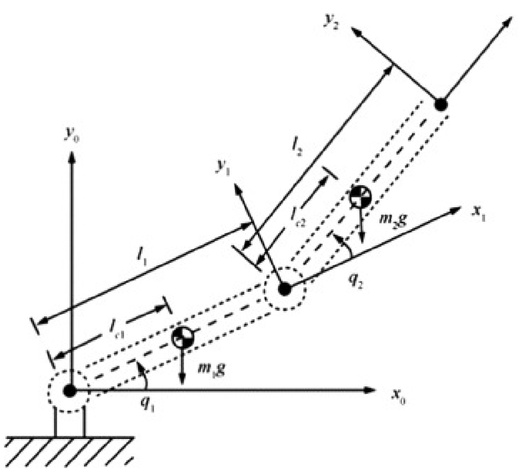
SC602
Simulations Report

Saad Khan
200100085

***Abstract*—The following report contains the simulation results and controller design subtleties for a two link manipulator tracking problem and a rigid body rotation problem. The control law for the two link manipulator is derived from the passivity based design in the first case and backstepping based design in the second case, whereas the control law for the rigid body rotation is also based on the passivity based design.**

I. Two Link Manipulator



Two Link Manipulator in generalized coordinate system

A. Model

The two link manipulator model is governed by the following dynamics:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} + g(q) = u$$

Where $q \in \mathbb{R}^2$ are the generalized coordinates, $\dot{q}$ are the generalized velocities. $0 < M(q) \in \mathbb{R}^{2\times 2}$, $C(q,\dot{q}) \in \mathbb{R}^{2\times 2}$, $0 \leq D \in \mathbb{R}^{2\times 2}$ and $g(q) \in \mathbb{R}^2$ are the coriolis, centrifugal, viscous damping and gravity respectively.

B. Control

Denote $e = q - q_r$, where q_r is a constant reference and $K_p \in \mathbb{R}^{2\times 2}$ being a symmetric positive definite matrix . Taking the control input $u = g(q) - K_p e + \nu$ and obtaining the modified dynamics as:

$$M(q)\ddot{e} + C(q,\dot{q})\dot{e} + D\dot{e} + K_p e = \nu$$

For a storage function $V(e,\dot{e})$ chosen as,

$$V(e,\dot{e}) = \frac{1}{2}\dot{e}^T M(q)\dot{e} + \frac{1}{2}e^T K_p e$$

we had proven that the modified dynamics is passive with the output $y = \dot{e}$, also that the (ν, y) pair is zero-state observable. We can use this property to choose a feedback $\nu = -\phi(y) = -\phi(\dot{e})$. We choose $\nu = -k_1 tanh(\dot{e})$ Because it satisfies the property that $\phi(0) = 0$ and also bounds the control. Also note that since the objective is to track **constant** reference, the derivatives of the error term e are just the derivatives of the generalized coordinates q itself, *i.e.* $e = q - q_r \implies \dot{e} = \dot{q} \implies \ddot{e} = \ddot{q}$. Taking all these subtleties into account we construct the final dynamics of the two link manipulator:

$$\ddot{q} = M(q)^{-1}(-k_1 tanh(\dot{q}) - C(q,\dot{q})\dot{q} - D\dot{q} - K_p e)$$

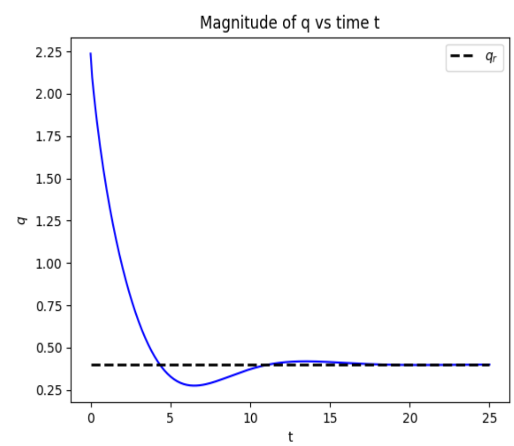
$$\dot{q}_{\text{next}} = \dot{q}_{\text{previous}} + \ddot{q}\Delta t$$

$$q_{\text{next}} = q_{\text{previous}} + \dot{q}\Delta t$$

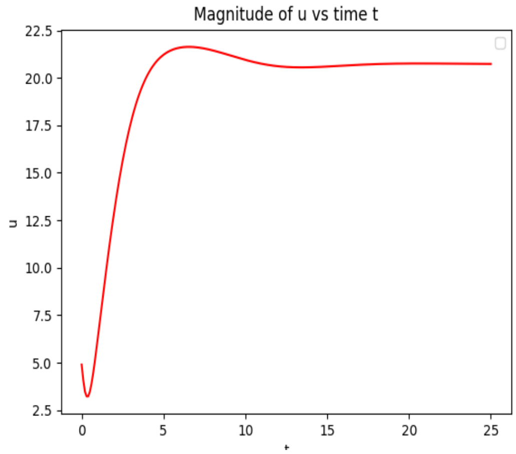
Now that we have laid down the underlying subtleties of the control design, we will see the simulation results.

C. Results

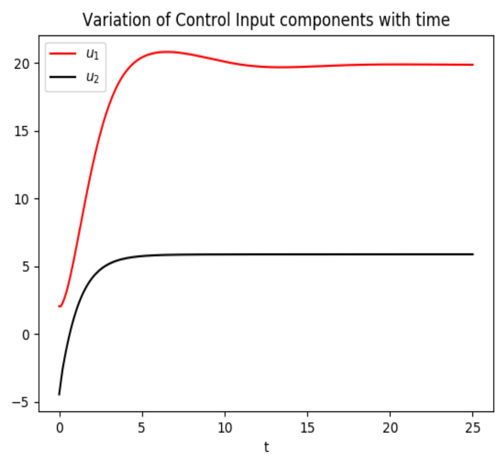
We simulate the model with the designed control using $m_1 = 2.5kg, m_2 = 1.5kg, l = 0.4m, g = 9.81m/s^2$ and the constant reference $q_r = \begin{bmatrix}0.4 & 0\end{bmatrix}^T$.



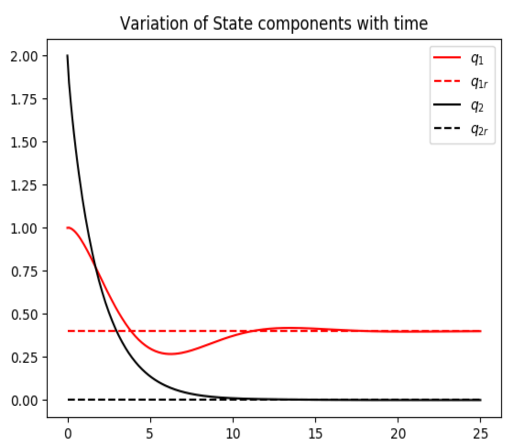
The above plot is when the initial state of the system $q_0 = \begin{bmatrix}1 & 2\end{bmatrix}^T$. It is evident from the plot that the state q converges to q_r in approximately 20 iterations.



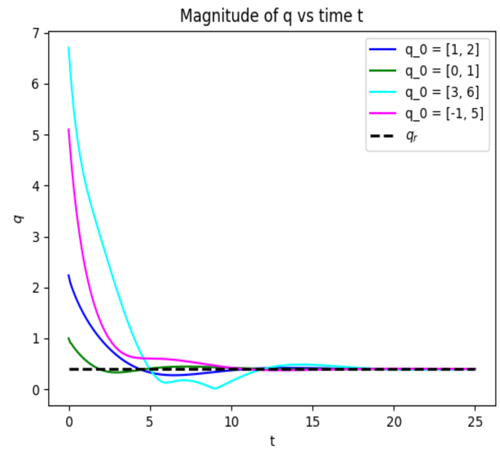
From the above plot we see that the control input rises and stabilizes at approximately 20 N-m, which is a fairly reasonable value of torque and can be provided by an actuator.



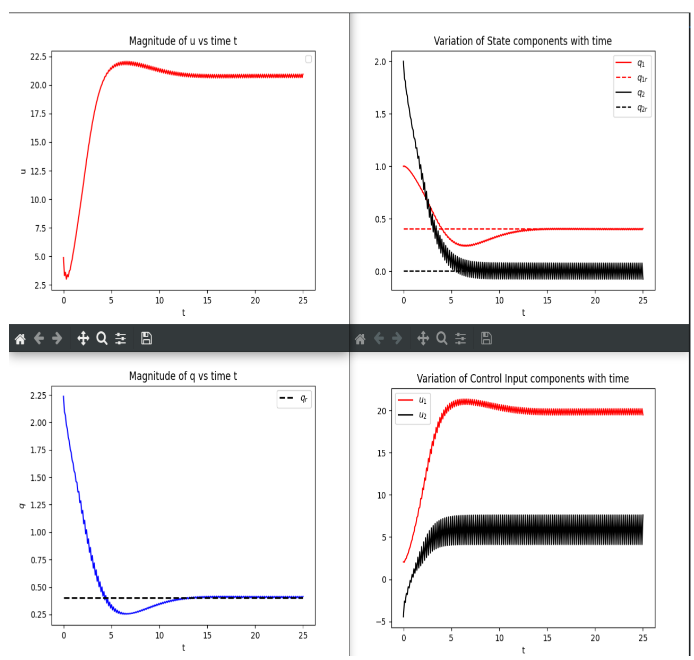
Above is the plot of the components of the control input u_1 and u_2 with respect to time. We can see that u_1 stabilizes at approximately 20 N-m and u_2 stabilizes at approximately 5 N-m.



Above is the plot of the components of the state q_1 and q_2 with respect to time. As we can see that the initial state of the system was $q = \begin{bmatrix} 1 & 2 \end{bmatrix}^T$, q_1 starts from 1 and q_2 starts from 2, and since the reference is $q_r = \begin{bmatrix} 0.4 & 0 \end{bmatrix}^T$, q_1 stabilizes at 0.4 and q_2 stabilizes at 0.

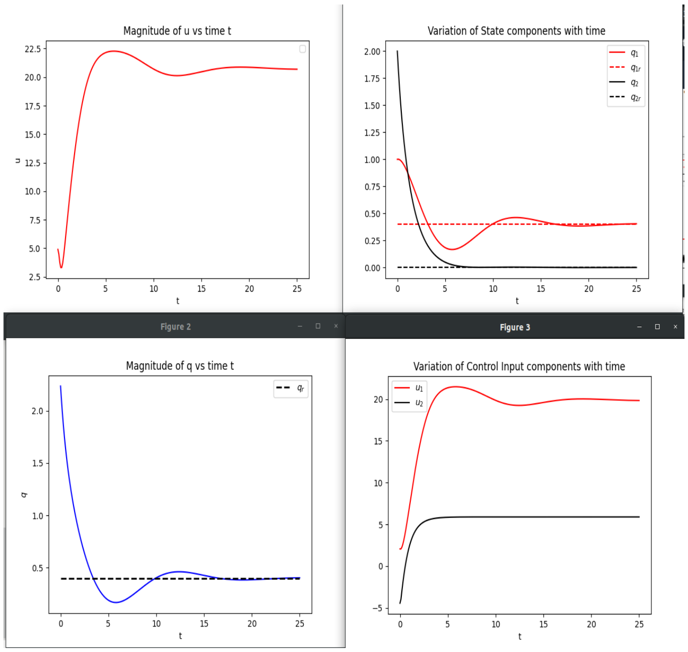


The above plot shows how different initial states all converge to the constant reference in approximately 20 iterations. Now that we have seen the convergence of the state in the effect of the controller, we now observe the effect of the constant k_1 used in the control design and determine what effect it has on the system. The response in all the previous figures was for $k_1 = 1$ which happens to produce quite a desirable performance.



Behaviour for $k_1 = 2$

Above is the plot when k_1 is double of the desirable gain, we see that the control input components oscillate at a very high frequency about some mean, this results in a high frequency vibration of the states as well. This is called **chattering** and this is undesirable as well as a prominent problem in many control designs. This leads us to conclude that increasing the value of the gain k_1 increases chattering in our system.



Behaviour for $k_1 = 0.5$

Above is the plot when k_1 is half of the desirable gain, we see that the response of system becomes sluggish as in the system takes more time to stabilize and there is significant overshoot in the states of the system. This again is not always desirable because we may have a restricted configuration space which would not always allow the room for overshoot. The settling time also is an important design consideration because faster the settling time the better behaviour of the system. This leads us to conclude that decreasing the value of the gain k_1 makes the system response sluggish, increases the settling time and the overshoot.

D. Conclusions

We saw that using the passivity based design we were successfully able to control the system states and make it track a constant reference, we also saw that various starting points all converge to the reference in a desirable time. In the end we saw the effect of the gain k_1 on the system.

II. Scalar Manipulator

A. Model

The model for the scalar manipulator as the name suggests is just the scalar version of the two link manipulator discussed in the previous section, in addition to that the system is augmented with an integrator and hence the dynamics of the system is given by:

$$m(q)\ddot{q} + c(q,\dot{q})\dot{q} = \xi$$

$$\dot{\xi} = \tau$$

where $q \in \mathbb{R}$ is the state (angle in this case) and $\tau \in \mathbb{R}$ is the control torque. Further, $m(q) = p_1 + 2p_3cos(q) > 0$ and $c(q,\dot{q}) = -p_3sin(q)\dot{q}$. We want this system to track a **constant** reference q_r. To do so we use the backstepping based design to derive a control torque from the non-augmented system.

B. Control

We start with the non-augmented system whose dynamics is given by:

$$m(q)\ddot{q} + c(q,\dot{q})\dot{q} = \tau$$

Since we want to track a reference we denote $e = q - q_r$, and choose the Control Lyapunov Function (CLF) as:

$$V_0(e,\dot{e}) = \frac{e^2}{2} + \frac{\dot{e}^2}{2}$$

The above CLF satisfies the properties it needs to. However note that since we want to track a **constant** reference the derivative of the error $\dot{e} = \dot{q}$. Which makes the CLF V_0:

$$V_0 = \frac{e^2}{2} + \frac{\dot{q}^2}{2}$$

For Lyapunov based design for the control τ we want $\dot{V_0} < 0$.

$$\therefore e\dot{e} + \dot{q}\ddot{q} < 0$$

$$\implies e\dot{q} + \dot{q}\left(\frac{\tau - c(q,\dot{q})}{m(q)}\right) < 0$$

$$\dot{q}\left(q - q_r + \frac{\tau - c(q,\dot{q})}{m(q)}\right) < 0$$

Note that if we choose a τ such that the term $q - q_r + \frac{\tau - c(q,\dot{q})}{m(q)} = -\dot{q}$, we get $\dot{V_0} = -\dot{q}^2 < 0$. Therefore for this to happend we get:

$$\tau = c(q,\dot{q})\dot{q} + m(q)(q_r - q - \dot{q}) = k_0$$

Now we augment the system with an integrator to obtain the original dynamics of the system,

$$m(q)\ddot{q} + c(q,\dot{q})\dot{q} = \xi$$

$$\dot{\xi} = \tau$$

For this system we choose the candidate CLF to be V_1, such that:

$$V_1 = V_0 + \frac{1}{2}(\xi - k_0)^2$$

Computing $\dot{V}_1$, we get:

$$\dot{V}_1 = \dot{V}_0 + (\xi - k_0)(\dot{\xi} - \dot{k}_0)$$
$$\implies \dot{V}_1 = e\dot{q} + \dot{q}\ddot{q} + (\xi - k_0)(\tau - \dot{k}_0)$$

Substituting $\dot{q}$ from the dynamics,

$$\implies \dot{V}_1 = \dot{q}\left(q - q_r + \frac{\xi - c(q,\dot{q})}{m(q)}\right) + \\ (\xi - c(q,\dot{q})\dot{q} - m(q)(q_r - q - \dot{q}))\,(\tau - \dot{k}_0)$$

$$\implies \dot{V}_1 = \dot{q}\left(q - q_r + \frac{\xi - c(q,\dot{q})}{m(q)}\right) + \\ m(q)\left(\frac{\xi - c(q,\dot{q})\dot{q}}{m(q)} - q_r + q + \dot{q}\right)(\tau - \dot{k}_0)$$

$$\implies \dot{V}_1 = \dot{q}\left(q - q_r + \frac{\xi - c(q,\dot{q})}{m(q)} + \dot{q} - \dot{q}\right) + \\ m(q)\left(\frac{\xi - c(q,\dot{q})\dot{q}}{m(q)} - q_r + q + \dot{q}\right)(\tau - \dot{k}_0)$$

$$\implies \dot{V}_1 = -\dot{q}^2 + \left(\frac{\xi - c(q,\dot{q})\dot{q}}{m(q)} + q - q_r + \dot{q}\right) \\ (\dot{q} + m(q)\tau - m(q)\dot{k}_0)$$

Now for Lyapunov based control design for this system we just need to ensure that the time derivative of the CLF $\dot{V}_1 < 0$, note that the first term in the above equation is $\dot{V}_0$ which is always negative, so we just need to choose a τ such that the second term also becomes negative. Therefore,

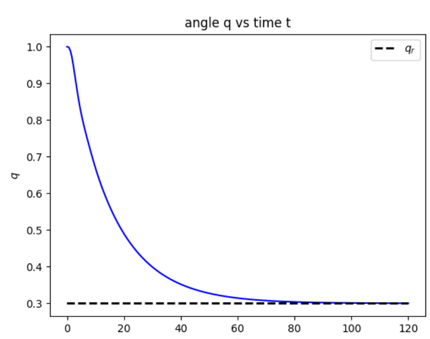
$$\dot{q} + m(q)\tau - m(q)\dot{k}_0 = -\left(\frac{\xi - c(q,\dot{q})\dot{q}}{m(q)} + q - q_r + \dot{q}\right)$$

ensures that $\dot{V}_1$ is always negative. Solving for the control torque τ from this equation gives us,

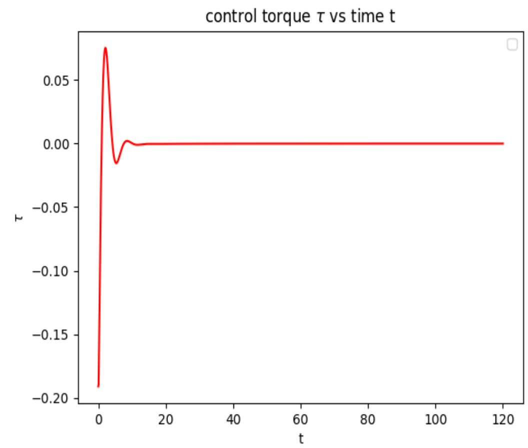
$$\tau = \dot{k}_0 - \frac{2\dot{q}}{m(q)^2} - \frac{\xi}{m(q)^2} + \frac{c(q,\dot{q})\dot{q}}{m(q)^2} + \frac{q_r - q}{m(q)^2}$$

The control obtained ensures that the time derivative of the CLF V_1 is negative definite and hence ensures Lyapunov stability.

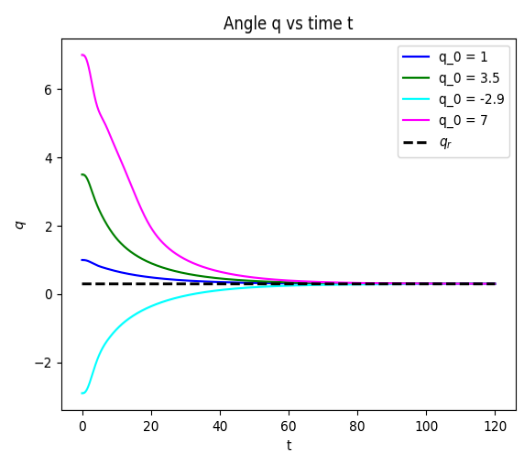
C. Results



We choose to track a constant reference $q_r = 0.3rad$. Assuming the initial state to be $q_0 = 1rad$. Above is the plot of the variation of q with respect to time. Note that q converges to the reference q_r in around 100 iterations, which is a fairly desirable performance. To obtain this behavious the variation of the control torque τ with respect to time is plotted below.



Note that τ starts of with a negative value in order to decrease $q(t)$ and in order to avoid the overshoot it increases to a positive value before stabilizing at 0 N-m. The control torque stays within a fairly reasonable margin of (-0.2 N-m,0.1 N-m) which is a small torque value and can be easily actuated by a commercial actuators. Next we see the convergence of the state from different initial conditions,

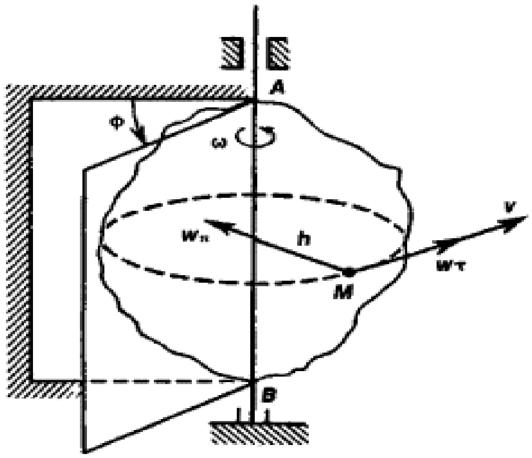


above is the plot of the angle q converging to the reference q_r for different values of the initial angle q_0.

D. Conclusions

In this section, we saw that using backstepping based design on the scalar manipulator model helped us derive a control torque τ that helps track a constant reference. We also saw how different initial conditions converge to the desired state in an acceptable number of iterations.

III. Rigid Body Rotation



Rigid Body Rotation

A. Model

The rigid body rotation is governed with the following dynamics:

$$\dot{\rho} = [I + [\rho]_\times + \rho\rho^T]\omega$$

$$J\dot{\omega} = -(\omega \times J\omega) + u$$

With

$$y = \omega$$

Where $\rho \in \mathbb{R}^3$ is the modified Rodrigues parameter, $J = J^T > 0$ is the inertia matrix, $\omega \in \mathbb{R}^3$ is the angular velocity and $u \in \mathbb{R}^3$ is the thrust. Note that the system is zero state observable.

B. Control

Previously, we had proven that with the candidate storage function $V(\omega) = \frac{1}{2}\omega^T J\omega$, the driver system is passive with (u, y). Note that the dynamics $\dot{\rho} = [I + [\rho]_\times + \rho\rho^T]\omega$ is cascaded with the passive driver system. In order for the whole system to be stable we need to find a Positive Definite Function $W(\rho)$ such that the whole system becomes passive with respect to $y = \omega$. The passive system output drives the ρ dynamics. For the selection of the control u, consider the total Lyapunov function $U(\omega, \rho)$ such that:

$$U(\omega, \rho) = V(\omega) + W(\rho)$$

Taking the time derivative on both sides,

$$\dot{U}(\omega, \rho) = \dot{V}(\omega) + \dot{W}(\rho)$$

We know that $\dot{V}(\omega) lequ^T y$ with $y = \omega$, since the driver system is passive. Hence we get,

$$\dot{U}(\omega, \rho) \leq uy^T + \frac{\partial W(\rho)}{\partial \rho}\left([I + [\rho]_\times + \rho\rho^T]\omega\right)$$

$$\therefore \dot{U}(\omega, \rho) \leq y^T\left(u + \left(\frac{\partial W(\rho)}{\partial \rho}[I + [\rho]_\times + \rho\rho^T]\right)^T\right)$$

Therefore for the entire system to be passive with respect to $y = \omega$, we need,

$$\dot{U}(\omega, \rho) \leq y^T\nu$$

For some control ν, Therefore comparing the two equations above, we obtain,

$$u = \nu - \left(\frac{\partial W(\rho)}{\partial \rho}[I + [\rho]_\times + \rho\rho^T]\right)^T$$

Now that we proved that the whole system is passive with the control ν we can further use the Barbashin-Krasovskii-LaSalle theorem in conjunction with zero state observability to prove that the system is indeed stable in the sense of Lyapunov. For this particular dynamics a choice of $W(\rho)$ that works is:

$$W(\rho) = k_1 ln(1 + \rho^T\rho)$$

Also we know that the system is passive with respect to $y = \omega$ and the (ν, y) is zero state observable. Therefore $\nu = -\phi(y)$, such that $\phi(0) = 0$. Hence we choose $\nu = -k_2 tanh(\omega)$. Therefore the control u becomes,

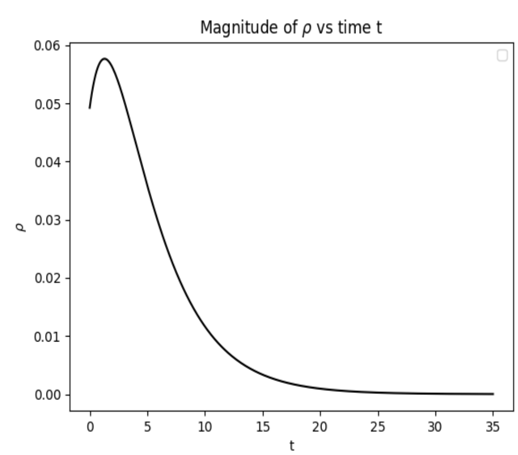
$$u = -k_2 tanh(\omega) - \left(\frac{\partial W(\rho)}{\partial \rho}[I + [\rho]_\times + \rho\rho^T]\right)^T$$

This completes the passivity based control law for the system.

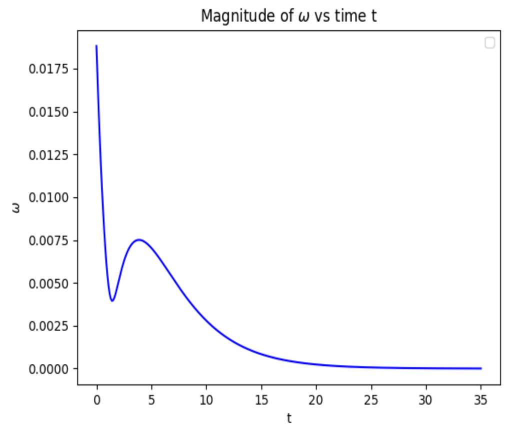
C. Results

The objective of the control is to stabilize the system to the zero state for any given initial state, we simulate the system and the designed control law for the initial conditions of $\rho(0) = \begin{bmatrix}-0.02 & 0 & 0.045\end{bmatrix}^T, \omega(0) = \begin{bmatrix}0.004 & -0.007 & 0.017\end{bmatrix}^T rad/s$. With,

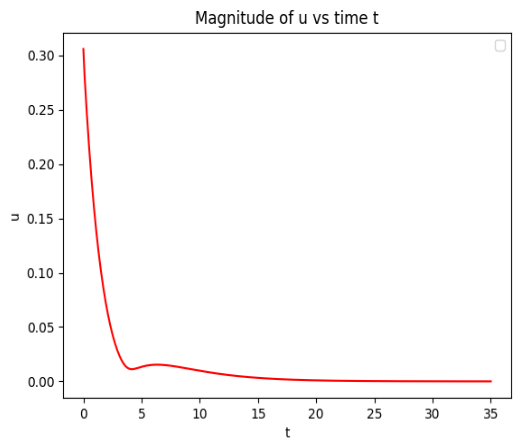
$$J = \begin{bmatrix} 20.85 & 1.2 & 0.9 \\ 1.2 & 17.85 & 1.4 \\ 0.9 & 1.4 & 15.85 \end{bmatrix}$$



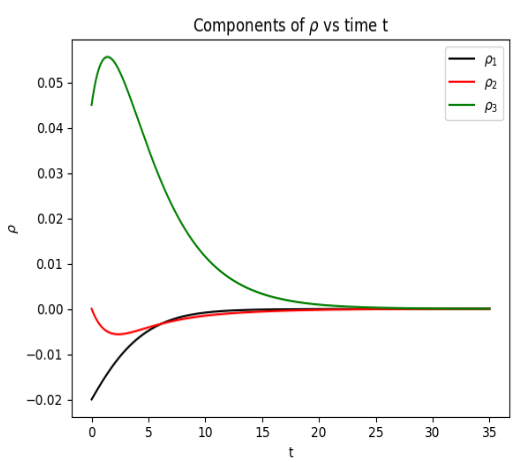
The plot above shows the stabilization of the magnitude of the modified Rodrigues parameter. Similarly the next plot shows the stabilization of the magnitude of the angular velocity vector.



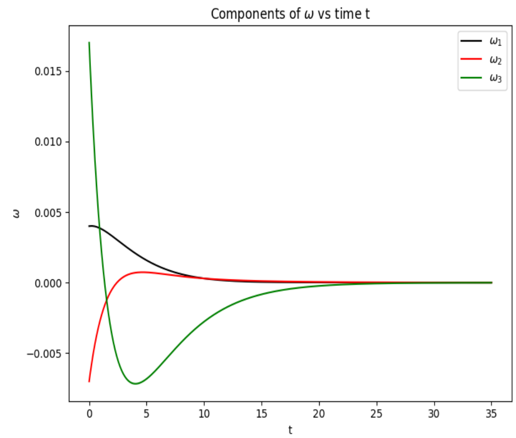
The variation of the magnitude of the control input u with respect to time is plotted below.



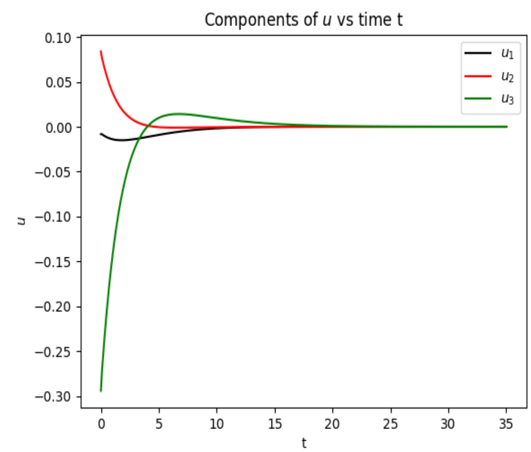
We now see the variation of the individual components of ρ, ω and u.



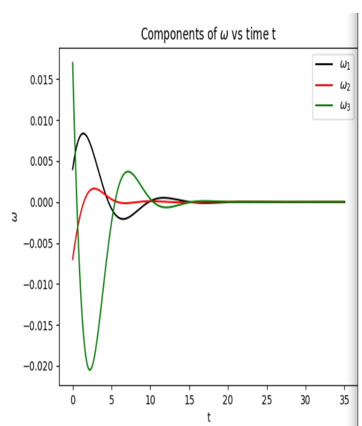
As we can see here the designed control stabilizes all the components of the modified Rodrigues parameter.

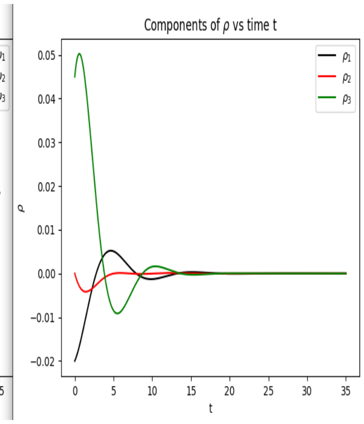


The above plot shows the stabilization of the components of the angular velocity vector.

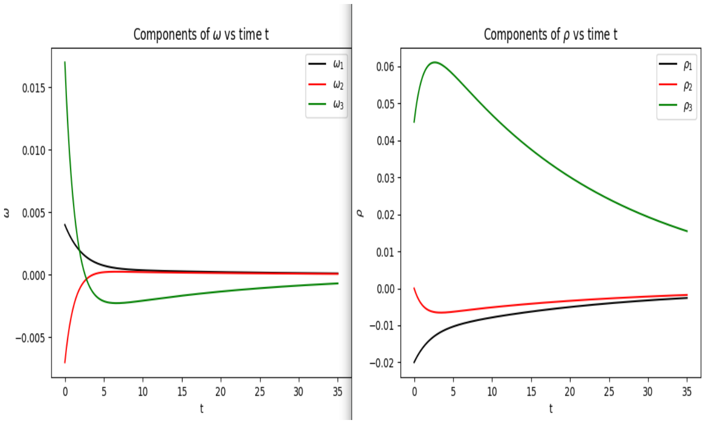


The variation of the components of the control also stabilizes at 0, which is the result of zero state observability that the control input stabilizes to 0 once the system parameters stabilize at 0. All the plots so far were for gain values $k_1 = 1$ and $k_2 = 12$. We now study the effect of changing these gain values. First we will only tweak k_1.

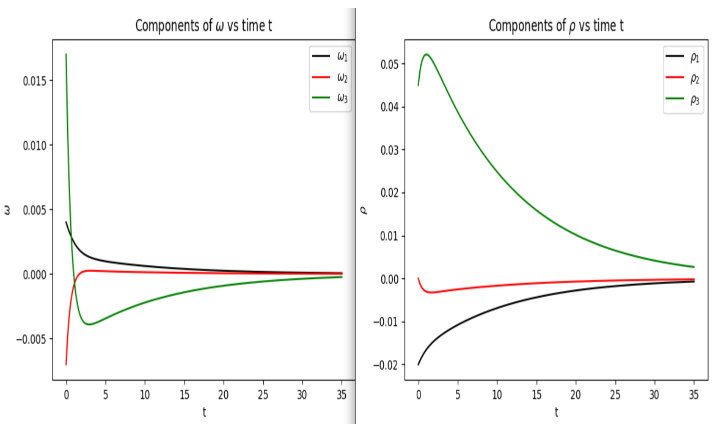




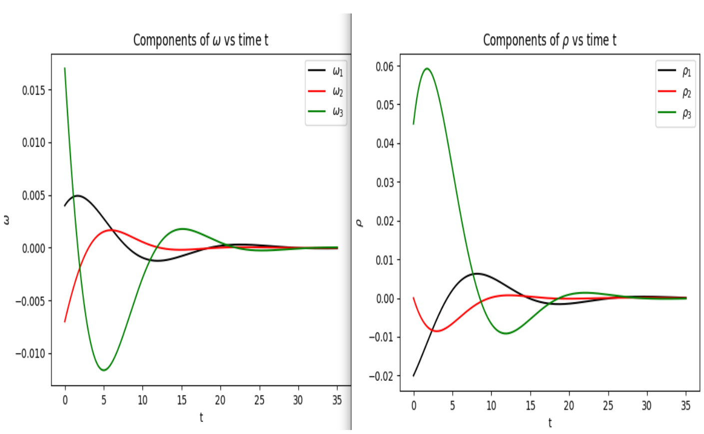
Increasing the value of k_1 to 4 times of it's original value reduces the settling time but increases the overshoot.



Reducing the value of k_1 to one-fourth it's original value reduces the overshoot but significantly increases the settling time. (Note that ρ_3 did not even converge even after 35 iterations). Now we will tweak only k_2.



Doubling the value of k_2 we observe that the overshoot decreases but the settling time increases. And making the value of k_2 half of the original value increases the overshoot but decreases the settling time.



It is really interesting to observe the pattern here, increasing only k_1 and decreasing only k_2 has the exact same effect on the system, *i.e.* increasing overshoot and reducing settling time. And decreasing only k_1 and increasing only k_2 also has the exact same effect on the system, *i.e.* decreasing the overshoot and increasing the settling time. Using this information we can find that the system's behaviour is desirable for $\frac{k_2}{k_1} \approx 10$.

D. *Conclusions*

We saw that using the passivity based design we can construct a control law that drives the system to the equilibrium state. We then saw the simulated plots of the various system states and their evolution over time. After that we observed the effect of tweaking the gains k_1 and k_2 on the system and after understanding these effects we were able to find these gain values that produces the most desirable performance.

IV. Sources and References

- Two Link Manipulator Image
- Rigid Body Rotation Image