Sentiment Booms and Information
Online Appendix

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A Online Appendix: Proofs and Derivations

A.1 Derivations

Every household i consists of many traders indexed by $ij \in [0,1]$. The information set of each trader consists of $\{s_{ijt}, \{z_{jt}\}, a_t, \varepsilon_t\}$, i.e., traders observe their private signal, all public signals, and the aggregate states. This setting allows that traders have rational expectations about aggregate states but still disagree about firm-specific variables, which motivates trade. I impose that $\kappa_L = 0$ and $\kappa_H = 2$ to avoid distortions in asset prices that stem from the choice of position limits.

The beliefs of traders about firm productivity A_{jt} are relevant for their trading decision. Trader ij's beliefs are given by

$$\tilde{\mathbb{E}}\{A_{jt}|s_{ijt}, z_{jt}\} = \exp\left\{\omega_{p,ijt}a_t + \omega_{s,ijt}s_{ijt} + \omega_{z,ijt}\left(z_{jt} - \frac{\varepsilon_t}{\sqrt{\beta_{jt}}}\right) + \frac{1}{2}\mathbb{V}_{ijt}\right\}. \tag{1}$$

Similarly, the beliefs of the marginal trader are

$$\tilde{\mathbb{E}}\{A_{jt}|s_{ijt} = z_{jt}, z_{jt}\} = \exp\left\{\omega_{p,jt}a_t + \omega_{s,jt}z_{jt} + \omega_{z,jt}\left(z_{jt} - \frac{\varepsilon_t}{\sqrt{\beta_{jt}}}\right) + \frac{1}{2}\mathbb{V}_{jt}\right\}, \tag{2}$$

where $a_{jt} \overset{iid}{\sim} \mathcal{N}(a_t, \sigma_a^2)$, $\varepsilon_{jt} \overset{iid}{\sim} \mathcal{N}(\varepsilon_t, \sigma_\varepsilon^2)$ and ω-terms are the corresponding Bayesian weights,

$$\omega_{z,ijt} = \frac{\beta_{jt}\sigma_\varepsilon^{-2}}{\sigma_a^{-2} + \beta_{ijt} + \beta_{jt}\sigma_\varepsilon^{-2}}, \quad \omega_{z,jt} = \frac{\beta_{jt}\sigma_\varepsilon^{-2}}{\sigma_a^{-2} + \beta_{jt} + \beta_{jt}\sigma_\varepsilon^{-2}} \tag{3}$$

$$\omega_{s,ijt} = \frac{\beta_{ijt}}{\sigma_a^{-2} + \beta_{ijt} + \beta_{jt}\sigma_\varepsilon^{-2}}, \quad \omega_{s,jt} = \frac{\beta_{jt}}{\sigma_a^{-2} + \beta_{jt} + \beta_{jt}\sigma_\varepsilon^{-2}} \tag{4}$$

$$\omega_{p,ijt} = \frac{\sigma_a^{-2}}{\sigma_a^{-2} + \beta_{ijt} + \beta_{jt}\sigma_\varepsilon^{-2}}, \quad \omega_{p,jt} = \frac{\sigma_a^{-2}}{\sigma_a^{-2} + \beta_{jt} + \beta_{jt}\sigma_\varepsilon^{-2}}, \tag{5}$$

and $\{\mathbb{V}_{jt}, \mathbb{V}_{ijt}\}$ stand for posterior uncertainty

$$\mathbb{V}_{ijt} = \frac{1}{\sigma_a^{-2} + \beta_{ijt} + \beta_{jt}\sigma_\varepsilon^{-2}}, \quad \mathbb{V}_{jt} = \frac{1}{\sigma_a^{-2} + \beta_{jt} + \beta_{jt}\sigma_\varepsilon^{-2}}. \tag{6}$$

The private information precision β_{ijt} is highlighted in blue and is part of the information production decision. Alternatively, the beliefs of the marginal trader who observed $s_{ijt} = z_{jt}$ can be expressed as a function of shocks,

$$\ln \tilde{\mathbb{E}}\{A_{jt}|s_{ijt} = z_{jt}, z_{jt}\} = \omega_{p,jt}a_t + \omega_{s\varepsilon,jt}\varepsilon_t + \omega_{a,jt}a_{jt} + \frac{\omega_{a,jt}}{\sqrt{\beta_{jt}}}(\varepsilon_{jt} - \varepsilon_t) + \frac{1}{2}\mathbb{V}_{jt}, \tag{7}$$

where the corresponding Bayesian weights are

$$\omega_{a,jt} = \omega_{z,jt} + \omega_{s,jt} \tag{8}$$

$$\omega_{\varepsilon,jt} = \omega_{a,jt}/\sqrt{\beta_{jt}} \tag{9}$$

$$\omega_{s\varepsilon,jt} = \omega_{s,jt}/\sqrt{\beta_{jt}}. \tag{10}$$

Trader ij buys shares of firm j whenever

$$\tilde{\mathbb{E}}\left\{\Pi_{jt+1}|s_{ijt}, z_{jt}\right\} \geq \tilde{\mathbb{E}}\left\{\Pi_{jt+1}|s_{ijt} = z_{jt}, z_{jt}\right\} \tag{11}$$

$$\iff \tilde{\mathbb{E}}\left\{A_{jt}|s_{ijt}, z_{jt}\right\} \geq \tilde{\mathbb{E}}\left\{A_{jt}|s_{ijt} = z_{jt}, z_{jt}\right\}. \tag{12}$$

The inequality can be expressed as a cutoff for the idiosyncratic noise,

$$\begin{aligned}\eta_{ijt} \geq \frac{\sqrt{\beta_{ijt}}}{\omega_{s,ijt}}\left(\omega_{p,ijt}a_t + \omega_{z,ijt}\left(z_{jt} - \frac{1}{\sqrt{\beta_{jt}}}\varepsilon_t\right) + \frac{\mathbb{V}_{ijt}}{2}\right) + \sqrt{\beta_{ijt}}a_{jt} \\ - \frac{\sqrt{\beta_{ijt}}}{\omega_{s,ijt}}\left(\omega_{p,jt}a_t + \omega_{s,jt}z_{jt} + \omega_{z,jt}\left(z_{jt} - \frac{1}{\sqrt{\beta_{jt}}}\varepsilon_t\right) + \frac{\mathbb{V}_{jt}}{2}\right).\end{aligned} \tag{13}$$

Since η_{ijt} is standard-normal, the perceived probability of buying is

$$\begin{aligned}\mathcal{P}\left\{x_{ijt} = 2\right\} = \Phi\Bigg[-\frac{\sqrt{\beta_{ijt}}}{\omega_{s,ijt}}\left(\omega_{p,ijt}a_t + \omega_{z,ijt}\left(z_{jt} - \frac{1}{\sqrt{\beta_{jt}}}\varepsilon_t\right) + \frac{1}{2}\mathbb{V}_{ijt}\right) + \sqrt{\beta_{ijt}}a_{jt} \\ - \frac{\sqrt{\beta_{ijt}}}{\omega_{s,ijt}}\left(\omega_{p,jt}a_t + \omega_{s,jt}z_{jt} + \omega_{z,jt}\left(z_{jt} - \frac{1}{\sqrt{\beta_{jt}}}\varepsilon_t\right) + \frac{1}{2}\mathbb{V}_{jt}\right)\Bigg],\end{aligned} \tag{14}$$

where $\Phi(\cdot)$ is the standard-normal cdf. For a symmetric information choice ($\beta_{ijt} = \beta_{jt}$), the buying probability can be simplified to

$$\mathcal{P}\left\{x_{ijt} = 2|a_{jt}, \varepsilon_{jt}, \beta_{ijt}, \beta_{jt}\right\}\big|_{\beta_{ijt}=\beta_{jt}} = \Phi\left(-\varepsilon_{jt}\right). \tag{15}$$

Traders think they are more likely to buy shares when the realization of the sentiment shock is relatively low, and shares are therefore cheap relative to their fundamental value.

Finally, traders choose their information precision, taking all other traders' symmetric choice as given. The derivative of the probability of buying with respect to β_{ijt} is

$$\begin{aligned}\frac{\partial\mathcal{P}\left\{x_{ijt}=2\right\}}{\partial\beta_{ijt}} &= \phi\left[-\frac{\sqrt{\beta_{ijt}}}{\omega_{s,ijt}}\left(\omega_{p,ijt}a_t+\omega_{z,ijt}\left(z_{jt}-\frac{1}{\sqrt{\beta_{jt}}}\varepsilon_t\right)+\frac{1}{2}\mathbb{V}_{ijt}\right)+\sqrt{\beta_{ijt}}a_{jt}\right.\\ &\quad\left.-\frac{\sqrt{\beta_{ijt}}}{\omega_{s,ijt}}\left(\omega_{p,jt}a_t+\omega_{s,jt}z_{jt}+\omega_{z,jt}\left(z_{jt}-\frac{1}{\sqrt{\beta_{jt}}}\varepsilon_t\right)+\frac{1}{2}\mathbb{V}_{jt}\right)\right]\\ &\quad*\left[-\frac{1}{2\beta_{ijt}^{3/2}}\left(\sigma_a^{-2}a_t+\beta_{jt}\sigma_\varepsilon^{-2}\left(z_{jt}-\frac{\varepsilon_t}{\sqrt{\beta_{jt}}}\right)+\frac{1}{2}\right)+\frac{a_{jt}}{2\sqrt{\beta_{ijt}}}\right.\\ &\quad\left.-\left(\frac{1}{\sqrt{\beta_{ijt}}}-\frac{1}{2\beta_{ijt}^{3/2}}\left(\mathbb{V}_{ijt}\right)^{-1}\right)*\left(\omega_{p,jt}a_t+\omega_{s,jt}z_{jt}+\omega_{z,jt}\left(z_{jt}-\frac{\varepsilon_t}{\sqrt{\beta_{jt}}}\right)+\frac{1}{2}\mathbb{V}_{jt}\right)\right] \qquad (16)\end{aligned}$$

where $\phi(\cdot)$ is the standard normal pdf. For a symmetric information choice ($\beta_{ijt}=\beta_{jt}$) this expression can be simplified to

$$\begin{aligned}&\left.\frac{\partial\mathcal{P}\left\{x_{ijt}=2\right\}}{\partial\beta_{ijt}}\right|_{\beta_{ijt}=\beta_{jt}} =\\ &\phi\left(\varepsilon_{jt}\right)\left[\frac{1}{2\sqrt{\beta}_{jt}}\left(a_{jt}+z_{jt}\right)-\frac{1}{\sqrt{\beta_{jt}}}\left(\omega_{p,jt}a_t+\omega_{s,jt}z_{jt}+\omega_{z,jt}\left(z_{jt}-\frac{\varepsilon_t}{\sqrt{\beta_{jt}}}\right)+\frac{1}{2}\mathbb{V}_{jt}\right)\right]. \qquad (17)\end{aligned}$$

A.2 Auxiliary Results

Lemma 1 (Auxiliary Results Market Allocation)**.** *Denote the Bayesian weights*

$$\omega_a=\frac{\beta_{jt}\left(1+\sigma_\varepsilon^{-2}\right)}{\sigma_a^{-2}+\beta_{jt}\left(1+\sigma_\varepsilon^{-2}\right)}, \qquad (18)$$

$$\omega_\varepsilon=\frac{\sqrt{\beta_{jt}}\left(1+\sigma_\varepsilon^{-2}\right)}{\sigma_a^{-2}+\beta_{jt}\left(1+\sigma_\varepsilon^{-2}\right)} \qquad (19)$$

$$\omega_{z\varepsilon}=\frac{\sqrt{\beta_{jt}}\sigma_\varepsilon^{-2}}{\sigma_a^{-2}+\sqrt{\beta_{jt}}\left(1+\sigma_\varepsilon^{-2}\right)}, \qquad (20)$$

and posterior uncertainty

$$\mathbb{V}=\frac{1}{\sigma_a^{-2}+\beta_t\left(1+\sigma_\varepsilon^{-2}\right)} \qquad (21)$$

then

(i) $\omega_a^2\sigma_a^2+\omega_\varepsilon^2\frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}+\mathbb{V}=\sigma_a^2$,

(ii) $\omega_a^2\sigma_a^2+\omega_\varepsilon^2\frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}=\sigma_a^2-\mathbb{V}=\omega_a\sigma_a^2$,

(iii) $\frac{\omega_\varepsilon}{1+\sigma_\varepsilon^2}=\omega_{z\varepsilon}$.

Proof. (i)

$$\begin{aligned}
\omega_a^2\sigma_a^2 + \omega_\varepsilon^2\frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2} + \mathbb{V} &= \frac{\sigma_a^2\beta^2\left(1+\sigma_\varepsilon^{-2}\right)^2}{\left(\sigma_a^{-2}+\beta\left(1+\sigma_\varepsilon^{-2}\right)\right)^2} + \frac{\beta\left(1+\sigma_\varepsilon^{-2}\right)}{\left(\sigma_a^{-2}+\beta\left(1+\sigma_\varepsilon^{-2}\right)\right)^2} \\
&\quad + \frac{\sigma_a^{-2}+\beta\left(1+\sigma_\varepsilon^{-2}\right)}{\left(\sigma_a^{-2}+\beta\left(1+\sigma_\varepsilon^{-2}\right)\right)^2} \\
&= \frac{\sigma_a^{-2}+2\beta\left(1+\sigma_\varepsilon^{-2}\right)+\sigma_a^2\beta^2\left(1+\sigma_\varepsilon^{-2}\right)^2}{\left(\sigma_a^{-2}+\beta\left(1+\sigma_\varepsilon^{-2}\right)\right)^2} \\
&= \frac{\sigma_a^{-4}+2\sigma_a^{-2}\beta\left(1+\sigma_\varepsilon^{-2}\right)+\beta^2\left(1+\sigma_\varepsilon^{-2}\right)^2}{\left(\sigma_a^{-2}+\beta\left(1+\sigma_\varepsilon^{-2}\right)\right)^2}\sigma_a^2 \\
&= \frac{\left(\sigma_a^{-2}+\beta\left(1+\sigma_\varepsilon^{-2}\right)\right)^2}{\left(\sigma_a^{-2}+\beta\left(1+\sigma_\varepsilon^{-2}\right)\right)^2}\sigma_a^2 \\
&= \sigma_a^2 \qquad (22)
\end{aligned}$$

(ii) The first equality follows from (i). Then

$$\begin{aligned}
\sigma_a^2 - \mathbb{V} &= \sigma_a^2 - \frac{1}{\sigma_a^{-2}+\beta\left(1+\sigma_\varepsilon^{-2}\right)} \\
&= \frac{\sigma_a^2\sigma_a^{-2}+\sigma_a^2\beta\left(1+\sigma_\varepsilon^{-2}\right)-1}{\sigma_a^{-2}+\beta\left(1+\sigma_\varepsilon^{-2}\right)} \\
&= \frac{\beta\left(1+\sigma_\varepsilon^{-2}\right)}{\sigma_a^{-2}+\beta\left(1+\sigma_\varepsilon^{-2}\right)}\sigma_a^2 \\
&= \omega_a\sigma_a^2. \qquad (23)
\end{aligned}$$

(iii)

$$\frac{\omega_\varepsilon}{1+\sigma_\varepsilon^2} = \frac{\omega_\varepsilon}{\sigma_\varepsilon^2\left(1+\sigma_\varepsilon^{-2}\right)} = \frac{\omega_\varepsilon\sigma_\varepsilon^{-2}}{\left(1+\sigma_\varepsilon^{-2}\right)} = \omega_{z\varepsilon}. \qquad (24)$$

□

Lemma 2. *Denote the efficient Bayesian weights*

$$\omega_a^{eff} = \frac{\beta_t\sigma_\varepsilon^{-2}}{\sigma_a^{-2}+\beta_t\sigma_\varepsilon^{-2}} \qquad (25)$$

$$\omega_\varepsilon^{eff} = \frac{\sqrt{\beta_t}\sigma_\varepsilon^{-2}}{\sigma_a^{-2}+\beta_t\sigma_\varepsilon^{-2}} \qquad (26)$$

and posterior uncertainty

$$\mathbb{V}^{eff} = \frac{1}{\sigma_a^{-2}+\beta_t\sigma_\varepsilon^{-2}}. \qquad (27)$$

These are the weights that a rational uninformed observer would use after observing the price *signal z_{jt}, in contrast to the Bayesian weights that traders that suffer from correlation neglect use as introduced in section A.1. Then the following relationships hold:*

(i)

$$\underbrace{\left(\omega_a^{eff}\right)^2\sigma_a^2+\left(\omega_\varepsilon^{eff}\right)^2\sigma_\varepsilon^2}_{=Var(\mathbb{E}\{a_{jt}|z_{jt}\})}+\underbrace{\mathbb{V}^{eff}}_{Var(a_{jt}|z_{jt})}=\underbrace{\sigma_a^2}_{Var(a_{jt})}. \tag{28}$$

(ii)

$$\sigma_a^2-\mathbb{V}^{eff}=\omega_a^{eff}\sigma_a^2 \tag{29}$$

(iii)

$$\theta\omega_a^{eff}\sigma_a^2+\mathbb{V}^{eff}=(\theta-1)\,\omega_a^{eff}\sigma_a^2+\sigma_a^2 \tag{30}$$

Proof. (i):

$$\begin{aligned}\left(\omega_a^{eff}\right)^2\sigma_a^2+\left(\omega_\varepsilon^{eff}\right)^2\sigma_\varepsilon^2+\mathbb{V}^{eff}&=\frac{\beta_t^2\sigma_\varepsilon^{-4}}{\left(\sigma_a^{-2}+\beta_t\sigma_\varepsilon^{-2}\right)^2}\sigma_a^2+\frac{\beta_t\sigma_\varepsilon^{-4}}{\left(\sigma_a^{-2}+\beta_t\sigma_\varepsilon^{-2}\right)^2}\sigma_\varepsilon^2\\&\quad+\frac{1}{\sigma_a^{-2}+\beta_t\sigma_\varepsilon^{-2}}\\&=\left(\mathbb{V}^{eff}\right)^2\left(\sigma_a^{-2}+2\beta_t\sigma_\varepsilon^{-2}+\beta_t^2\sigma_\varepsilon^{-4}\sigma_a^2\right)\\&=\left(\mathbb{V}^{eff}\right)^2\left(\sigma_a^{-4}+2\beta_t\sigma_\varepsilon^{-2}\sigma_a^{-2}+\beta_t^2\sigma_\varepsilon^{-4}\right)\sigma_a^2\\&=\left(\mathbb{V}^{eff}\right)^2\left(\sigma_a^{-2}+\beta_t\sigma_\varepsilon^{-2}\right)^2\sigma_a^2\\&=\sigma_a^2.\end{aligned} \tag{31}$$

(ii):

$$\begin{aligned}\sigma_a^2-\mathbb{V}^{eff}&=\sigma_a^2-\frac{1}{\sigma_a^2+\beta\sigma_\varepsilon^{-2}}\\&=\frac{\sigma_a^2\sigma_a^{-2}+\sigma_a^2\beta\sigma_\varepsilon^{-2}-1}{\sigma_a^{-2}+\beta\sigma_\varepsilon^{-2}}\\&=\frac{\beta\sigma_\varepsilon^{-2}}{\sigma_a^{-2}+\beta\sigma_\varepsilon^{-2}}\sigma_a^2\\&=\omega_a^{eff}\sigma_a^2\end{aligned} \tag{32}$$

(iii):

$$\begin{aligned}
\theta\omega_a^{eff}\sigma_a^2+\mathbb{V}^{eff} &= \frac{\theta\beta_{t-1}\sigma_\varepsilon^{-2}\sigma_a^2}{\sigma_a^{-2}+\beta_{t-1}\sigma_\varepsilon^{-2}}+\frac{1}{\sigma_a^{-2}+\beta_{t-1}\sigma_\varepsilon^{-2}} \\
&= \frac{\theta\beta_{t-1}\sigma_\varepsilon^{-2}+\sigma_a^{-2}}{\sigma_a^{-2}+\beta_{t-1}\sigma_\varepsilon^{-2}}\sigma_a^2 \\
\text{Add and subtract} &= \frac{\theta\beta_{t-1}\sigma_\varepsilon^{-2}+\beta_{t-1}\sigma_\varepsilon^{-2}-\beta_{t-1}\sigma_\varepsilon^{-2}+\sigma_a^{-2}}{\sigma_a^{-2}+\beta_{t-1}\sigma_\varepsilon^{-2}}\sigma_a^2 \\
\text{Split} &= (\theta-1)\frac{\beta_{t-1}\sigma_\varepsilon^{-2}}{\sigma_a^{-2}+\beta_{t-1}\sigma_\varepsilon^{-2}}\sigma_a^2+\frac{\sigma_a^{-2}+\beta_{t-1}\sigma_\varepsilon^{-2}}{\sigma_a^{-2}+\beta_{t-1}\sigma_\varepsilon^{-2}}\sigma_a^2 \\
&= (\theta-1)\,\omega_a^{eff}\sigma_a^2+\sigma_a^2 \qquad (33)
\end{aligned}$$

□

Lemma 3 (Joining two Normal PDFs)**.** *Let $f(\varepsilon_{jt})$ be the pdf of $\varepsilon_{jt}\sim\mathcal{N}(\varepsilon_t,\sigma_\varepsilon^2)$ and $\phi(\cdot)$ the standard-normal pdf. Then*

$$f(\varepsilon_{jt})\,\phi(\varepsilon_{jt})=\exp\left\{-\frac{\varepsilon_t^2}{2(1+\sigma_\varepsilon^2)}\right\}\sqrt{\frac{1}{2\pi(1+\sigma_\varepsilon^2)}}\tilde{f}(\varepsilon_{jt}) \qquad (34)$$

where $\tilde{f}(\varepsilon_{jt})$ is the transformed pdf of $\varepsilon_{jt}\sim\mathcal{N}\left(\frac{\varepsilon_t}{1+\sigma_\varepsilon^2},\frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}\right)$.

Proof. Write out the pdfs explicitly,

$$\phi(\varepsilon_{jt})=\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{\varepsilon_{jt}^2}{2}\right\} \qquad (35)$$

$$f(\varepsilon_{jt})=\frac{1}{\sqrt{2\pi\sigma_\varepsilon^2}}\exp\left\{-\frac{(\varepsilon_{jt}-\varepsilon_t)^2}{2\sigma_\varepsilon^2}\right\} \qquad (36)$$

$$f(\varepsilon_{jt})\,\phi(\varepsilon_{jt})=\frac{1}{\sqrt{2\pi\sigma_\varepsilon^2}}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{\varepsilon_{jt}^2}{2}-\frac{(\varepsilon_{jt}-\varepsilon_t)^2}{2\sigma_\varepsilon^2}\right\}. \qquad (37)$$

Rearranging the term inside the exponential function,

$$\begin{aligned}
\frac{(\varepsilon_{jt}-\varepsilon_t)^2}{\sigma_\varepsilon^2}+\varepsilon_{jt}^2 &= \frac{\varepsilon_{jt}^2-2\varepsilon_{jt}\varepsilon_t+\varepsilon_t^2}{\sigma_\varepsilon^2}+\varepsilon_{jt}^2\\
\text{join fractions} &= \frac{\left(1+\sigma_\varepsilon^2\right)\varepsilon_{jt}^2-2\varepsilon_t\varepsilon_{jt}+\varepsilon_t^2}{\sigma_\varepsilon^2}\\
\text{divide by } \left(1+\sigma_\varepsilon^2\right) &= \frac{\varepsilon_{jt}^2-2\varepsilon_{jt}\frac{\varepsilon_t}{1+\sigma_\varepsilon^2}+\frac{\varepsilon_t^2}{1+\sigma_\varepsilon^2}}{\frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}}\\
\text{add and subtract} &= \frac{\varepsilon_{jt}^2-2\varepsilon_{jt}\frac{\varepsilon_t}{1+\sigma_\varepsilon^2}+\frac{\varepsilon_t^2}{1+\sigma_\varepsilon^2}}{\frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}}+\frac{\left(\frac{\varepsilon_t}{1+\sigma_\varepsilon^2}\right)^2-\left(\frac{\varepsilon_t}{1+\sigma_\varepsilon^2}\right)^2}{\frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}}\\
\text{exchange terms} &= \frac{\varepsilon_{jt}^2-2\varepsilon_{jt}\frac{\varepsilon_t}{1+\sigma_\varepsilon^2}+\left(\frac{\varepsilon_t}{1+\sigma_\varepsilon^2}\right)^2}{\frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}}+\frac{\frac{\varepsilon_t^2}{1+\sigma_\varepsilon^2}-\left(\frac{\varepsilon_t}{1+\sigma_\varepsilon^2}\right)^2}{\frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}}\\
\text{binomial} &= \frac{\left(\varepsilon_{jt}-\frac{\varepsilon_t}{1+\sigma_\varepsilon^2}\right)^2}{\frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}}+\frac{\frac{\varepsilon_t^2}{1+\sigma_\varepsilon^2}-\left(\frac{\varepsilon_t}{1+\sigma_\varepsilon^2}\right)^2}{\frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}}\\
&= \frac{\left(\varepsilon_{jt}-\frac{\varepsilon_t}{1+\sigma_\varepsilon^2}\right)^2}{\frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}}+\frac{\varepsilon_t^2}{1+\sigma_\varepsilon^2} \qquad (38)
\end{aligned}$$

This allows to write

$$\begin{aligned}
f\left(\varepsilon_{jt}\right)\phi\left(\varepsilon_{jt}\right) &= \frac{1}{\sqrt{2\pi\sigma_\varepsilon^2}}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{\varepsilon_{jt}}{2}-\frac{\left(\varepsilon_{jt}-\varepsilon_t\right)^2}{2\sigma_\varepsilon^2}\right\}\\
&= \frac{1}{\sqrt{\sigma_\varepsilon^2}}\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\frac{\left(\varepsilon_{jt}-\frac{\varepsilon_t}{1+\sigma_\varepsilon^2}\right)^2}{\frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}}\right\}\exp\left\{-\frac{1}{2}\left(\frac{\varepsilon_t^2}{1+\sigma_\varepsilon^2}\right)\right\}\\
&= \frac{1}{\sqrt{\sigma_\varepsilon^2}}\sqrt{\frac{\sigma_\varepsilon^2}{2\pi\left(1+\sigma_\varepsilon^2\right)}}\frac{1}{\sqrt{2\pi\frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}}}\exp\left\{-\frac{1}{2}\frac{\left(\varepsilon_{jt}-\frac{\varepsilon_t}{1+\sigma_\varepsilon^2}\right)^2}{\frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}}\right\}\\
&\quad * \exp\left\{-\frac{1}{2}\left(\frac{\varepsilon_t^2}{1+\sigma_\varepsilon^2}\right)\right\}\\
&= \exp\left\{-\frac{1}{2}\left(\frac{\varepsilon_t^2}{1+\sigma_\varepsilon^2}\right)\right\}\sqrt{\frac{1}{2\pi\left(1+\sigma_\varepsilon^2\right)}}\tilde{f}(\varepsilon_{jt}), \qquad (39)
\end{aligned}$$

where $\tilde{f}\left(\varepsilon_{jt}\right)$ is the transformed pdf of $\varepsilon_{jt}\sim\mathcal{N}\left(\frac{\varepsilon_t}{1+\sigma_\varepsilon^2},\frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}\right)$. $\square$

A.3 Main Proofs

Proof of Proposition 1. This proof follows the same steps as the proof for Proposition 1 in Albagli, Hellwig, and Tsyvinski (2017), since the financial market in this model is isomorphic to their model. Their proof is repeated here for completeness. The only difference is that K_{jt+1} depends on the price signal z_{jt}, whereas k in Albagli, Hellwig, and Tsyvinski (2017) is determined before trading takes place. Therefore, it is necessary to assume that $K_{jt+1}(z_{jt})$ is non-decreasing in z_{jt} as the price might otherwise be not invertible, which is confirmed ex-post. The proof begins in the following.

There must be a threshold $\hat{s}(P_{jt})$ such that all households with $s_{ijt} \geq \hat{s}(P_{jt})$ find it profitable to buy two units of share j and otherwise abstain from trading. It follows that the price must be equal to the valuation of the trader who is merely indifferent between buying and not buying,

$$P_{jt} = \frac{1}{R_{t+1}} \tilde{\mathbb{E}} \left\{ \Pi \left(A_{jt}, K_{jt+1} \right) | s_{ijt} = \hat{s}\left(P_{jt} \right), P_{jt} \right\}. \tag{40}$$

This monotone demand schedule leads to total demand

$$D(\theta, \varepsilon, P) = 2 \left(1 - \Phi \left(\sqrt{\beta_{jt}} \left(\hat{s}\left(P_{jt} \right) - a_{jt} \right) - \varepsilon_{jt} \right) \right). \tag{41}$$

Equalizing total demand with a normalized supply of one leads to the market-clearing condition

$$2 \left(1 - \Phi \left(\sqrt{\beta_{jt}} \left(\hat{s}\left(P_{jt} \right) - a_{jt} \right) - \varepsilon_{jt} \right) \right) = 1, \tag{42}$$

with the unique solution $\hat{s}(P_{jt}) = z_{jt} = a_{jt} + \frac{\varepsilon_{jt}}{\sqrt{\beta_{jt}}}$. If P_{jt} is pinned down by z_{jt}, then P_{jt} is invertible, given that K_{jt+1} is non-decreasing in z_{jt}. It follows, then, that observing P_{jt} is equivalent to observing $z_{jt} \sim \mathcal{N}\left(a_{jt}, \beta_{jt}^{-1}\sigma_\varepsilon^2\right)$. Traders treat the signal z_{jt} and their private signal $s_{ijt} \sim \mathcal{N}\left(a_{jt}, \beta_{ijt}^{-1}\right)$ as mutually independent. Using this result, the price can be restated as

$$P\left(z_{jt}, K_{jt+1} \right) = \frac{1}{R_{t+1}} \tilde{\mathbb{E}} \left\{ \Pi \left(A_{jt}, K_{jt+1} \right) | s_{ijt} = z_{jt}, z_{jt} \right\}, \tag{43}$$

where posterior expectations of trader ij are given by

$$a_{jt} | s_{ijt}, z_{jt} \sim \mathcal{N} \left(\frac{\sigma_a^{-2} a_t + \beta_{ijt} s_{ijt} + \beta_{jt} \sigma_\varepsilon^{-2} z_{jt}}{\sigma_a^{-2} + \beta_{ijt} + \beta_{jt} \sigma_\varepsilon^{-2}}, \frac{1}{\sigma_a^{-2} + \beta_{ijt} + \beta_{jt} \sigma_\varepsilon^{-2}} \right). \tag{44}$$

Using the result that the firm invests all proceeds into capital ($K_{jt+1} = P_{jt}$), it follows indeed that K_{jt+1} is non-decreasing in z_{jt} and P_{jt} is an invertible function of the price signal z_{jt}. It remains to show the uniqueness of the above solution. Begin with the assumption that demand $x(s_{ijt}, P_{jt})$ is non-increasing in P_{jt}. It follows that $\hat{s}(P_{jt})$ is non-decreasing in P_{jt}.

There are two cases to differentiate. First, if $\hat{s}(P_{jt})$ is strictly increasing in P_{jt}, then the price is indeed uniquely pinned-down by z_{jt} and invertible; it can be expressed like above. Secondly, assume that the threshold is flat over some interval, such that $\hat{s}(P_{jt}) = \hat{s}$ over some interval $P_{jt} \in (P', P'')$ for $P' \neq P''$. Furthermore, choose $\epsilon > 0$ small enough such that $\hat{s}(P_{jt})$ is increasing to the left and the right of the interval, i.e., over $P_{jt} \in (P' - \epsilon, P')$ and $P_{jt} \in (P'', P'' + \epsilon)$. In these regions, $\hat{s}(P_{jt})$ is monotonically increasing in P_{jt}, which is uniquely pinned down by z_{jt} and invertible; observing the price is equivalent to observing the signal z_{jt}. In this case the price can be expressed as before for $z_{jt} \in (\hat{s}(P' - \varepsilon), \hat{s})$ and $z_{jt} \in (\hat{s}, \hat{s}(P'' + \epsilon))$. This leads to a contradiction in the assumption that $P' \neq P''$, because $P(z_{jt}, K_{jt+1})$ is both continuous and monotonically increasing in z_{jt}. Therefore, $\hat{s}(P_{jt})$ cannot be flat, and the above solution is indeed unique. □

Proof of Proposition 2. (i) Using (7) in (26) leads to the expression for firm capital

$$K_{jt+1} = \left(\frac{\alpha Y_{t+1}^{\alpha_Y}}{R_{t+1}} \tilde{\mathbb{E}} \left\{ A_{jt} | s_{ijt} = z_{jt}, z_{jt} \right\} \right)^{\theta}. \tag{45}$$

Plugging R_{t+1} from (27) into (26) using (45) leads to

$$K_{jt+1} = \frac{\tilde{\mathbb{E}} \left\{ A_{jt} | s_{ijt} = z_{jt}, z_{jt} \right\}^{\theta}}{\int_0^1 \tilde{\mathbb{E}} \left\{ A_{jt} | s_{ijt} = z_{jt}, z_{jt} \right\}^{\theta} dj} K_{t+1}, \tag{46}$$

which finishes the proof.

(ii) Plugging the above expression for firm capital (29) into the aggregate production function (5) leads to

$$\begin{aligned}
Y_t &= \left(\int_0^1 Y_{jt}^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\alpha\theta}{\theta-1}} \\
&= \left(\int_0^1 A_{jt-1} K_{jt}^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\alpha\theta}{\theta-1}} \\
&= \frac{\left(\int_0^1 A_{jt-1} \tilde{\mathbb{E}} \left\{ A_{jt-1} | s_{ijt-1} = z_{jt-1}, z_{jt-1} \right\}^{\theta-1} dj \right)^{\frac{\alpha\theta}{\theta-1}}}{\left(\int_0^1 \tilde{\mathbb{E}} \left\{ A_{jt-1} | s_{ijt-1} = z_{jt-1}, z_{jt-1} \right\}^{\theta} dj \right)^{\alpha}} K_t^{\alpha} \\
&= A_{t-1} L^{1-\alpha} K_t^{\alpha}
\end{aligned} \tag{47}$$

where total factor productivity is

$$
\begin{aligned}
A_{t-1} &= \frac{\left(\int_0^1 A_{jt-1}\tilde{\mathbb{E}}\left\{A_{jt-1}|s_{ijt-1}=z_{jt-1},z_{jt-1}\right\}^{\theta-1}dj\right)^{\frac{\alpha\theta}{\theta-1}}}{\left(\int_0^1 \tilde{\mathbb{E}}\left\{A_{jt-1}|s_{ijt-1}=z_{jt-1},z_{jt-1}\right\}^{\theta}dj\right)^{\alpha}} \\
&= \frac{\exp\left\{\theta a_{t-1}+\left(\left(\theta-1\right)\omega_a+1\right)^2\frac{\sigma_a^2}{2}+\left(\theta-1\right)^2\omega_\varepsilon^2\frac{\sigma_\varepsilon^2}{2}+\left(\theta-1\right)\omega_{s\varepsilon}\varepsilon_{t-1}+\frac{(\theta-1)}{2}\mathbb{V}\right\}^{\frac{\alpha\theta}{\theta-1}}}{\exp\left\{\theta a_{t-1}+\theta^2\omega_a^2\frac{\sigma_a^2}{2}+\theta^2\omega_\varepsilon^2\frac{\sigma_\varepsilon^2}{2}+\theta\omega_{s\varepsilon}\varepsilon_{t-1}+\frac{\theta}{2}\mathbb{V}\right\}^{\alpha}} \\
&= \exp\left(\frac{\alpha\theta}{\theta-1}a_{t-1}+\left(\frac{\alpha\theta}{\theta-1}\left(\left(\theta-1\right)\omega_a+1\right)^2-\alpha\theta^2\omega_a^2\right)\frac{\sigma_a^2}{2}+\left(\alpha\theta\left(\theta-1\right)-\alpha\theta^2\right)\omega_\varepsilon^2\frac{\sigma_\varepsilon^2}{2}\right) \\
&= \exp\left(\frac{\alpha\theta}{\theta-1}a_{t-1}+\alpha\theta\left(\left(\theta-1\right)\omega_a^2+2\omega_a+\frac{1}{\theta-1}-\theta\omega_a^2\right)\frac{\sigma_a^2}{2}-\alpha\theta\omega_\varepsilon^2\frac{\sigma_\varepsilon^2}{2}\right) \\
&= \exp\left(\frac{1}{\theta-1}\left(a_{t-1}+\frac{\sigma_a^2}{2}\right)+\omega_a\left(2-\omega_a\right)\frac{\sigma_a^2}{2}-\omega_\varepsilon^2\frac{\sigma_\varepsilon^2}{2}\right)^{\alpha\theta}.
\end{aligned}
\tag{48}
$$

The weights $\{\omega_a,\omega_\varepsilon,\omega_{s\varepsilon}\}$ and $\mathbb{V}$ are derived in section A.1. Finally, total factor productivity can be expressed as

$$
\ln A_{t-1}\left(a_{t-1},\beta_{t-1}\right)=\underbrace{\frac{\alpha\theta}{\theta-1}\left(a_{t-1}+\frac{\sigma_a^2}{2}\right)}_{\text{exogenous}}+\underbrace{\kappa^a\left(\beta_{t-1}\right)\sigma_a^2-\kappa^\varepsilon\left(\beta_{t-1}\right)\sigma_\varepsilon^2}_{\text{allocative efficiency}}
\tag{49}
$$

where $\kappa^a\left(\beta_{t-1}\right)=\frac{\alpha\theta}{2}\omega_a\left(2-\omega_a\right)$ and $\kappa^\varepsilon\left(\beta_{t-1}\right)=\frac{\alpha\theta}{2}\omega_\varepsilon^2$.

(iii) I will show that the allocative efficiency component of TFP takes its minimum for $\beta_{t-1}>0$ if $\sigma_\varepsilon^2>1$. The allocational efficiency component is proportional to

$$
\begin{aligned}
&\omega_a\left(2-\omega_a\right)\sigma_a^2-\omega_\varepsilon^2\sigma_\varepsilon^2 \\
&= \frac{2\beta_{t-1}\left(1+\sigma_\varepsilon^{-2}\right)}{\sigma_a^{-2}+\beta_{t-1}\left(1+\sigma_\varepsilon^{-2}\right)}\sigma_a^2-\frac{\beta_{t-1}^2\left(1+\sigma_\varepsilon^{-2}\right)^2}{\left(\sigma_a^{-2}+\beta_{t-1}\left(1+\sigma_\varepsilon^{-2}\right)\right)^2}\sigma_a^2-\frac{\beta_{t-1}\left(1+\sigma_\varepsilon^{-2}\right)^2}{\left(\sigma_a^{-2}+\beta_{t-1}\left(1+\sigma_\varepsilon^{-2}\right)\right)^2}\sigma_\varepsilon^2 \\
&= \frac{2\beta_{t-1}\left(1+\sigma_\varepsilon^{-2}\right)+2\beta_{t-1}^2\left(1+\sigma_\varepsilon^{-2}\right)^2\sigma_a^2-\beta_{t-1}^2\left(1+\sigma_\varepsilon^{-2}\right)^2\sigma_a^2-\beta_{t-1}\left(1+\sigma_\varepsilon^{-2}\right)^2\sigma_\varepsilon^2}{\left(\sigma_a^{-2}+\beta_{t-1}\left(1+\sigma_\varepsilon^{-2}\right)\right)^2} \\
&= \frac{2\beta_{t-1}\left(1+\sigma_\varepsilon^{-2}\right)+\beta_{t-1}^2\left(1+\sigma_\varepsilon^{-2}\right)^2\sigma_a^2-\beta_{t-1}\left(1+\sigma_\varepsilon^{-2}\right)^2\sigma_\varepsilon^2}{\left(\sigma_a^{-2}+\beta_{t-1}\left(1+\sigma_\varepsilon^{-2}\right)\right)^2} \\
&= \frac{\beta_{t-1}^2\sigma_a^2\left(1+\sigma_\varepsilon^{-2}\right)^2+\beta_{t-1}\left(\sigma_\varepsilon^{-2}-\sigma_\varepsilon^2\right)}{\left(\sigma_a^{-2}+\beta_{t-1}\left(1+\sigma_\varepsilon^{-2}\right)\right)^2},
\end{aligned}
\tag{50}
$$

which is weakly positive for all values of β_{t-1} if $\sigma_\varepsilon^2<1$. Since the allocative efficiency component is zero for $\beta_{t-1} = 0$, it must be that the minimum is attained for some $\beta_{t-1} > 0$ when $\sigma_\varepsilon^2 > 1$.

(iv) Using the previous result, it remains to take the derivative of (50) with respect to β_{t-1}. Denote

$$a = \beta_{t-1}^2 \sigma_a^2 \left(1 + \sigma_\varepsilon^{-2}\right)^2 + \beta_{t-1}\left(\sigma_\varepsilon^{-2} - \sigma_\varepsilon^2\right) \tag{51}$$

$$b = \left(\sigma_a^{-2} + \beta_{t-1}\left(1 + \sigma_\varepsilon^{-2}\right)\right)^2 \tag{52}$$

After some algebra,

$$\begin{aligned} \frac{\partial a}{\partial \beta_{t-1}} b = {} & 2\beta_{t-1}^3 \sigma_a^2 \left(1 + \sigma_\varepsilon^{-2}\right)^4 + 4\beta_{t-1}^2 \left(1 + \sigma_\varepsilon^{-2}\right)^3 + 2\beta_{t-1}\sigma_a^{-2}\left(1 + \sigma_\varepsilon^{-2}\right)^2 \\ & + \beta_{t-1}^2 \left(1 + \sigma_\varepsilon^{-2}\right)^2 \left(\sigma_\varepsilon^{-2} - \sigma_\varepsilon^2\right) + 2\beta_{t-1}\sigma_a^{-2}\left(1 + \sigma_\varepsilon^{-2}\right)\left(\sigma_\varepsilon^{-2} - \sigma_\varepsilon^2\right) \\ & + \sigma_a^{-4}\left(\sigma_\varepsilon^{-2} - \sigma_\varepsilon^2\right) \end{aligned} \tag{53}$$

and

$$\begin{aligned} \frac{\partial b}{\partial \beta_{t-1}} a = {} & 2\beta_{t-1}^3 \sigma_a^2 \left(1 + \sigma_\varepsilon^{-2}\right)^4 + 2\beta_{t-1}^2 \left(1 + \sigma_\varepsilon^{-2}\right)^3 \\ & + 2\beta_{t-1}^2 \left(\sigma_\varepsilon^{-2} - \sigma_\varepsilon^2\right)\left(1 + \sigma_\varepsilon^{-2}\right)^2 + 2\beta_{t-1}\sigma_a^{-2}\left(\sigma_\varepsilon^{-2} - \sigma_\varepsilon^2\right)\left(1 + \sigma_\varepsilon^{-2}\right) \end{aligned} \tag{54}$$

Dropping the positive denominator, the derivative of (50) is $\frac{\partial a}{\partial \beta_{t-1}} b - \frac{\partial b}{\partial \beta_{t-1}} a$, which is after dividing through $(1 + \sigma_\varepsilon^{-2})^2$,

$$2\beta_{t-1}^2 + \beta_{t-1}^2 \sigma_\varepsilon^{-2} + 2\beta_{t-1}\sigma_a^{-2} + \sigma_a^{-4} \frac{\sigma_\varepsilon^{-2} - \sigma_\varepsilon^2}{\left(1 + \sigma_\varepsilon^{-2}\right)^2} + \beta_{t-1}^2 \sigma_\varepsilon^2, \tag{55}$$

which is positive for all values of β_{t-1} if $\sigma_\varepsilon^2 < 1$. □

Lemma 4. *In the symmetric equilibrium with $\beta_{ijt} = \beta_{jt}$ and $K_{t+1} < W_t$,*

(i) Sentiment shocks ε_t affect the marginal benefit of information production through three channels,

$$\widetilde{MB}\left(\beta_{ijt}, \beta_{jt}\right)\Big|_{\beta_{ijt} = \beta_{jt}} \propto \exp\left\{ \underbrace{-\frac{\varepsilon_t^2}{2\left(1 + \sigma_\varepsilon^2\right)}}_{\textit{Information-Sensitivity}} \underbrace{-\left(\theta - 1\right)\omega_{s\varepsilon}\varepsilon_t}_{\textit{Relative Size}} + \underbrace{\frac{\alpha}{1 - \alpha}\omega_{s\varepsilon}\varepsilon_t}_{\textit{Absolute Size}} \right\}. \tag{56}$$

(ii) Productivity shocks a_t affect the marginal benefit of information production through

three channels,

$$\widetilde{MB}\left(\beta_{ijt},\beta_{jt}\right)\Big|_{\beta_{ijt}=\beta_{jt}} \propto \exp\left\{\left(\underbrace{\frac{\alpha\theta-\theta+1}{\left(1-\alpha\right)\alpha\theta}}_{\text{Aggregate Demand}}+\underbrace{\frac{\alpha}{1-\alpha}}_{\text{Absolute Size}}+\underbrace{1}_{\text{Log-Normal Scaling}}\right)a_t\right\}, \tag{57}$$

with an overall positive effect for $\theta < \frac{1}{1-2\alpha}$.

Proof of Lemma 4. *(i)* Assume $a_t = 0$ without loss of generality. The marginal benefit to increasing β_{ijt} is

$$\widetilde{MB}\left(\beta_{ijt},\beta_{jt}\right)\Big|_{\beta_{ijt}=\beta_{jt}} = \int_{-\infty}^{\infty} g\left(a_{jt}\right)\int_{-\infty}^{\infty} f\left(\varepsilon_{jt}\right)\frac{\partial\mathcal{P}\left\{x_{ijt}=2\right\}}{\partial\beta_{ijt}}\Big|_{\beta_{ijt}=\beta_{jt}}\alpha A_t^{\alpha_Y} \tag{58}$$
$$*\,\frac{\left(A_{jt}-\mathbb{E}\left\{A_{jt}|s_{ijt}=z_{jt}z_{jt}\right\}\right)\tilde{\mathbb{E}}\left\{A_{jt}|s_{ijt}=z_{jt},z_{jt}\right\}^{\theta-1}}{\left(\int_0^1\tilde{\mathbb{E}}\left\{A_{jt}|s_{ijt}=z_{jt},z_{jt}\right\}^{\theta}dj\right)^{\frac{\theta-1}{\theta}}}K_{t+1}^{\alpha}d\varepsilon_{jt}da_{jt},$$

where $g\left(a_{jt}\right)$ is the pdf of $a_{jt}\sim\mathcal{N}\left(0,\sigma_a^2\right)$ and $f\left(\varepsilon_{jt}\right)$ is the pdf of $\varepsilon_{jt}\sim\mathcal{N}\left(\varepsilon_t,\sigma_\varepsilon^2\right)$. The most immediate effect comes from changes to aggregate investment K_{t+1}^{α}. For $\delta R_{t+1}=1$,

$$K_{t+1}^{\alpha}=\left(\alpha\delta A_t^{\alpha_Y}\left(\int_0^1\tilde{\mathbb{E}}\left\{A_{jt}|s_{ijt}=z_{jt},z_{jt}\right\}^{\theta}dj\right)^{\frac{1}{\theta}}\right)^{\frac{\alpha}{1-\alpha}}\propto\exp\left\{\frac{\alpha}{1-\alpha}\omega_{s\varepsilon}\varepsilon_t\right\}. \tag{59}$$

The *Absolute Size* channel is summarized by $\exp\left\{\frac{\alpha}{1-\alpha}\omega_{s\varepsilon}\varepsilon_t\right\}$. Next, the derivative of the probability of buying at $\beta_{ijt}=\beta_{jt}$ is

$$\frac{\partial\mathcal{P}\left\{x_{ijt}=2\right\}}{\partial\beta_{ijt}}\Big|_{\beta_{ijt}=\beta_{jt}}=\phi\left(\varepsilon_{jt}\right)\left(\frac{\omega_{p,jt}}{\sqrt{\beta_{jt}}}a_{jt}+\frac{\varepsilon_{jt}}{2\beta_{jt}}-\frac{\omega_{\varepsilon,jt}\varepsilon_{jt}-\omega_{z\varepsilon,jt}\varepsilon_t}{\sqrt{\beta_{jt}}}+\frac{1}{2}\frac{\mathbb{V}_{jt}}{\sqrt{\beta_{jt}}}\right), \tag{60}$$

where $\phi\left(\cdot\right)$ is the standard-normal pdf. Combine $f\left(\varepsilon_{jt}\right)$ with $\phi\left(\varepsilon_{jt}\right)$ using Lemma 3,

$$\phi\left(\varepsilon_{jt}\right)f\left(\varepsilon_{jt}\right)=\exp\left\{-\frac{\varepsilon_t^2}{2\left(1+\sigma_\varepsilon^2\right)}\right\}\sqrt{\frac{1}{2\pi\left(1+\sigma_\varepsilon^2\right)}}\tilde{f}(\varepsilon_{jt}), \tag{61}$$

where $\tilde{f}\left(\varepsilon_{jt}\right)$ is the pdf of a fictional variable $\varepsilon_{jt}\sim\mathcal{N}\left(\frac{\varepsilon_t}{1+\sigma_\varepsilon^2},\frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}\right)$. The *Information-*

Sensitivity channel is summarized by $\exp\left\{-\frac{\varepsilon_t^2}{2(1+\sigma_\varepsilon^2)}\right\}$. For the rest of the proof, use

$$\varepsilon_{jt} = \sqrt{\frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}}x + \frac{\varepsilon_t}{1+\sigma_\varepsilon^2} \tag{62}$$

$$d\varepsilon_{jt} = \sqrt{\frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}}dx. \tag{63}$$

Substitute ε_{jt} out of the terms in parenthesis for $\left.\frac{\partial \mathcal{P}\{x_{ijt}=2\}}{\partial \beta_{ijt}}\right|_{\beta_{ijt}=\beta_{jt}}$ leads to

$$\frac{\omega_{p,jt}}{\sqrt{\beta_{jt}}}a_{jt} + \left(\frac{1}{2\beta_{jt}} - \frac{\omega_{\varepsilon,jt}}{\sqrt{\beta_{jt}}}\right)\sqrt{\frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}}x + \frac{\varepsilon_t}{2\beta_{jt}\left(1+\sigma_\varepsilon^2\right)} + \frac{1}{2}\frac{\mathbb{V}_{jt}}{\sqrt{\beta_{jt}}}. \tag{64}$$

Substituting ε_{jt} out of $\tilde{\mathbb{E}}\left\{A_{jt}|s_{ijt}=z_{jt},z_{jt}\right\}$,

$$\begin{aligned}
\tilde{\mathbb{E}}\left\{A_{jt}|s_{ijt}=z_{jt},z_{jt}\right\} &= \exp\left\{\omega_a a_{jt} + \omega_\varepsilon \varepsilon_{jt} - \omega_{z\varepsilon}\varepsilon_t + \frac{1}{2}\mathbb{V}\right\} \\
&= \exp\left\{\omega_a a_{jt} + \omega_\varepsilon\sqrt{\frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}}x + \cancel{\omega_\varepsilon\frac{\varepsilon_t}{1+\sigma_\varepsilon^2} - \omega_{z\varepsilon}\varepsilon_t} + \frac{1}{2}\mathbb{V}\right\} \\
&\overset{\text{Lemma 1 (iii)}}{=} \exp\left\{\omega_a a_{jt} + \omega_\varepsilon\sqrt{\frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}}x + \frac{1}{2}\mathbb{V}\right\}.
\end{aligned} \tag{65}$$

Substitute ε_{jt} out of the firm-specific multiplier for firm capital,

$$\begin{aligned}
&\frac{\tilde{\mathbb{E}}\left\{A_{jt}|s_{ijt}=z_{jt},z_{jt}\right\}^{\theta-1}}{\left(\int_0^1 \tilde{\mathbb{E}}\left\{A_{jt}|s_{ijt}=z_{jt},z_{jt}\right\}^{\theta} dj\right)^{\frac{\theta-1}{\theta}}} \\
&= \exp\left\{(\theta-1)\,\omega_a a_{jt} + (\theta-1)\,\omega_\varepsilon\sqrt{\frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}}x + (\theta-1)\,\omega_\varepsilon\frac{\varepsilon_t}{1+\sigma_\varepsilon^2} - (\theta-1)\,\omega_\varepsilon\varepsilon_t - \frac{(\theta-1)\,\theta}{2}\left(\omega_a^2\sigma_a^2 + \omega_\varepsilon^2\sigma_\varepsilon^2\right)\right\} \\
&\propto \exp\left\{(\theta-1)\,\omega_a a_{jt} + (\theta-1)\,\omega_\varepsilon\sqrt{\frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}}x - (\theta-1)\,\omega_\varepsilon\frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}\varepsilon_t\right\} \\
&= \exp\left\{(\theta-1)\,\omega_a a_{jt} + (\theta-1)\,\omega_\varepsilon\sqrt{\frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}}x - (\theta-1)\,\omega_{s\varepsilon}\varepsilon_t\right\} \\
&= \exp\left\{(\theta-1)\,\omega_a a_{jt} + (\theta-1)\,\omega_\varepsilon\sqrt{\frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}}x\right\}\exp\left\{-(\theta-1)\,\omega_{s\varepsilon}\varepsilon_t\right\},
\end{aligned} \tag{66}$$

where I used Lemma 1 (iii) repeatedly. The *Relative Size* channel is summarized through $\exp\left\{-(\theta-1)\,\omega_{s\varepsilon}\varepsilon_t\right\}$. It remains to show that there are no other terms in $\widetilde{MB}\left(\beta_{ijt},\beta_{jt}\right)$ that depend on ε_t. It is sufficient to show that

$$\int_{-\infty}^{\infty} g\left(a_{jt}\right) \int_{-\infty}^{\infty} \tilde{f}\left(\varepsilon_{jt}\right) \frac{\left(A_{jt} - \mathbb{E}\left\{A_{jt}|s_{ijt} = z_{jt}z_{jt}\right\}\right) \tilde{\mathbb{E}}\left\{A_{jt}|s_{ijt} = z_{jt}, z_{jt}\right\}^{\theta-1}}{\left(\int_0^1 \tilde{\mathbb{E}}\left\{A_{jt}|s_{ijt} = z_{jt}, z_{jt}\right\}^{\theta} dj\right)^{\frac{\theta-1}{\theta}}} d\varepsilon_{jt} da_{jt} \stackrel{!}{=} 0. \tag{67}$$

Substituting ε_{jt} out leads to

$$\begin{aligned}
&\int_{-\infty}^{\infty} g\left(a_{jt}\right) \int_{-\infty}^{\infty} \frac{1}{\cancel{\sqrt{\frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}}}} \phi\left(x\right) \frac{\left(A_{jt} - \mathbb{E}\left\{A_{jt}|s_{ijt} = z_{jt}z_{jt}\right\}\right) \tilde{\mathbb{E}}\left\{A_{jt}|s_{ijt} = z_{jt}, z_{jt}\right\}^{\theta-1}}{\left(\int_0^1 \tilde{\mathbb{E}}\left\{A_{jt}|s_{ijt} = z_{jt}, z_{jt}\right\}^{\theta} dj\right)^{\frac{\theta-1}{\theta}}} \cancel{\sqrt{\frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}}} dx da_{jt} \\
\propto &\int_{-\infty}^{\infty} g\left(a_{jt}\right) \int_{-\infty}^{\infty} \phi\left(x\right) \left(A_{jt} \tilde{\mathbb{E}}\left\{A_{jt}|s_{ijt} = z_{jt}, z_{jt}\right\}^{\theta-1} - \mathbb{E}\left\{A_{jt}|s_{ijt} = z_{jt}z_{jt}\right\}^{\theta}\right) dx da_{jt} \\
= &\int_{-\infty}^{\infty} g\left(a_{jt}\right) \int_{-\infty}^{\infty} \phi\left(x\right) \left(\exp\left\{\left(\left(\theta-1\right)\omega_a + 1\right) a_{jt} + \left(\theta-1\right)\omega_\varepsilon \sqrt{\frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}} x + \frac{\left(\theta-1\right)}{2}\mathbb{V}\right\}\right. \\
&\left. - \exp\left\{\theta\omega_a a_{jt} + \theta\omega_\varepsilon \sqrt{\frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}} x + \frac{\theta}{2}\mathbb{V}\right\}\right) dx da_{jt} \\
\propto &\int_{-\infty}^{\infty} g\left(a_{jt}\right) \int_{-\infty}^{\infty} \phi\left(x\right) \left(\exp\left\{\left(\left(\theta-1\right)\omega_a + 1\right) a_{jt} + \left(\theta-1\right)\omega_\varepsilon \sqrt{\frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}} x - \frac{\mathbb{V}}{2}\right\}\right. \\
&\left. - \exp\left\{\theta\omega_a a_{jt} + \theta\omega_\varepsilon \sqrt{\frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}} x\right\}\right) dx da_{jt} \\
= &\int_{-\infty}^{\infty} g\left(a_{jt}\right) \left(\exp\left\{\left(\left(\theta-1\right)\omega_a + 1\right) a_{jt} + \frac{\left(\theta-1\right)^2 \omega_\varepsilon^2}{2} \frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2} - \frac{\mathbb{V}}{2}\right\} - \exp\left\{\theta\omega_a a_{jt} + \frac{\theta^2\omega_\varepsilon^2}{2} \frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}\right\}\right) da_{jt} \\
= &\exp\left\{\frac{\left(\left(\theta-1\right)\omega_a + 1\right)^2}{2}\sigma_a^2 + \frac{\left(\theta-1\right)^2 \omega_\varepsilon^2}{2} \frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2} - \frac{\mathbb{V}}{2}\right\} - \exp\left\{\frac{\theta^2\omega_a^2}{2}\sigma_a^2 + \frac{\theta^2\omega_\varepsilon^2}{2} \frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}\right\}.
\end{aligned} \tag{68}$$

It remains to show that

$$\left(\left(\theta-1\right)\omega_a + 1\right)^2 \sigma_a^2 + \left(\theta-1\right)^2 \omega_\varepsilon^2 \frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2} - \mathbb{V} \stackrel{!}{=} \theta^2\omega_a^2\sigma_a^2 + \theta^2\omega_\varepsilon^2 \frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}. \tag{69}$$

Using Lemma 1 (i), the LHS is equal to

$$\begin{aligned}
&\left(\left(\theta^2 - 2\theta + 1\right)\omega_a^2 + 2\left(\theta-1\right)\omega_a \cancel{+1}\right)\sigma_a^2 + \left(\theta^2 - 2\theta + 1\right)\omega_\varepsilon^2 \frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2} \cancel{-\sigma_a^2} + \omega_a^2\sigma_a^2 + \omega_\varepsilon^2 \frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2} \\
&= \left(\left(\theta^2 - 2\theta + 2\right)\omega_a^2 + 2\left(\theta-1\right)\omega_a\right)\sigma_a^2 + \left(\theta^2 - 2\theta + 2\right)\omega_\varepsilon^2 \frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2} \\
&= 2\left(\theta-1\right)\omega_a\sigma_a^2 + \left(\theta^2 + 2\left(1-\theta\right)\right)\left(\omega_a^2\sigma_a^2 + \omega_\varepsilon^2 \frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}\right) \\
&\stackrel{\text{Lemma 1 (ii)}}{=} 2\left(\theta-1\right)\omega_a\sigma_a^2 + \left(\theta^2 + 2\left(1-\theta\right)\right)\left(\omega_a\sigma_a^2\right) \\
&= \theta^2\omega_a\sigma_a^2.
\end{aligned} \tag{70}$$

Using Lemma 1 (ii), the RHS is equal to

$$\theta^2\left(\omega_a^2\sigma_a^2 + \omega_\varepsilon^2 \frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}\right) = \theta^2\omega_a\sigma_a^2. \tag{71}$$

Combining both confirms the conjecture. The marginal benefit of information production depends on ε_t only through the multiplicative effects in (59), (61) and (66), such that

$$\widetilde{MB}\left(\beta_{ijt},\beta_{jt}\right)\Big|_{\beta_{ijt}=\beta_{jt}} \propto \exp\left\{\underbrace{\frac{\alpha}{1-\alpha}\omega_{s\varepsilon}\varepsilon_t}_{\text{Absolute Size}}\ \underbrace{-\frac{\varepsilon_t^2}{2\left(1+\sigma_\varepsilon^2\right)}}_{\text{Information Sensitivity}}\ \underbrace{-\left(\theta-1\right)\omega_{s\varepsilon}\varepsilon_t}_{\text{Relative Size}}\right\}. \tag{72}$$

(ii) Follow the same strategy as in *(i)* and use the same expression for the marginal benefit of information production (58). Start with the expressions for aggregate investment, K_{t+1}^{α}, and productivity $A_t^{\alpha_Y}$ in (58). For $\delta R_{t+1}=1$, they are equal to

$$A_t^{\alpha_Y}K_{t+1}^{\alpha} = A_t^{\alpha_Y}\left(\alpha\delta A_t^{\alpha_Y}\left(\int_0^1 \tilde{\mathbb{E}}\left\{A_{jt}|s_{ijt}=z_{jt},z_{jt}\right\}^{\theta}dj\right)^{\frac{1}{\theta}}\right)^{\frac{\alpha}{1-\alpha}}, \tag{73}$$

Recall that $\alpha_Y=\frac{\alpha\theta-\theta+1}{\alpha\theta}$. Then

$$A_t^{\alpha_Y} \propto \exp\left\{\frac{\alpha\theta-\theta+1}{\alpha\theta}a_t\right\} \tag{74}$$

$$\left(\int_0^1 \tilde{\mathbb{E}}\left\{A_{jt}|s_{ijt}=z_{jt},z_{jt}\right\}^{\theta}dj\right)^{\frac{1}{\theta}} \propto \exp\left\{a_t\right\}. \tag{75}$$

Putting both together yields

$$\begin{aligned} A_t^{\alpha_Y}K_{t+1}^{\alpha} &\propto \exp\left\{\frac{\alpha\theta-\theta+1}{\alpha\theta}\left(1+\frac{\alpha}{1-\alpha}\right)a_t+\frac{\alpha}{1-\alpha}a_t\right\} \\ &= \exp\left\{\left(\frac{\alpha\theta-\theta+1}{\left(1-\alpha\right)\alpha\theta}+\frac{\alpha}{1-\alpha}\right)a_t\right\}. \end{aligned} \tag{76}$$

Again, using substitution with

$$a_{jt} = \sqrt{\sigma_a^2}y + a_t \tag{77}$$

$$da_{jt} = \sqrt{\sigma_a^2}dy \tag{78}$$

it follows that

$$A_{jt} \propto \exp\left\{a_t\right\} \tag{79}$$

$$\mathbb{E}\left\{A_{jt}|s_{ijt}=z_{jt}z_{jt}\right\} \propto \exp\left\{a_t\right\}, \tag{80}$$

which yields

$$\frac{\left(A_{jt} - \mathbb{E}\left\{A_{jt}|s_{ijt} = z_{jt}z_{jt}\right\}\right)\tilde{\mathbb{E}}\left\{A_{jt}|s_{ijt} = z_{jt}, z_{jt}\right\}^{\theta-1}}{\left(\int_0^1 \tilde{\mathbb{E}}\left\{A_{jt}|s_{ijt} = z_{jt}, z_{jt}\right\}^{\theta} dj\right)^{\frac{\theta-1}{\theta}}} \propto \exp\left\{a_t\right\}. \tag{81}$$

The change in the buying probability does *not* depend on a_t

$$\left.\frac{\partial \mathcal{P}\left\{x_{ijt} = 2\right\}}{\partial \beta_{ijt}}\right|_{\beta_{ijt}=\beta_{jt}} = \phi\left(\varepsilon_{jt}\right)\left(\frac{\omega_p}{\sqrt{\beta_{jt}}}\sqrt{\sigma_a^2}y + \frac{\varepsilon_{jt}}{2\beta_{jt}} - \frac{\omega_\varepsilon \varepsilon_{jt} - \omega_{z\varepsilon}\varepsilon_t}{\sqrt{\beta_{jt}}} + \frac{\mathbb{V}_{jt}}{2\sqrt{\beta_{jt}}}\right). \tag{82}$$

Since no other terms depend on a_t and I substituted a_{jt} out, the individual terms can be added up,

$$\frac{\alpha\theta - \theta + 1}{(1-\alpha)\alpha\theta} + \frac{\alpha}{1-\alpha} + 1 = \frac{2\alpha\theta - \theta + 1}{(1-\alpha)\alpha\theta}, \tag{83}$$

such that the marginal benefit of information production is

$$\left.\widetilde{MB}\left(\beta_{ijt}, \beta_{jt}\right)\right|_{\beta_{ijt}=\beta_{jt}} \propto \exp\left\{\left(\frac{2\alpha\theta - \theta + 1}{(1-\alpha)\alpha\theta}\right)a_t\right\}, \tag{84}$$

and the term inside the parenthesis is positive whenever $\theta < \frac{1}{1-2\alpha}$. □

Proof of Proposition 3. The cutoff can be derived by using the result from Lemma 4 (i) and taking the derivative with respect to ε_t of the following expression,

$$\frac{\partial}{\partial \varepsilon_t}\left(-\frac{\varepsilon_t^2}{2\left(1+\sigma_\varepsilon^2\right)} - (\theta-1)\,\omega_{s\varepsilon}\varepsilon_t + \frac{\alpha}{1-\alpha}\omega_{s\varepsilon}\varepsilon_t\right) \stackrel{!}{=} 0. \tag{85}$$

Denote $\bar{\varepsilon}$ as the value of ε_t for which the above expression is maximized. Then,

$$\begin{aligned} -\frac{\bar{\varepsilon}}{1+\sigma_\varepsilon^2} - (\theta-1)\,\omega_{s\varepsilon} + \frac{\alpha}{1-\alpha}\omega_{s\varepsilon} = 0 \\ \iff \frac{\bar{\varepsilon}}{1+\sigma_\varepsilon^2} = \left(\frac{1}{1-\alpha} - \theta\right)\omega_{s\varepsilon} \\ \iff \bar{\varepsilon} = \left(1+\sigma_\varepsilon^2\right)\left(\frac{1}{1-\alpha} - \theta\right)\omega_{s\varepsilon}, \end{aligned} \tag{86}$$

where $\omega_{s\varepsilon} = \frac{\sqrt{\beta_t}}{\sigma_a^{-2}+\beta_t\left(1+\sigma_\varepsilon^{-2}\right)}$. For $\varepsilon_t < \bar{\varepsilon}$, information production β_t is increasing in ε_t. For $\varepsilon_t > \bar{\varepsilon}$, information production β_t is decreasing in ε_t. □

Proof of Proposition 4. Follows from Lemma 4 (ii). □

Proof of Proposition 5. *(i)* Using the result from Proposition 3 (ii) and the assumption that $\theta > \frac{1}{1-\alpha}$, it must be that positive sentiment shocks crowd out information production. Moreover, as $\beta^* < \frac{\sigma_a^{-2}}{1+\sigma_\varepsilon^{-2}}$, it must be that the pass-through of aggregate sentiment shocks $\omega_{s\varepsilon}$ decreases in information production. Therefore, the pass-through is smaller when information is endogenous than if information production is fixed at β^*. As a result, sentiment shocks are dampened by information production in financial markets, as less precise information by itself leads to less investment and lowers the pass-through of sentiment shocks.

(ii) $\lim_{\varepsilon_t\to\infty}\sqrt{\beta_t(\varepsilon_t)}\varepsilon_t = 0$ guarantees that the pass-through of sentiment shocks goes faster to zero than the sentiment shock goes to infinity, i.e., the direct effect of sentiment shocks on investment disappears as shocks become arbitrarily large. Moreover, Lemma 4 (i) shows that through the information-sensitivity effect $\lim_{\varepsilon_t\to\infty}\beta_t(\varepsilon_t) = 0$. □

Proof of Proposition 6. Follows from Proposition 4 and Assumption 2. An increase in aggregate productivity encourages more information production, leading to more investment. As a result, productivity shocks are amplified. □

Proof of Proposition 7. Since $\widetilde{MB}\left(\beta_{ijt},\beta_{jt}\right) \propto \kappa_H - \kappa_L$ whereas $MB^{SP}\left(\beta_t\right)$ is not a function of position limits $\{\kappa_H,\kappa_L\}$, it must that $\widetilde{MB}\left(\beta_{ijt},\beta_{jt}\right)\Big|_{\beta_{ijt}=\beta_{jt}=\beta_t} \neq MB^{SP}\left(\beta_t\right)$ for almost all values of β_t. Therefore, the information production in the competitive economy and social planner allocation do not coincide almost everywhere. □

Proof of Proposition 8. The social planner's allocation is given by equalizing the marginal products of capital for each firm given the market signals $\{z_{jt}\}$. The maximization problem of the social planner for firm capital allocation is

$$\max_{K_{j1}} \mathbb{E}\left\{\left(\int_0^1 Y_{j1}^{\frac{\theta-1}{\theta}}dj\right)^{\frac{\alpha\theta}{\theta-1}} | z_{jt}\right\} - R_1^{SP}K_{j1}, \tag{87}$$

for some interest rate R_1^{SP}. The resulting first-order condition for firm capital is

$$K_{j1}^{SP} = \left(\alpha Y_1^{\alpha_Y}\frac{\mathbb{E}\left\{A_{j0}|z_{j0}\right\}}{R_1^{SP}}\right)^{\theta}. \tag{88}$$

Integrating on both sides yields

$$R_1^{SP} = \alpha Y_1^{\alpha_Y}\left(\int_0^1 \mathbb{E}\left\{A_{j0}|z_{j0}\right\}^{\theta}dj\right)^{\frac{1}{\theta}}\left(K_1^{SP}\right)^{-\frac{1}{\theta}}. \tag{89}$$

Substituting R_1^{SP} out of K_{j1}^{SP} yields (45). Plugging (45) into (5) leads to

$$Y_1^{SP} = A_0^{SP}K_1^{\alpha} \tag{90}$$

where

$$A_0^{SP} = \frac{\left(\int_0^1 A_0 \mathbb{E}\left\{A_0|z_0\right\}^{\theta-1} dj\right)^{\frac{\alpha\theta}{\theta-1}}}{\left(\int_0^1 \mathbb{E}\left\{A_0|z_0\right\}^{\theta} dj\right)^{\alpha}} \overset{\text{L.I.E.}}{=} \left(\int_0^1 \mathbb{E}\left\{A_0|z_0\right\}^{\theta} dj\right)^{\frac{\alpha}{\theta-1}}. \tag{91}$$

The analytical expression can be obtained by evaluating the conditional expectations and using the constrained efficient Bayesian weights and posterior uncertainty,

$$\omega_p^{eff} = \frac{\sigma_a^{-2}}{\sigma_a^{-2} + \beta_0\sigma_\varepsilon^{-2}}, \quad \omega_a^{eff} = \frac{\beta_0\sigma_\varepsilon^{-2}}{\sigma_a^{-2} + \beta_0\sigma_\varepsilon^{-2}} \tag{92}$$

$$\omega_\varepsilon^{eff} = \frac{\sqrt{\beta_0}\sigma_\varepsilon^{-2}}{\sigma_a^{-2} + \beta_0\sigma_\varepsilon^{-2}}, \quad \mathbb{V}^{eff} = \frac{1}{\sigma_a^{-2} + \beta_0\sigma_\varepsilon^{-2}} \tag{93}$$

leading to

$$\begin{aligned}
A_0^{SP} &= \left(\int_0^1 \mathbb{E}\left\{A_{j0}|z_{j0}\right\}^{\theta} dj\right)^{\frac{\alpha}{\theta-1}} \\
&= \left(\int_0^1 \left\{\exp\left\{\theta\omega_p^{eff} a_0 + \theta\omega_a^{eff} a_{j0} + \theta\omega_\varepsilon^{eff}(\varepsilon_{j0} - \varepsilon_0) + \theta\frac{\mathbb{V}^{eff}}{2}\right\}\right\} dj\right)^{\frac{\alpha}{\theta-1}} \\
&= \exp\left\{\theta\omega_p^{eff} a_0 + \theta\omega_a^{eff} a_0 + \frac{\theta^2}{2}\left(\omega_a^{eff}\right)^2\sigma_a^2 + \frac{\theta^2}{2}\left(\omega_\varepsilon^{eff}\right)^2\sigma_\varepsilon^2 + \theta\frac{\mathbb{V}^{eff}}{2}\right\}^{\frac{\alpha}{\theta-1}} \\
&= \exp\left\{a_0 + \frac{\theta}{2}\left(\omega_a^{eff}\right)^2\sigma_a^2 + \frac{\theta}{2}\left(\omega_\varepsilon^{eff}\right)^2\sigma_\varepsilon^2 + \frac{\mathbb{V}^{eff}}{2}\right\}^{\frac{\alpha\theta}{\theta-1}} \\
&\overset{\text{Lemma 2 (i) and (ii)}}{=} \exp\left\{a_0 + \frac{1}{2}\left(\theta\omega_a^{eff}\sigma_a^2 + \mathbb{V}^{eff}\right)\right\}^{\frac{\alpha\theta}{\theta-1}} \\
&\overset{\text{Lemma 2 (iii)}}{=} \exp\left\{a_0 + \frac{1}{2}\left(\sigma_a^2 + (\theta-1)\,\omega_a^{eff}\sigma_a^2\right)\right\}^{\frac{\alpha\theta}{\theta-1}} \\
&= \exp\left\{\frac{1}{1-\theta}\left(a_0 + \frac{\sigma_a^2}{2}\right) + \omega_a^{eff}\frac{\sigma_a^2}{2}\right\}^{\alpha\theta}.
\end{aligned} \tag{94}$$

TFP under the efficient allocation of capital can be decomposed into two expressions,

$$\ln A_0^{SP} = \underbrace{\frac{\alpha\theta}{\theta-1}\left(a_0 + \frac{\sigma_a^2}{2}\right)}_{\text{exogenous}} + \underbrace{\alpha\theta\omega_a^{eff}\frac{\sigma_a^2}{2}}_{\text{allocative efficiency}}. \tag{95}$$

It follows that

$$\frac{\partial\omega_a^{eff}}{\partial\beta_0} > 0 \Rightarrow \frac{\partial A_0^{SP}}{\partial\beta_0} > 0. \tag{96}$$

Substituting Y_1^{SP} out of the expression for R_1^{SP} then leads to the interest rate

$$R_1^{SP} = \alpha A_0^{SP} \left(K_1^{SP}\right)^{\alpha-1}, \tag{97}$$

and aggregate investment

$$K_1^{SP} = \min\left\{\left(\alpha\delta A_0^{SP}\right)^{\frac{1}{1-\alpha}}, W_0\right\}. \tag{98}$$

Consumption follows using (98) in (41). Finally, taking K_1^{SP} as given and plugging aggregate capital investment in in Y_1^{SP} in (46) in (SP), (47) follows after taking the derivative of (46) with respect to β_0. □

Proof of Proposition 9. I will show that the decentralized allocations coincide with the social planner's allocations. Households receive from firm j the dividend

$$\hat{\Pi}_{j1} = \tau(z_{j0})\,\Pi_{j1} = \frac{\mathbb{E}\{A_{j0}|z_{j0}\}}{\tilde{\mathbb{E}}\{A_{j0}|s_{ij0}=z_{j0}, z_{j0}\}}\alpha Y_1^{\alpha_Y} A_{j0} K_{j1}^{\frac{\theta-1}{\theta}} \tag{99}$$

and the marginal trader expects the dividend to be

$$\tilde{\mathbb{E}}\left\{\hat{\Pi}_{j1}|s_{ij0}=z_{j0}, z_{j0}\right\} = \alpha Y_1^{\alpha_Y}\mathbb{E}\{A_{j0}|z_{j0}\}\, K_{j1}^{\frac{\theta-1}{\theta}}. \tag{100}$$

The price is using $P_{j0} = K_{j1}$,

$$P_{j0} = \frac{1}{R_1}\tilde{\mathbb{E}}\left\{\hat{\Pi}_{j1}|s_{ij0}=z_{j0}, z_{j0}\right\} = \left(\frac{1}{R_1}\alpha Y_1^{\alpha_Y}\mathbb{E}\{A_{j0}|z_{j0}\}\right)^{\theta}. \tag{101}$$

Integrating on both sides yields the interest rate

$$R_1 = \frac{\int_0^1 \tilde{\mathbb{E}}\left\{\hat{\Pi}_{j1}|s_{ij0}=z_{j0}, z_{j0}\right\} dj}{\int_0^1 P_{j0}dj} = \alpha Y_1^{\alpha_Y}\left(\int_0^1 \mathbb{E}\{A_{j0}|z_{j0}\}^{\theta}\, dj\right)^{\frac{1}{\theta}} K_1^{-\frac{1}{\theta}}, \tag{102}$$

plugging in the interest rate in the expression for the price yields

$$K_{j1}^{DE} = \frac{\mathbb{E}\{A_{j0}|z_{j0}\}^{\theta}}{\int_0^1 \mathbb{E}\{A_{j0}|z_{j0}\}^{\theta}\, dj} K_1 = K_{j1}^{SP}, \tag{103}$$

Finally, following the same steps as in Proposition 8 shows that the interest rate and aggregate investment replicate the social planner's allocation for a given information precision β_0.

□

Proof of Proposition 10. Let the social planner buy $d^{SP} \in (-1,1)$ units of shares in all markets. The market-clearing condition for market j becomes

$$2\left(1-\Phi\left(\sqrt{\beta_{j0}}\left(\hat{s}\left(P_{j0}\right)-a_{j0}\right)-\varepsilon_{j0}\right)\right)=1-d^{SP}. \tag{104}$$

Keeping position limits fixed, the social planner's demand d^{SP} changes the identity of the marginal trader. The marginal trader becomes more optimistic on average if the social planner purchases more assets. The threshold signal becomes,

$$\hat{s}\left(P_{j0},d^{SP}\right)=a_{j0}+\frac{\varepsilon_{j0}+\Phi^{-1}\left(\frac{1+d^{SP}}{2}\right)}{\sqrt{\beta_{j0}}}. \tag{105}$$

It follows immediately that asset purchases or sales with $d^{SP}=2\Phi\left(-\varepsilon_0\right)-1$ ensure that the marginal trader holds unbiased beliefs,

$$\hat{s}\left(P_{j0},d^{SP}\right)=a_{j0}+\frac{\varepsilon_{j0}-\varepsilon_0}{\sqrt{\beta_{j0}}}. \tag{106}$$

It follows that prices are unbiased, and aggregate investment is as if the sentiment shock was absent.

Traders expect to buy in equilibrium whenever $s_{ijt}>\hat{s}\left(P_{j0},d^{SP}\right)$. Asset purchases/sales set the threshold $\hat{s}\left(P_{j0},d^{SP}\right)$ at a level as if the aggregate sentiment shock was $\varepsilon_0=0$, effectively undoing any change to the incentive to produce information. Because the trader thinks that she is unaffected by the sentiment shock and asset purchases / sales correct the sentiment shock, information production reverts to the previous level □

Proof of Proposition 11. (i) Denote $k_{jt+1}=\ln K_{jt+1}$. Using (29) allows writing the variance of the log of firm capital stocks as

$$Var\left(k_{jt+1}\right)=\theta^2 Var\left(\ln\tilde{\mathbb{E}}\left\{A_{jt}|s_{ijt}=z_{jt},z_{jt}\right\}\right)=\frac{\theta^2}{2}\left(\omega_a^2\sigma_a^2+\omega_\varepsilon^2\sigma_\varepsilon^2\right). \tag{107}$$

Which can be expressed as

$$\begin{aligned}\omega_a^2\sigma_a^2+\omega_\varepsilon^2\sigma_\varepsilon^2&=\frac{\beta_{t-1}^2\left(1+\sigma_\varepsilon^{-2}\right)^2\sigma_a^2+\beta_{t-1}\left(1+\sigma_\varepsilon^{-2}\right)^2\sigma_\varepsilon^2}{\left(\sigma_a^{-2}+\beta_{t-1}\left(1+\sigma_\varepsilon^{-2}\right)\right)^2}\\&\propto\frac{\beta_{t-1}^2\sigma_a^2+\beta_{t-1}\sigma_\varepsilon^2}{\sigma_a^{-4}+2\sigma_a^{-2}\beta_{t-1}\left(1+\sigma_\varepsilon^{-2}\right)+\beta_{t-1}^2\left(1+\sigma_\varepsilon^{-2}\right)^2}.\end{aligned} \tag{108}$$

Taking the derivative with respect to β_{t-1} and dropping the denominator leads to the simplified expression

$$2\beta_{t-1}\sigma_a^{-2} + \sigma_a^{-4}\sigma_\varepsilon^2 + \beta_{t-1}^2\left(1-\sigma_\varepsilon^2\right)\left(1+\sigma_\varepsilon^{-2}\right) \tag{109}$$

which is positive for all values of β_{t-1} for $\sigma_\varepsilon^2 \leq 1$.

(ii) Denote $\Delta k_{jt+1} = k_{jt+1} - k_{jt}$. Then

$$\Delta k_{jt+1} = \Delta\theta \ln \tilde{\mathbb{E}}\left\{A_{jt}|s_{ijt} = z_{jt}, z_{jt}\right\} + \Delta \ln \int_0^1 \tilde{\mathbb{E}}\left\{A_{jt}|s_{ijt} = z_{jt}, z_{jt}\right\} dj + \Delta K_{t+1}. \tag{110}$$

Deriving the variance of Δk_{jt+1} across firms yields

$$Var\left(\Delta k_{jt+1}\right) = \theta^2\left(\omega_a^2\sigma_a^2 + \omega_\varepsilon^2\sigma_\varepsilon^2\right) \tag{111}$$

which is monotonically increasing in β_{t-1} for $\sigma_\varepsilon^2 \leq 1$ as in (i). □