
Introduction to Information Retrieval

— Latent Semantic Indexing —

Outline

- ① **Latent semantic indexing**
- ② Dimensionality reduction
- ③ LSI in information retrieval

Recall: Term-document matrix

	Anthony and Cleopatra	Julius Caesar	The Tempest	Hamlet	Othello	Macbeth
anthony	5.25	3.18	0.0	0.0	0.0	0.35
brutus	1.21	6.10	0.0	1.0	0.0	0.0
caesar	8.59	2.54	0.0	1.51	0.25	0.0
calpurnia	0.0	1.54	0.0	0.0	0.0	0.0
cleopatra	2.85	0.0	0.0	0.0	0.0	0.0
mercy	1.51	0.0	1.90	0.12	5.25	0.88
worser	1.37	0.0	0.11	4.15	0.25	1.95
...						

This matrix is the basis for computing the similarity between documents and queries. Today: Can we transform this matrix, so that we get a better measure of similarity between documents and queries?

Latent semantic indexing: Overview

- We will decompose the term-document matrix into a product of matrices.
- The particular decomposition we'll use: singular value decomposition (SVD).
- SVD: $C = U\Sigma V^T$ (where C = term-document matrix)
- We will then use the SVD to compute a new, improved term-document matrix C' .
- We'll get better similarity values out of C' (compared to C).
- Using SVD for this purpose is called latent semantic indexing or LSI.

Example of $C = U\Sigma V^T$: The matrix C

C	d_1	d_2	d_3	d_4	d_5	d_6
ship	1	0	1	0	0	0
boat	0	1	0	0	0	0
ocean	1	1	0	0	0	0
wood	1	0	0	1	1	0
tree	0	0	0	1	0	1

This is a standard term-document matrix. Actually, we use a non-weighted matrix here to simplify the example.

Example of $C = U\Sigma V^T$: The matrix U

U	1	2	3	4	5
ship	−0.44	−0.30	0.57	0.58	0.25
boat	−0.13	−0.33	−0.59	0.00	0.73
ocean	−0.48	−0.51	−0.37	0.00	−0.61
wood	−0.70	0.35	0.15	−0.58	0.16
tree	−0.26	0.65	−0.41	0.58	−0.09

- One row per term, one column per $\min(M,N)$ where M is the number of terms and N is the number of documents.
- This is an orthonormal matrix.
- Row vectors are unit length vectors, orthogonal to each other.
- Dimensions are “Semantic” dimensions, capture distinct topics like politics, sports, economics.
- Each number u_{ij} in the matrix indicates how strongly related term i is to the topic represented by semantic dimension j .

Example of $C = U\Sigma V^T$: The matrix Σ

Σ	1	2	3	4	5
1	2.16	0.00	0.00	0.00	0.00
2	0.00	1.59	0.00	0.00	0.00
3	0.00	0.00	1.28	0.00	0.00
4	0.00	0.00	0.00	1.00	0.00
5	0.00	0.00	0.00	0.00	0.39

- This is a square, diagonal matrix of dimensionality $\min(M,N) \times \min(M,N)$.
- The diagonal consists of the singular values of C .
- The magnitude of the singular value measures the importance of the corresponding semantic dimension. We'll make use of this by omitting unimportant dimensions.

Example of $C = U\Sigma V^T$: The matrix V^T

V^T	d_1	d_2	d_3	d_4	d_5	d_6
1	-0.75	-0.28	-0.20	-0.45	-0.33	-0.12
2	-0.29	-0.53	-0.19	0.63	0.22	0.41
3	0.28	-0.75	0.45	-0.20	0.12	-0.33
4	0.00	0.00	0.58	0.00	-0.58	0.58
5	-0.53	0.29	0.63	0.19	0.41	-0.22

- One column per document, one row per $\min(M,N)$ where M is the number of terms and N is the number of documents.
- This is an orthonormal matrix:
 - Column vectors have unit length.
 - Any two distinct column vectors are orthogonal to each other.
- These are again the semantic dimensions from the term matrix U that capture distinct topics like politics, sports, economics.
- Each number v_{ji} in the matrix indicates how strongly related document i is to the topic represented by semantic dimension j .

Example of $C = U\Sigma V^T$: All four matrices

C	d_1	d_2	d_3	d_4	d_5	d_6	
ship	1	0	1	0	0	0	
boat	0	1	0	0	0	0	
ocean	1	1	0	0	0	0	=
wood	1	0	0	1	1	0	
tree	0	0	0	1	0	1	
U	1	2	3	4	5		
ship	-0.44	-0.30	0.57	0.58	0.25		
boat	-0.13	-0.33	-0.59	0.00	0.73		
ocean	-0.48	-0.51	-0.37	0.00	-0.61		×
wood	-0.70	0.35	0.15	-0.58	0.16		
tree	-0.26	0.65	-0.41	0.58	-0.09		
Σ	1	2	3	4	5		
1	2.16	0.00	0.00	0.00	0.00		
2	0.00	1.59	0.00	0.00	0.00		
3	0.00	0.00	1.28	0.00	0.00		×
4	0.00	0.00	0.00	1.00	0.00		
5	0.00	0.00	0.00	0.00	0.39		
V^T	d_1	d_2	d_3	d_4	d_5	d_6	
1	-0.75	-0.28	-0.20	-0.45	-0.33	-0.12	
2	-0.29	-0.53	-0.19	0.63	0.22	0.41	
3	0.28	-0.75	0.45	-0.20	0.12	-0.33	
4	0.00	0.00	0.58	0.00	-0.58	0.58	
5	-0.53	0.29	0.63	0.19	0.41	-0.22	

LSI: Summary

- We've decomposed the term-document matrix C into a product of three matrices.
- The term matrix U – consists of one (row) vector for each term
- The document matrix V^T – consists of one (column) vector for each document
- The singular value matrix Σ – diagonal matrix with singular values, reflecting importance of each dimension
- Next: Why are we doing this?

Outline

- ① Latent semantic indexing
- ② **Dimensionality reduction**
- ③ LSI in information retrieval

How we use the SVD in LSI

- Key property: Each singular value tells us how important its dimension is.
- By setting less important dimensions to zero, we keep the important information, but get rid of the “details”.
- These details may
 - be noise – in that case, reduced LSI is a better representation because it is less noisy
 - make things dissimilar that should be similar – again reduced LSI is a better representation because it represents similarity better.
- Analogy for “fewer details is better”
 - Image of a bright red flower
 - Image of a black and white flower
 - Omitting color makes it easier to see similarity

Reducing the dimensionality to 2

U	1	2	3	4	5	
ship	-0.44	-0.30	0.00	0.00	0.00	
boat	-0.13	-0.33	0.00	0.00	0.00	
ocean	-0.48	-0.51	0.00	0.00	0.00	
wood	-0.70	0.35	0.00	0.00	0.00	
tree	-0.26	0.65	0.00	0.00	0.00	
Σ_2	1	2	3	4	5	
1	2.16	0.00	0.00	0.00	0.00	
2	0.00	1.59	0.00	0.00	0.00	
3	0.00	0.00	0.00	0.00	0.00	
4	0.00	0.00	0.00	0.00	0.00	
5	0.00	0.00	0.00	0.00	0.00	
V^T	d_1	d_2	d_3	d_4	d_5	d_6
1	-0.75	-0.28	-0.20	-0.45	-0.33	-0.12
2	-0.29	-0.53	-0.19	0.63	0.22	0.41
3	0.00	0.00	0.00	0.00	0.00	0.00
4	0.00	0.00	0.00	0.00	0.00	0.00
5	0.00	0.00	0.00	0.00	0.00	0.00

- Actually, we only zero out singular values in Σ .
- This has the effect of setting the corresponding dimensions in U and V^T to zero when computing the product $C = U\Sigma V^T$.

Reducing the dimensionality to 2

C_2	d_1	d_2	d_3	d_4	d_5	d_6
ship	0.85	0.52	0.28	0.13	0.21	-0.08
boat	0.36	0.36	0.16	-0.20	-0.02	-0.18
ocean	1.01	0.72	0.36	-0.04	0.16	-0.21
wood	0.97	0.12	0.20	1.03	0.62	0.41
tree	0.12	-0.39	-0.08	0.90	0.41	0.49
U	1	2	3	4	5	
ship	-0.44	-0.30	0.57	0.58	0.25	
boat	-0.13	-0.33	-0.59	0.00	0.73	
ocean	-0.48	-0.51	-0.37	0.00	-0.61	×
wood	-0.70	0.35	0.15	-0.58	0.16	
tree	-0.26	0.65	-0.41	0.58	-0.09	
Σ_2	1	2	3	4	5	
1	2.16	0.00	0.00	0.00	0.00	
2	0.00	1.59	0.00	0.00	0.00	
3	0.00	0.00	0.00	0.00	0.00	×
4	0.00	0.00	0.00	0.00	0.00	
5	0.00	0.00	0.00	0.00	0.00	
V^T	d_1	d_2	d_3	d_4	d_5	d_6
1	-0.75	-0.28	-0.20	-0.45	-0.33	-0.12
2	-0.29	-0.53	-0.19	0.63	0.22	0.41
3	0.28	-0.75	0.45	-0.20	0.12	-0.33
4	0.00	0.00	0.58	0.00	-0.58	0.58
5	-0.53	0.29	0.63	0.19	0.41	-0.22

Recall unreduced decomposition $C = U \Sigma V^T$

C	d_1	d_2	d_3	d_4	d_5	d_6	
ship	1	0	1	0	0	0	
boat	0	1	0	0	0	0	
ocean	1	1	0	0	0	0	=
wood	1	0	0	1	1	0	
tree	0	0	0	1	0	1	
U	1	2	3	4	5		
ship	-0.44	-0.30	0.57	0.58	0.25		
boat	-0.13	-0.33	-0.59	0.00	0.73		
ocean	-0.48	-0.51	-0.37	0.00	-0.61		×
wood	-0.70	0.35	0.15	-0.58	0.16		
tree	-0.26	0.65	-0.41	0.58	-0.09		
Σ	1	2	3	4	5		
1	2.16	0.00	0.00	0.00	0.00		
2	0.00	1.59	0.00	0.00	0.00		
3	0.00	0.00	1.28	0.00	0.00		×
4	0.00	0.00	0.00	1.00	0.00		
5	0.00	0.00	0.00	0.00	0.39		
V^T	d_1	d_2	d_3	d_4	d_5	d_6	
1	-0.75	-0.28	-0.20	-0.45	-0.33	-0.12	
2	-0.29	-0.53	-0.19	0.63	0.22	0.41	
3	0.28	-0.75	0.45	-0.20	0.12	-0.33	
4	0.00	0.00	0.58	0.00	-0.58	0.58	
5	-0.53	0.29	0.63	0.19	0.41	-0.22	

Original matrix C vs. reduced $C_2 = U\Sigma_2 V^T$

C	d_1	d_2	d_3	d_4	d_5	d_6
ship	1	0	1	0	0	0
boat	0	1	0	0	0	0
ocean	1	1	0	0	0	0
wood	1	0	0	1	1	0
tree	0	0	0	1	0	1
C_2	d_1	d_2	d_3	d_4	d_5	d_6
ship	0.85	0.52	0.28	0.13	0.21	-0.08
boat	0.36	0.36	0.16	-0.20	-0.02	-0.18
ocean	1.01	0.72	0.36	-0.04	0.16	-0.21
wood	0.97	0.12	0.20	1.03	0.62	0.41
tree	0.12	-0.39	-0.08	0.90	0.41	0.49

- We can view C_2 as a two-dimensional representation of the matrix.
- We have performed a dimensionality reduction to two dimensions.

Why the reduced matrix is “better”

C	d_1	d_2	d_3	d_4	d_5	d_6
ship	1	0	1	0	0	0
boat	0	1	0	0	0	0
ocean	1	1	0	0	0	0
wood	1	0	0	1	1	0
tree	0	0	0	1	0	1
C_2	d_1	d_2	d_3	d_4	d_5	d_6
ship	0.85	0.52	0.28	0.13	0.21	-0.08
boat	0.36	0.36	0.16	-0.20	-0.02	-0.18
ocean	1.01	0.72	0.36	-0.04	0.16	-0.21
wood	0.97	0.12	0.20	1.03	0.62	0.41
tree	0.12	-0.39	-0.08	0.90	0.41	0.49

- Similarity of d_2 and d_3 in the original space:
0.
- Similarity of d_2 and d_3 in the reduced space:

$$0.52 * 0.28 + 0.36 * 0.16 + 0.72 * 0.36 + 0.12 * 0.20 + -0.39 * -0.08 \approx 0.52$$

Why the reduced matrix is “better”

C	d_1	d_2	d_3	d_4	d_5	d_6
ship	1	0	1	0	0	0
boat	0	1	0	0	0	0
ocean	1	1	0	0	0	0
wood	1	0	0	1	1	0
tree	0	0	0	1	0	1
C_2	d_1	d_2	d_3	d_4	d_5	d_6
ship	0.85	0.52	0.28	0.13	0.21	-0.08
boat	0.36	0.36	0.16	-0.20	-0.02	-0.18
ocean	1.01	0.72	0.36	-0.04	0.16	-0.21
wood	0.97	0.12	0.20	1.03	0.62	0.41
tree	0.12	-0.39	-0.08	0.90	0.41	0.49

- “boat” and “ship” are semantically similar.
- The “reduced” similarity measure reflects this.
- What property of the SVD reduction is responsible for improved similarity?

Why the reduced matrix is “better”

C	d_1	d_2	d_3	d_4	d_5	d_6
ship	1	0	1	0	0	0
boat	0	1	0	0	0	0
ocean	1	1	0	0	0	0
wood	1	0	0	1	1	0
tree	0	0	0	1	0	1
C_2	d_1	d_2	d_3	d_4	d_5	d_6
ship	0.85	0.52	0.28	0.13	0.21	-0.08
boat	0.36	0.36	0.16	-0.20	-0.02	-0.18
ocean	1.01	0.72	0.36	-0.04	0.16	-0.21
wood	0.97	0.12	0.20	1.03	0.62	0.41
tree	0.12	-0.39	-0.08	0.90	0.41	0.49

Outline

- ① Latent semantic indexing
- ② Dimensionality reduction
- ③ **LSI in information retrieval**

Implementation

- Compute SVD of term-document matrix
- Reduce the space and compute reduced document representations

Reducing the dimensionality to 2

C_2	d_1	d_2	d_3	d_4	d_5	d_6
ship	0.85	0.52	0.28	0.13	0.21	-0.08
boat	0.36	0.36	0.16	-0.20	-0.02	-0.18
ocean	1.01	0.72	0.36	-0.04	0.16	-0.21
wood	0.97	0.12	0.20	1.03	0.62	0.41
tree	0.12	-0.39	-0.08	0.90	0.41	0.49
U	1	2	3	4	5	
ship	-0.44	-0.30	0.57	0.58	0.25	
boat	-0.13	-0.33	-0.59	0.00	0.73	
ocean	-0.48	-0.51	-0.37	0.00	-0.61	\times
wood	-0.70	0.35	0.15	-0.58	0.16	
tree	-0.26	0.65	-0.41	0.58	-0.09	
Σ_2	1	2	3	4	5	
1	2.16	0.00	0.00	0.00	0.00	
2	0.00	1.59	0.00	0.00	0.00	
3	0.00	0.00	0.00	0.00	0.00	\times
4	0.00	0.00	0.00	0.00	0.00	
5	0.00	0.00	0.00	0.00	0.00	
V^T	d_1	d_2	d_3	d_4	d_5	d_6
1	-0.75	-0.28	-0.20	-0.45	-0.33	-0.12
2	-0.29	-0.53	-0.19	0.63	0.22	0.41
3	0.28	-0.75	0.45	-0.20	0.12	-0.33
4	0.00	0.00	0.58	0.00	-0.58	0.58
5	-0.53	0.29	0.63	0.19	0.41	-0.22

$$C = U\Sigma V^T$$

Implementation

- Compute SVD of term-document matrix
- Reduce the space and compute reduced document representations
- Map the query into the reduced space $\vec{q}_2^T = \Sigma_2^{-1} U_2^T \vec{q}^T$.
- This follows from: $C_2 = U \Sigma_2 V^T \Rightarrow \Sigma_2^{-1} U^T C = V_2^T$
- Compute similarity of q_2 with all reduced documents in V_2 .
- Output ranked list of documents as usual
- Exercise: What is the fundamental problem with this approach?

Why we use LSI in information retrieval

- LSI takes documents that are semantically similar (= talk about the same topics), . . .but are not similar in the vector space (because they use different words) and re-represents them in a reduced vector space in which they have higher similarity.
- Thus, LSI addresses the problems of synonymy and semantic relatedness.
- Standard vector space: Synonyms contribute nothing to document similarity.
- Desired effect of LSI: Synonyms contribute strongly to document similarity.

How LSI addresses synonymy and semantic relatedness

- The dimensionality reduction forces us to omit a lot of “detail”.
- We have to map different words (= different dimensions of the full space) to the same dimension in the reduced space.
- The “cost” of mapping synonyms to the same dimension is much less than the cost of collapsing unrelated words.
- SVD maps synonyms to the same dimension.
- But it will avoid doing that for unrelated words.

LSI: Comparison to other approaches

- Recap: Relevance feedback and query expansion are used to increase recall in information retrieval – if query and documents have (in the extreme case) no terms in common.
- LSI increases recall and hurts precision.
- Thus, it addresses the same problems as (pseudo) relevance feedback and query expansion . . .

LSI: Comparison to other approaches

- As we reduce k , recall tends to increase, as expected.
- Most surprisingly, a value of k in the low hundreds can actually increase precision on some query benchmarks. This appears to suggest that for a suitable value of k , LSI addresses some of the challenges of synonymy.
- Computational Cost of LSI is high.

Optimality

- SVD is optimal in the following sense.
- Keeping the k largest singular values and setting all others to zero gives you the optimal approximation of the original matrix C . (Eckart-Young theorem)
- Optimal: no other matrix of the same rank (= with the same underlying dimensionality) approximates C better.
$$\|C\|_F = \sqrt{\sum_i \sum_j c_{ij}^2}$$
- Measure of approximation is Frobenius norm:
for the error $(C - C_k)$
- So LSI uses the “best possible” matrix.
- Caveat: There is only a tenuous relationship between the Frobenius norm and cosine similarity between documents.