MATH50003 Numerical Analysis

V.2 Discrete Fourier Transform

Dr Sheehan Olver

Part V
Approximation Theory

1. Fourier Expansions and approximating Fourier coefficients
2. Discrete Fourier Transforms and interpolation

Not at all the same as Fourier Transform.

Is the same as Fast Fourier Transform (FFT)

V.2.1 The discrete Fourier transform

Map from values to approximate Fourier coefficients

$\theta_j = \frac{2\pi j}{n}$

Definition 34 (DFT)**.** The *Discrete Fourier Transform (DFT)* is defined as:

$$Q_n := \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-i\theta_1} & e^{-i\theta_2} & \cdots & e^{-i\theta_{n-1}} \\ 1 & e^{-i2\theta_1} & e^{-i2\theta_2} & \cdots & e^{-i2\theta_{n-1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-i(n-1)\theta_1} & e^{-i(n-1)\theta_2} & \cdots & e^{-i(n-1)\theta_{n-1}} \end{bmatrix}$$

$$= \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \cdots & \omega^{-(n-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \cdots & \omega^{-2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(n-1)} & \omega^{-2(n-1)} & \cdots & \omega^{-(n-1)^2} \end{bmatrix}$$

for the n-th root of unity $\omega = e^{2\pi i/n}$.

Note $Q_n = Q_n^T \neq Q_n^*$

We will show $Q_n^* Q_n = I$.

Note

$$\begin{bmatrix} \hat{f}_0^n \\ \vdots \\ \hat{f}_{n-1}^n \end{bmatrix} = \frac{1}{n} \begin{bmatrix} f(\theta_0) + f(\theta_1) + \cdots + f(\theta_{n-1}) \\ f(\theta_0) + f(\theta_1)\underbrace{e^{-i\theta_1}}_{\omega^{-1}} + \cdots + f(\theta_{n-1})\underbrace{e^{-i\theta_{n-1}}}_{\omega^{-(n-1)}} \\ \vdots \\ f(\theta_0) + f(\theta_1)\underbrace{e^{-i(n-1)\theta_1}}_{\omega^{-(n-1)}} + \cdots + f(\theta_{n-1})\underbrace{e^{-i(n-1)\theta_{n-1}}}_{\omega^{-(n-1)^2}} \end{bmatrix}$$

$$= \underbrace{\frac{1}{\sqrt{n}} Q_n}_{} \begin{bmatrix} f(\theta_0) \\ \vdots \\ f(\theta_{n-1}) \end{bmatrix}$$

linear map from samples to coeffs.

Proposition 1 (DFT is Unitary) $Q_n \in U(n)$, that is, $Q_n^\star Q_n = Q_n Q_n^\star = I$.

Proof

$$Q_n Q_n^* = \frac{1}{n}\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \cdots & \omega^{-(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(n-1)} & \cdots & \omega^{-(n-1)^2} \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega & \cdots & \omega^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \cdots & \omega^{(n-1)^2} \end{bmatrix}$$

$$= \frac{1}{n}\begin{bmatrix} \sum_{j=0}^{n-1} 1 & \sum_{j=0}^{n-1} \omega^{j} & \sum_{j=0}^{n-1} \omega^{2j} & \cdots & \sum_{j=0}^{n-1} \omega^{(n-1)j} \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

$$\sum_{j=0}^{n-1} 1 \Bigg]$$

$$= \begin{bmatrix} \Sigma_n[1] & \Sigma_n[e^{i\theta}] & \cdots & \Sigma_n[e^{i(n-1)\theta}] \\ \Sigma_n[e^{-i\theta}] & \Sigma_n[1] & \ddots & \vdots \\ \vdots & \ddots & \ddots & \end{bmatrix}$$

$$\underset{\text{Lemma 8}}{=} \begin{bmatrix} 1 & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{bmatrix} = I$$

Example 21 (Computing Sum).

V.2.2 Interpolation

Approximate Fourier series interpolates at sample points

Corollary 4 (Interpolation).

$$f_n(\theta) := \sum_{k=0}^{n-1} \hat{f}_k^n \mathrm{e}^{\mathrm{i}k\theta}$$

interpolates f at θ_j:

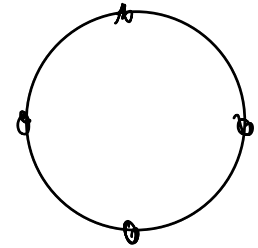
$$f_n(\theta_j) = f(\theta_j)$$

<u>Proof</u>

$$f_n(\theta_j) = \sum_{k=0}^{n-1} \hat{f}_k^n \underbrace{e^{ik\theta_j}}_{e^{ik2\pi j/n} = \omega^{kj}}$$

$$= \underbrace{\begin{bmatrix} 1 & w^{j} & w^{2j} & \cdots & w^{(n-1)j} \end{bmatrix}}_{\sqrt{n}\, e_j^T \; Q_n^*} \underbrace{\begin{bmatrix} \hat{f}_0^{\,n} \\ \hat{f}_1^{\,n} \\ \hat{f}_L^{\,n} \\ \vdots \\ \hat{f}_{n-1}^{\,n} \end{bmatrix}}_{\frac{1}{\sqrt{n}} Q_n \begin{bmatrix} f(\theta_0) \\ \vdots \\ f(\theta_{n-1}) \end{bmatrix}}$$

$$= e_j^T \; \cancel{Q_n^* \, Q_n} \begin{bmatrix} f(\theta_0) \\ \vdots \\ f(\theta_{n-1}) \end{bmatrix} = f(\theta_j)$$



Example 22 (DFT versus Lagrange).

Interpolate $f(z) = e^z$ at $[1, i, -1, -i]$

Method 1: Use Lagrange:

$$\ell_1(z) = \frac{(z-i)(z+1)(z+i)}{(1-i)\;2\;(1+i)}$$

$$\ell_2(z) = \frac{(z-1)(z+1)(z+i)}{(i-1)(i+1)\;2i}$$

$$\ell_3(z) = \_\_\_\_$$

$$\ell_4(z) = \_\_\_\_$$

Then $p(z) = e\,\ell_1(z) + e^{i}\,\ell_2(z) + e^{-1}\,\ell_3(z) + e^{-i}\,\ell_4(z)$

Method 2: DFT

$$\begin{bmatrix} \hat{f}^4_0 \\ \vdots \\ \hat{f}^4_3 \end{bmatrix} = \frac{1}{\sqrt{4}} Q_n \begin{bmatrix} e \\ e^{i} \\ e^{-1} \\ e^{-i} \end{bmatrix}$$

$$= \frac{1}{4} \begin{array}{c} \begin{array}{cccc} & W^{-1} & & \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \end{array} \begin{bmatrix} e \\ e^{i} \\ e^{-1} \\ e^{-i} \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} e + e^{i} + e^{-1} + e^{-i} \\ e - ie^{i} - e^{-1} + ie^{-i} \\ e - e^{i} + e^{-1} - e^{-i} \\ e + ie^{i} - e^{-1} - ie^{-i} \end{bmatrix}$$

$$\Rightarrow \quad p(z) = \frac{1}{2}(\cosh 1 + \cos 1) +$$

$$\frac{1}{2}(\sinh 1 + \sin 1)\, z +$$

$$\frac{1}{2}(\cosh 1 - \cos 1)\, z^2 +$$

$$\frac{1}{2}(\sinh 1 - \sin 1) z^3$$