Homework #1

Illya Starikov

Due Date: February 3, 2016

1 Insertion Sort

1.1 Operations of Insertion Sort

3 $\underline{41}$ 52 26 38 57 9 49	1 Comparison(s).
3 41 $\underline{52}$ 26 38 57 9 49	1 Comparison(s).
3 41 52 $\underline{26}$ 38 57 9 49	3 Comparison(s).
3 26 41 52 $\underline{38}$ 57 9 49	3 Comparison(s).
3 26 38 41 52 $\underline{57}$ 9 49	1 Comparison(s).
3 26 38 41 52 57 $\underline{9}$ 49	6 Comparison(s).
3 9 26 38 41 52 57 $\underline{49}$	3 Comparison(s).
3 9 26 38 41 49 52 57	0 Comparison(s).

1.2 Number of Comparisons

18 total operations.

2 Question 2

B	A	O	o	Ω	ω	Θ
2^n	$2^{\frac{n}{2}}$	no	no	yes	yes	no
$n^{\lg c}$	$c^{\lg n}$	yes	no	yes	no	yes

2.1 Justification

To determine larger asymptotic growth, take the limit of one function over the other. Arbitrarily choosing 2^n for the numerator, we see that:

$$\lim_{n\to\infty} \frac{2^n}{2^{\frac{n}{2}}} = \lim_{n\to\infty} 2^{\frac{n}{2}} = \infty \tag{1}$$

Alternatively, if we chose 2^n as the denominator, we notice that $\lim_{n\to\infty} \frac{2^{\frac{n}{2}}}{2^n} = 0$, so we know that the condition $2^n > 2^{\frac{n}{2}}$ holds, and for certain conditions the functions are equal (take $n = 0, 2^0 = 2^{\frac{0}{2}}$).

$\therefore 2^n$ is $\omega(2^{\frac{n}{2}})$ and $\Omega(2^{\frac{n}{2}})$. QED.

2.2 Justification II

By the properties of logarithms, we will show that $n^{\lg c} = c^{\lg n}$.

$$\begin{aligned}
n^{\lg c} &= c^{\lg n} && (2)\\
&= c^{\log_2 n} && (3)\\
&= c^{\frac{\ln n}{\ln 2}} && (4)\\
&= e^{\ln c \times \frac{\ln n}{\ln 2}} && (5)\\
&= e^{\frac{\ln c}{\ln 2} \times \ln n} && (6)\\
&= e^{\lg c \ln n} && (7)\\
&= n^{\lg c} && (8)
\end{aligned}$$

$\therefore n^{\lg c}$ is $O(c^{\lg n})$, $\Omega(c^{\lg n})$ and $\Theta(c^{\lg n})$. QED.

3 Big-O Implies Big-Ω

Theorem: Let f(n) and g(n) be asymptotically positive functions, $O(g(n))$ be the set $\{f(n) : \exists c, n_0 \in \mathbb{R}^+, \forall n, n_0 \in \mathbb{R}, 0 \leq f(n) \leq c\ g(n) \wedge n > n_0\}$, and $\Omega(g(n))$ be the set $\{f(n) : \exists c, n_0 \in \mathbb{R}^+, \forall n, n_0 \in \mathbb{R}^+, 0 \leq cg(n) \leq f(n) \wedge n > n_0\}$ Then,

$$f(n) = O(g(n)) \Rightarrow g(n) = \Omega(f(n)) \tag{9}$$

[*We will prove so by contradiction.*]

Proof: Suppose not. That is, suppose

$$\begin{aligned}
\sim [f(n) = O(g(n)) \quad &\Rightarrow \quad g(n) = \Omega(f(n))] && (10)\\
f(n) = O(g(n)) \quad &\wedge \quad \sim [g(n) = \Omega(f(n))] && (11)
\end{aligned}$$

Using the formal definition of $\Omega(g(n))$, we can see the negation is as follows:

$$\sim \Omega(g(n)) = \{f(n) : \forall c, n_0 \in \mathbb{R}^+, \exists n, n_0 \in \mathbb{R}^+ 0, > cg(n) > f(n) \wedge n \leq n_0\} \tag{12}$$

This is a contradiction, for there are no c, n_0 such that $0 \leq cg(n) \leq f(n) \wedge 0 > cf(n) > g(n)$ because c, n_0 are defined as *positive* constants, and $f(n), g(n)$ are defined as *positive* functions. Because no such positive function and positive constants exists to satisfy

$$0 > cf(n) > g(n) \tag{13}$$

This has led us to a contradiction. QED.

4 Asymptotic Proofs

4.1 Proof I

Assuming $n > 1$,

$$\begin{aligned} n^2 &\leq 20n^2 + 2n + 5 && (14)\\ &\leq 20n^2 + 2n^2 + 5 && (15)\\ &\leq 27n^2 && (16) \end{aligned}$$

$\therefore$ For $C = 27, n_0 = 1$, $20n^2 + 2n + 5 = O(n^2)$. QED.

4.2 Proof II

Assume $C = 1, n_0 = 1$. $\therefore$This satisfies the condition that c and n_0 are positive constants such that $0 \leq Cn^2 \leq 5n^2 - 15n + 100\ \forall n \geq n_0$. QED.

4.3 Proof III

4.3.1 Lower Bound

Assume $C = 1, n_0 = 1$. $\therefore$This satisfies the condition that c and n_0 are positive constants such that $0 \leq Cn^2 \leq 5n^2 + 2n\ \forall n \geq n_0$. QED.

4.3.2 Upper Bound

Assuming $n > 1$,

$$\begin{aligned} n^2 &\leq 5n^2 + 2 && (17)\\ &\leq 5n^2 + 2n^2 && (18)\\ &\leq 7n^2 && (19) \end{aligned}$$

$\therefore C = 7, n_0 = 1, 5n^2 + 2n = \Theta(n^2)$. QED.

4.4 Proof IV

To prove that $5n + 7 = o(n^2)$ we must show that $\exists c, n_0 \in \mathbb{R}^+, 0 \leq f(n) < Cg(n) \land n > n_0$. We do so by showing that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$. [*with the help of L'Hôpital's Rule*]

$$
\begin{aligned}
& \lim_{n\to\infty} \frac{5n+1}{n^2} && (20) \\
= \quad & \lim_{n\to\infty} \frac{5}{2n} && (21) \\
= \quad & 0 && (22)
\end{aligned}
$$

QED.