MATH50003 Numerical Analysis

VI.2 Classical Orthogonal Polynomials

Dr Sheehan Olver

Part VI

Orthogonal Polynomials

1. General Orthogonal Polynomials and basic properties
2. Classical Orthogonal Polynomials with special structure
3. Gaussian Quadrature for high-accuracy integration

VI.2 Classical Orthogonal Polynomials
Special families of orthogonal polynomials

- There are 3 classical weights: Jacobi, Laguerre, and Hermite
  - With 3 special types of Jacobi: Ultraspherical, Chebyshev, Legendre
- Classical OPs can be defined by special properties:
  - Derivatives are also OPs
  - Eigenfunctions of simple differential operators
- Classical OPs are fundamental in computations
  - Especially Chebyshev!

Table 18.3.1: Orthogonality properties for classical OP's: intervals, weight functions, normalizations, leading coefficients, and parameter constraints. In the second row $\mathcal{A}_n$ denotes $2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\big/((2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)n!)$, with $\mathcal{A}_0 = 2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)\big/\Gamma(\alpha+\beta+2)$. For further implications of the parameter constraints see the *Note* in §18.5(iii).

weight   normalisation

Name	$p_n(x)$	(a,b)	$w(x)$	h_n	k_n	$\widetilde{k}_n/k_n$	Constraints
Jacobi	$P_n^{(\alpha,\beta)}(x)$	$(-1,1)$	$(1-x)^\alpha(1+x)^\beta$	$\mathcal{A}_n$	$\dfrac{(n+\alpha+\beta+1)_n}{2^n n!}$	$\dfrac{n(\alpha-\beta)}{2n+\alpha+\beta}$	$\alpha,\beta>-1$
Ultraspherical (Gegenbauer)	$C_n^{(\lambda)}(x)$	$(-1,1)$	$(1-x^2)^{\lambda-\frac12}$	$\dfrac{2^{1-2\lambda}\pi\Gamma(n+2\lambda)}{(n+\lambda)(\Gamma(\lambda))^2 n!}$	$\dfrac{2^n(\lambda)_n}{n!}$	0	$\lambda>-\frac12, \lambda\neq 0$
Chebyshev of first kind	$T_n(x)$	$(-1,1)$	$(1-x^2)^{-\frac12}$	$\begin{cases}\frac12\pi, & n>0\\ \pi, & n=0\end{cases}$	$\begin{cases}2^{n-1}, & n>0\\ 1, & n=0\end{cases}$	0	
Chebyshev of second kind	$U_n(x)$	$(-1,1)$	$(1-x^2)^{\frac12}$	$\frac12\pi$	2^n	0	
Chebyshev of third kind	$V_n(x)$	$(-1,1)$	$(1-x)^{-\frac12}(1+x)^{\frac12}$	π	2^n	$-\frac12$	
Chebyshev of fourth kind	$W_n(x)$	$(-1,1)$	$(1-x)^{\frac12}(1+x)^{-\frac12}$	π	2^n	$\frac12$	
Shifted Chebyshev of first kind	$T_n^*(x)$	$(0,1)$	$(x-x^2)^{-\frac12}$	$\begin{cases}\frac12\pi, & n>0\\ \pi, & n=0\end{cases}$	$\begin{cases}2^{2n-1}, & n>0\\ 1, & n=0\end{cases}$	$-\frac12 n$	
Shifted Chebyshev of second kind	$U_n^*(x)$	$(0,1)$	$(x-x^2)^{\frac12}$	$\frac18\pi$	2^{2n}	$-\frac12 n$	
Legendre	$P_n(x)$	$(-1,1)$	1	$2/(2n+1)$	$2^n\left(\frac12\right)_n\big/n!$	0	
Shifted Legendre	$P_n^*(x)$	$(0,1)$	1	$1/(2n+1)$	$2^{2n}\left(\frac12\right)_n\big/n!$	$-\frac12 n$	
Laguerre	$L_n^{(\alpha)}(x)$	$(0,\infty)$	$e^{-x}x^\alpha$	$\Gamma(n+\alpha+1)\big/n!$	$(-1)^n/n!$	$-n(n+\alpha)$	$\alpha>-1$
Hermite	$H_n(x)$	$(-\infty,\infty)$	e^{-x^2}	$\pi^{\frac12}2^n n!$	2^n	0	
Hermite	$He_n(x)$	$(-\infty,\infty)$	$e^{-\frac12 x^2}$	$(2\pi)^{\frac12}n!$	1	0	

$U_n(x) = 2^n x^n + O(x^{n-1})$

Past exam (unseen)

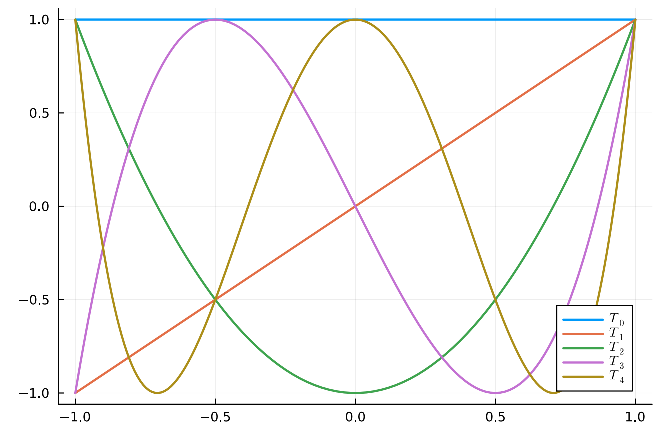
PS

VI.2.1 Chebyshev polynomials

Simple OPs that are Fourier in disguise

Definition 39 (Chebyshev polynomials, 1st kind)**.** $T_n(x)$ are orthogonal with respect to $\underbrace{1/\sqrt{1-x^2}}_{w(x)}$ and satisfy:

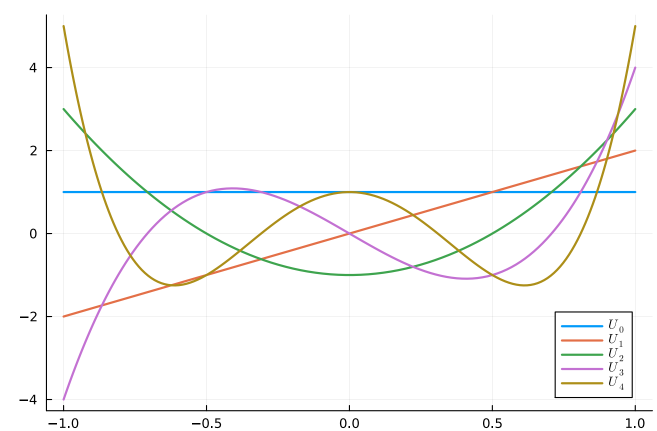
$$
\begin{aligned}
T_0(x) &= 1, \\
T_n(x) &= \underbrace{2^{n-1}}_{k_n} x^n + O(x^{n-1})
\end{aligned}
$$



w, k_n uniquely define $T_n(x)$

Definition 40 (Chebyshev polynomials, 2nd kind)**.** $U_n(x)$ are orthogonal with respect to $\sqrt{1-x^2}$ and satisfy:

$$U_n(x) = 2^n x^n + O(x^{n-1})$$



Theorem 16 (Chebyshev T are cos). *For* $-1 \leq x \leq 1$

$$T_n(x) = \cos(n \operatorname{acos} x)$$

In other words

$$T_n(\cos\theta) = \cos(n\theta).$$

If $f(x) = \sum_{k=0}^{\infty} \check{f}_k T_k(x)$

then

$$f(\cos\theta) = \sum_{k=0}^{\infty} \check{f}_k \cos k\theta$$

<u>Proof</u> <u>Theorem 14</u> (Ortho to lower degree) says if we show

$$P_n(x) := \cos n \operatorname{acos} x$$

are OPs if

(1) graded polys

(2) ortho wrt $\int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}} dx$

(3) $k_n = 2^{n-1}$

$k_0 = 1$

2 $\langle P_n, P_m \rangle = \int_{-1}^{1} \frac{P_n(x) P_m(x)}{\sqrt{1-x^2}}\,dx = -\int_{\pi}^{0} \frac{\cos n\theta \cos m\theta}{\cancel{\sqrt{1-\cos^2\theta}}} \cancel{\sin\theta}\, d\theta$

$\cancel{\sin\theta}$

$x = \cos\theta$
$dx = -\sin\theta\, d\theta$

$= 0$ if $n \neq m$ (can show via $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$)

1/3 Via 3-term recurrence:

$\underbrace{x P_0(x)}_{\cos(0 \operatorname{acos} x) = 1} = \underbrace{P_1(x)}_{\cos \operatorname{acos} x = x} = x$

trig identity

$x P_n(x) = \cos\theta \cos n\theta = \dfrac{\cos(n-1)\theta + \cos(n+1)\theta}{2}$

$= \dfrac{P_{n-1}(x) + P_{n+1}(x)}{2}$

$$\Rightarrow \quad P_{n+1}(x) = 2x\, P_n(x) - P_{n-1}(x)$$

If $\quad P_n(x) = \underbrace{2^{n-1}}_{k_n} x^n + O(x^{n-1}) \quad$ then

$$P_{n+1}(x) = 2x\left(2^{n-1} x^n + O(x^{n-1})\right) - O(x^{n-1})$$

$$= \underbrace{2^n}_{k_{n+1}} x^{n+1} + O(x^n)$$

Corollary 10 (Chebyshev 3-term recurrence)**.**

$$xT_0(x) = T_1(x)$$
$$xT_n(x) = \frac{T_{n-1}(x) + T_{n+1}(x)}{2}$$

Explicit

Theorem 17 (Chebyshev U are sin)**.** *For* $x = \cos\theta$*,*

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}$$

which satisfy:

$$xU_0(x) = U_1(x)/2$$
$$xU_n(x) = \frac{U_{n-1}(x)}{2} + \frac{U_{n+1}(x)}{2}.$$

$W(x) = \sqrt{1-x^2}$

See PS.

We can build:

$T_0(x) = 1$

$T_1(x) = x$

$T_2(x) = 2x\,T_1(x) - T_0(x)$

$= 2x^2 - 1$

$T_3(x) = 2x\,T_2(x) - T_1(x)$

$= 4x^3 - 3x$

$T_4(x) = 8x^4 - 6x^2$

$- 2x^2 + 1$

VI.2.2 Legendre polynomials

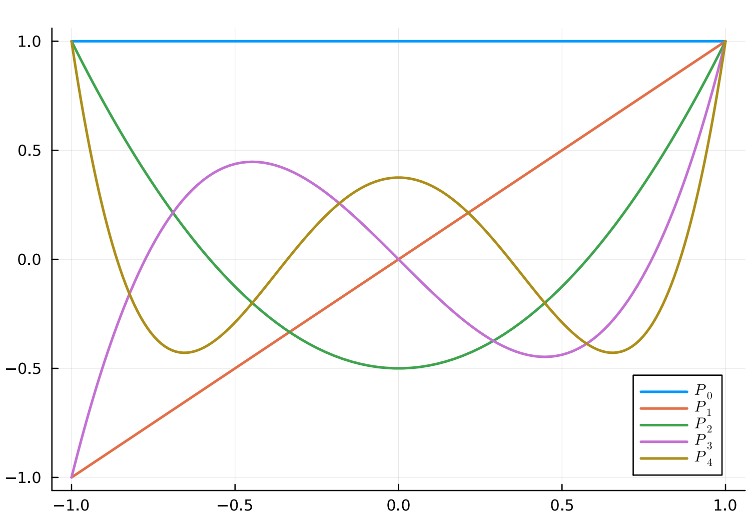
OPs with respect to uniform weight

Definition 41 (Legendre)**.** Legendre polynomials $P_n(x)$ are orthogonal polynomials with respect to $w(x) = 1$ on $[-1, 1]$, with

$$k_n = \frac{1}{2^n}\binom{2n}{n} = \frac{(2n)!}{2^n (n!)^2}$$

Ie

$$P_n(x) = k_n x^n + O(x^{n-1})$$



Lemma 9 (Legendre Rodriguez formula).

$$P_n(x) = \frac{1}{(-2)^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} (1 - x^2)^n = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

Proof

Theorem 12: need to show

(1) graded

(2) orthogonal to all lower degree polys.

(3) $k_n = \frac{1}{2^n} \binom{2n}{n}$

1/3

$$\left(\frac{d}{dx}\right)^n \underbrace{(x^2 - 1)^n}_{x^{2n} + O(x^{2n-2})} = \left(\frac{d}{dx}\right)^{n-1} \left(2n x^{2n-1} + O(x^{2n-3})\right)$$

$$= 2n(2n-1) \cdots (n+1) x^n + O(x^{n-2})$$

$\uparrow$

n-1 more derivatives

$$\frac{(2n)!}{n!} = 2^n n! \, k_n$$

$\checkmark$

(2)

$$\int_{-1}^{1} \left(\frac{d}{dx}\right)^n \underbrace{(x^2-1)^n}_{\substack{\text{vanishes}\\ \text{to } (n-1)\\ \text{derivatives at}\\ \pm 1}} \underbrace{r_m(x)}_{\substack{\text{degree } m<n\\ \text{poly}}} \, dx$$

$$= \cancel{\left[\left(\frac{d}{dx}\right)^{n-1} (x^2-1)^n \, r_m(x) \right]_{-1}^{1}} - \int_{-1}^{1} \left(\frac{d}{dx}\right)^{n-1} r_m'(x)\, dx$$

vanishes at ± 1

$$= (-1)^n \int_{-1}^{1} (x^2-1)^n \underbrace{r_m^{(n)}(x)}_{=0 \text{ since } r_m \text{ is}} \, dx = 0$$

n-1 more times

degree $m<n$.

Lemma 10 (Legendre monomial coefficients).

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_n(x) = \underbrace{\frac{(2n)!}{2^n (n!)^2}}_{k_n} x^n - \underbrace{\frac{(2n-2)!}{2^n (n-2)!(n-1)!}}_{k_n^{(2)}} x^{n-2} + O(x^{n-4}).$$

Proof

$$\left(\frac{d}{dx}\right)^n (x^2-1)^n \underset{\text{Binomial Theorem}}{=} \left(\frac{d}{dx}\right)^n \left[x^{2n} - n x^{2n-2} + O(x^{2n-4})\right]$$

$$= \frac{(2n)!}{n!} x^n - \underbrace{n(2n-2)(2n-3)\cdots(n-1)}_{\frac{n(2n-2)!}{(n-2)!} = k_n^{(2)} 2^n n!} x^{n-2} + O(x^{n-4})$$

Theorem 18 (Legendre 3-term recurrence). (simple rational 3-term recurrence)

$$xP_0(x) = P_1(x)$$
$$(2n+1)xP_n(x) = nP_{n-1}(x) + (n+1)P_{n+1}(x)$$

<u>Proof</u> Consider

$$r_n(x) := (2n+1)\,x\,P_n(x) - n\,P_{n-1}(x) - (n+1)P_{n+1}(x)$$

First show degree $n-2$. Because then

$$\langle r_n, P_k\rangle = (2n+1)\langle P_n, \overset{\text{degree } k+1}{\overbrace{x P_k}}\rangle - n\langle P_{n-1}, P_k\rangle - (n+1)\langle P_{n+1}, P_k\rangle$$

$$= 0 \text{ if } k \leq n-1 \Rightarrow r_n \equiv 0.$$

We have $\underbrace{\qquad\qquad\qquad}_{=0}$

$$r_n(x) = \left[(2n+1)k_n - (n+1)k_{n+1}\right]x^{n+1} +$$

$$\underbrace{\left[(2n+1)k_n^{(2)} - n k_{n-1} - (n+1)k_{n+1}^{(2)}\right]}_{=0 \quad \text{(exercise)}} x^{n-1} + O(x^{n-2})$$

$\Rightarrow$ r_n is degree $n-2$ $\blacksquare$

Q: How to compute

$$f(x) = \sum_{k=0}^{\infty} c_k P_k(x) \; ?$$

$$c_k = \frac{\langle f, P_k \rangle}{\|P_k\|^2}$$