Math 231 Lecture 2 8/29/18

- Preclass Tutorial 1 & 2 and HW 1 & 2. (hard deadline: 9/7 Friday 8:30AM)
- Head TA: Benjamin Wright, bwwrigh2@illinois.edu
- Tutoring room starts Tuesday, Sept 4. 5-8pm M,Tu,W,Th in 347 AH.
- Extra office hours today: Wednesday 1-2pm (024 Illini Hall).

Tool 1: Substitution

$$\boxed{\int_a^b f(u(x))\,u'(x)\,\mathrm{d}x = \int_{u(a)}^{u(b)} f(u)\,\mathrm{d}u}$$

$$\frac{d}{dx}\left[F(u(x))\right] = F'(u(x))\cdot u'(x)$$

$$\int F'(u(x))\cdot u'(x)\,dx = F(u)+C$$

$$\uparrow \; f \qquad = \int F'(u)\,du = \int f(u)\,du$$

iClicker 1: The substitution rule follows from:

(A) Product Rule (B) Quotient Rule (C) Chain Rule

Tool 2: Integration by Parts

$$\boxed{\int_a^b u\,\mathrm{d}v = uv\Big|_a^b - \int_a^b v\,\mathrm{d}u}$$

Integration by Parts follows from the Product Rule:

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right] = f'(x)\cdot g(x) + f(x)\cdot g'(x)$$

Integrate

$$f(x)\cdot g(x) = \int f'(x)\cdot g(x)\,dx + \int f(x)\cdot g'(x)\,dx$$

Rearrange

$$\int \underbrace{f(x)}_{u}\cdot\underbrace{g'(x)\,dx}_{dv} = \underbrace{f(x)}_{u}\cdot\underbrace{g(x)}_{v} - \int \underbrace{g(x)}_{v}\cdot\underbrace{f'(x)\,dx}_{du}$$

Let $u = f(x)$, $v = g(x)$; then $du = f'(x)\,dx$, $dv = g'(x)\,dx$

$$\int u\cdot dv = u\cdot v - \int v\cdot du$$

iClicker 2: $\int \underbrace{x}_{u}\cdot\underbrace{e^x dx}_{dv} = \oplus\, x\cdot e^x \ominus \int e^x\,dx = x\cdot e^x - e^x + C$

$u = x \quad \oplus \quad dv = e^x dx$

$du = dx \quad \ominus \quad v = e^x$

(A) $x\cdot e^x + C$

(B) $x\cdot e^x - x + C$

(C) $x\cdot e^x - e^x + C$

(D) $e^x + C$

Strategy for integration

If you cannot integrate directly, try the "process of elimination":

- Does substitution work? What can u be?
- Does integration by parts work? What can u, dv be?

For integration by parts, you can use the following acronym to decide which part should be $\underline{u}$ and which one $\underline{dv}$:

L I A T E

- L → Log
- I → Inverse trig
- A → Algebraic $\left(x^3, \sqrt{x^2-1}, \frac{x-1}{\sqrt{x}+x^{5/2}}, \dots\right)$
- T → Trig
- E → Exponential

Whichever function comes first should be $\underline{u}$.

Examples:

- $\int \underbrace{x}_{u} \cdot \underbrace{\sin x \, dx}_{dv}$ x is **A**lgebraic ← u; $\sin x$ is **T**rigonometric

- $\int \overbrace{x^3 \cdot \underbrace{\ln x}_{u} \, dx}^{dv}$ x^3 is **A**lgebraic; $\ln x$ is **L**ogarithmic ← u

This technique is not perfect, but works most of the time.
You don't need to remember / use it if you don't want to.
(you will NOT be tested on this)

Ex 1: $\int \ln x \, dx$

• <u>Try substitution</u>: $u = \ln x \longrightarrow e^u = x$

$du = \frac{1}{x} \, dx$

see iClicker 2

$$\int \ln x \, dx = \text{"}\int u \cdot x \, du\text{"} = \int u \cdot e^u \, du \overset{\downarrow}{=} u \cdot e^u - e^u + C$$

eliminate x

$$= \boxed{\ln x \cdot x - x + C}$$

always give an answer in terms of the original variable (here x)

• <u>Try parts</u>:

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x} \, dx \qquad v = x$

$$\int \underbrace{\ln x}_{u} \, \underbrace{dx}_{dv} = \oplus \ln x \cdot x \ominus \int \frac{1}{x} \cdot x \, dx$$

$$= \boxed{\ln x \cdot x - x + C}$$

Moral: knowing derivative helps you find integral.

Ex 2: $\int \frac{x^3}{\sqrt{1+x^2}}\,\mathrm{d}x$

- Try substitution

$u = \sqrt{1+x^2}$ $\quad u^2 = 1+x^2 \longrightarrow x^2 = u^2-1$

$2u\,du = 2x\,dx \Longrightarrow u\,du = x\,dx$

$$\int \frac{x^3}{\sqrt{1+x^2}}\,dx = \int \frac{x^2}{u}\cdot u\,du = \int \frac{u^2-1}{\cancel{u}}\cdot \cancel{u}\,du$$

eliminate x^2

$$= \frac{u^3}{3} - u + C = \boxed{\frac{(1+x^2)^{3/2}}{3} - (1+x^2)^{1/2} + C}$$

- OR : $u = 1+x^2 \longrightarrow x^2 = u-1$

$du = 2x\,dx \longrightarrow x\,dx = \frac{1}{2}\,du$

$$\int \frac{x^2}{\sqrt{1+x^2}}\cdot x\,dx = \frac{1}{2}\int \frac{u-1}{\sqrt{u}}\,du = \frac{1}{2}\int \left(u^{1/2} - u^{-1/2}\right)du$$

$$= \frac{1}{2}\left(\frac{2}{3}u^{3/2} - 2\cdot u^{1/2}\right) + C = \frac{1}{3}u^{3/2} - u^{1/2} + C$$

$$= \boxed{\frac{1}{3}\cdot(1+x^2)^{3/2} - (1+x^2)^{1/2} + C}$$

Ex 2: $\displaystyle\int \frac{x^3}{\sqrt{1+x^2}}\,dx = \int \frac{x}{\sqrt{1+x^2}} \cdot x^2\,dx$

- Try by parts

$u = x^2 \qquad dv = \dfrac{x}{\sqrt{1+x^2}}\,dx$

$du = 2x \qquad v = (1+x^2)^{1/2}$

$$\int \frac{x}{\sqrt{1+x^2}}\,dx = \frac{1}{2}\int u^{-1/2}\,du$$

$$\boxed{\begin{aligned} u &= 1+x^2 \\ du &= 2x\,dx \end{aligned}}$$

$$= u^{1/2} + C = (1+x^2)^{1/2} + C$$

$$\int \underbrace{x^2}_{u} \cdot \underbrace{\frac{x}{\sqrt{1+x^2}}\,dx}_{dv} = x^2 \cdot (1+x^2)^{1/2} - \int 2x\,(1+x^2)^{1/2}\,dx$$

do substitution $= 1 + x^2$

$$= \boxed{x^2 \cdot (1+x^2)^{1/2} - \frac{2}{3}(1+x^2)^{3/2} + C}$$

IS THIS THE SAME AS BEFORE ?

$(1+x^2-1)$

$$= \underbrace{(1+x^2)\cdot(1+x^2)^{1/2}}\, - 1\,(1+x^2)^{1/2} - \frac{2}{3}(1+x^2)^{3/2} + C$$

$$= (1+x^2)^{3/2} - (1+x^2)^{1/2} - \frac{2}{3}(1+x^2)^{3/2} + C$$

$$= \boxed{\frac{1}{3}(1+x^2)^{3/2} - (1+x^2)^{1/2} + C}$$

Yes, same as before.

Tabular method for integration by parts

Integration by Parts: $\int u \, dv = uv - \int v \, du$

taking derivatives → d   I ← taking Integrals

d		I
u	$+$	dv
du	$-$	v

$+uv - \int v \, du$

Ex 3: $\int x^2 \cos(x) \, dx$ $= +x^2 \sin x - 2x \cdot (-\cos x) + 2 \cdot (-\sin x)$

$- \int \cancel{0 \cdot (-\sin x)} \, dx + C$

d		I
x^2	$+$	$\cos x$
$2x$	$-$	$\sin x$
2	$+$	$-\cos x$
0	$-$	$-\sin x$

d		I
x^2	$+$	$\cos x$
$2x$	$-$	$\sin x$
2	$+$	$-\cos x$

$= x^2 \sin x - 2x(-\cos x) + \int 2(-\cos x) \, dx$