Dummit–Foote Exercises

January 31, 2025

1 Chapter 1: Introduction to Groups

1.1 Basic Axioms and Examples

1.2 Diherdral Groups

1.3 Symmetric Groups

1.4 Matrix Groups

Problem 1.4.1. $|\mathrm{GL}_2(\mathbb{F}_2)| = 6$.

Problem 1.4.2.

Problem 1.4.3.

Problem 1.4.4.

Problem 1.4.5.

Problem 1.4.6.

Problem 1.4.7. *For a prime p, $|\mathrm{GL}_2(\mathbb{F}_p)| = p^4 - p^3 - p^2 + p$.*

1.5 The Quaternion Group

1.6 Homomorphisms and Isomorphisms

1.7 Group Actions

2 Chapter 2: Subgroups

3 Chapter 3: Quotient Groups and Homomorphisms

4 Chapter 4: Group Actions

4.1 Group Actions and Permutation Representations

4.2 Groups Acting on Themselves by Left Multiplication - Cayley's Theorem

Problem 4.2.1.

Problem 4.2.2.

Problem 4.2.3.

Problem 4.2.4.

Problem 4.2.5.

Problem 4.2.6.

Problem 4.2.7.

Problem 4.2.8. *Let G be a group. Assume that H is a subgroup of G with index n. Prove that there exists a normal subgroup K of G such that K is a subgroup of H and $|G : K| \leq n!$.*

5 Chapter 5: Direct and Semidirect Products and Abelian Groups

6 Chapter 6: Further Topics in Group Theory

7 Chapter 7: Introduction to Rings

7.1 Basic Definitions and Examples

Problem 7.1.1. *Let R be a ring with 1. Then $(-1)^2 = 1$ in R.*

Problem 7.1.2. *Let R be a ring with 1. If $u \in R$ is a unit, then $-u$ is also a unit.*

8 Chapter 8: Euclidean Domains, Principal Ideal Domains, and Unique Factorization Domains

9 Chapter 9: Polynomial Rings

10 Chapter 10: Introduction to Module Theory

11 Chapter 11: Vector Spaces

12 Chapter 12: Module Theory over Principal Ideal Domains

13 Chapter 13: Field Theory

14 Chapter 14; Galois Theory

15 Chapter 15: Commutative Rings and Algebraic Geometry

16 Chapter 16: Artinian Rings, Discrete Valuation Rings, and Dedekind Domains

17 Chapter 17: Introduction to Homological Algebra and Group Cohomology

18 Chapter 18: Representation Theory and Character Theory

18.1 Linear Actions and Modules over Group Rings

Problem 18.1.1. *Let G be a finite group and F be a field. Prove that if $\rho : G \to \mathrm{GL}(V)$ is a representation, then ρ gives a faithful representation of $G/\ker\rho$.*

18.2 Wedderburn's Theorem and Some Consequeces

18.3 Character Theory and the Orthogonality Relations

Problem 18.3.1. *Prove that* $\operatorname{tr} AB = \operatorname{tr} BA$ *for $n \times n$ matrices A and B with entries from any commutative ring.*

Problem 18.3.2.

Problem 18.3.3.

Problem 18.3.4.

Problem 18.3.5.

Problem 18.3.6. *Let G be a finite group. Let $\phi : G \to \mathrm{GL}(V)$ be a representation with character ψ over $\mathbb{C}$. Let W be the subspace $\{v \in V \mid \phi(g)(v) = v \text{ for all } g \in G\}$ of V fixed pointwise by all elements of G. Prove that* $\dim W = (\psi, \chi_1)$*, where χ_1 is the principal character of G.*

19 Chapter 19: Examples and Applications of Character Theory