**Numerical Analysis MATH50003 (2024–25) Problem Sheet 8**

**Problem 1** Give explicit formulae for $\hat{f}_k$ and $\hat{f}_k^n$ for the following functions:

$$\cos\theta, \cos 4\theta, \sin^4\theta, \frac{3}{3-\mathrm{e}^{\mathrm{i}\theta}}, \frac{1}{1-2\mathrm{e}^{\mathrm{i}\theta}}$$

**SOLUTION**

(1) Just expand in complex exponentials to find that

$$\cos\theta = \frac{\exp(\mathrm{i}\theta)+\exp(-\mathrm{i}\theta)}{2}$$

that is $\hat{f}_1 = \hat{f}_{-1} = 1/2$, $\hat{f}_k = 0$ otherwise. Therefore for $p \in \mathbb{Z}$ we have

$$\begin{aligned}
\hat{f}_k^1 &= \hat{f}_1 + \hat{f}_{-1} = 1 \\
\hat{f}_{2p}^2 &= 0, \hat{f}_{2p+1}^2 = \hat{f}_1 + \hat{f}_{-1} = 1 \\
\hat{f}_{1+np}^n &= \hat{f}_{-1+np}^n = 1/2, \hat{f}_k^n = 0
\end{aligned}$$

for $n = 3, 4, \ldots$.

(2) Similarly

$$\cos 4\theta = \frac{\exp(4\mathrm{i}\theta)+\exp(-4\mathrm{i}\theta)}{2}$$

that is $\hat{f}_4 = \hat{f}_{-4} = 1/2$, $\hat{f}_k = 0$ otherwise. Therefore for $p \in \mathbb{Z}$ we have

$$\begin{aligned}
\hat{f}_p^1 &= \hat{f}_4 + \hat{f}_{-4} = 1 \\
\hat{f}_{2p}^2 &= \hat{f}_4 + \hat{f}_{-4} = 1, \hat{f}_{2p+1}^2 = 0 \\
\hat{f}_{3p}^3 &= 0, \hat{f}_{3p\pm1}^3 = \hat{f}_{\pm4} = 1/2 \\
\hat{f}_{4p}^4 &= \hat{f}_{-4} + \hat{f}_4 = 1, \hat{f}_{4p\pm1}^4 = 0, \hat{f}_{4p+2}^4 = 0 \\
\hat{f}_{5p}^5 &= 0, \hat{f}_{5p+1}^5 = \hat{f}_{-4} = 1/2, \hat{f}_{5p-1}^5 = \hat{f}_4 = 1/2, \hat{f}_{5p\pm2}^5 = 0 \\
\hat{f}_{6p}^6 &= 0, \hat{f}_{6p\pm1}^6 = 0, \hat{f}_{6p+2}^6 = \hat{f}_{-4} = 1/2, \hat{f}_{6p-2}^6 = \hat{f}_4 = 1/2, \hat{f}_{6p+3}^6 = 0 \\
\hat{f}_{7p}^7 &= 0, \hat{f}_{7p\pm1}^7 = 0, \hat{f}_{7p\pm2}^7 = 0, \hat{f}_{7p\pm3}^7 = \hat{f}_{\mp4} = 1/2 \\
\hat{f}_{8p}^8 &= \hat{f}_{8p\pm1}^8 = \hat{f}_{8p\pm2}^8 = \hat{f}_{8p\pm3}^8 = 0, \hat{f}_{8p+4}^8 = \hat{f}_4 + \hat{f}_{-4} = 1 \\
\hat{f}_{k+pn}^n &= \hat{f}_k \text{ for } -4 \le k \le 4, \text{ 0 otherwise.}
\end{aligned}$$

(3) Here we have:

$$\begin{aligned}
(\sin\theta)^4 &= \left(\frac{\exp(\mathrm{i}\theta)-\exp(-\mathrm{i}\theta)}{2\mathrm{i}}\right)^4 = \left(\frac{\exp(2\mathrm{i}\theta)-2+\exp(-2\mathrm{i}\theta)}{-4}\right)^2 \\
&= \frac{\exp(4\mathrm{i}\theta)-4\exp(2\mathrm{i}\theta)+6-4\exp(-2\mathrm{i}\theta)+\exp(-4\mathrm{i}\theta)}{16}
\end{aligned}$$

that is $\hat{f}_{-4} = \hat{f}_4 = 1/16$, $\hat{f}_{-2} = \hat{f}_2 = -1/4$, $f_0 = 3/8$, $\hat{f}_k = 0$ otherwise. Therefore for $p \in \mathbb{Z}$

we have

$$
\begin{aligned}
&\hat f^1_p = \hat f_{-4} + \hat f_{-2} + \hat f_0 + \hat f_2 + \hat f_4 = 0\\
&\hat f^2_k = 0\\
&\hat f^3_{3p} = \hat f_0 = 3/8, \hat f^3_{3p+1} = \hat f_{-2} + \hat f_4 = -3/16, \hat f^3_{3p-1} = \hat f_2 + \hat f_{-4} = -3/16\\
&\hat f^4_{4p} = \hat f_0 + \hat f_{-4} + \hat f_4 = 1/2, \hat f^4_{4p\pm 1} = 0, \hat f^4_{4p+2} = \hat f_2 + \hat f_{-2} = -1/2\\
&\hat f^5_{5p} = \hat f_0 = 3/8, \hat f^5_{5p+1} = \hat f_{-4} = 1/16, \hat f^5_{5p-1} = \hat f_4 = 1/16, \hat f^5_{5p+2} = \hat f_2 = -1/4, \hat f^5_{5p-2} = \hat f_{-2} = -1/4\\
&\hat f^6_{6p} = \hat f_0 = 3/8, \hat f^6_{6p\pm 1} = 0, \hat f^6_{6p+2} = \hat f_2 + \hat f_{-4} = -3/16, \hat f^6_{6p-2} = \hat f_{-2} + \hat f_4 = -3/16, \hat f^6_{6p+3} = 0\\
&\hat f^7_{7p} = \hat f_0 = 3/8, \hat f^7_{7p\pm 1} = 0, \hat f^7_{7p\pm 2} = \hat f_{\pm 2} = -1/4, \hat f^7_{7p\pm 3} = \hat f_{\mp 4} = 1/16\\
&\hat f^8_{8p} = \hat f_0 = 3/8, \hat f^8_{8p\pm 1} = 0, \hat f^8_{8p\pm 2} = \hat f_{\pm 2} = -1/4, \hat f^8_{8p\pm 3} = 0, \hat f^8_{8p+4} = \hat f_4 + \hat f_{-4} = 1/8
\end{aligned}
$$

$\hat f^n_{k+pn} = \hat f_k$ for $-4 \le k \le 4$, 0 otherwise.

(4) Under the change of variables $z = \mathrm{e}^{\mathrm{i}\theta}$ we can use Geoemtric series to determine

$$\frac{3}{3-z} = \frac{1}{1-z/3} = \sum_{k=0}^{\infty} \frac{z^k}{3^k}$$

That is $\hat f_k = 1/3^k$ for $k \ge 0$, and $\hat f_k = 0$ otherwise. We then have for $0 \le k \le n-1$

$$\hat f^n_{k+pn} = \sum_{\ell=0}^{\infty} \frac{1}{3^{k+\ell n}} = \frac{1}{3^k}\frac{1}{1-1/3^n} = \frac{3^n}{3^{n+k} - 3^k}$$

(5) Now make the change of variables $z = \mathrm{e}^{-\mathrm{i}\theta}$ to get:

$$\frac{1}{1-2/z} = \frac{1}{-2/z}\frac{1}{1-z/2} = \frac{1}{-2/z}\sum_{k=0}^{\infty}\frac{z^k}{2^k} = -\sum_{k=1}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}k\theta}}{2^k}$$

That is $\hat f_k = -1/2^{-k}$ for $k \le -1$ and 0 otherwise. We then have for $-n \le k \le -1$

$$\hat f^n_{k+pn} = -\sum_{\ell=0}^{\infty}\frac{1}{2^{-k+\ell n}} = -\frac{1}{2^{-k}}\frac{1}{1-1/2^n} = -\frac{2^{n+k}}{2^n - 1}$$

**END**

**Problem 2** Prove that if the first $\lambda - 1$ derivatives $f(\theta), f'(\theta), \ldots, f^{(\lambda-1)}(\theta)$ are $2\pi$-periodic and $f^{(\lambda)}$ is uniformly bounded that

$$|\hat f_k| = O(|k|^{-\lambda}) \qquad \text{as } |k| \to \infty$$

Use this to show for the Taylor case $(0 = \hat f_{-1} = \hat f_{-2} = \cdots)$ that

$$|f(\theta) - \sum_{k=0}^{n-1} \hat f_k \mathrm{e}^{\mathrm{i}k\theta}| = O(n^{1-\lambda}) \qquad \text{as } n \to \infty$$

**SOLUTION** A straightforward application of integration by parts yields the result

$$\hat f_k = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\mathrm{e}^{-ik\theta}d\theta = \frac{(-i)^\lambda}{2\pi k^\lambda}\int_0^{2\pi} f^{(\lambda)}(\theta)\mathrm{e}^{-ik\theta}d\theta$$

given that $f^{(\lambda)}$ is uniformly bounded, the second part follows directly from this result

$$|\sum_{k=n}^{\infty} \hat{f}_k \mathrm{e}^{\mathrm{i}k\theta}| \leq \sum_{k=n}^{\infty} |\hat{f}_k| \leq C \sum_{k=n}^{\infty} k^{-\lambda}$$

for some constant $C$. The result then follows by the dominant convergence test:

$$\sum_{k=n}^{\infty} k^{-\lambda} \leq \int_{n-1}^{\infty} k^{-\lambda} \mathrm{d}k = O(n^{1-\lambda}).$$

**END**

**Problem 3(a)** If $f$ is a trigonometric polynomial ($\hat{f}_k = 0$ for $|k| > m$) show for $n \geq 2m + 1$ that we can exactly recover $f$:

$$f(\theta) = \sum_{k=-m}^{m} \hat{f}_k^n \mathrm{e}^{\mathrm{i}k\theta}$$

**SOLUTION** This follows from

$$\hat{f}_k^n = \sum_{p=-\infty}^{\infty} \hat{f}_{k+pn} = \hat{f}_k$$

if $-m \leq k \leq m$ as if $p > 0$ we have $k + pn \geq k + 2m + 1 \geq m + 1$ hence $\hat{f}_{k+pn} = 0$ and if $k < 0$ we have $k - pn \leq k - 2m - 1 \leq -m - 1$ hence $\hat{f}_{k+pn} = 0$.

**END**

**Problem 3(b)** For the general (non-Taylor) case and $n = 2m + 1$, prove convergence for

$$f_{-m:m}(\theta) := \sum_{k=-m}^{m} \hat{f}_k^n \mathrm{e}^{\mathrm{i}k\theta}$$

to $f(\theta)$ as $n \to \infty$. What is the rate of convergence if we know that the first $\lambda - 1$ derivatives $f(\theta), f'(\theta), \ldots, f^{(\lambda-1)}(\theta)$ are $2\pi$-periodic and $f^{(\lambda)}$ is uniformly bounded?

**SOLUTION**

Observe that by aliasing (see corollary in lecture notes) and triangle inequality we have the following

$$|\hat{f}_k^n - \hat{f}_k| \leq \sum_{p=1}^{\infty} (|\hat{f}_{k+pn}| + |\hat{f}_{k-pn}|)$$

Using the result from Problem 2 yields

$$|\hat{f}_k^n - \hat{f}_k| \leq \frac{C}{n^\lambda} \sum_{p=1}^{\infty} \frac{1}{\left(p + \frac{k}{n}\right)^\lambda} + \frac{1}{\left(p - \frac{k}{n}\right)^\lambda}$$

now we pick $|q| < \frac{1}{2}$ (such that the estimate below will hold for both summands above) and construct an integral with convex and monotonocally decreasing integrand such that

$$(p + q)^{-\lambda} < \int_{p-\frac{1}{2}}^{p+\frac{1}{2}} (x + q)^{-\lambda} dx$$

more over summing over the left-hand side from 1 to $\infty$ yields a bound by the integral:

$$\int_{\frac{1}{2}}^{\infty} (x + q)^{-\lambda} dx = \frac{1}{\lambda} (\frac{1}{2} + q)^{-\lambda+1}$$

Finally let $q = \pm\frac{k}{n}$ to achieve the rate of convergence

$$|\hat{f}_k^n - \hat{f}_k| \leq \frac{C_\lambda}{n^\lambda}\left(\left(\frac{1}{2} + k/n\right)^{-\lambda+1} + \left(\left(\frac{1}{2} - k/n\right)\right)^{-\lambda+1}\right)$$

where $C_\lambda$ is a constant depending on $\lambda$. Note that it is indeed important to split the $n$ coefficients equally over the negative and positive coefficients as stated in the notes, due to the estatime we used above.

Finally, we have:

$$\begin{aligned}
|f(\theta) - f_{-m:m}(\theta)| &= |\sum_{k=-m}^{m}(\hat{f}_k - \hat{f}_k^n)z^k + \sum_{k=m+1}^{\infty}\hat{f}_k z^k + \sum_{k=-\infty}^{-m-1}\hat{f}_k z^k| \\
&\leq \sum_{k=-m}^{m}|\hat{f}_k - \hat{f}_k^n| + \sum_{k=m+1}^{\infty}|\hat{f}_k| + \sum_{k=-\infty}^{-m-1}|\hat{f}_k| \\
&\leq \sum_{k=-m}^{m}\frac{C_\lambda}{n^\lambda}\left(\left(\frac{1}{2} + k/n\right)^{-\lambda+1} + \left(\left(\frac{1}{2} - k/n\right)\right)^{-\lambda+1}\right) + \sum_{k=m+1}^{\infty}|\hat{f}_k| + \sum_{k=-\infty}^{-m-1}|\hat{f}_k| \\
&= \frac{C_\lambda}{n^\lambda}2^\lambda + \sum_{k=m+1}^{\infty}|\hat{f}_k| + \sum_{k=-\infty}^{-m-1}|\hat{f}_k| \\
&= O(n^{-\lambda}) + O(n^{1-\lambda}) + O(n^{1-\lambda}) \\
&= O(n^{1-\lambda})
\end{aligned}$$

**END**

**Problem 3(c)** Show that $f_{-m:m}(\theta)$ interpolates $f$ at $\theta_j = 2\pi j/n$ for $n = 2m+1$.

**SOLUTION** Note from the aliasing property we have

$$\begin{aligned}
\hat{f}_k^n e^{ik\theta_j} &= \hat{f}_k^n e^{2\pi ikj/n} = \hat{f}_{k+n}^n e^{2\pi i(k+n)j/n} \\
&= \hat{f}_{k+n}^n e^{i(k+n)\theta_j}
\end{aligned}$$

Thus we have

$$\begin{aligned}
f_{-m:m}(\theta_j) &= \sum_{k=-m}^{-1}\hat{f}_k^n e^{ik\theta_j} + \sum_{k=0}^{m}\hat{f}_k^n e^{ik\theta_j} \\
&= \sum_{k=-m}^{-1}\hat{f}_{k+n}^n e^{i(k+n)\theta_j} + \sum_{k=0}^{m}\hat{f}_k^n e^{ik\theta_j} \\
&= \sum_{k=n-m}^{n-1}\hat{f}_k^n e^{i(k)\theta_j} + \sum_{k=0}^{m}\hat{f}_k^n e^{ik\theta_j} = f_n(\theta_j) = f(\theta_j)
\end{aligned}$$

**END**

**Problem 4(a)** Show for $0 \leq k, \ell \leq n-1$

$$\frac{1}{n}\sum_{j=1}^{n}\cos k\theta_j \cos \ell\theta_j = \begin{cases} 1 & k = \ell = 0 \\ 1/2 & k = \ell \\ 0 & \text{otherwise} \end{cases}$$

for $\theta_j = \pi(j - 1/2)/n$. Hint: Be careful as the $\theta_j$ differ from before, and only cover half the period, $[0, \pi]$. Using symmetry may help. You may also consider replacing cos with complex exponentials:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

**SOLUTION** The case $k = l = 0$ is immediate. Otherwise, we have,

$$\frac{1}{n}\sum_{j=1}^{n}\cos(k\theta_j)\cos(l\theta_j) = \frac{1}{4n}\sum_{j=1}^{n}\left[e^{i(k+l)\theta_j} + e^{-i(k+l)\theta_j} + e^{i(k-l)\theta_j} + e^{-i(k-l)\theta_j}\right].$$

For $\omega = \exp(i\pi/n)$ and any $m$ not a multiple of $2n$ we have

$$\begin{aligned}\sum_{j=1}^{n} e^{im\theta_j} &= \sum_{j=0}^{n-1} e^{im\pi(j+1/2)/n} = e^{im\pi/(2n)}\sum_{j=0}^{n-1} e^{im\pi j/n} = \omega^{m/2}\sum_{j=0}^{n-1}\omega^{mj} \\ &= \omega^{m/2}\frac{\omega^{nm}-1}{\omega^m-1} = \omega^{m/2}\frac{(-1)^m-1}{\omega^m-1}\end{aligned}$$

and hence

$$\begin{aligned}\sum_{j=1}^{n}[e^{im\theta_j} + e^{-im\theta_j}] &= \omega^{m/2}\frac{(-1)^m-1}{\omega^m-1} + \omega^{-m/2}\frac{(-1)^m-1}{\omega^{-m}-1} \\ &= \omega^{m/2}\frac{(-1)^m-1}{\omega^m-1} + \omega^{m/2}\frac{(-1)^m-1}{1-\omega^m} = 0.\end{aligned}$$

Since $0 < k + l \leq 2n - 2$ we know $k + l$ is not a multiple of $2n$ hence

$$\sum_{j=1}^{n}\left[e^{i(k+l)\theta_j} + e^{-i(k+l)\theta_j}\right] = 0.$$

Now if $k = l$ we have

$$\sum_{j=1}^{n} e^{i(k-l)\theta_j} = \sum_{j=1}^{n} e^{-i(k-l)\theta_j} = n.$$

Otherwise $k - l \neq 0$ but also $1 - n \leq k - l \leq n - 1$ hence $k - l$ cannot be a multiple of $2n$. And thus

$$\sum_{j=1}^{n}\left[e^{i(k-l)\theta_j} + e^{i(l-k)\theta_j}\right] = 0.$$

**END**

**Problem 4(b)** Consider the Discrete Cosine Transform (DCT)

$$C_n := \begin{bmatrix}\sqrt{1/n} & & & \\ & \sqrt{2/n} & & \\ & & \ddots & \\ & & & \sqrt{2/n}\end{bmatrix}\begin{bmatrix}1 & \cdots & 1 \\ \cos\theta_1 & \cdots & \cos\theta_n \\ \vdots & \ddots & \vdots \\ \cos(n-1)\theta_1 & \cdots & \cos(n-1)\theta_n\end{bmatrix}$$

for $\theta_j = \pi(j - 1/2)/n$. Prove that $C_n$ is orthogonal: $C_n^\top C_n = C_n C_n^\top = I$.

**SOLUTION**

The components of $C_n$ are

$$\boldsymbol{e}_k^\top C_n \boldsymbol{e}_j = \frac{1}{\sqrt{n}}\begin{cases}1 & k = 1 \\ \sqrt{2} & k \neq 1\end{cases}\cos((k-1)\theta_j),$$

where $\theta_j = \pi(j-1/2)/n$. We find using the previous part:

$$\begin{aligned}
\boldsymbol{e}_k^\top C_n C_n^\top \boldsymbol{e}_\ell &= \left(\begin{cases} 1 & k=\ell=1 \\ \sqrt{2} & k,\ell=1, k\neq \ell \\ 2 & k,\ell \neq 1 \end{cases}\right) \frac{1}{n}\sum_{j=1}^n \cos((k-1)\theta_j)\cos((\ell-1)\theta_j) \\
&= \left(\begin{cases} 1 & k=\ell=1 \\ \sqrt{2} & k,\ell=1, k\neq \ell \\ 2 & k,\ell \neq 1 \end{cases}\right)\left(\begin{cases} 1 & k=\ell=1 \\ 1/2 & k=\ell \\ 0 & \text{otherwise} \end{cases}\right) = \delta_{k\ell}.
\end{aligned}$$

**END**

**Problem 5** What polynomial interpolates $\cos z$ at $1, \exp(2\pi\mathrm{i}/3)$ and $\exp(-2\pi\mathrm{i}/3)$?

**SOLUTION** For $\omega = \exp(2\pi\mathrm{i}/3)$, we use the DFT:

$$\begin{aligned}
\begin{pmatrix} \hat{f}_0^3 \\ \hat{f}_1^3 \\ \hat{f}_2^3 \\ \hat{f}_2^3 \end{pmatrix} &= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \exp(-2\pi\mathrm{i}/3) & \exp(2\pi\mathrm{i}/3) \\ 1 & \exp(2\pi\mathrm{i}/3) & \exp(-2\pi\mathrm{i}/3) \end{bmatrix}\begin{pmatrix} \cos(1) \\ \cos(\exp(2\pi\mathrm{i}/3)) \\ \cos(\exp(-2\pi\mathrm{i}/3)) \end{pmatrix} \\
&= \frac{1}{3}\begin{pmatrix} \cos(1) + \cos(\exp(2\pi\mathrm{i}/3)) + \cos(\exp(-2\pi\mathrm{i}/3)) \\ \cos(1) + \exp(-2\pi\mathrm{i}/3)\cos(\exp(2\pi\mathrm{i}/3)) + \exp(2\pi\mathrm{i}/3)\cos(\exp(-2\pi\mathrm{i}/3)) \\ \cos(1) + \exp(2\pi\mathrm{i}/3)\cos(\exp(2\pi\mathrm{i}/3)) + \exp(-2\pi\mathrm{i}/3)\cos(\exp(-2\pi\mathrm{i}/3)) \end{pmatrix}
\end{aligned}$$

That is, the polynomial is

$$\begin{aligned}
&\frac{\cos(1) + \cos(\exp(2\pi\mathrm{i}/3)) + \cos(\exp(-2\pi\mathrm{i}/3))}{3} \\
&+ \frac{\cos(1) + \exp(-2\pi\mathrm{i}/3)\cos(\exp(2\pi\mathrm{i}/3)) + \exp(2\pi\mathrm{i}/3)\cos(\exp(-2\pi\mathrm{i}/3))}{3} z \\
&+ \frac{\cos(1) + \exp(2\pi\mathrm{i}/3)\cos(\exp(2\pi\mathrm{i}/3)) + \exp(-2\pi\mathrm{i}/3)\cos(\exp(-2\pi\mathrm{i}/3))}{3} z^2.
\end{aligned}$$

**END**