Chapter 14: Multiple Integration

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14.1 Double Integrals over Rectangular Regions

Volumes and Double Integrals

A function f defined on a rectangular region R in the xy-plane is **integratabtle** on R if $\lim\limits_{\Delta\to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$ exists fora ll partitions of R and for all choices of (x_k^*, y_k^*) within those partitions. The limit is the **double integral of** f **over** R, which we write

$$\iint\limits_R f(x, y)\, dA = \lim_{\Delta\to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k \tag{1}$$

If f is nonnegative on R, then the double integral equals the volume of the solid bounded by $z = f(x, y)$ and the xy-plane over R.

Double Integrals on Rectangular Regions

Let f be continuous on the rectangular region $R = \{(x, y) : a \le x \le b,\, c \le y \le d\}$. The double integral of f over R may be evaluated by either of two iterated integrals:

$$\iint\limits_R f(x, y)\, dA = \int_c^d \int_a^b dx\, dy = \int_a^b \int_c^d dy\, dx \tag{2}$$

Average Value of a Function over a Plane Region

The **average value** of an integrable function f over a region R is

$$\bar{f} = \frac{1}{\text{area of } R} \iint\limits_R f(x, y)\, dA \tag{3}$$

14.2 Double Integrals over General Regions

Let R be a region bounded below and above by the graphs of the continuous functions $y = g(x)$ and $y = h(x)$, respectively, and by the lines $x = a$ and $x = b$. If f is continuous on R, then

$$\iint\limits_R f(x, y)\, dA = \int_a^b \int_{g(x)}^{h(x)} f(x, y)\, dy\, dx \tag{4}$$

Let R be a region bounded on the left and right by the graphs of the continuous functions $x = g(y)$ and $x = h(y)$, respectively, and the lines $y = c$ and $y = d$. If f is continuous on R, then

$$\iint\limits_R f(x,\, y)\, dA = \int_c^d \int_{g(y)}^{h(y)} f(x,\, y)\, dx\, dy \tag{5}$$

Areas of Regions by Double Integrals

Let R be a region in the xy-plane. Then

$$\text{area of } R = \iint\limits_R \tag{6}$$

14.3 Double Integrals in Polar Coordinates

Double Integrals over Polar Rectangular Region

Let f be continuous on the region in the xy-plane $R = \{(r,\theta) : 0 \le a \le r \le b, \alpha \le \theta \le \beta\}$, where $\beta - \alpha \le 2\pi$. Then

$$\iint\limits_R f(r,\theta)\, dA = \int_\alpha^\beta \int_a^b f(r,\theta)\; r\, dr\, d\theta \tag{7}$$

Double Integrals over More General Polar Regions

Let f be continuous on the region in the xy-plane

$$R = \{(r,\theta) : 0 \le g(\theta) \le r \le h(\theta), \alpha \le \theta \le \beta\} \tag{8}$$

where $0 < \beta - \alpha \le 2\pi$. Then

$$\iint\limits_R f(r,\theta)\, dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r,\theta) r\, dr\, d\theta. \tag{9}$$

Area of Polar Regions

The area of the region $R = \{(r,\theta) : 0 \le g(\theta) \le r \le h(\theta), \alpha \le \theta \le \beta\}$, where $0 < \beta - \alpha \le 2\pi$, is

$$A = \iint\limits_R dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} r\, dr\, d\theta \tag{10}$$

14.4 Triple Integrals

Let f be continuous over the region

$$D = \{(x,\, y,\, z) : a \leq x \leq b, g(x) \leq y \leq h(x),\, G(x,\, y) \leq z \leq H(x,\, y)\} \quad (11)$$

where g, h, G, and H are continuous functions. Then f is integrable over D and the triple integral is evaluated as the iterated integral

$$\iiint\limits_D f(x,\, y,\, z)\, dV = \int_a^b \int_{g(x)}^{h(x)} \int_{G(x,y)}^{H(x,y)} f(x,\, y,\, z)\, dz\, dy\, dx. \quad (12)$$

Average Value of a Function of Three Variables

If f is continuous on a region D of $\mathbb{R}^3$, then the average value of over D is

$$\bar{f} = \frac{1}{\text{volume } (D)} \iiint\limits_D f(x,\, y,\, z)\, dV \quad (13)$$

14.5 Triple Integrals in Cylindrical and Spherical Coordinates

Transformations Between Cylindrical and Rectangular Coordinates

Rectangular → Cylindrical	**Cylindrical → Rectangular**
$r^2 = x^2 + y^2$	$x = r\cos\theta$
$\tan\theta = {}^{y}\!/_{x}$	$y = r\sin\theta$
$z = z$	$z = z$

Triple Integrals in Cylindrical Coordinates

Let f be continuous over the region

$$D = \{(r,\, \theta,\, z) : g(\theta) \le r \le h(\theta),\, \alpha \le \theta \le \beta,\, G(x,\, y) \le z \le H(x,\, y).\} \tag{14}$$

Then f is integrable over D and the triple integral of f over D in cylindrical coordinates is

$$\iiint\limits_D f(r,\, \theta,\, z)\, dV = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} \int_{G(r\cos\theta,\, r\sin\theta)}^{H(r\cos\theta,\, r\sin\theta)} f(r,\, \theta,\, z)\, dz\ r\, dr\, d\theta \tag{15}$$

Tranformations Between Spherical and Rectangular Coordinates

Rectangular → Spherical	**Spherical → Rectangular**
$\rho^2 = x^2 + y^2 + z^2$	$x = \rho\sin\varphi\cos\theta$
Use trigonometry to find φ and θ	$y = \rho\sin\varphi\sin\theta$
	$z = \rho\cos\varphi$

Triple Integrals in Spherical Coordinates

Let f be continuous over the region

$$D = \{(\rho,\, \varphi,\, \theta) : g(\varphi,\, \theta) \le \rho \le h(\varphi,\, \theta),\, a \le \varphi b,\, \alpha \le \theta \le \beta\} \tag{16}$$

Then f is integrable over D, and the triple integral of f over D in spherical coordinates is

$$\iiint\limits_D f(\rho,\,\varphi,\,\theta)dV = \int_{\alpha}^{\beta}\int_a^b\int_{g(\varphi,\,\theta)}^{h(\varphi,\,\theta)} f(\rho,\,\varphi,\,\theta)\rho^2\sin\varphi\;d\rho\,d\varphi\,d\theta \tag{17}$$

14.6 Integrals for Mass Calculations

Center of Mass in One Dimension

Let ρ be an integrable density function on the interval $[a,\, b]$ (which represents a thin rod or wire). The **center of mass** is location at the point $\bar{x} = \frac{M}{m}$, where the **total moment** M and mass m are

$$M = \int_a^b x\rho(x)\,dx \quad \text{and} \quad m = \int_a^b \rho(x)\,dx \tag{18}$$

Center of Mass in Two Dimensions

Let ρ be integrable density function defined over a closed bounded region R in $\mathbb{R}^2$. The coordinates of the center of mass of the object represented by R are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint\limits_R x\rho(x,\, y)\,dA \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = \frac{1}{m} = \iint\limits_R y\rho(x,\, y)\,dA \tag{19}$$

Center of Mass in Three Dimensions

Let ρ be integrable density function on a closed bounded region D in $\mathbb{R}^3$. The coordinates of the center of mass of the region are

$$\bar{x} = \frac{M_{yz}}{m} = \frac{1}{m}\iiint\limits_R x\rho(x,\, y,\, z)\,dV \tag{20}$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{1}{m}\iiint\limits_R y\rho(x,\, y,\, z)\,dV \tag{21}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{1}{m}\iiint\limits_R z\rho(x,\, y,\, z)\,dV \tag{22}$$

where $m = \iiint_D \rho(x,\, y,\, z)\,dV$ is the mass, and M_{yz}, M_{xz} and M_{xy} are the moments with respect to the coordinate planes.