图像处理：第一次作业

Illusionna　2025xxxxxxx004　人工智能学院

19:00, Wednesday 5th November, 2025

目录

1　灰度级变换函数

(a)　对比度展宽变换连续函数

$$s = T(r) = \frac{r^E}{r^E + m^E} = \frac{1}{1 + \left(\dfrac{m}{r}\right)^E}$$

其中，r 的取值范围在 $[0,\ L-1]$ 之内，m 是灰度级数中值 $\dfrac{L}{2}$，E 是控制曲线陡峭程度。对变换函数求导可得：

$$\frac{\mathrm{d}s}{\mathrm{d}r} = \frac{\mathrm{d}T(r)}{\mathrm{d}r} = \frac{E r^{E-1} m^E}{\left(r^E + m^E\right)^2}$$

则：

$$\left.\frac{\mathrm{d}s}{\mathrm{d}r}\right|_{r=m} = \frac{E}{4m}$$

即在 $r = m$ 的时候，变换函数的斜率达到 $\dfrac{E}{4m}$。

(b)　关于参数 E 的变换函数

$$T(r; E) = \frac{r^E}{r^E + m^E} \qquad E > 0$$

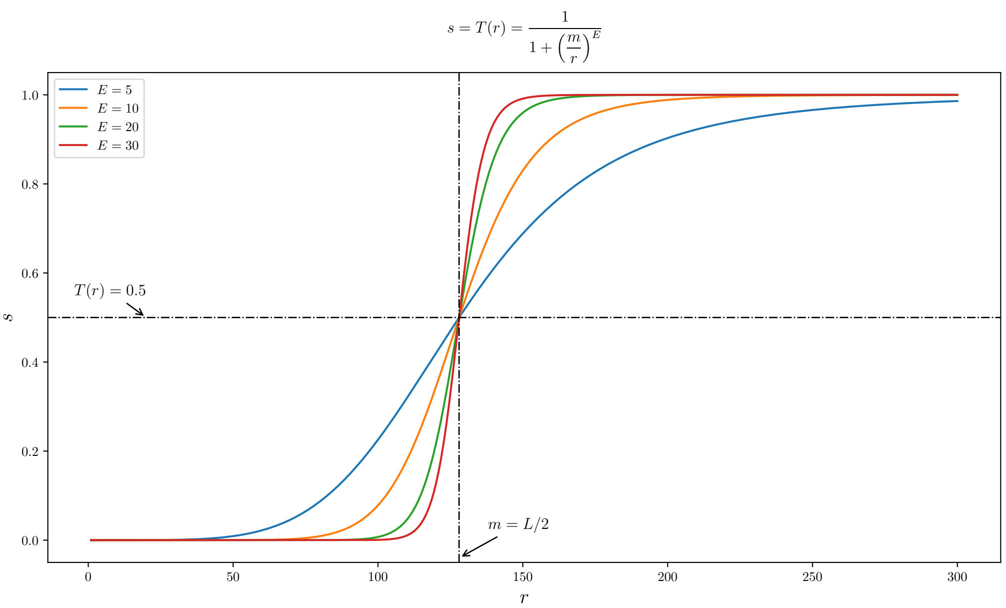

图 1: 不同参数 E 下的变换函数

```python
import numpy as np
import matplotlib.pyplot as plt
plt.rcParams['font.family'] = 'Times New Roman'
plt.rcParams['text.usetex'] = True
plt.rcParams['text.latex.preamble'] = r'\usepackage{amsmath}'
L = 256
m = L / 2
T = lambda r, E: 1 / (1 + (m / r) ** E)
X = np.linspace(1, 300, 1000)
plt.plot(X, T(X, 5), label = r'$E=5$')
plt.plot(X, T(X, 10), label = r'$E=10$')
plt.plot(X, T(X, 20), label = r'$E=20$')
plt.plot(X, T(X, 30), label = r'$E=30$')
plt.axvline(x = m, color = 'black', linestyle = '-.', linewidth =
    1)
plt.axhline(y = 0.5, color = 'black', linestyle = '-.', linewidth =
    1)
plt.annotate(
    r'$T(r)=0.5$',
    xy = (20, 0.5), xytext = (20, 0.55),
    arrowprops = dict(arrowstyle = '->', color = 'black'),
    ha = 'right', fontsize = 12
)
plt.annotate(
    r'$m=L/2$',
    xy = (L / 2, -0.04), xytext = (L / 2 + 10, 0.025),
    arrowprops = dict(arrowstyle = '->', color = 'black'),
    ha = 'left', fontsize = 12
)
plt.xlabel(r'$r$', fontsize = 15)
plt.ylabel(r'$s$', fontsize = 15)
plt.title(r'$s=T(r)=\dfrac{1}{1+\left(\dfrac{m}{r}\right)^E}$', pad
    = 30)
plt.legend()
plt.show()
```

2 直方图

$$\texttt{data} = \{0:0.17,\ 1:0.25,\ 2:0.21,\ 3:0.16,\ 4:0.07,\ 5:0.08,\ 6:0.04,\ 7:0.02\}$$

根据累计直方图公式，可得表 1 所示。

$$s_n = \sum_{i=0}^{n} \Pr(r_i)$$

其中，灰度均衡化公式：

$$g = \lfloor (L-1) \times s_n + 0.5 \rfloor$$

表 1: 解题步骤

灰度级	0	1	2	3	4	5	6	7
直方图概率	0.17	0.25	0.21	0.16	0.07	0.08	0.04	0.02
累计概率	0.17	0.42	0.63	0.79	0.86	0.94	0.98	1
均衡化	1	3	4	6	6	7	7	7
映射	$0 \to 1$	$1 \to 3$	$2 \to 4$	$3 \to 6$	$4 \to 6$	$5 \to 7$	$6 \to 7$	$7 \to 7$
新直方图		0.17		0.25	0.21		0.23	0.14

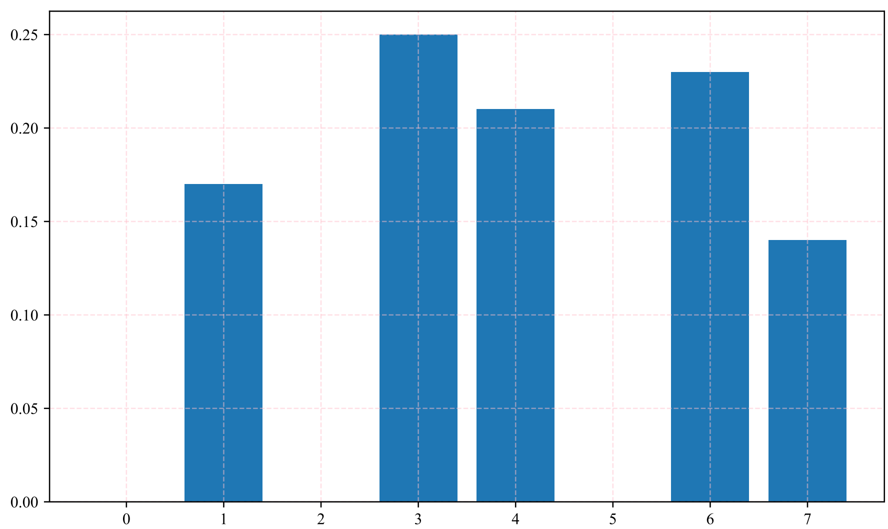

图 2: 均衡化后的直方图

3 高斯概率密度函数均衡化的变换函数

概率密度函数的累积是分布函数，高斯概率密度函数取值范围是 $(-\infty, +\infty)$，而灰度值 r 只有正实数，由于正态分布对称轴是 $r = m$，方差是 σ^2，所以当 $r < 0$ 时，面积 $\displaystyle\int_{-\infty}^{0} \mathrm{Pr}(x)\mathrm{d}x \approx 0$ 即可被忽略。

$$\begin{aligned}
s = T(r) &= \int_{-\infty}^{r} \mathrm{Pr}(x)\mathrm{d}x \\
&= \int_{-\infty}^{0} \mathrm{Pr}(x)\mathrm{d}x + \int_{0}^{r} \mathrm{Pr}(x)\mathrm{d}x \\
&\approx \frac{1}{\sqrt{2\pi}\sigma} \int_{0}^{r} \mathrm{e}^{-\dfrac{(x-m)^2}{2\sigma^2}} \mathrm{d}x
\end{aligned}$$

由于高斯概率密度直方图是通过对连续函数采样得到的，为了将累积分布函数应用到图像像素值，需要将它归一化到 $[0, 255]$ 范围内，以确保其适配灰度级范围，用于映射原始像素值:

$$r_{\mathrm{new}} = s_{\mathrm{normalized}} = \lfloor 255 \times s + 0.5 \rfloor = \left\lfloor \frac{255}{\sqrt{2\pi}\sigma} \int_{0}^{r} \mathrm{e}^{-\dfrac{(x-m)^2}{2\sigma^2}} \mathrm{d}x + 0.5 \right\rfloor$$

4 求解灰度变换表达式

$$\begin{aligned}
\mathrm{Pr}(r) = -2r + 2 &\Longrightarrow \int_{0}^{r} (-2x+2)\mathrm{d}x = -r^2 + 2r \\
\mathrm{Pr}(z) = 2z &\Longrightarrow \int_{0}^{z} 2x\mathrm{d}x = z^2
\end{aligned}$$

$$\begin{aligned}
&\because \quad \int_{0}^{r} (-2x+2)\mathrm{d}x = \int_{0}^{z} 2x\mathrm{d}x \\
&\therefore \quad z^2 = -r^2 + 2r \qquad (r, z > 0) \\
&\therefore \quad z = \sqrt{-r^2 + 2r}
\end{aligned}$$

5 卷积

定义 向量 f 和向量 g 在实数域空间 $\mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$ 上的卷积运算 $\circledast$

$$(f \circledast g)_k \triangleq \sum_{i=0}^{n-1} f_i \cdot g_{k-i} \qquad k = 0, 1, 2, \ldots, n-1$$

(a) 向量卷积

step 1： 卷积序列颠倒。

step 2： 从左向右单位平滑卷积序列，与输入序列进行乘法叠加运算。

$$[1,2,3,4,5,4,3,2,1] \circledast [2,0,-2] = \begin{cases} (2,4,4,4,4,0,-4,-4,-4,-4,-2) & \text{full} \\ (4,4,4,4,0,-4,-4,-4,-4) & \text{same} \end{cases}$$

(b) 矩阵卷积

step 1： 卷积滤核颠倒。

step 2： 从左向右从上到下单位平滑卷积核，与输入矩阵进行乘法叠加运算。

$$\begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} \circledast \begin{bmatrix} 1 & 3 & 2 & 0 & 4 \\ 1 & 0 & 3 & 2 & 3 \\ 0 & 4 & 1 & 0 & 5 \\ 2 & 3 & 2 & 1 & 4 \\ 3 & 1 & 0 & 4 & 2 \end{bmatrix} = \begin{cases} \begin{bmatrix} -1 & -3 & -1 & 3 & -2 & 0 & 4 \\ -3 & -6 & -4 & 4 & -4 & 2 & 11 \\ -3 & -7 & -6 & 3 & -6 & 4 & 15 \\ -3 & -11 & -4 & 8 & -10 & 3 & 17 \\ -7 & -11 & 2 & 5 & -10 & 6 & 15 \\ -8 & -5 & 6 & -4 & -6 & 9 & 8 \\ -3 & -1 & 3 & -3 & -2 & 4 & 2 \end{bmatrix} & \text{full} \\ \begin{bmatrix} -6 & -4 & 4 & -4 & 2 \\ -7 & -6 & 3 & -6 & 4 \\ -11 & -4 & 8 & -10 & 3 \\ -11 & 2 & 5 & -10 & 6 \\ -5 & 6 & -4 & -6 & 9 \end{bmatrix} & \text{same} \end{cases}$$

6 证明拉普拉斯算子旋转不变性

定义 任意阶微分都是线性操作，所以拉普拉斯变换也是一个线性操作，各向同性微分的拉普拉斯算子可以作用在矩阵上，二元图像函数 $f(x,y)$ 的拉普拉斯变换

$$\boldsymbol{\nabla}^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

轴旋转 θ 角坐标方程：

$$\begin{cases} x = x^{'}\cos\theta - y^{'}\sin\theta \\ y = x^{'}\sin\theta + y^{'}\cos\theta \end{cases}$$

其中 $(x,\ y)$ 是非旋转坐标，而 $(x^{'},\ y^{'})$ 是旋转坐标。

$$\begin{aligned}
\boldsymbol{\nabla}^2 f &= \frac{\partial^2 f}{\partial x'^2} + \frac{\partial^2 f}{\partial y'^2} \\
&= \frac{\partial}{\partial x'}\left(\frac{\partial f}{\partial x'}\right) + \frac{\partial}{\partial y'}\left(\frac{\partial f}{\partial y'}\right) \\
&= \frac{\partial}{\partial x'}\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x'}\right) + \frac{\partial}{\partial y'}\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial y'}\right) \\
&= \frac{\partial}{\partial x'}\left(\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\right) + \frac{\partial}{\partial y'}\left(-\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\right)\frac{\partial x}{\partial x'} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\right)\frac{\partial y}{\partial x'} \\
&\quad + \frac{\partial}{\partial x}\left(-\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta\right)\frac{\partial x}{\partial y'} + \frac{\partial}{\partial y}\left(-\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta\right)\frac{\partial y}{\partial y'} \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\right)\cos\theta + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\right)\sin\theta \\
&\quad - \frac{\partial}{\partial x}\left(-\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta\right)\sin\theta + \frac{\partial}{\partial y}\left(-\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta\right)\cos\theta \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\cos^2\theta + \frac{\partial f}{\partial y}\sin\theta\cos\theta\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\cos\theta\sin\theta + \frac{\partial f}{\partial y}\sin^2\theta\right) \\
&\quad + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\sin^2\theta - \frac{\partial f}{\partial y}\cos\theta\sin\theta\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\cos^2\theta - \frac{\partial f}{\partial x}\sin\theta\cos\theta\right) \\
&= \frac{\partial^2 f}{\partial x^2}\cos^2\theta + \frac{\partial^2 f}{\partial x^2}\sin^2\theta + \frac{\partial^2 f}{\partial y^2}\sin^2\theta + \frac{\partial^2 f}{\partial y^2}\cos^2\theta \\
&= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \\
&= \boldsymbol{\nabla}^2 f
\end{aligned}$$