Chapter V

Numerical Fourier series

So far, we have seen intuitive numerical methods for computing derivatives, integrals, and solving differential equations, primarily based on representing functions by their values at a grid of points. But by using more sophisticated mathematical tools, we can achieve much more accurate and reliable numerical methods. In particular, we can effectively use Fourier series for computing very accurately with periodic functions, and orthogonal polynomials for non-periodic functions that are smooth within an interval. Here we introduce these fundamental tools and explore applications to quadrature (computing integrals) where they produce incredibly accurate approximations, ones that converge exponentially (or faster) for analytic functions.

1. IV.1 Fourier Expansions: we discuss Fourier series and their usage in approximating periodic functions, using the Trapezium rule to compute the Fourier coefficients.

2. IV.2 Discrete Fourier Transform: The Trapezium rule approximation can be recast as a unitary matrix, known as the Discrete Fourier Transform (DFT). This is used to prove interpolation properties.

V.1 Fourier Expansions

Fourier series are a powerful tool in wide areas of mathematics, including solving partial differential equations, signal processing, and elsewhere. They are also very useful in computational methods, particularly for problems that have periodicity. Periodicity arises naturally when solving problems in radial coordinates, or when approximating a problem on the real line by a periodic problem with a large period. Fourier series are also related to orthogonal polynomials, which can be used for non-periodic problems.

The most fundamental basis is (complex) Fourier: we have $e^{ik\theta}$ are orthogonal with respect to the inner product

$$\langle f, g \rangle := \frac{1}{2\pi} \int_0^{2\pi} \bar{f}(\theta) g(\theta) \mathrm{d}\theta,$$

where we conjugate the first argument to be consistent with the vector inner product $\boldsymbol{x}^\star \boldsymbol{y}$. We will use the notation $\mathbb{T} := [0, 2\pi)$ (typically this has the topology of a circle attached but we do not need to worry about that here). We can (typically) expand functions in this basis:

Definition 29 (Fourier)**.** A function f has a Fourier expansion if

$$f(\theta) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{ik\theta}$$

where

$$\hat{f}_k := \langle e^{ik\theta}, f\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\theta} f(\theta) d\theta$$

A basic observation is if a Fourier expansion has no negative terms it is equivalent to a Taylor series in disguise:

Definition 30 (Fourier-Taylor)**.** A function f has a Fourier–Taylor expansion if

$$f(\theta) = \sum_{k=0}^{\infty} \hat{f}_k e^{ik\theta} = \sum_{k=0}^{\infty} \hat{f}_k z^k$$

where $\hat{f}_k := \langle e^{ik\theta}, f\rangle$, and $z = e^{i\theta}$.

In numerical analysis we try to build on the analogy with linear algebra as much as possible. Therefore we can write this this as:

$$f(\theta) = \underbrace{[1|e^{i\theta}|e^{2i\theta}|\cdots]}_{T(\theta)} \underbrace{\begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \\ \vdots \end{bmatrix}}_{\hat{\boldsymbol{f}}}.$$

Essentially, expansions in bases are viewed as a way of turning *functions* into (infinite) *vectors*. And (differential) *operators* into *matrices*.

V.1.1 Convergence of Fourier series

In analysis one typically works with continuous functions and relates results to continuity. In numerical analysis we inherently have to work with *vectors*, so it is more natural to focus on the case where the *Fourier coefficients* $\hat{f}_k$ are *absolutely convergent*:

Definition 31 (absolute convergent)**.** We write $\hat{\boldsymbol{f}} \in \ell^1$ if it is absolutely convergent, or in otherwords, the 1-norm of $\hat{\boldsymbol{f}}$ is bounded:

$$\|\hat{\boldsymbol{f}}\|_1 := \sum_{k=-\infty}^{\infty} |\hat{f}_k| < \infty.$$

We first state a basic results (whose proof is beyond the scope of this module):

Theorem 10 (Fourier series equivalence)**.** *If $f, g : \mathbb{T} \to \mathbb{C}$ are periodic and continuous and $\hat{f}_k = \hat{g}_k$ for all $k \in \mathbb{Z}$ then $f = g$.*

Proof See Körner 2022 (Theorem 2.4). ■

This allows us to prove the following:

Theorem 11 (Absolute converging Fourier series)**.** *If $\hat{\boldsymbol{f}} \in \ell^1$ then*

$$f(\theta) = \sum_{k=-\infty}^{\infty} \hat{f}_k \mathrm{e}^{\mathrm{i}k\theta},$$

which converges uniformly.

Proof

Note that

$$g_n(\theta) := \sum_{k=-n}^{n} \hat{f}_k \mathrm{e}^{\mathrm{i}k\theta}$$

is uniformly-absolutely convergent as $n \to \infty$, that is,

$$\sum_{k=-n}^{n} |\hat{f}_k \mathrm{e}^{\mathrm{i}k\theta}| = \sum_{k=-n}^{n} |\hat{f}_k| \to \|\hat{\boldsymbol{f}}\|_1.$$

This guarantees that $g_n(\theta)$ converges uniformly to a continuous function $g(\theta)$. We have for $n > k$, that the k-th Fourier coefficient of $g_n(\theta)$ equals $\hat{f}_k$. Thus, by the properties of uniform convergence,

$$\hat{f}_k = \lim_{n\to\infty} \hat{f}_k = \lim_{n\to\infty} \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}k\theta} g_n(\theta) \mathrm{d}\theta = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}k\theta} \lim_{n\to\infty} g_n(\theta) \mathrm{d}\theta = \hat{g}_k.$$

Since f and g are continuous and share the same Fourier coefficients, they are equal.

■

When does a function have absolutely convergent Fourier coefficients? We can deduce it from periodic differentiability of the function:

Lemma 7 (differentiability and absolutely convergence)**.** *If $f : \mathbb{T} \to \mathbb{C}$ and f' are periodic and f'' is uniformly bounded, then $\hat{\boldsymbol{f}} \in \ell^1$.*

Proof Integrate by parts twice using the fact that $f(0) = f(2\pi)$, $f'(0) = f'(2\pi)$:

$$\begin{aligned} 2\pi \hat{f}_k &= \int_0^{2\pi} f(\theta) \mathrm{e}^{-\mathrm{i}k\theta} \mathrm{d}\theta = [f(\theta) \frac{\mathrm{e}^{-ik\theta}}{-ik}]_0^{2\pi} + \frac{1}{\mathrm{i}k} \int_0^{2\pi} f'(\theta) \mathrm{e}^{-\mathrm{i}k\theta} \mathrm{d}\theta \\ &= [f'(\theta) \frac{\mathrm{e}^{-\mathrm{i}k\theta}}{(-\mathrm{i}k)^2}]_0^{2\pi} - \frac{1}{k^2} \int_0^{2\pi} f''(\theta) \mathrm{e}^{-\mathrm{i}k\theta} \mathrm{d}\theta \\ &= -\frac{1}{k^2} \int_0^{2\pi} f''(\theta) \mathrm{e}^{-\mathrm{i}k\theta} \mathrm{d}\theta. \end{aligned}$$

Thus uniform boundedness of f'' guarantees $|\hat{f}_k| \leq M|k|^{-2}$ for some M, and we have

$$\sum_{k=-\infty}^{\infty} |\hat{f}_k| \leq |\hat{f}_0| + 2M \sum_{k=1}^{\infty} |k|^{-2} < \infty$$

using the dominant convergence test.

■

This condition can be weakened to Lipschitz continuity but the proof is beyond the scope of this module. Of more practical importance is the other direction: the more times differentiable a function the faster the coefficients decay, and thence the faster Fourier expansions converge. In fact, if a function is smooth and 2π-periodic its Fourier coefficients decay faster than algebraically: they decay like $O(k^{-\lambda})$ for any λ. This will be explored in the problem sheet.

V.1.2 Trapezium rule and discrete Fourier coefficients

Definition 32 (Periodic Trapezium Rule)**.** Let $\theta_j = 2\pi j/n$ for $j = 0, 1, \ldots, n$ denote $n+1$ evenly spaced points over $[0, 2\pi]$. Recall that the *Trapezium rule* over $[0, 2\pi]$ is the approximation:

$$\int_0^{2\pi} f(\theta)\mathrm{d}\theta \approx \frac{2\pi}{n}\left[\frac{f(0)}{2} + \sum_{j=1}^{n-1} f(\theta_j) + \frac{f(2\pi)}{2}\right]$$

But if f is periodic we have $f(0) = f(2\pi)$ and we get the *periodic Trapezium rule*:

$$\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\mathrm{d}\theta \approx \underbrace{\frac{1}{n}\sum_{j=0}^{n-1} f(\theta_j)}_{\Sigma_n[f]}$$

We know that $\mathrm{e}^{\mathrm{i}k\theta}$ are orthogonal with respect to the continuous inner product. The following says that this property is maintained (up to "aliasing") when we replace the continuous integral with a trapezium rule approximation:

Lemma 8 (Discrete orthogonality)**.** *We have:*

$$\sum_{j=0}^{n-1} \mathrm{e}^{\mathrm{i}k\theta_j} = \begin{cases} n & k = \ldots, -2n, -n, 0, n, 2n, \ldots \\ 0 & \textit{otherwise} \end{cases}$$

In other words,

$$\Sigma_n[\mathrm{e}^{\mathrm{i}(k-\ell)\theta}] = \begin{cases} 1 & k - \ell = \ldots, -2n, -n, 0, n, 2n, \ldots \\ 0 & \textit{otherwise} \end{cases}.$$

Proof

Consider $\omega := \mathrm{e}^{\mathrm{i}\theta_1} = \mathrm{e}^{\frac{2\pi\mathrm{i}}{n}}$. This is an n-th root of unity: $\omega^n = 1$. Note that $\mathrm{e}^{\mathrm{i}\theta_j} = \mathrm{e}^{\frac{2\pi\mathrm{i}j}{n}} = \omega^j$.

(Case 1: $k = pn$ for an integer p) We have

$$\sum_{j=0}^{n-1} \mathrm{e}^{\mathrm{i}k\theta_j} = \sum_{j=0}^{n-1} \omega^{kj} = \sum_{j=0}^{n-1} (\omega^{pn})^j = \sum_{j=0}^{n-1} 1 = n$$

(Case 2: $k \neq pn$ for an integer p) Recall that (via a telescoping sum argument)

$$\sum_{j=0}^{n-1} z^j = \frac{z^n - 1}{z - 1}.$$

Then we have

$$\sum_{j=0}^{n-1} \mathrm{e}^{\mathrm{i}k\theta_j} = \sum_{j=0}^{n-1} (\omega^k)^j = \frac{\omega^{kn} - 1}{\omega^k - 1} = 0.$$

where we use the fact that k is not a multiple of n to guarantee that $\omega^k \neq 1$.

■

V.1.3 Convergence of Approximate Fourier expansions

We will now use the Trapezium rule to approximate Fourier coefficients and expansions:

Definition 33 (Discrete Fourier coefficients)**.** Define the Trapezium rule approximation to the Fourier coefficients by:

$$\hat{f}_k^n := \Sigma_n[\mathrm{e}^{-ik\theta} f(\theta)] = \frac{1}{n}\sum_{j=0}^{n-1} \mathrm{e}^{-ik\theta_j} f(\theta_j)$$

A remarkable fact is that the discete Fourier coefficients can be expressed as a sum of the true Fourier coefficients:

Theorem 12 (discrete Fourier coefficients)**.** *If $\hat{\boldsymbol{f}} \in \ell^1$ (absolutely convergent Fourier coefficients) then*

$$\hat{f}_k^n = \cdots + \hat{f}_{k-2n} + \hat{f}_{k-n} + \hat{f}_k + \hat{f}_{k+n} + \hat{f}_{k+2n} + \cdots$$

Proof

$$\hat{f}_k^n = \Sigma_n[f(\theta)\mathrm{e}^{-\mathrm{i}k\theta}] = \sum_{\ell=-\infty}^{\infty} \hat{f}_\ell \Sigma_n[\mathrm{e}^{\mathrm{i}(\ell-k)\theta}]$$

$$= \sum_{\ell=-\infty}^{\infty} \hat{f}_\ell \begin{cases} 1 & \ell - k = \ldots, -2n, -n, 0, n, 2n, \ldots \\ 0 & \text{otherwise} \end{cases}$$

■

Example 20 (Taylor coefficients via Geometric series)**.** Consider the function

$$f(\theta) = \frac{2}{2 - \mathrm{e}^{\mathrm{i}\theta}}$$

Under the change of variables $z = \mathrm{e}^{\mathrm{i}\theta}$ we know for z on the unit circle this becomes (using the geometric series with $z/2$)

$$\frac{2}{2-z} = \sum_{k=0}^{\infty} \frac{z^k}{2^k}$$

i.e., $\hat{f}_k = 1/2^k$ which is absolutely summable:

$$\sum_{k=0}^{\infty} |\hat{f}_k| = f(0) = 2.$$

If we use an n point discretisation we get for $0 \leq k \leq n-1$ (using the geoemtric series with 2^{-n})

$$\hat{f}_k^n = \hat{f}_k + \hat{f}_{k+n} + \hat{f}_{k+n} + \cdots = \sum_{p=0}^{\infty} \frac{1}{2^{k+pn}} = \frac{2^{n-k}}{2^n - 1}$$

Note that as $n \to \infty$, we have $\hat{f}_k^n \to \hat{f}_k$.

Note that there is redundancy:

Corollary 3 (aliasing)**.** *For all $p \in \mathbb{Z}$, $\hat{f}_k^n = \hat{f}_{k+pn}^n$.*

Proof Follows immediately:

$$\hat{f}^n_{k+pn} = \sum_{j=-\infty}^{\infty} \hat{f}_{k+(p+j)n} = \sum_{j=-\infty}^{\infty} \hat{f}_{k+jn} = \hat{f}^n_k.$$

■

In other words if we know $\hat{f}^n_0, \ldots, \hat{f}^n_{n-1}$, we know $\hat{f}^n_k$ for all k via a permutation, for example if $n = 2m+1$ we have

$$\begin{bmatrix} \hat{f}^n_{-m} \\ \vdots \\ \hat{f}^n_{-1} \\ \hat{f}^n_0 \\ \vdots \\ \hat{f}^n_m \end{bmatrix} = \underbrace{\begin{bmatrix} & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \\ 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \end{bmatrix}}_{P_\sigma} \begin{bmatrix} \hat{f}^n_0 \\ \vdots \\ \hat{f}^n_m \\ \hat{f}^n_{m+1} \\ \vdots \\ \hat{f}^n_{n-1} \end{bmatrix}$$

where σ has Cauchy notation (*Careful*: we are using 1-based indexing here):

$$\begin{pmatrix} 1 & 2 & \cdots & m & m+1 & m+2 & \cdots & n \\ m+2 & m+3 & \cdots & n & 1 & 2 & \cdots & m+1 \end{pmatrix}.$$

We can prove *convergence* whenever f has absolutely summable coefficients. We will prove the result here in the special case where the negative coefficients are zero. That is, $\hat{f}^n_0, \ldots, \hat{f}^n_{n-1}$ are approximations of the Fourier–Taylor coefficients.

Theorem 13 (Approximate Fourier-Taylor expansions converge)**.** *If* $0 = \hat{f}_{-1} = \hat{f}_{-2} = \cdots$ *and* $\hat{\boldsymbol{f}}$ *is absolutely convergent then*

$$f_n(\theta) = \sum_{k=0}^{n-1} \hat{f}^n_k \mathrm{e}^{\mathrm{i}k\theta}$$

converges uniformly to $f(\theta)$*.*

Proof

$$|f(\theta) - f_n(\theta)| = |\sum_{k=0}^{n-1} (\hat{f}_k - \hat{f}^n_k)\mathrm{e}^{\mathrm{i}k\theta} + \sum_{k=n}^{\infty} \hat{f}_k \mathrm{e}^{\mathrm{i}k\theta}| = |\sum_{k=n}^{\infty} \hat{f}_k(\mathrm{e}^{\mathrm{i}k\theta} - \mathrm{e}^{\mathrm{i}\mathrm{mod}(k,n)\theta})| \leq 2\sum_{k=n}^{\infty} |\hat{f}_k|$$

which goes to zero as $n \to \infty$. ■

For the general case we need to choose a range of coefficients that includes roughly an equal number of negative and positive coefficients (preferring negative over positive in a tie as a convention):

$$f_n(\theta) = \sum_{k=-\lceil n/2 \rceil}^{\lfloor n/2 \rfloor} \hat{f}_k \mathrm{e}^{\mathrm{i}k\theta}$$

In the problem sheet we will prove this converges provided the coefficients are absolutely convergent.

V.2 Discrete Fourier Transform

In the previous section we explored using the trapezium rule for approximating Fourier coefficients. This is a linear map from function values to coefficients and thus can be reinterpreted as a matrix-vector product, called the the Discrete Fourier Transform. It turns out the matrix is unitary which leads to important properties including interpolation.

Remark A clever way of decomposing the DFT leads to a fast way of applying and inverting it, which is one of the most influencial algorithms of the 20th century: the Fast Fourier Transform. But this is beyond the scope of this module.

V.2.1 The Discrete Fourier transform

Definition 34 (DFT)**.** The *Discrete Fourier Transform (DFT)* is defined as:

$$\begin{aligned}
Q_n &:= \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \mathrm{e}^{-\mathrm{i}\theta_1} & \mathrm{e}^{-\mathrm{i}\theta_2} & \cdots & \mathrm{e}^{-\mathrm{i}\theta_{n-1}} \\ 1 & \mathrm{e}^{-\mathrm{i}2\theta_1} & \mathrm{e}^{-\mathrm{i}2\theta_2} & \cdots & \mathrm{e}^{-\mathrm{i}2\theta_{n-1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \mathrm{e}^{-\mathrm{i}(n-1)\theta_1} & \mathrm{e}^{-\mathrm{i}(n-1)\theta_2} & \cdots & \mathrm{e}^{-\mathrm{i}(n-1)\theta_{n-1}} \end{bmatrix} \\
&= \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \cdots & \omega^{-(n-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \cdots & \omega^{-2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(n-1)} & \omega^{-2(n-1)} & \cdots & \omega^{-(n-1)^2} \end{bmatrix}
\end{aligned}$$

for the n-th root of unity $\omega = \mathrm{e}^{2\pi\mathrm{i}/n}$.

Note that

$$\begin{aligned}
Q_n^\star &= \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \mathrm{e}^{\mathrm{i}\theta_1} & \mathrm{e}^{\mathrm{i}2\theta_1} & \cdots & \mathrm{e}^{\mathrm{i}(n-1)\theta_1} \\ 1 & \mathrm{e}^{\mathrm{i}\theta_2} & \mathrm{e}^{\mathrm{i}2\theta_2} & \cdots & \mathrm{e}^{\mathrm{i}(n-1)\theta_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \mathrm{e}^{\mathrm{i}\theta_{n-1}} & \mathrm{e}^{\mathrm{i}2\theta_{n-1}} & \cdots & \mathrm{e}^{\mathrm{i}(n-1)\theta_{n-1}} \end{bmatrix} \\
&= \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{1} & \omega^{2} & \cdots & \omega^{(n-1)} \\ 1 & \omega^{2} & \omega^{4} & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{(n-1)} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)^2} \end{bmatrix}
\end{aligned}$$

Hence we have

$$\underbrace{\begin{bmatrix} \hat{f}_0^n \\ \vdots \\ \hat{f}_{n-1}^n \end{bmatrix}}_{\hat{\boldsymbol{f}}^n} = \frac{1}{\sqrt{n}} Q_n \underbrace{\begin{bmatrix} f(\theta_0) \\ \vdots \\ f(\theta_{n-1}) \end{bmatrix}}_{\boldsymbol{f}^n}$$

The choice of normalisation constant is motivated by the following:

Proposition 1 (DFT is Unitary) $Q_n \in U(n)$, that is, $Q_n^\star Q_n = Q_n Q_n^\star = I$.

Proof

$$Q_n Q_n^\star = \begin{bmatrix} \Sigma_n[1] & \Sigma_n[e^{i\theta}] & \cdots & \Sigma_n[e^{i(n-1)\theta}] \\ \Sigma_n[e^{-i\theta}] & \Sigma_n[1] & \cdots & \Sigma_n[e^{i(n-2)\theta}] \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_n[e^{-i(n-1)\theta}] & \Sigma_n[e^{-i(n-2)\theta}] & \cdots & \Sigma_n[1] \end{bmatrix} = I$$

■

In other words, Q_n is easily inverted and we also have a map from discrete Fourier coefficients back to values:

$$\sqrt{n} Q_n^\star \hat{\boldsymbol{f}}^n = \boldsymbol{f}^n$$

Example 21 (Computing Sum)**.** Define the following infinite sum (which has no name apparently, according to Mathematica):

$$S_n(k) := \sum_{p=0}^{\infty} \frac{1}{(k+pn)!}$$

We can use the DFT to compute $S_n(0), \ldots, S_n(n-1)$. Consider

$$f(\theta) = \exp(e^{i\theta}) = \sum_{k=0}^{\infty} \frac{e^{ik\theta}}{k!}$$

where we know the Fourier coefficients from the Taylor series of e^z. The discrete Fourier coefficients satisfy for $0 \le k \le n-1$:

$$\hat{f}_k^n = \hat{f}_k + \hat{f}_{k+n} + \hat{f}_{k+2n} + \cdots = S_n(k)$$

Thus we have

$$\begin{bmatrix} S_n(0) \\ \vdots \\ S_n(n-1) \end{bmatrix} = \frac{1}{\sqrt{n}} Q_n \begin{bmatrix} e \\ \exp(e^{2i\pi/n}) \\ \vdots \\ \exp(e^{2i(n-1)\pi/n}) \end{bmatrix}$$

V.2.2 Interpolation

We investigated interpolation and least squares using polynomials at evenly spaced points, observing that there were issues with stability. We now show that the DFT actually gives coefficients that interpolate using Fourier expansions. As the DFT is a unitary matrix multiplication is "stable", i.e. it preserves norms and hence we know it cannot cause the same huge blow-up we saw for polynomials. That is: whilst polynomials are bad for interpolation at evenly spaced points, trigonometric polynomials are great.

The following guarantees that our approximate Fourier series actually interpolates the data:

Corollary 4 (Interpolation)**.**

$$f_n(\theta) := \sum_{k=0}^{n-1} \hat{f}_k^n e^{ik\theta}$$

interpolates f at θ_j:

$$f_n(\theta_j) = f(\theta_j)$$

Proof We have

$$f_n(\theta_j) = \sum_{k=0}^{n-1} \hat{f}_k^n \mathrm{e}^{\mathrm{i}k\theta_j} = \sqrt{n}\boldsymbol{e}_{j+1}^\top Q_n^\star \hat{\boldsymbol{f}}^n = \boldsymbol{e}_{j+1}^\top Q_n^\star Q_n \boldsymbol{f}^n = f(\theta_j).$$

■

Example 22 (DFT versus Lagrange)**.** Consider interpolating $\exp z$ by a polynomial at the points $1, \mathrm{i}, -1, -\mathrm{i}$. We can use Lagrange polynomials:

$$\begin{aligned}
\ell_1(z) &= \frac{(z-\mathrm{i})(z+1)(z+\mathrm{i})}{2(1-\mathrm{i})(1+\mathrm{i})} = \frac{z^3+z^2+z+1}{4} \\
\ell_2(z) &= \frac{(z-1)(z+1)(z+\mathrm{i})}{(\mathrm{i}-1)(\mathrm{i}+1)2\mathrm{i}} = \frac{\mathrm{i}z^3-z^2-\mathrm{i}z+1}{4} \\
\ell_3(z) &= \frac{(z-1)(z-\mathrm{i})(z+\mathrm{i})}{-2(-1-\mathrm{i})(-1+\mathrm{i})} = \frac{-z^3+z^2-z+1}{4} \\
\ell_4(z) &= \frac{(z-1)(z-\mathrm{i})(z+1)}{(-\mathrm{i}-1)(-2\mathrm{i})(-\mathrm{i}+1)} = \frac{-\mathrm{i}z^3-z^2+\mathrm{i}z+1}{4}
\end{aligned}$$

So we get the interpolant:

$$\begin{aligned}
&\mathrm{e}\ell_1(z) + \mathrm{e}^{\mathrm{i}}\ell_2(z) + \mathrm{e}^{-1}\ell_3(z) + \mathrm{e}^{-\mathrm{i}}\ell_4(z) \\
&= \frac{\mathrm{e}+\mathrm{e}^{\mathrm{i}}+\mathrm{e}^{-1}+\mathrm{e}^{-\mathrm{i}}}{4} + \frac{\mathrm{e}-\mathrm{i}\mathrm{e}^{\mathrm{i}}-\mathrm{e}^{-1}+\mathrm{i}\mathrm{e}^{-\mathrm{i}}}{4}z + \frac{\mathrm{e}-\mathrm{e}^{\mathrm{i}}+\mathrm{e}^{-1}-\mathrm{i}\mathrm{e}^{-\mathrm{i}}}{4}z^2 + \frac{\mathrm{e}+\mathrm{i}\mathrm{e}^{\mathrm{i}}-\mathrm{e}^{-1}-\mathrm{i}\mathrm{e}^{-\mathrm{i}}}{4}z^3
\end{aligned}$$

Alternatively we could have deduced this directly from the DFT. In particular, we know the coefficients of the interpolating polynomial must be, for $\omega = \mathrm{i}$ and $f(\theta) = \exp(\mathrm{e}^{\mathrm{i}\theta})$,

$$\begin{bmatrix} \hat{f}_0^4 \\ \hat{f}_1^4 \\ \hat{f}_2^4 \\ \hat{f}_3^4 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\mathrm{i} & -1 & \mathrm{i} \\ 1 & -1 & 1 & -1 \\ 1 & \mathrm{i} & -1 & -\mathrm{i} \end{bmatrix}\begin{bmatrix} \mathrm{e} \\ \mathrm{e}^{\mathrm{i}} \\ \mathrm{e}^{-1} \\ \mathrm{e}^{-\mathrm{i}} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} \mathrm{e}+\mathrm{e}^{\mathrm{i}}+\mathrm{e}^{-1}+\mathrm{e}^{-\mathrm{i}} \\ \mathrm{e}-\mathrm{i}\mathrm{e}^{\mathrm{i}}-\mathrm{e}^{-1}+\mathrm{i}\mathrm{e}^{-\mathrm{i}} \\ \mathrm{e}-\mathrm{e}^{\mathrm{i}}+\mathrm{e}^{-1}-\mathrm{e}^{-\mathrm{i}} \\ \mathrm{e}+\mathrm{i}\mathrm{e}^{\mathrm{i}}-\mathrm{e}^{-1}-\mathrm{i}\mathrm{e}^{-\mathrm{i}} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \cosh 1 + \cos 1 \\ \sinh 1 + \sin 1 \\ \cosh 1 - \cos 1 \\ \sinh 1 - \sin 1 \end{bmatrix}.$$

The interpolation property also applies to the approximation

$$f_n(\theta) = \sum_{k=-\lceil n/2 \rceil}^{\lfloor n/2 \rfloor} \hat{f}_k \mathrm{e}^{\mathrm{i}k\theta}$$

for general Fourier series, which is investigated in the problem sheet.