Fast and Guaranteed Safe Controller Synthesis for Nonlinear Vehicle Models

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Appendix

A System models for testing

A.1 Lyapunov controller for a bijective mobile robot

The bijective mobile robot is one of the models used in testing the 2D scenarios. The kinematics for a mobile robot are given by

$$\dot{q} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}. \tag{1}$$

The model can be made bijective by using the states $s = \sin\theta$ and $c = \cos\theta$ in place of θ. The kinematic equation becomes

$$\dot{q} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{s} \\ \dot{c} \end{bmatrix} = \begin{bmatrix} c & 0 \\ s & 0 \\ 0 & c \\ 0 & s \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}. \tag{2}$$

When a reference trajectory is introduced, the error states are given by

$$\begin{aligned} e_x &= c(x_{\mathsf{ref}} - x) + s(y_{\mathsf{ref}} - y) \\ e_y &= -s(x_{\mathsf{ref}} - x) + c(y_{\mathsf{ref}} - y) \\ e_s &= \sin(\theta_{\mathsf{ref}} - \theta) = s_{\mathsf{ref}}c - c_{\mathsf{ref}}s \\ e_c &= \cos(\theta_{\mathsf{ref}} - \theta) = c_{\mathsf{ref}}c + s_r s - 1. \end{aligned} \tag{3}$$

From [?], the following Lyapunov function is proposed:

$$V = \frac{k}{2}(e_x^2 + e_y^2) + \frac{1}{2(1 + \frac{e_c}{a})}(e_s^2 + e_c^2) \tag{4}$$

with $k > 0$ and $a > 2$ are constants. The range of e_c is $[-2, 0]$ and therefore $0 < \frac{a-2}{a} \leq 1 + \frac{e_c}{a} \leq q$ and $1 \leq \frac{1}{1+\frac{e_c}{a}} \leq \frac{a}{a-2}$. The Lyapunov function in 4 has the derivative

$$\dot{V} = -ke_x v_b + e_s \left(kv_r e_y - \frac{w_b}{(1+\frac{e_c}{a})^2} \right) \tag{5}$$

which is negative semi-definite with the control law

$$\begin{aligned} v_b &= k_x e_x \\ w_b &= kv_r e_y (1 + \frac{e_c}{a})^2 + k_s e_s \left[\left(1 + \frac{e_c}{a}\right)^2 \right]^n . \end{aligned} \tag{6}$$

It can be checked that $e_s^2 + e_c^2 = -2e_c \in [0, 4]$. The term $\frac{1}{2(1+\frac{e_c}{a})}(e_s^2 + e_c^2)$ in V can also be bounded with $\frac{1}{2(1+\frac{e_c}{a})}(e_s^2 + e_c^2) \in [2, \frac{2a}{a-2}]$.

A.2 Lyapunov-based controller for an autonomous underwater vehicle

The autonomous underwater vehicle is one of the models used for testing the 3D scenarios. The position of the AUV is $\mathbf{x} = \begin{bmatrix} x\ y\ z \end{bmatrix}^\top$, and the Euler angles (roll, pitch, and yaw respectively) are $\boldsymbol{\theta} = \begin{bmatrix} \phi\ \theta\ \psi \end{bmatrix}^\top$. The equations of motion for the position and Euler angles are given by

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \cos\psi\cos\theta \\ \sin\psi\cos\theta \\ \sin\theta \end{bmatrix} v \tag{7}$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} . \tag{8}$$

The angular acceleration $\boldsymbol{\omega} = \begin{bmatrix} \omega_x\ \omega_y\ \omega_z \end{bmatrix}^\top$ is given in the local frame. By combining the equations of motion for the position and Euler angles, the total kinematis are given by

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\boldsymbol{\theta}} \end{bmatrix} = \begin{bmatrix} \mathbf{b_1} & 0_{3\times3} \\ 0_{1\times3} & \mathbf{B_2} \end{bmatrix} \begin{bmatrix} u_1 \\ \mathbf{u_2} \end{bmatrix} \tag{9}$$

where the $m \times n$ zero matrix is denoted by $0_{m\times n}$ and

$$\begin{aligned} \mathbf{b_1} &= \begin{bmatrix} \cos\psi\cos\theta \\ \sin\psi\cos\theta \\ -\sin\theta \end{bmatrix} \\ \mathbf{B_2} &= \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{bmatrix} . \end{aligned} \tag{10}$$

The error is defined as

$$\mathbf{x_e} = \mathbf{R}^\top(\mathbf{x_{ref}} - \mathbf{x}) \tag{11}$$

$$\boldsymbol{\theta}_e = \boldsymbol{\theta}_{\mathsf{ref}} - \boldsymbol{\theta} \tag{12}$$

where $\mathbf{R}$ is the rotation matrix

$$\mathbf{R} = \begin{bmatrix} c\psi c\theta & c\psi s\theta s\phi - s\psi c\phi & c\psi s\theta c\phi + s\psi s\phi \\ s\psi c\theta & s\psi s\theta s\phi + c\psi c\phi & s\psi s\theta c\phi - c\psi s\phi \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix}. \tag{13}$$

Note that $c\theta$ and $s\theta$ denote $\cos\theta$ and $\sin\theta$ respectively.

Consider the Lyapunov function proposed in [**?**].

$$V = \frac{1}{2}\mathbf{x_e}^\top\mathbf{x_e} + \mathbf{k}^\top\mathbf{f}(\boldsymbol{\theta}_e) \tag{14}$$

where $\mathbf{k} = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}^\top$ is the controller gains vector and $\mathbf{f}(\boldsymbol{\theta}_e) = \begin{bmatrix} 1-\cos\phi_e & 1-\cos\theta_e & 1-\cos\psi_e \end{bmatrix}^\top$ is a vector-valued function. The time derivative of 14 is

$$\dot{V} = \mathbf{x_e}^\top\dot{\mathbf{x}}_e + \mathbf{k}^\top\frac{d\mathbf{f}}{d\boldsymbol{\theta}_e}\dot{\boldsymbol{\theta}}_e. \tag{15}$$

The error dynamics are given as

$$\begin{aligned} \dot{\mathbf{x}}_e &= \mathbf{b}_{1e}u_{1d} - \mathbf{R}^\top\mathbf{b}_1u_1 - \boldsymbol{\omega}\times\mathbf{x}_e \\ \dot{\boldsymbol{\theta}}_e &= \dot{\boldsymbol{\theta}}_{\mathsf{ref}} - \dot{\boldsymbol{\theta}} = \mathbf{B}_{2ref}\mathbf{u}_2d - \mathbf{B}_2\mathbf{u}_2. \end{aligned} \tag{16}$$

When 16 is substituted into 15, it becomes

$$\begin{aligned} \dot{V} = \mathbf{p_e}^\top\{&\mathbf{q} + (\mathbf{B_{2ref}} - \mathbf{B_2})\mathbf{u_{2ref}} - \mathbf{B_2}\mathbf{u_{2ref}}\} \\ &+ \mathbf{x_e}\{v_{\mathsf{ref}}(\cos\psi_e\cos\theta_e - 1) - v_b\} \end{aligned} \tag{17}$$

The feedback control law is chosen to be

$$\begin{aligned} u_1 &= v_{\mathsf{ref}} + v_b \\ \mathbf{u_2} &= \mathbf{u_{2ref}} + \mathbf{u_{2b}} \end{aligned} \tag{18}$$

where the subscript b denotes the feedback terms. The feedback terms are chosen to be

$$\begin{aligned} v_b &= v_{\mathsf{ref}}(\cos\psi_e\cos\theta_e - 1) + \gamma^2x_e \\ \mathbf{u_{2b}} &= \mathbf{B_2}^{-1}\{\mathbf{q} + (\mathbf{B_{2ref}} - \mathbf{B_2})\mathbf{u_{2ref}} + \mathbf{p_e}\} \end{aligned} \tag{19}$$

where γ is a chosen constant, x_e is the error in the local x position, $\mathbf{q} = \begin{bmatrix} 0 & -z_ev_{\mathsf{ref}}/k_2 & y_ev_{\mathsf{ref}}\cos\theta_e/k_3 \end{bmatrix}^\top$, and $\mathbf{p_e} = \begin{bmatrix} k_1\sin\phi_e & k_2\sin\theta_e & k_3\sin\psi_e \end{bmatrix}^\top$. When the inputs are substituted into 17, the time derivative becomes

$$\dot{V} = \mathbf{p}_e^\top\mathbf{p}_e - \gamma^2x_e^2 \tag{20}$$

which is negative semi-definite. We could also see that term $\mathbf{k}^\top\mathbf{f}(\boldsymbol{\theta}_e) \in [0, 2(k_1 + k_2 + k_3)]$ for any $\boldsymbol{\theta}_e$.

A.3 Lyapunov controller for a hovering kinematic car

The hovering kinematic car is one of the models used in testing the 3D scenarios. The x, y, and θ states are the same as the kinematic car. A fourth state z is added to allow the car to hover. The kinematics for the hovering kinematic car are given by

$$\dot{q} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & 0 \\ \sin\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ v_z \\ \omega \end{bmatrix} \tag{21}$$

where v is the velocity in the xy-plane, v_z is the velocity along the z-axis, and ω is the rate of turning. When a reference trajectory is introduced, the error states are given by

$$\begin{aligned} e_x &= \cos\theta(x_r - x) + \sin\theta(y_r - y) \\ e_y &= -\sin\theta(x_r - x) + \cos\theta(y_r - y) \\ e_z &= z_r - z \\ e_\theta &= \theta_r - \theta. \end{aligned} \tag{22}$$

The following Lyapunov function is proposed:

$$V = \frac{1}{2}(x_e^2 + y_e^2 + z_e^2) + \frac{(1 - \cos\theta_e)}{k_2} \tag{23}$$

with the time derivative

$$\dot{V} = -k_1 x_e^2 - k_4 z_e^2 - \frac{v_{\mathsf{ref}} k_3 \sin^2\theta_e}{k_2}. \tag{24}$$

This time derivative is negative semi-definite when $k_1, k_2, k_3, k_4 > 0$ and the control law is given by:

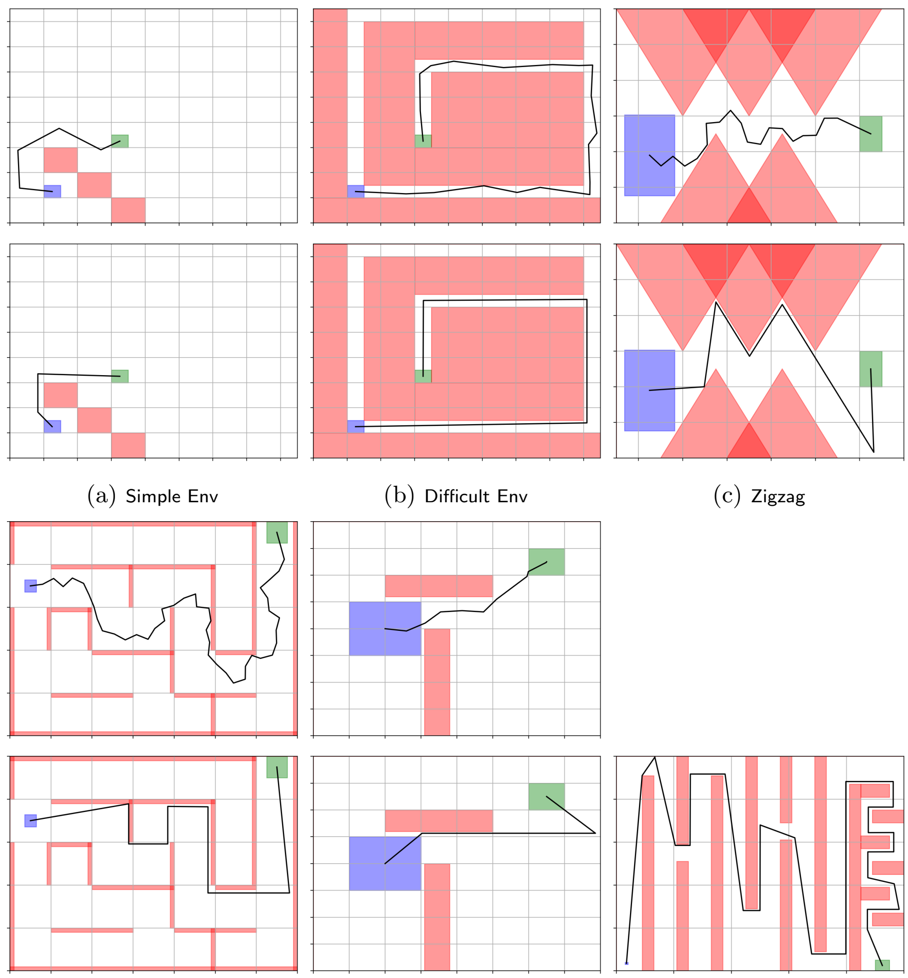
$$\begin{aligned} v &= v_{\mathsf{ref}} \cos\theta_e + k_1 x_e \\ \omega &= \omega_{\mathsf{ref}} + v_{\mathsf{ref}}(k_2 y_e + k_3 \sin\theta_e) \\ v_z &= v_{z,\mathsf{ref}} + k_4 z_e \end{aligned} \tag{25}$$

We can also see that the term $\frac{(1-\cos\theta_e)}{k_2} \in [0, \frac{2}{k_2}]$ for any θ_e.

B Example solutions to various scenarios

In this appendix there are some example solutions to the various scenarios found in this paper.

The comparison of RRT to SAT-Plan was run without bloating the polytopes to see if a solution could be found. It should be noted that the RRT algorithm used was a very basic algorithm from [**?**]. The paths for RRT could be optimized if a different algorithm such as RRT* was used; however, this would increase the running time of the algorithm.

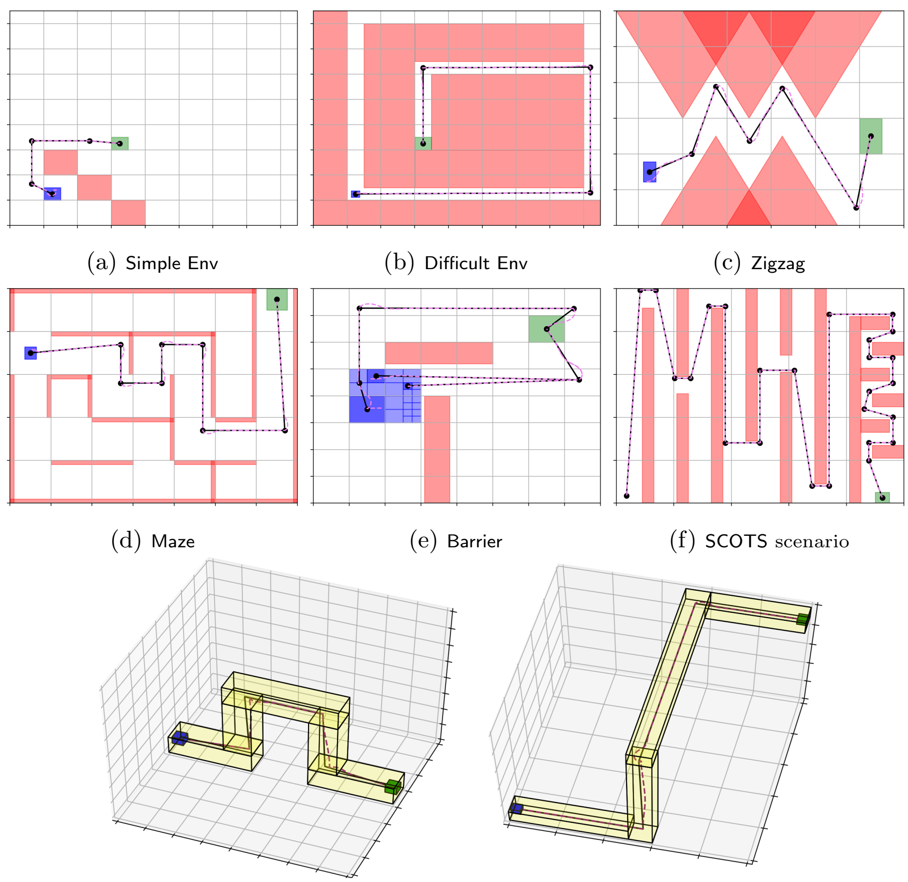


(a) Simple Env (b) Difficult Env (c) Zigzag

(d) Maze (e) Barrier (f) SCOTS scenario

Fig. 1: Example solutions using RRT vs. SAT-Plan. The top figures show the RRT solution and the bottom figures show the SAT-Plan solution. Note that RRT timed out in the SCOTS scenario.

Below are some example solutions to the reach-avoid scenarios. The scenarios run in the two-dimensional $\mathcal{W}$ use the car model. The scenarios run in the three dimensional $\mathcal{W}$ use the hovercraft model. The black lines denote ξ_{ref} and the dotted violet lines denote ξ_g.



(a) Simple Env (b) Difficult Env (c) Zigzag (d) Maze (e) Barrier (f) SCOTS scenario (g) L-tunnel (h) Z-tunnel

Fig. 2: Example solutions to the scenarios using FACTEST. The initial set is seen in blue, the goal set is seen in green, and the obstacle set is shown in red.