dfrac{\Gamma, \phi \ \vdash\ \theta}{\Gamma \ \vdash\ \phi \to \theta}}{\Gamma, \psi \ \vdash\ \phi \to \theta}}{\Gamma, \psi \ \vdash\ \theta}$$

□

$\boxed{((A \vee C) \wedge (B \vee C)) \ \vdash\ (A \wedge B) \vee C}$:
$\phi = ((A \wedge C) \vee (B \wedge C))$
$\psi = (A \wedge B) \vee C)$

$$
\begin{prooftree}
\AxiomC{$\phi \vdash \phi$}
\LeftLabel{(ут)}
\UnaryInfC{$\phi, \neg\psi \vdash \phi$}
\AxiomC{$\neg(A \wedge B) \wedge \neg C \vdash \neg(A \wedge B) \wedge \neg C$}
\LeftLabel{(уд $\wedge$ слев)}
\UnaryInfC{$\neg(A \wedge B) \wedge \neg C \vdash \neg C$}
\LeftLabel{(введ $\vee$ слев)}
\UnaryInfC{$\neg(A \wedge B) \wedge \neg C \vdash \neg A \vee \neg C$}
\LeftLabel{(лем2, морган)}
\UnaryInfC{$\neg(A \wedge B) \wedge \neg C \vdash \neg(A \wedge C)$}
\AxiomC{$\neg(A \wedge B) \wedge \neg C \vdash \neg(A \wedge B) \wedge \neg C$}
\RightLabel{(уд $\wedge$ слев)}
\UnaryInfC{$\neg(A \wedge B) \wedge \neg C \vdash \neg C$}
\RightLabel{(введ $\vee$ слев)}
\UnaryInfC{$\neg(A \wedge B) \wedge \neg C \vdash \neg B \vee \neg C$}
\RightLabel{(лем2, морган)}
\UnaryInfC{$\neg(A \wedge B) \wedge \neg C \vdash \neg(B \wedge C)$}
\RightLabel{(введ $\wedge$)}
\BinaryInfC{$\neg(A \wedge B) \wedge \neg C \vdash \neg(A \wedge C) \wedge \neg(B \wedge C)$}
\RightLabel{(лем1, морган)}
\UnaryInfC{$\neg\psi \vdash \neg(A \wedge C) \wedge \neg(B \wedge C)$}
\RightLabel{(лем2, морган)}
\UnaryInfC{$\neg\psi \vdash \neg\phi$}
\RightLabel{(ут, пер)}
\UnaryInfC{$\phi, \neg\psi \vdash \neg\phi$}
\RightLabel{(не)}
\BinaryInfC{$\phi \vdash \psi$}
\end{prooftree}
$$