HW 1 Group 7 MLP & Backpropagation

1.1 and 1.2 Forward Pass

First Layer:

Drive:

$$\begin{aligned}\vec{d}^{(1)} &= W^{(1)} \cdot \vec{x} + \vec{b} \\ &= \begin{pmatrix} -2 & 3 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} (-2) \cdot 2 + 3 \cdot 3 + 0 \\ 2 \cdot 2 + 1 \cdot 3 + 0 \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ 7 \end{pmatrix}\end{aligned}$$

Activation:

$$\vec{a}^{(1)} = \sigma(\vec{d}^{(1)}) = \begin{pmatrix} 5 \\ 7 \end{pmatrix}^2 = \begin{pmatrix} 25 \\ 49 \end{pmatrix}$$

Final Layer:

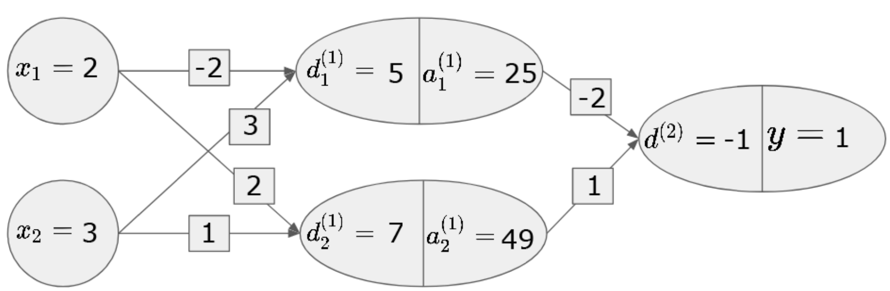
Drive:

$$d^{(2)} = \vec{w}^{(2)} \cdot \vec{a}^{(1)} = \begin{pmatrix} -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 25 \\ 49 \end{pmatrix} = (-2) \cdot 25 + 1 \cdot 49 = -50 + 49 = -1$$

Activation:

$$y = \sigma(d^{(2)}) = (-1)^2 = 1$$

Illustration:



1.3 Backpropagation

a) Calculate the error:

$$L(t; y) = \frac{1}{2}(t - y)^2 = \frac{1}{2}(3 - 1)^2 = \frac{1}{2}(2)^2 = \frac{1}{2} \cdot 4 = 2$$

b) Derive the activation function:

$$\begin{aligned} \sigma(x) &= x^2 \\ \sigma'(x) &= 2x \end{aligned}$$

c) Compute the partial derivatives of the loss function with regard to each weight, using the backpropagation alogrithm:

General rule *(see lecture slides)*:

$$\frac{\partial L(\vec{t}; \vec{y})}{\partial w_{ji}^{(K)}} = \delta_j^{(K)} \cdot a_i^{(K-1)}$$

where:

$$\delta_j^{(K)} = \begin{cases} -(t_j - y_j) \cdot \sigma'(net_j^{(K)}), & \text{if } K = L + 1 \\ \sum_k \delta_k^{(K+1)} \cdot w_{kj}^{(K+1)} \cdot \sigma'(net_j^{(K)}), & \text{otherwise} \end{cases}$$

Precalculations:

$$\delta_1^{(2)} = -(t_1 - y_1) \cdot \sigma'(net_1^{(2)}) = -(3 - 1) \cdot 2(-1) = -2 \cdot -2 = 4$$

$$\delta_1^{(1)} = \sum_{k=1}^{1} \delta_k^{(2)} \cdot w_{k1}^{(2)} \cdot \sigma'(net_1^{(1)}) = \delta_1^{(2)} \cdot w_{11}^{(2)} \cdot 2(5) = 4 \cdot (-2) \cdot 10 = -80$$

$$\delta_2^{(1)} = \sum_{k=1}^{1} \delta_k^{(2)} \cdot w_{k2}^{(2)} \cdot \sigma'(net_2^{(1)}) = \delta_1^{(2)} \cdot w_{12}^{(2)} \cdot 2(7) = 4 \cdot 1 \cdot 14 = 56$$

Partial derivatives of the loss function with regard to each weight:

$$\frac{\partial L(\vec{t};\vec{y})}{\partial w_{11}^{(2)}} = \delta_1^{(2)} \cdot a_1^{(1)} = 4 \cdot 25 = 100$$
$$\frac{\partial L(\vec{t};\vec{y})}{\partial w_{12}^{(2)}} = \delta_1^{(2)} \cdot a_2^{(1)} = 4 \cdot 49 = 196$$
$$\frac{\partial L(\vec{t};\vec{y})}{\partial w_{11}^{(1)}} = \delta_1^{(1)} \cdot a_1^{(0)} = -80 \cdot 2 = -160$$
$$\frac{\partial L(\vec{t};\vec{y})}{\partial w_{12}^{(1)}} = \delta_1^{(1)} \cdot a_2^{(0)} = -80 \cdot 3 = -240$$
$$\frac{\partial L(\vec{t};\vec{y})}{\partial w_{21}^{(1)}} = \delta_2^{(1)} \cdot a_1^{(0)} = 56 \cdot 2 = 112$$
$$\frac{\partial L(\vec{t};\vec{y})}{\partial w_{22}^{(1)}} = \delta_2^{(1)} \cdot a_2^{(0)} = 56 \cdot 3 = 168$$

d) Compute the weight update for each weight

General update rule:

$$\theta_{new} = \theta_{old} - \gamma \nabla_\theta L(\vec{t};\vec{y})$$

Individual weight updates:

$$\begin{aligned}
w_{11}^{(2)} &= -2 - 0.01 \cdot 100 &&= -3 \\
w_{12}^{(2)} &= 1 - 0.01 \cdot 196 &&= -0.96 \\
w_{11}^{(1)} &= -2 - 0.01 \cdot (-160) &&= -0.4 \\
w_{12}^{(1)} &= 3 - 0.01 \cdot (-240) &&= 5.4 \\
w_{21}^{(1)} &= 2 - 0.01 \cdot 112 &&= 0.88 \\
w_{22}^{(1)} &= 1 - 0.01 \cdot 168 &&= -0.68
\end{aligned}$$