Office Hours Next Week:

Tuesday 4-5pm

Wednesday 10-11am

(No Thursday)

MATH50003 Numerical Analysis

V.1 Fourier Expansions

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Part V
Numerical Fourier series

1. Fourier Expansions and approximating Fourier coefficients
2. Discrete Fourier Transforms and interpolation

V.1.1 Basics of Fourier series

Expanding functions in trigonometric polynomials

Definition 29 (Fourier)**.** A function f has a Fourier expansion if

$$f(\theta) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{ik\theta}$$

How to truncate?

where

$$\hat{f}_k := \langle e^{ik\theta}, f \rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta} f(\theta) d\theta$$

How to compute?

for

$$\langle f, g \rangle := \frac{1}{2\pi} \int_0^{2\pi} \overline{f(\theta)} g(\theta) d\theta$$

conjugate 1st argument to match

$\vec{x}^* \vec{y}$

Definition 30 (Fourier-Taylor)**.** A function f has a Fourier–Taylor expansion if

(for $0 = \hat{f}_{-1} = \hat{f}_{-2} = \cdots$)

$$f(\theta) = \sum_{k=0}^{\infty} \hat{f}_k \mathrm{e}^{\mathrm{i}k\theta} = \underbrace{\sum_{k=0}^{\infty} \hat{f}_k z^k}_{\text{Taylor on unit circle.}}$$

where $\hat{f}_k := \langle \mathrm{e}^{\mathrm{i}k\theta}, f \rangle$, and $z = \mathrm{e}^{\mathrm{i}\theta}$.

When does the series converge uniformly to $f(\theta)$?

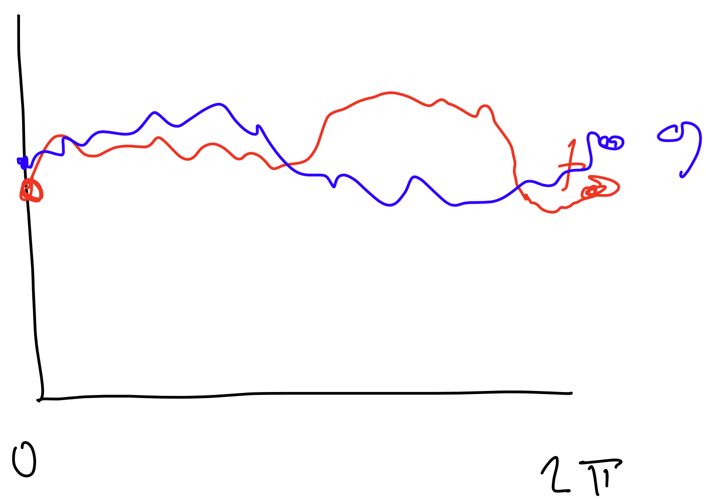
$$[\,-\,\hat{f}_{-2}, \hat{f}_{-1}, \hat{f}_0, \hat{f}_1, \hat{f}_2, \,-\,]^T$$

Definition 31 (absolute convergent)**.** We write $\hat{\boldsymbol{f}} \in \ell^1$ if it is absolutely convergent, or in otherwords, the 1-norm of $\hat{\boldsymbol{f}}$ is bounded:

$$\|\hat{\boldsymbol{f}}\|_1 := \sum_{k=-\infty}^{\infty} |\hat{f}_k| < \infty.$$

Theorem 10 (Fourier series equivalence)**.** *If $f, g : \mathbb{T} \to \mathbb{C}$ are periodic and continuous and $\hat{f}_k = \hat{g}_k$ for all $k \in \mathbb{Z}$ then $f = g$.*

Proof See Körner 2022 (Theorem 2.4). ■



Theorem 11 (Absolute converging Fourier series)**.** *If* $\hat{\boldsymbol{f}} \in \ell^1$ *then*

$$f(\theta) = \sum_{k=-\infty}^{\infty} \hat{f}_k \mathrm{e}^{\mathrm{i}k\theta},$$

which converges uniformly.

Proof

Consider

$$g_n(\theta) := \sum_{k=-n}^{n} \hat{f}_k e^{ik\theta}$$

This is uniformly - absolutely convergent! $\forall \theta \in [0, 2\pi)$

$$|g_n(\theta)| \leq \sum_{k=-n}^{n} |\hat{f}_k \cancel{e^{ik\theta}}| \xrightarrow[n\to\infty]{} \|\hat{f}\|_1 < \infty$$

by assumption,

$$\Rightarrow \quad g_n(\theta) \xrightarrow[\substack{\text{uniform} \\ n\to\infty}]{} \underbrace{g(\theta)}_{\text{some cont. function,}}$$

Note for $n > k$,

$$\hat{(g_n)}_k = \Big\langle e^{ik\theta},\ \underbrace{\sum_{l=-n}^{n} \hat{f}_l e^{il\theta}}_{g_n(\theta)} \Big\rangle$$

$$= \sum_{l=-n}^{n} \hat{f}_l \underbrace{\langle e^{ik\theta}, e^{il\theta} \rangle}_{=\delta_{kl}} = \hat{f}_k$$

$$\Rightarrow \quad \hat{f}_k = \lim_{n\to\infty} \underbrace{\hat{f}_k}_{\hat{(g_n)}_k} = \lim_{n\to\infty} \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta} g_n(\theta)\, d\theta$$

due to uniform convergence

$$= \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta} \, g(\theta) \, d\theta = \hat{g}_k$$

$$\Rightarrow \quad f = g \qquad \blacksquare$$

When is $\|\hat{f}\|_1 < \infty$?

Lemma 7 (differentiability and absolutely convergence)**.** *If $f : \mathbb{T} \to \mathbb{C}$ and f' are periodic and f'' is uniformly bounded, then $\hat{\boldsymbol{f}} \in \ell^1$.*

$f(0) = f(2\pi)$

$f'(0) = f'(2\pi)$

Proof

$$2\pi \hat{f}_k = \int_0^{2\pi} f(\theta) \underbrace{e^{-ik\theta}}_{\frac{d}{d\theta}\left[\frac{e^{-ik\theta}}{-ik}\right]} d\theta$$

$$= \underbrace{\cancel{\left[f(\theta) \frac{e^{-ik\theta}}{-ik} \right]_0^{2\pi}}}_{=0 \text{ since } f(0)=f(2\pi)} + \frac{1}{ik} \int_0^{2\pi} f'(\theta) e^{-ik\theta} d\theta$$

$$\underset{\text{same logic}}{=} -\frac{1}{k^2} \int_0^{2\pi} f''(\theta) e^{-ik\theta} d\theta$$

$$\Rightarrow$$

$$|\hat{f}_k| \le \frac{1}{\cancel{2\pi}} \underbrace{\cancel{2\pi}}_{\text{width}} \frac{1}{k^2} \underbrace{\sup |f''(\theta)\, \cancel{e^{-ik\theta}}|}_{M} \Rightarrow$$

$$\|\hat{f}\|_1 = \sum_{k=-\infty}^{\infty} |\hat{f}_k| \le |\hat{f}_0| + \sum_{k=1}^{\infty} \frac{M}{k^2} + \sum_{k=-\infty}^{-1} \frac{M}{k^2}$$

$$< \infty$$

?

dominant convergence theorem

V.1.2 Trapezium rule and discrete Fourier coefficients

Using the Trapezium rule to approximate coefficients has nice structure

Definition 32 (Periodic Trapezium Rule). (For periodic functions)

$0 = \theta_0 \quad \theta_1 \quad \theta_2 \quad \cdots \quad \theta_n = 2\pi$

$h = \dfrac{2\pi}{n}$ i.e. $\theta_j = jh = \dfrac{2\pi j}{n}$.

Recall $$\int_0^{2\pi} f(\theta)\,d\theta \approx \frac{2\pi}{n}\left[\frac{f(0)}{2} + \sum_{j=1}^{n-1} f(\theta_j) + \frac{f(2\pi)}{2}\right].$$

$= \dfrac{f(0)}{2}$

So define

$$\Sigma_n[f] := \frac{1}{n}\sum_{j=0}^{n-1} f(\theta_j) \approx \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta.$$

Lemma 8 (Discrete orthogonality)**.** *We have:*

$$\sum_{j=0}^{n-1} e^{ik\theta_j} = \begin{cases} n & k = \ldots, -2n, -n, 0, n, 2n, \ldots \\ 0 & \textit{otherwise} \end{cases}$$

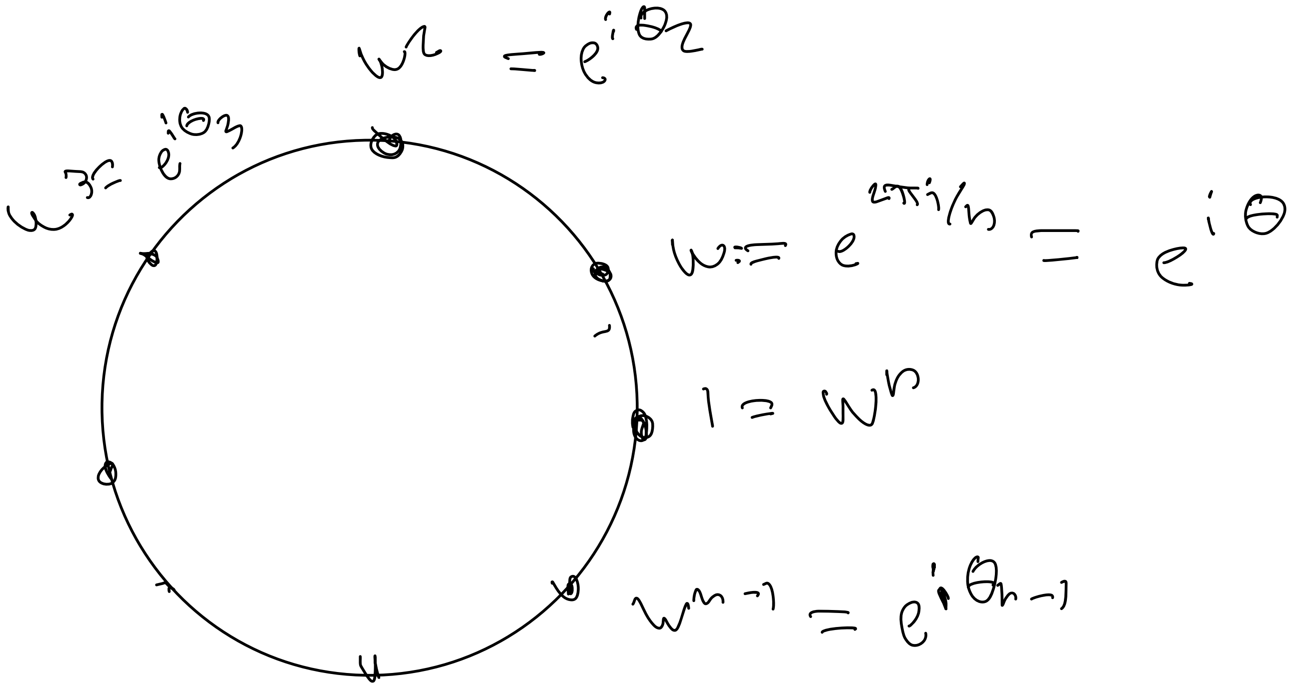
I.e.

$$\delta_{k\ell} = \underbrace{\langle e^{i\ell\theta}, e^{ik\theta} \rangle}_{\int_0^{2\pi} e^{i(k-\ell)\theta}\, d\theta} \approx \Sigma_n\left[e^{i(k-\ell)\theta}\right]$$

$$= \frac{1}{n}\sum_{j=0}^{n-1} e^{i(k-\ell)\theta_j}$$

$$= \begin{cases} 1 & k = \ldots, \ell-n, \ell, \ell+n, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

Proof



Case 1: $k = pn$ for $p \in \mathbb{Z} \Rightarrow$

$$\sum_{j=0}^{n-1} \underbrace{e^{ik\theta_j}}_{(e^{i\theta_j})^k} = \sum_{j=0}^{n-1} (w^j)^k = n.$$

$= w^{jk}$

$= w^{jpn}$

$= (w^p)^{jn} = 1$

Case 2: $k \neq pn$. Recall

$$\sum_{j=0}^{n-1} z^j = \frac{z^n - 1}{z - 1} \Rightarrow$$

$$\sum_{j=0}^{n-1} e^{ik\theta j} = \sum_{j=0}^{n-1} (w^k)^j = \frac{\overbrace{w^{kn}}^{(w^n)^k} - 1}{\underbrace{w^k}_{\neq 1} - 1} = 0$$

$\blacksquare$

V.1.3 Convergence of Approximate Fourier coefficients

Using Trapezium rule leads to a convergent approximation

Definition 33 (Discrete Fourier coefficients)**.** Define the Trapezium rule approximation to the Fourier coefficients by:

$\frac{1}{2\pi}\int_0^{2\pi} e^{-ik\theta} f(\theta)d\theta = \hat{f}_k \approx$

$$\hat{f}_k^n := \Sigma_n[\mathrm{e}^{-ik\theta} f(\theta)] = \frac{1}{n}\sum_{j=0}^{n-1} \mathrm{e}^{-ik\theta_j} f(\theta_j)$$

Then approximate

Fourier - Taylor series! $f(\theta) \approx \underbrace{\sum_{k=0}^{n-1} \hat{f}_k^n e^{ik\theta}}_{f_n(\theta)}$

Fourier series! $f(\theta) \approx \sum^{m} \hat{f}_k^n e^{ik\theta}$

$n = 2m+1$

$\underbrace{\sum_{k=-m}}_{f_{-m:m}(\theta)}$

Theorem 12 (discrete Fourier coefficients). *If $\hat{\boldsymbol{f}} \in \ell^1$ (absolutely convergent Fourier coefficients) then*

$$\underbrace{\hat f_k}_{\text{not computable}} \approx \underbrace{\hat f_k^n}_{\text{computable}} = \cdots + \hat f_{k-2n} + \hat f_{k-n} + \hat f_k + \hat f_{k+n} + \hat f_{k+2n} + \cdots$$

$$\sum_{\ell=-\infty}^{\infty} \hat f_\ell e^{i\ell\theta}$$

Proof

$$\hat f_k^n := \underbrace{\Sigma_n \left[f e^{-ik\theta} \right]}_{= \frac{1}{n}\sum_{j=1}^{n} f(\theta_j) e^{-ik\theta_j}} = \sum_{\ell=-\infty}^{\infty} \hat f_\ell \underbrace{\Sigma_n \left[e^{i(\ell-k)\theta} \right]}_{= \begin{cases} 1 & \ell-k=pn \\ 0 & \text{otherwise} \end{cases}}$$

$$= \cdots + \hat f_{k-n} + \hat f_k + \hat f_{k+n} + \hat f_{k+2n} + \cdots$$

For Fourier–Taylor case ($0 = \hat f_{-1} = \hat f_{-2} = \cdots$), and $0 \le k \le n-1$, then

$$\hat{f}^n_k = \hat{f}_k + \hat{f}_{k+n} + \hat{f}_{k+2n} + \cdots$$

Example 20 (Taylor coefficients via Geometric series).

Consider

$$f(\theta) = \frac{2}{2 - e^{i\theta}} \underset{z = e^{i\theta}}{=} \frac{2}{2 - z}$$

$$= \frac{1}{1 - \underbrace{z/2}_{|z/2| = 1/2}} \underset{\text{geom series}}{=} \sum_{k=0}^{\infty} \left(\frac{z}{2}\right)^k = \sum_{k=0}^{\infty} \underbrace{\frac{1}{2^k}}_{\hat{f}_k} e^{ik\theta}$$

Then for $0 \le k \le n-1$

$$\frac{1}{n} \sum_{j=0}^{n-1} \frac{2}{2 - e^{i\theta_j}} e^{ik\theta_j} = \hat{f}^n_k \underset{\text{Theorem}}{=} \hat{f}_k + \hat{f}_{k+n} + \hat{f}_{k+2n} + \cdots$$

$$= \sum_{p=0}^{\infty} \underbrace{\frac{1}{2^{k+pn}}}_{f_{k+pn}} = \frac{1}{2^k} \frac{1}{1-2^{-n}} = \frac{2^{n-k}}{2^n - 1}$$

$$\Big\downarrow n \to \infty$$

$$2^{-k} = \hat{f}_k$$

Theorem 13 (Approximate Fourier-Taylor expansions converge). *If* $0 = \hat{f}_{-1} = \hat{f}_{-2} = \cdots$ *and* $\hat{\boldsymbol{f}}$ *is absolutely convergent then*

$$f_n(\theta) = \sum_{k=0}^{n-1} \hat{f}_k^n \mathrm{e}^{\mathrm{i}k\theta}$$

converges uniformly to $f(\theta)$.

$\hat{f}_k + \hat{f}_{k+n} + \longrightarrow$

Proof $\quad \forall\, \theta \in [0, 2\pi)$

$$\Big|\underbrace{f(\theta)}_{\sum_{k=0}^{\infty} \hat{f}_k e^{ik\theta}} - f_n(\theta)\Big| = \Big|\sum_{k=0}^{n-1} (\hat{f}_k - \hat{f}_k^n)\, e^{ik\theta}$$

$$+ \sum_{k=n}^{\infty} \hat{f}_k\, e^{ik\theta}\Big|$$

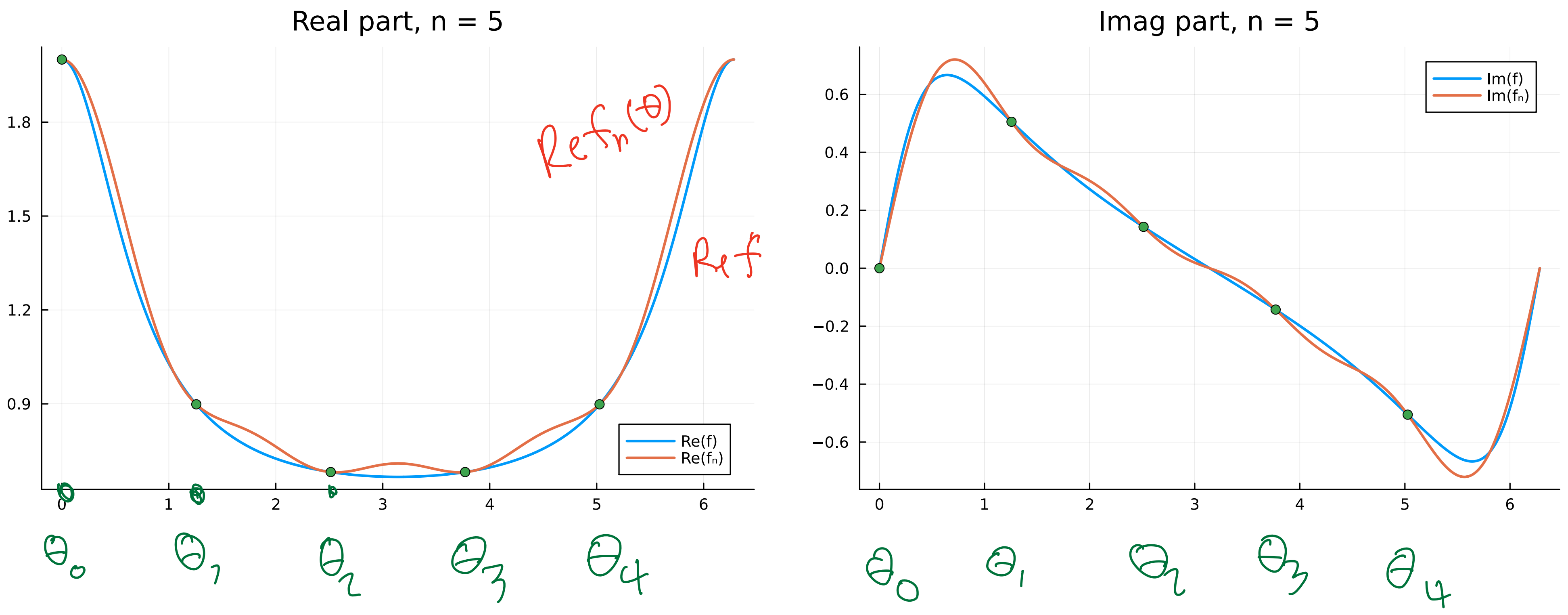
$$= \left| \hat{f}_n \left(e^{in\theta} - 1 \right) + \hat{f}_{n+1} \left(e^{i(n+1)\theta} - e^{i\theta} \right) + \hat{f}_{n+2} \left(e^{i(n+2)\theta} - e^{i2\theta} \right) + \text{———} \right|$$

$$\leq |\hat{f}_n| \underbrace{\left| e^{in\theta} - 1 \right|}_{\leq 2} + |\hat{f}_{n+1}| \underbrace{\left| e^{i(n+1)\theta} - e^{i\theta} \right|}_{\leq 2} + \text{——}$$

$$\leq 2 \sum_{k=n}^{\infty} |\hat{f}_k| \xrightarrow[n \to \infty]{} 0 \quad \text{because}$$

$$\sum_{k=0}^{\infty} |\hat{f}_k| < \infty$$

$$f(\theta) = \frac{2}{2 - \underbrace{e^{i\theta}}_{z}} = \frac{2}{2 - z} = \sum_{k=0}^{\infty} \frac{1}{2^k} z^k$$

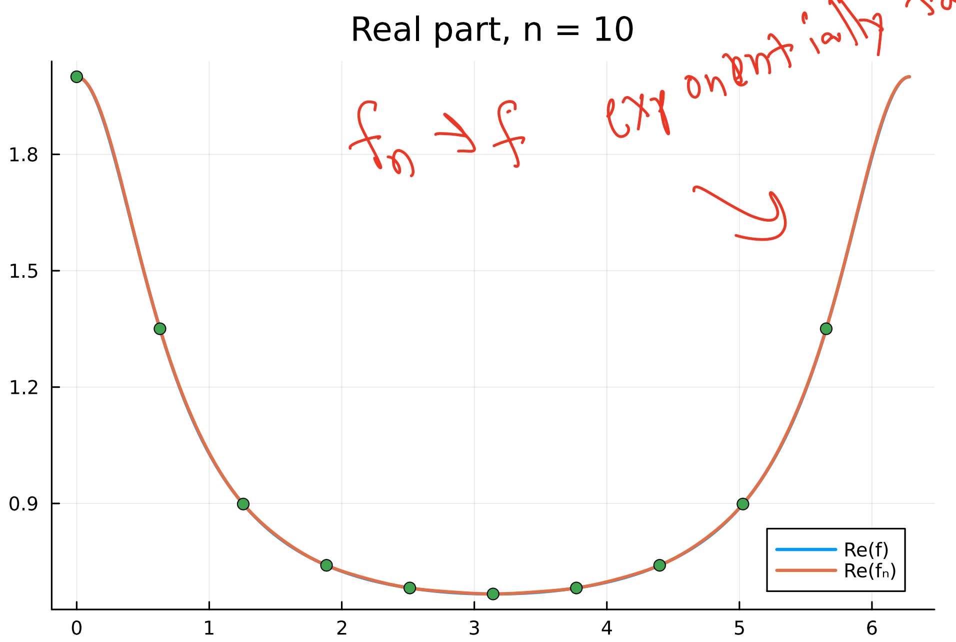


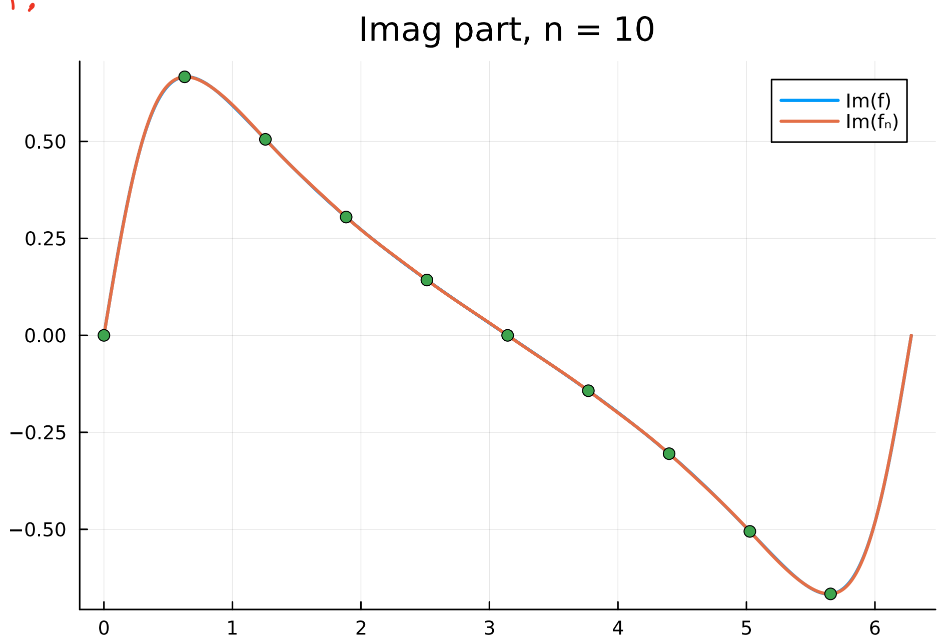
$f_n(\theta)$ interpolates $f(\theta)$ at θ_j.

WHY?

$$f(\theta) = \frac{2}{2 - e^{i\theta}}$$

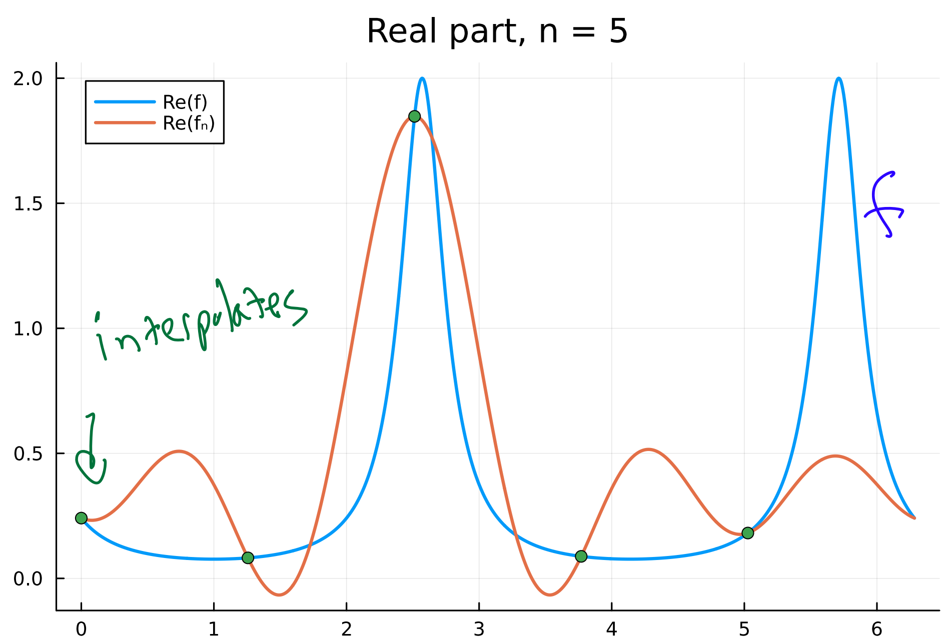
$f_n \to f$ exponentially fast.

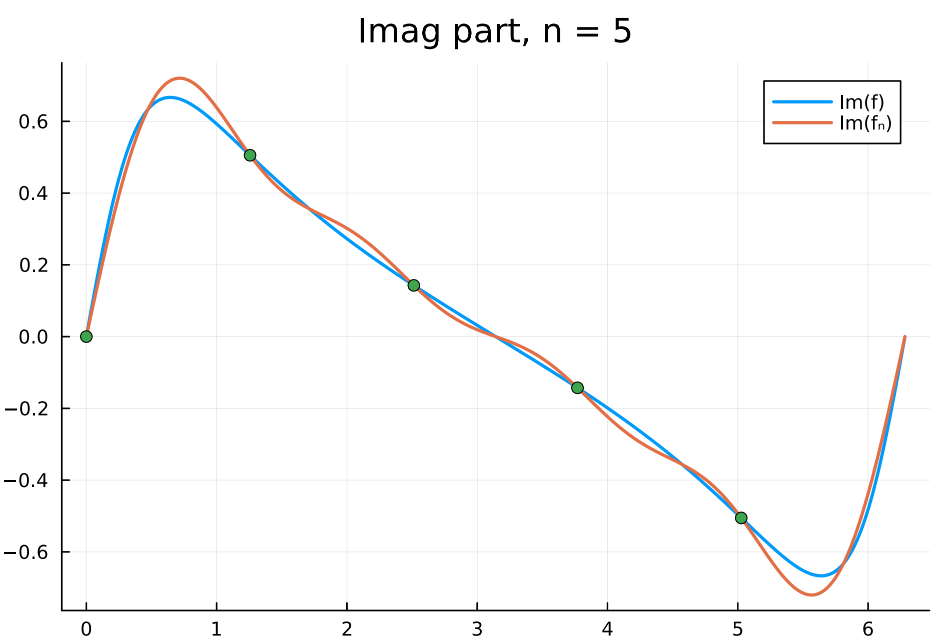




Not Taylor

$$f(\theta) = \frac{2}{25\cos(\theta-1)^2+1} = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{ik\theta}$$

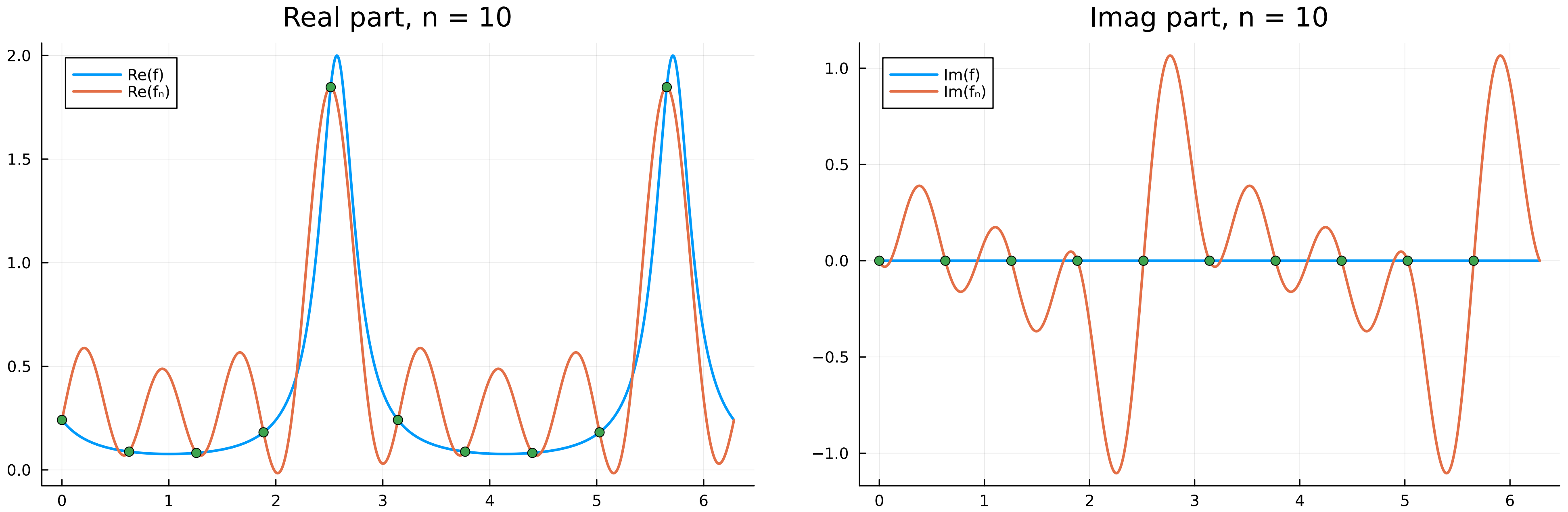




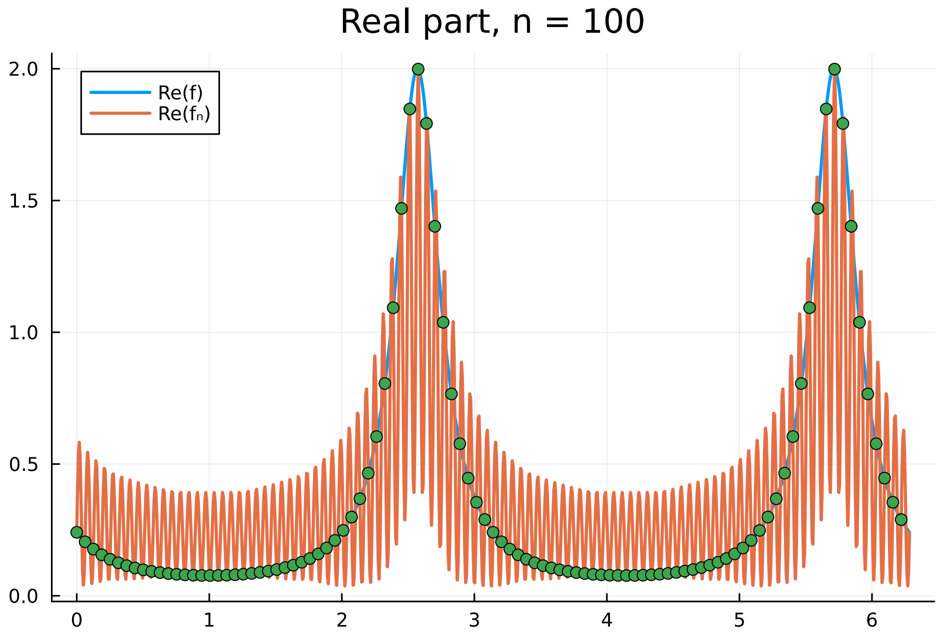
$$f_n(\theta) = \sum_{k=0}^{n-1} \hat{f}_k^n e^{ik\theta}$$

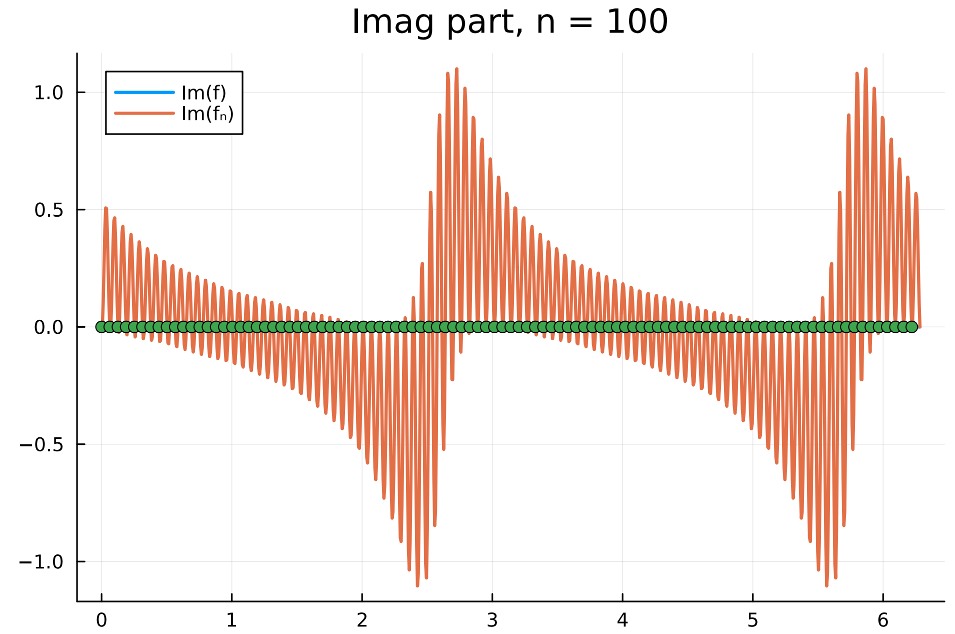
only positive k

$$f(\theta) = \frac{2}{25\cos(\theta - 1)^2 + 1}$$



$$f(\theta) = \frac{2}{25\cos(\theta - 1)^2 + 1}$$





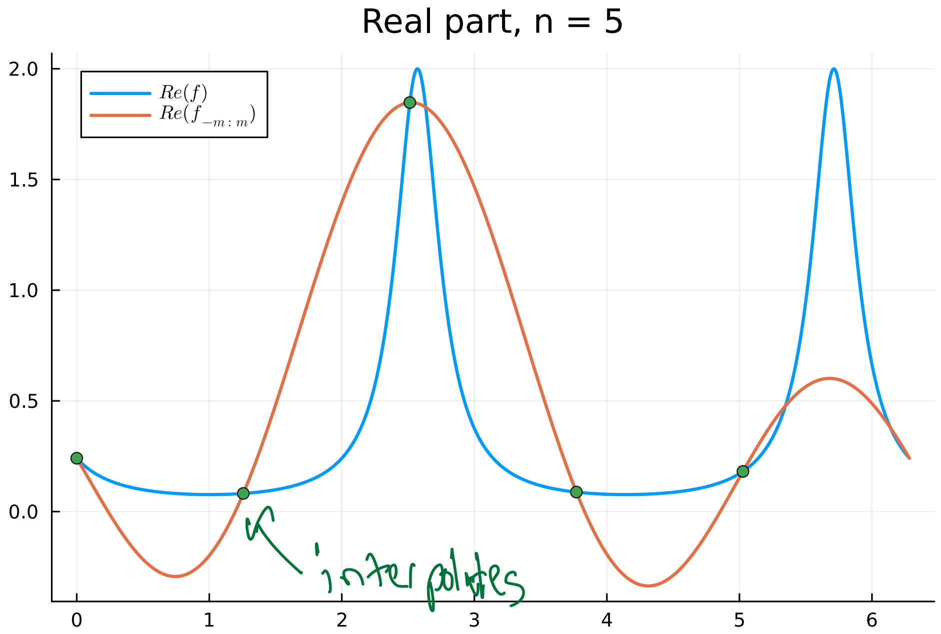
Corollary 3 (aliasing). *For all $p \in \mathbb{Z}$, $\hat{f}^n_k = \hat{f}^n_{k+pn}$.* $= \cdots + \hat{f}_{k-n} + \hat{f}_k + \hat{f}_{k+n} + \cdots$

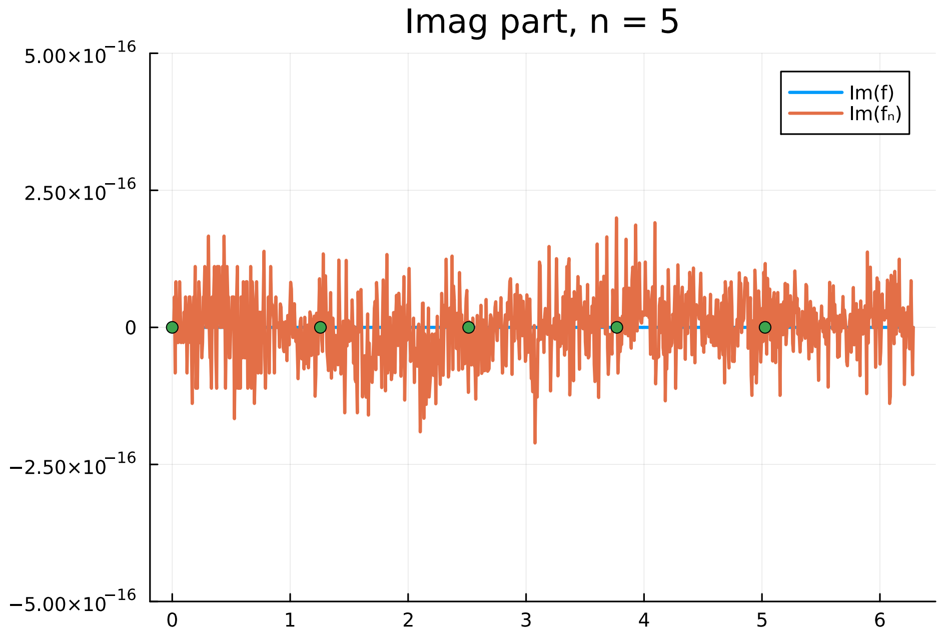
Eg: $\hat{f}^n_{-1} = \hat{f}^n_{n-1}$ For $n = 2m+1$:

$$\underbrace{\begin{bmatrix} \hat{f}^n_{-m} \\ \vdots \\ \hat{f}^n_{-1} \\ \hline \hat{f}^n_0 \\ \vdots \\ \hat{f}^n_m \end{bmatrix}}_{\text{Fourier}} = \left[\begin{array}{ccc|ccc} & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \\ \hline 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \end{array}\right] \underbrace{\begin{bmatrix} \hat{f}^n_0 \\ \hat{f}^n_1 \\ \vdots \\ \hat{f}^n_m \\ \hline \hat{f}^n_{m+1} \\ \vdots \\ \hat{f}^n_{n-1} \end{bmatrix}}_{\text{Taylor}}$$

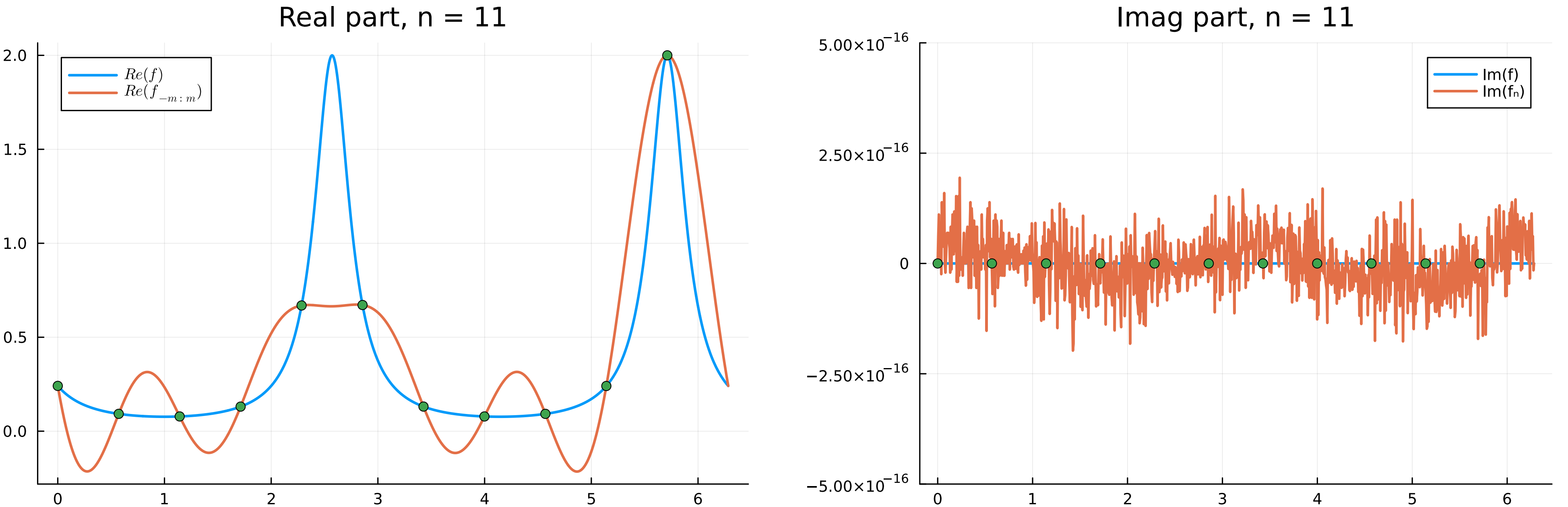
Consider $f_{-m:m}(\theta) := \sum_{k=-m}^{m} \hat{f}_k^n e^{ik\theta}$ where $n = 2m + 1$.

$$f(\theta) = \frac{2}{25\cos(\theta - 1)^2 + 1}$$

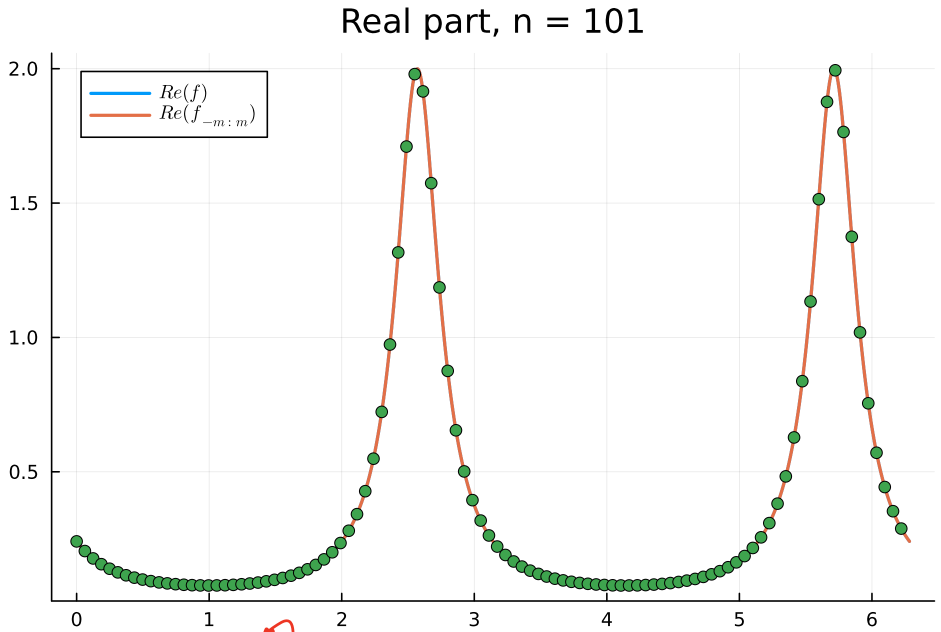


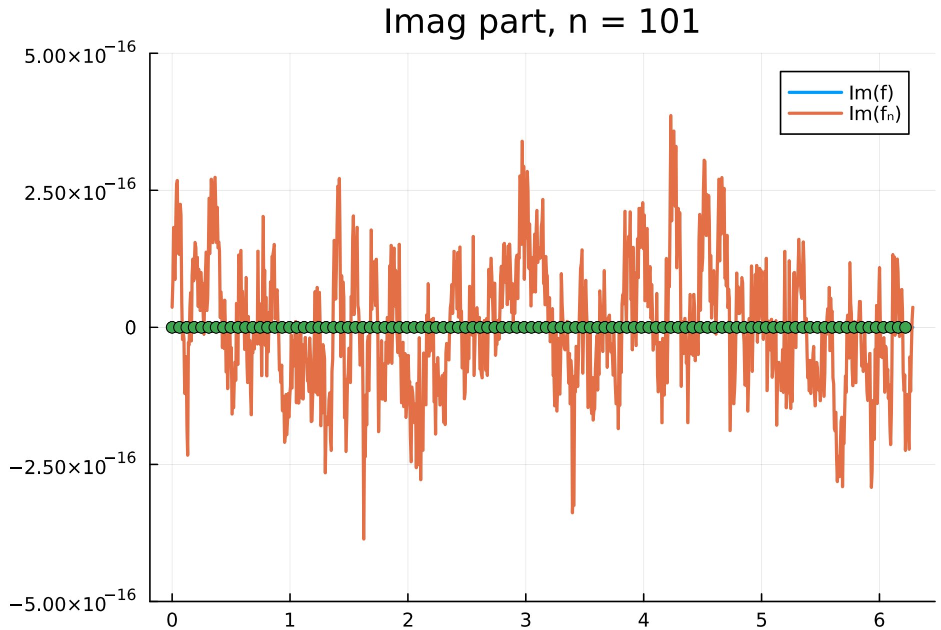


$$f(\theta) = \frac{2}{25\cos(\theta - 1)^2 + 1}$$



$$f(\theta) = \frac{2}{25\cos(\theta - 1)^2 + 1}$$





Converges