$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A, (A \to B) \vdash A}\,(\text{утнч})}{(A \to B), A \vdash A}\,(\text{перест})}{(A \to B), A \vdash B}\,(\text{mod pon}) \quad \dfrac{\dfrac{A \to B \vdash A \to B}{(A \to B), A \vdash A \to B}\,(\text{утнч})}{(A \to B), A, (B \to C) \vdash B}\,(\text{утнч})}{(A \to B), (B \to C), A \vdash B}\,(\text{перест}) \quad \dfrac{\dfrac{\dfrac{B \to C \vdash B \to C}{(B \to C), (A \to B) \vdash B \to C}\,(\text{утнч})}{(B \to C), (A \to B), A \vdash B \to C}\,(\text{утнч})}{(A \to B), (B \to C), A \vdash B \to C}\,(\text{перест})}{(A \to B), (B \to C), A \vdash C}\,(\text{mod pon})}{(A \to B), (B \to C) \vdash (A \to C)}\,(\to \text{введ})$$

$$\dfrac{\dfrac{A \wedge B \vdash A \wedge B}{A \wedge B \vdash A}\,(\wedge \text{удал прав})}{\vdash (A \wedge B) \to A}\,(\to \text{введ})$$

$\boxed{\neg A \vee \neg B \vdash \neg(A \wedge B)}$:

$$\dfrac{\dfrac{(1)}{\neg A \vee \neg B, \neg A \vdash \neg(A \wedge B)} \quad \dfrac{(2)}{\neg A \vee \neg B, \neg B \vdash \neg(A \wedge B)}}{\neg A \vee \neg B \vdash \neg(A \wedge B)}\,(\text{уд} \vee)$$

(1):

$$\dfrac{\dfrac{\neg A \vdash \neg A}{\neg A \vee \neg B, \neg A, \neg\neg(A \wedge B) \vdash \neg A}\,(\text{ут, ут, пер}) \quad \dfrac{\dfrac{\neg\neg(A \wedge B) \vdash A \wedge B \text{ - д.а.}}{\neg\neg(A \wedge B) \vdash A}\,(\text{уд} \wedge \text{прав})}{\neg A \vee \neg B, \neg A, \neg\neg(A \wedge B) \vdash A}\,(\text{ут, ут, пер, пер})}{\neg A \vee \neg B, \neg A \vdash \neg(A \wedge B)}\,(\text{не})$$

(2):

$$\dfrac{\dfrac{\neg B \vdash \neg B}{\neg A \vee \neg B, \neg B, \neg\neg(A \wedge B) \vdash \neg B}\,(\text{ут, ут, пер}) \quad \dfrac{\dfrac{\neg\neg(A \wedge B) \vdash A \wedge B \text{ - д.а.}}{\neg\neg(A \wedge B) \vdash B}\,(\text{уд} \wedge \text{слев})}{\neg A \vee \neg B, \neg B, \neg\neg(A \wedge B) \vdash B}\,(\text{ут, ут, пер, пер})}{\neg A \vee \neg B, \neg A \vdash \neg(A \wedge B)}\,(\text{не})$$

$\boxed{\neg A \wedge \neg B \vdash \neg(A \vee B)}$:

$$\dfrac{\dfrac{\neg A \wedge \neg B \vdash \neg A \wedge \neg B}{\neg A \wedge \neg B, \neg\neg(A \vee B) \vdash \neg A \wedge \neg B}\,(\text{ут}) \quad \dfrac{\dfrac{(1)}{\neg\neg(A \vee B) \vdash \neg(\neg A \wedge \neg B)}}{\neg A \wedge \neg B, \neg\neg(A \vee B) \vdash \neg(\neg A \wedge \neg B)}\,(\text{ут, пер})}{\neg A \wedge \neg B \vdash \neg(A \vee B)}\,(\text{не})$$

(1):

$$\dfrac{\neg\neg(A \vee B) \vdash A \vee B \text{ д.а.} \quad \dfrac{(1.1)}{\neg\neg(A \vee B), A \vdash \neg(\neg A \wedge \neg B)} \quad \dfrac{(1.2)}{\neg\neg(A \vee B), B \vdash \neg(\neg A \wedge \neg B)}}{\neg\neg(A \vee B) \vdash \neg(\neg A \wedge \neg B)}\,(\text{уд} \vee)$$

(1.1):

$$\dfrac{\dfrac{}{\neg\neg(A\vee B), A, \neg\neg(\neg A\wedge\neg B)\ \vdash\ A}\ (\text{ут, ут, пер}) \quad \dfrac{\dfrac{}{\neg\neg(\neg A\wedge\neg B)\ \vdash\ \neg A\wedge\neg B}\ \text{д.а}}{\neg\neg(\neg A\wedge\neg B)\ \vdash\ \neg A}\ (\text{уд}\wedge\text{справ})}{\neg\neg(A\vee B), A, \neg\neg(\neg A\wedge\neg B)\ \vdash\ \neg A}\ (\text{ут, ут,пер, пер})}{\neg\neg(A\vee B), A\ \vdash\ \neg(\neg A\wedge\neg B)}\ (\text{не})$$

(1.2):

$$\dfrac{\dfrac{}{\neg\neg(A\vee B), B, \neg\neg(\neg A\wedge\neg B)\ \vdash\ B}\ (\text{ут, ут, пер}) \quad \dfrac{\dfrac{}{\neg\neg(\neg A\wedge\neg B)\ \vdash\ \neg A\wedge\neg B}\ \text{д.а.}}{\neg\neg(\neg A\wedge\neg B)\ \vdash\ \neg B}\ (\text{уд}\wedge\text{справ})}{\neg\neg(A\vee B), B, \neg\neg(\neg A\wedge\neg B)\ \vdash\ \neg B}\ (\text{ут, ут,пер, пер})}{\neg\neg(A\vee B), B\ \vdash\ \neg(\neg A\wedge\neg B)}\ (\text{не})$$

$\boxed{(A\wedge B)\vee C\ \vdash\ ((A\vee C)\wedge(B\vee C))}$:

$$\dfrac{\dfrac{}{(A\wedge B)\vee C\ \vdash\ (A\wedge B)\vee C} \quad \dfrac{\dfrac{}{(A\wedge B)\vee C, A\wedge B\ \vdash\ \ldots}\ (1) \quad \dfrac{}{(A\wedge B)\vee C, C\ \vdash\ \ldots}\ (2)}{(A\wedge B)\vee C\ \vdash\ ((A\vee C)\wedge(B\vee C))}\ (\text{уд}\vee)$$

(1):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A\wedge B\ \vdash\ A\wedge B}\ (\text{ут, пер})}{(A\wedge B)\vee C, A\wedge B\ \vdash\ A\wedge B}\ (\text{уд}\wedge\text{слев})}{(A\wedge B)\vee C, A\wedge B\ \vdash\ B}\ (\text{введ}\vee\text{справ}) \quad \dfrac{\dfrac{\dfrac{}{A\wedge B\ \vdash\ A\wedge B}\ (\text{ут, пер})}{(A\wedge B)\vee C, A\wedge B\ \vdash\ A\wedge B}\ (\text{уд}\wedge\text{справ})}{(A\wedge B)\vee C, A\wedge B\ \vdash\ A}\ (\text{введ}\vee\text{справ})}{(A\wedge B)\vee C, A\wedge B\ \vdash\ B\vee C \quad (A\wedge B)\vee C, A\wedge B\ \vdash\ A\vee C}\ (\text{введ}\wedge)$$

(2):

$$\dfrac{\dfrac{\dfrac{}{C\ \vdash\ C}\ (\text{ут, пер})}{(A\wedge B)\vee C, C\ \vdash\ C}\ (\text{введ}\vee\text{слев}) \quad \dfrac{\dfrac{}{C\ \vdash\ C}\ (\text{ут, пер})}{(A\wedge B)\vee C, C\ \vdash\ C}\ (\text{введ}\vee\text{слев})}{(A\wedge B)\vee C, C\ \vdash\ B\vee C \quad (A\wedge B)\vee C, C\ \vdash\ A\vee C}\ (\text{введ}\wedge)$$

$\boxed{(A\vee B)\wedge C\ \vdash\ ((A\wedge C)\vee(B\wedge C))}$:

$\phi = ((A\wedge C)\vee(B\wedge C))$

$\psi = (A\wedge B)\vee C)$

$$\dfrac{\dfrac{}{\psi\ \vdash\ \psi}\ (\text{уд}\wedge\text{пр}) \quad \dfrac{\dfrac{}{(A\vee B)\wedge C, A\ \vdash\ (A\wedge C)}\ (1)}{(A\vee B)\wedge C, A\ \vdash\ \phi}\ (\text{в}\vee\text{пр}) \quad \dfrac{\dfrac{}{(A\vee B)\wedge C, B\ \vdash\ (B\wedge C)}\ (2)}{(A\vee B)\wedge C, B\ \vdash\ \phi}\ (\text{в}\vee\text{лев})}{\psi\ \vdash\ A\vee B \quad (A\vee B)\wedge C\ \vdash\ ((A\wedge C)\vee(B\wedge C))}\ (\text{уд}\vee)$$

(1):

$$\dfrac{\dfrac{}{A\ \vdash\ A}\ (\text{введ}\wedge) \quad \dfrac{\dfrac{}{(A\vee B)\wedge C\ \vdash\ (A\vee B)\wedge C}\ (\text{ут})}{(A\vee B)\wedge C, A\ \vdash\ (A\vee B)\wedge C}\ (\text{уд}\wedge\text{лев})}{(A\vee B)\wedge C, A\ \vdash\ A \quad (A\vee B)\wedge C, A\ \vdash\ C}\ (\text{введ}\wedge)$$

(2):

$$\dfrac{\dfrac{B \vdash B}{(A \vee B) \wedge C, B \vdash B} \quad \dfrac{\dfrac{(A \vee B) \wedge C \vdash (A \vee B) \wedge C}{(A \vee B) \wedge C, B \vdash (A \vee B) \wedge C}\,(\text{ут})}{(A \vee B) \wedge C, B \vdash C}\,(\text{уд} \wedge \text{лев})}{(A \vee B) \wedge C, B \vdash (B \wedge C)}\,(\text{введ} \wedge)$$

$\boxed{((A \wedge C) \vee (B \wedge C)) \vdash (A \vee B) \wedge C}$:
$\phi = ((A \wedge C) \vee (B \wedge C))$

$$\dfrac{\phi \vdash \phi \quad \dfrac{(1)}{\phi, A \wedge C \vdash (A \vee B) \wedge C} \quad \dfrac{(2)}{\phi, B \wedge C \vdash (A \vee B) \wedge C}}{\phi \vdash (A \vee B) \wedge C}\,(\text{уд} \vee)$$

(1):

$$\dfrac{\dfrac{\dfrac{\dfrac{A \wedge C \vdash A \wedge C}{((A \wedge C) \vee (B \wedge C)), A \wedge C \vdash A \wedge C}\,(\text{ут, пер})}{((A \wedge C) \vee (B \wedge C)), A \wedge C \vdash A}\,(\text{уд} \wedge \text{прав})}{((A \wedge C) \vee (B \wedge C)), A \wedge C \vdash A \vee B}\,(\text{введ} \vee \text{лев}) \quad \dfrac{\dfrac{\dfrac{A \wedge C \vdash A \wedge C}{((A \wedge C) \vee (B \wedge C)), A \wedge C \vdash A \wedge C}\,(\text{ут, пер})}{((A \wedge C) \vee (B \wedge C)), A \wedge C \vdash C}\,(\text{уд} \wedge \text{лев})}{((A \wedge C) \vee (B \wedge C)), A \wedge C \vdash (A \vee B) \wedge C}\,(\text{введ} \wedge)$$

(2):

$$\dfrac{\dfrac{\dfrac{\dfrac{B \wedge C \vdash B \wedge C}{((A \wedge C) \vee (B \wedge C)), B \wedge C \vdash B \wedge C}\,(\text{ут, пер})}{((A \wedge C) \vee (B \wedge C)), B \wedge C \vdash B}\,(\text{уд} \wedge \text{прав})}{((A \wedge C) \vee (B \wedge C)), B \wedge C \vdash A \vee B}\,(\text{введ} \vee \text{лев}) \quad \dfrac{\dfrac{\dfrac{B \wedge C \vdash B \wedge C}{((A \wedge C) \vee (B \wedge C)), B \wedge C \vdash B \wedge C}\,(\text{ут, пер})}{((A \wedge C) \vee (B \wedge C)), B \wedge C \vdash C}\,(\text{уд} \wedge \text{лев})}{((A \wedge C) \vee (B \wedge C)), B \wedge C \vdash (A \vee B) \wedge C}\,(\text{введ} \wedge)$$

Лемма 1. *Если* $\phi \vdash \psi$ *выводима, то из* $\Gamma \vdash \phi$ *можно вывести* $\Gamma \vdash \psi$

Доказательство.
$$\dfrac{\dfrac{\dfrac{\phi \vdash \psi}{\vdash \phi \to \psi}}{\Gamma \vdash \phi \to \psi} \quad \Gamma \vdash \phi}{\Gamma \vdash \psi}$$

□

Лемма 2. *Если* $\psi \vdash \phi$ *выводима, то из* $\Gamma, \phi \vdash \theta$ *можно вывести* $\Gamma, \psi \vdash \theta$

Доказательство.
$$\dfrac{\dfrac{\psi \vdash \phi}{\Gamma, \psi \vdash \phi} \quad \dfrac{\Gamma, \phi \vdash \theta}{\Gamma \vdash \phi \to \theta}}{\Gamma, \psi \vdash \theta}$$

□

$\boxed{((A \vee C) \wedge (B \vee C)) \vdash (A \wedge B) \vee C}$:
$\phi = ((A \wedge C) \vee (B \wedge C))$
$\psi = (A \wedge B) \vee C)$

$$\dfrac{\dfrac{}{\phi \vdash \phi}\,(\text{ут})}{\phi, \neg\psi \vdash \phi}\qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(A \wedge B) \wedge \neg C \vdash \neg(A \wedge B) \wedge \neg C}\,(\text{уд} \wedge \text{слев})}{\neg(A \wedge B) \wedge \neg C \vdash \neg C}\,(\text{введ} \vee \text{слев})}{\neg(A \wedge B) \wedge \neg C \vdash \neg A \vee \neg C}\,(\text{лем2, морган})}{\neg(A \wedge B) \wedge \neg C \vdash \neg(A \wedge C)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\neg(A \wedge B) \wedge \neg C \vdash \neg(A \wedge B) \wedge \neg C}\,(\text{уд} \wedge \text{слев})}{\neg(A \wedge B) \wedge \neg C \vdash \neg C}\,(\text{введ} \vee \text{слев})}{\neg(A \wedge B) \wedge \neg C \vdash \neg B \vee \neg C}\,(\text{лем2, морган})}{\neg(A \wedge B) \wedge \neg C \vdash \neg(B \wedge C)}}{\neg(A \wedge B) \wedge \neg C \vdash \neg(A \wedge C) \wedge \neg(B \wedge C)}\,(\text{введ} \wedge)}{\neg\psi \vdash \neg(A \wedge C) \wedge \neg(B \wedge C)}\,(\text{лем1, морган})}{\neg\psi \vdash \neg\phi}\,(\text{лем2, морган})}{\phi, \neg\psi \vdash \neg\phi}\,(\text{ут, пер})$$

$$\phi \vdash \psi\,(\text{не})$$