

$$\begin{array}{c}
\text{(утнч)} \frac{A \vdash A}{A, (A \rightarrow B) \vdash A} \\
\text{(перест)} \frac{A, (A \rightarrow B) \vdash A}{(A \rightarrow B), A \vdash A} \\
\text{(mod pon)} \frac{(A \rightarrow B), A \vdash A}{(A \rightarrow B), A \vdash B} \\
\text{(утнч)} \frac{A \rightarrow B \vdash A \rightarrow B}{(A \rightarrow B), A \vdash A \rightarrow B} \\
\text{(перест)} \frac{(A \rightarrow B), A, (B \rightarrow C) \vdash B}{(A \rightarrow B), (B \rightarrow C), A \vdash B} \\
\text{(утнч)} \frac{B \rightarrow C \vdash B \rightarrow C}{(B \rightarrow C), (A \rightarrow B) \vdash B \rightarrow C} \\
\text{(перест)} \frac{(B \rightarrow C), (A \rightarrow B), A \vdash B \rightarrow C}{(A \rightarrow B), (B \rightarrow C), A \vdash B \rightarrow C} \\
\text{(mod pon)} \frac{(A \rightarrow B), (B \rightarrow C), A \vdash B \rightarrow C}{(A \rightarrow B), (B \rightarrow C), A \vdash C} \\
\text{(\(\rightarrow\) введ)} \frac{(A \rightarrow B), (B \rightarrow C), A \vdash C}{(A \rightarrow B), (B \rightarrow C) \vdash (A \rightarrow C)}
\end{array}$$

$$\begin{array}{c}
\frac{A \wedge B \vdash A \wedge B}{A \wedge B \vdash A} \text{ (\(\wedge\) удал прав)} \\
\vdash (A \wedge B) \rightarrow A \text{ (\(\rightarrow\) введ)}
\end{array}$$

$$\boxed{\neg A \vee \neg B \vdash \neg(A \wedge B)} :$$

$$\begin{array}{c}
(1) \quad \frac{\neg A \vee \neg B \vdash \neg A \vee \neg B}{\neg A \vee \neg B \vdash \neg(A \wedge B)} \\
(2) \quad \frac{\neg A \vee \neg B, \neg A \vdash \neg(A \wedge B) \quad \neg A \vee \neg B, \neg B \vdash \neg(A \wedge B)}{\neg A \vee \neg B \vdash \neg(A \wedge B)} \text{ (уд \(\vee\))}
\end{array}$$

(1):

$$\begin{array}{c}
\text{(ут, ут, пер)} \frac{\neg A \vdash \neg A}{\neg A \vee \neg B, \neg A, \neg\neg(A \wedge B) \vdash \neg A} \\
\frac{\neg A \vee \neg B, \neg A, \neg\neg(A \wedge B) \vdash \neg A}{\neg A \vee \neg B, \neg A \vdash \neg(A \wedge B)} \\
\frac{\neg\neg(A \wedge B) \vdash A \wedge B - \text{д.а.}}{\neg\neg(A \wedge B) \vdash A} \text{ (уд \(\wedge\) прав)} \\
\frac{\neg A \vee \neg B, \neg A, \neg\neg(A \wedge B) \vdash A}{\neg A \vee \neg B, \neg A \vdash \neg(A \wedge B)} \text{ (ут, ут, пер, пер) (не)}
\end{array}$$

(2):

$$\begin{array}{c}
\text{(ут, ут, пер)} \frac{\neg B \vdash \neg B}{\neg A \vee \neg B, \neg B, \neg\neg(A \wedge B) \vdash \neg B} \\
\frac{\neg A \vee \neg B, \neg B, \neg\neg(A \wedge B) \vdash \neg B}{\neg A \vee \neg B, \neg A \vdash \neg(A \wedge B)} \\
\frac{\neg\neg(A \wedge B) \vdash A \wedge B - \text{д.а.}}{\neg\neg(A \wedge B) \vdash B} \text{ (уд \(\wedge\) лев)} \\
\frac{\neg A \vee \neg B, \neg B, \neg\neg(A \wedge B) \vdash B}{\neg A \vee \neg B, \neg A \vdash \neg(A \wedge B)} \text{ (ут, ут, пер, пер) (не)}
\end{array}$$

$$\boxed{\neg A \wedge \neg B \vdash \neg(A \vee B)} :$$

$$\begin{array}{c}
(1) \quad \frac{\neg A \wedge \neg B \vdash \neg A \wedge \neg B}{\neg A \wedge \neg B, \neg\neg(A \vee B) \vdash \neg A \wedge \neg B} \\
\frac{\neg A \wedge \neg B, \neg\neg(A \vee B) \vdash \neg A \wedge \neg B}{\neg A \wedge \neg B \vdash \neg(A \vee B)} \text{ (ут, пер)} \\
\frac{\neg\neg(A \vee B) \vdash \neg(\neg A \wedge \neg B)}{\neg A \wedge \neg B, \neg\neg(A \vee B) \vdash \neg(\neg A \wedge \neg B)} \text{ (не)}
\end{array}$$

(1):

$$\begin{array}{c}
(1.1) \quad \frac{\neg\neg(A \vee B) \vdash A \vee B \text{ д.а.}}{\neg\neg(A \vee B) \vdash \neg(\neg A \wedge \neg B)} \\
(1.2) \quad \frac{\neg\neg(A \vee B), B \vdash \neg(\neg A \wedge \neg B)}{\neg\neg(A \vee B) \vdash \neg(\neg A \wedge \neg B)} \text{ (уд \(\vee\))}
\end{array}$$

(1.1):

$$\begin{array}{c} \text{(ут, ут, пер)} \frac{\frac{A \vdash A}{\neg\neg(A \vee B), A, \neg\neg(\neg A \wedge \neg B) \vdash A} \quad \frac{\frac{\neg\neg(\neg A \wedge \neg B) \vdash \neg A \wedge \neg B \text{ д.а.}}{\neg\neg(\neg A \wedge \neg B) \vdash \neg A} \text{(уд } \wedge \text{ справ)}}{\neg\neg(A \vee B), A, \neg\neg(\neg A \wedge \neg B) \vdash \neg A} \text{(ут, ут, пер, пер)} \\ \hline \neg\neg(A \vee B), A \vdash \neg(\neg A \wedge \neg B) \text{(не)} \end{array}$$

(1.2):

$$\begin{array}{c} \text{(ут, ут, пер)} \frac{\frac{B \vdash B}{\neg\neg(A \vee B), B, \neg\neg(\neg A \wedge \neg B) \vdash B} \quad \frac{\frac{\neg\neg(\neg A \wedge \neg B) \vdash \neg A \wedge \neg B \text{ д.а.}}{\neg\neg(\neg A \wedge \neg B) \vdash \neg B} \text{(уд } \wedge \text{ справ)}}{\neg\neg(A \vee B), B, \neg\neg(\neg A \wedge \neg B) \vdash \neg B} \text{(ут, ут, пер, пер)} \\ \hline \neg\neg(A \vee B), B \vdash \neg(\neg A \wedge \neg B) \text{(не)} \end{array}$$

$$\boxed{(A \wedge B) \vee C \vdash ((A \vee C) \wedge (B \vee C))} :$$

$$\frac{(A \wedge B) \vee C \vdash (A \wedge B) \vee C \quad \frac{(1)}{(A \wedge B) \vee C, A \wedge B \vdash \dots} \quad \frac{(2)}{(A \wedge B) \vee C, C \vdash \dots}}{(A \wedge B) \vee C \vdash ((A \vee C) \wedge (B \vee C))} \text{(уд } \vee \text{)}$$

(1):

$$\begin{array}{c} \text{(ут, пер)} \frac{A \wedge B \vdash A \wedge B}{(A \wedge B) \vee C, A \wedge B \vdash A \wedge B} \quad \frac{A \wedge B \vdash A \wedge B}{(A \wedge B) \vee C, A \wedge B \vdash A \wedge B} \text{(ут, пер)} \\ \text{(уд } \wedge \text{ слев)} \frac{(A \wedge B) \vee C, A \wedge B \vdash A \wedge B}{(A \wedge B) \vee C, A \wedge B \vdash B} \quad \frac{(A \wedge B) \vee C, A \wedge B \vdash A \wedge B}{(A \wedge B) \vee C, A \wedge B \vdash A} \text{(уд } \wedge \text{ справ)} \\ \text{(введ } \vee \text{ справ)} \frac{(A \wedge B) \vee C, A \wedge B \vdash B}{(A \wedge B) \vee C, A \wedge B \vdash B \vee C} \quad \frac{(A \wedge B) \vee C, A \wedge B \vdash A}{(A \wedge B) \vee C, A \wedge B \vdash A \vee C} \text{(введ } \vee \text{ справ)} \\ \hline (A \wedge B) \vee C, A \wedge B \vdash ((A \vee C) \wedge (B \vee C)) \text{(введ } \wedge \text{)}$$

(2):

$$\begin{array}{c} \text{(ут, пер)} \frac{C \vdash C}{(A \wedge B) \vee C, C \vdash C} \quad \frac{C \vdash C}{(A \wedge B) \vee C, C \vdash C} \text{(ут, пер)} \\ \text{(введ } \vee \text{ слев)} \frac{(A \wedge B) \vee C, C \vdash C}{(A \wedge B) \vee C, C \vdash B \vee C} \quad \frac{(A \wedge B) \vee C, C \vdash C}{(A \wedge B) \vee C, C \vdash A \vee C} \text{(введ } \vee \text{ слев)} \\ \hline (A \wedge B) \vee C, C \vdash ((A \vee C) \wedge (B \vee C)) \text{(введ } \wedge \text{)}$$

$$\boxed{(A \vee B) \wedge C \vdash ((A \wedge C) \vee (B \wedge C))} :$$

$$\phi = ((A \wedge C) \vee (B \wedge C))$$

$$\psi = (A \wedge B) \vee C$$

$$\begin{array}{c} \text{(уд } \wedge \text{ пр)} \frac{\psi \vdash \psi}{\psi \vdash A \vee B} \quad \frac{(1)}{(A \vee B) \wedge C, A \vdash (A \wedge C)} \quad \frac{(2)}{(A \vee B) \wedge C, B \vdash (B \wedge C)} \\ \text{(в } \vee \text{ пр)} \frac{\psi \vdash A \vee B \quad (A \vee B) \wedge C, A \vdash \phi \quad (A \vee B) \wedge C, B \vdash \phi}{(A \vee B) \wedge C \vdash ((A \wedge C) \vee (B \wedge C))} \text{(уд } \vee \text{)}$$

(1):

$$\begin{array}{c} \text{(введ } \wedge \text{)} \frac{A \vdash A}{(A \vee B) \wedge C, A \vdash A} \quad \frac{(A \vee B) \wedge C \vdash (A \vee B) \wedge C}{(A \vee B) \wedge C, A \vdash (A \vee B) \wedge C} \text{(ут)} \\ \text{(уд } \wedge \text{ лев)} \frac{(A \vee B) \wedge C, A \vdash A \quad (A \vee B) \wedge C, A \vdash C}{(A \vee B) \wedge C, A \vdash (A \wedge C)} \text{(введ } \wedge \text{)}$$

(2):

$$\frac{\text{(введ } \wedge) \frac{B \vdash B}{(A \vee B) \wedge C, B \vdash B} \quad \frac{\frac{(A \vee B) \wedge C \vdash (A \vee B) \wedge C}{(A \vee B) \wedge C, B \vdash (A \vee B) \wedge C} \text{(ут)} \quad \frac{(A \vee B) \wedge C, B \vdash C}{(A \vee B) \wedge C, B \vdash C} \text{(уд } \wedge \text{ лев)}}{(A \vee B) \wedge C, B \vdash (B \wedge C)} \text{(введ } \wedge)$$

$$\boxed{((A \wedge C) \vee (B \wedge C)) \vdash (A \vee B) \wedge C} : \\ \phi = ((A \wedge C) \vee (B \wedge C))$$

$$\frac{\phi \vdash \phi \quad \frac{(1)}{\phi, A \wedge C \vdash (A \vee B) \wedge C} \quad \frac{(2)}{\phi, B \wedge C \vdash (A \vee B) \wedge C}}{\phi \vdash (A \vee B) \wedge C} \text{(уд } \vee)$$

(1):

$$\frac{\text{(ут, пер)} \frac{A \wedge C \vdash A \wedge C}{((A \wedge C) \vee (B \wedge C)), A \wedge C \vdash A \wedge C} \quad \text{(уд } \wedge \text{ прав)} \frac{((A \wedge C) \vee (B \wedge C)), A \wedge C \vdash A \wedge C}{((A \wedge C) \vee (B \wedge C)), A \wedge C \vdash A} \quad \text{(введ } \vee \text{ лев)} \frac{((A \wedge C) \vee (B \wedge C)), A \wedge C \vdash A \vee B}{((A \wedge C) \vee (B \wedge C)), A \wedge C \vdash A \vee B} \quad \frac{\text{(ут, пер)} \frac{A \wedge C \vdash A \wedge C}{((A \wedge C) \vee (B \wedge C)), A \wedge C \vdash A \wedge C} \quad \text{(уд } \wedge \text{ лев)} \frac{((A \wedge C) \vee (B \wedge C)), A \wedge C \vdash A \wedge C}{((A \wedge C) \vee (B \wedge C)), A \wedge C \vdash C} \quad \text{(введ } \wedge) \frac{((A \wedge C) \vee (B \wedge C)), A \wedge C \vdash A \vee B \quad ((A \wedge C) \vee (B \wedge C)), A \wedge C \vdash C}{((A \wedge C) \vee (B \wedge C)), A \wedge C \vdash (A \vee B) \wedge C}$$

(2):

$$\frac{\text{(ут, пер)} \frac{B \wedge C \vdash B \wedge C}{((A \wedge C) \vee (B \wedge C)), B \wedge C \vdash B \wedge C} \quad \text{(уд } \wedge \text{ прав)} \frac{((A \wedge C) \vee (B \wedge C)), B \wedge C \vdash B \wedge C}{((A \wedge C) \vee (B \wedge C)), B \wedge C \vdash B} \quad \text{(введ } \vee \text{ лев)} \frac{((A \wedge C) \vee (B \wedge C)), B \wedge C \vdash B}{((A \wedge C) \vee (B \wedge C)), B \wedge C \vdash A \vee B} \quad \frac{\text{(ут, пер)} \frac{B \wedge C \vdash B \wedge C}{((A \wedge C) \vee (B \wedge C)), B \wedge C \vdash B \wedge C} \quad \text{(уд } \wedge \text{ лев)} \frac{((A \wedge C) \vee (B \wedge C)), B \wedge C \vdash B \wedge C}{((A \wedge C) \vee (B \wedge C)), B \wedge C \vdash C} \quad \text{(введ } \wedge) \frac{((A \wedge C) \vee (B \wedge C)), B \wedge C \vdash A \vee B \quad ((A \wedge C) \vee (B \wedge C)), B \wedge C \vdash C}{((A \wedge C) \vee (B \wedge C)), B \wedge C \vdash (A \vee B) \wedge C}$$

Лемма 1. Если $\phi \vdash \psi$ выводима, то из $\Gamma \vdash \phi$ можно вывести $\Gamma \vdash \psi$

$$\text{Доказательство.} \quad \frac{\frac{\phi \vdash \psi}{\vdash \phi \rightarrow \psi}}{\Gamma \vdash \phi \rightarrow \psi} \quad \frac{\Gamma \vdash \phi \rightarrow \psi \quad \Gamma \vdash \phi}{\Gamma \vdash \psi}$$

□

Лемма 2. Если $\psi \vdash \phi$ выводима, то из $\Gamma, \phi \vdash \theta$ можно вывести $\Gamma, \psi \vdash \theta$

$$\text{Доказательство.} \quad \frac{\frac{\psi \vdash \phi}{\Gamma, \psi \vdash \phi} \quad \frac{\Gamma, \phi \vdash \theta}{\Gamma \vdash \phi \rightarrow \theta}}{\Gamma, \psi \vdash \theta}$$

□

$$\boxed{((A \vee C) \wedge (B \vee C)) \vdash (A \wedge B) \vee C} : \\ \phi = ((A \wedge C) \vee (B \wedge C)) \\ \psi = (A \wedge B) \vee C$$

$$\begin{array}{c}
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\begin{array}{c}
\text{(уд } \wedge \text{ слев)} \frac{\neg(A \wedge B) \wedge \neg C \vdash \neg(A \wedge B) \wedge \neg C}{\neg(A \wedge B) \wedge \neg C \vdash \neg C} \quad \frac{\neg(A \wedge B) \wedge \neg C \vdash \neg(A \wedge B) \wedge \neg C}{\neg(A \wedge B) \wedge \neg C \vdash \neg C} \text{(уд } \wedge \text{ слев)} \\
\text{(введ } \vee \text{ слев)} \frac{\neg(A \wedge B) \wedge \neg C \vdash \neg C}{\neg(A \wedge B) \wedge \neg C \vdash \neg A \vee \neg C} \quad \frac{\neg(A \wedge B) \wedge \neg C \vdash \neg C}{\neg(A \wedge B) \wedge \neg C \vdash \neg B \vee \neg C} \text{(введ } \vee \text{ слев)} \\
\text{(лем2, морган)} \frac{\neg(A \wedge B) \wedge \neg C \vdash \neg A \vee \neg C}{\neg(A \wedge B) \wedge \neg C \vdash \neg(A \wedge C)} \quad \frac{\neg(A \wedge B) \wedge \neg C \vdash \neg B \vee \neg C}{\neg(A \wedge B) \wedge \neg C \vdash \neg(B \wedge C)} \text{(лем2, морган)} \\
\frac{\neg(A \wedge B) \wedge \neg C \vdash \neg(A \wedge C) \quad \neg(A \wedge B) \wedge \neg C \vdash \neg(B \wedge C)}{\neg(A \wedge B) \wedge \neg C \vdash \neg(A \wedge C) \wedge \neg(B \wedge C)} \text{(введ } \wedge) \\
\frac{\neg(A \wedge B) \wedge \neg C \vdash \neg(A \wedge C) \wedge \neg(B \wedge C)}{\neg\psi \vdash \neg(A \wedge C) \wedge \neg(B \wedge C)} \text{(лем1, морган)} \\
\frac{\neg\psi \vdash \neg(A \wedge C) \wedge \neg(B \wedge C)}{\neg\psi \vdash \neg\phi} \text{(лем2, морган)}
\end{array} \\
\frac{\phi \vdash \phi}{\phi, \neg\psi \vdash \phi} \text{(ут)} \quad \frac{\neg\psi \vdash \neg\phi}{\phi, \neg\psi \vdash \neg\phi} \text{(ут, пер)} \\
\hline
\phi \vdash \psi \text{(не)}
\end{array}
\end{array}$$