Chapter 13: Functions of Several Variables

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Contents

13.1 Planes and Surfaces

Plane in $\mathbb{R}^3$

Given a fixed point P_0 and a nonzero **normal vector n**, the set of points P in $\mathbb{R}^3$ for which $\vec{P_0P}$ is orthogonal to **n** is called a **plane**.

General Equation of a Plane in $\mathbb{R}^3$

The plane passing through the point $P_0(x_0, y_0, z_0)$ with a nonzero normal vector $\mathbf{n} = \langle a, b, c\rangle$ is described by the equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad \text{or} \quad ax + by + cz = d \tag{1}$$

where $d = ax_0 + by_0 + cz_0$.

Parallel and Orthogonal Planes

Two distinct planes are **parallel** if their respective normal vectors are parallel (that is, the normal vectors are scalar multiples of each other). Two planes are **orthogonal** if their respective normal vectors are orthogonal (that is, the dot product of the normal vectors is zero).

Cylinder

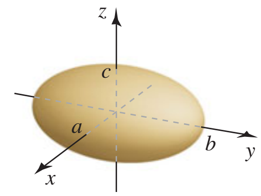
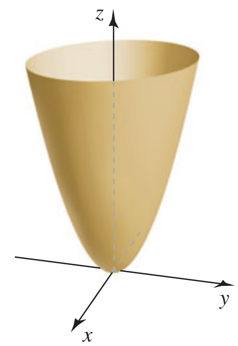
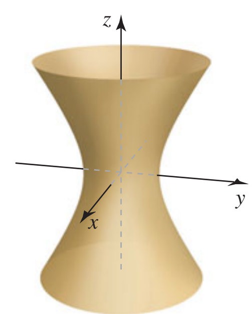
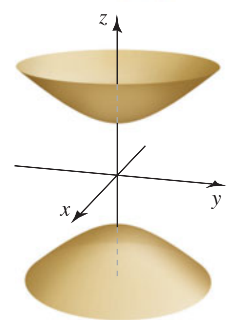
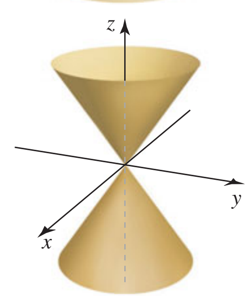
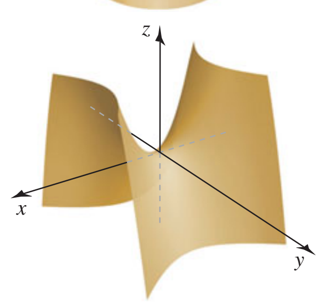
Given a curve C in a plane P and a line ℓ not in P, a **cylinder** is the surface consisting of all lines parallel to ℓ that pass through C.

Trace

A **trace** of a surface is the set of points at which the surface intersects a plane that is parallel to one of the coordinate planes. The trace in the coordinate planes are called **xy-trace**, the **xz-trace**, and the **yz-trace**.

Quadratic Surfaces

Name	Standard Equation	Features
Ellipsoid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	All traces are ellipses.
Elliptic paraboloid	$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$	Traces with $z=z_0>0$ are ellipses. Traces with $x=x_0$ or $y=y_0$ are parabolas.
Hyperboloid of one sheet	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	Traces with $z=z_0$ are ellipses for all z_0. Traces with $x=x_0$ or $y=y_0$ are hyperbolas.
Hyperboloid of two sheets	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	Traces with $z=z_0$ with $\|z_0\|>\|c\|$ are ellipses. Traces with $x=x_0$ and $y=y_0$ are hyperbolas.
Elliptic cone	$-\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$	Traces with $z=z_0\neq 0$ are ellipses. Traces with $x=x_0$ or $y=y_0$ are hyperbolas or intersecting lines.
Hyperbolic paraboloid	$z=\frac{x^2}{a^2}-\frac{y^2}{b^2}$	Traces with $z=z_0\neq 0$ are hyperbolas. Traces with $x=x_0$ or $y=y_0$ are parabolas.

Name	**Standard Equation**	**Features**	**Graph**
Ellipsoid	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$	All traces are ellipses.	
Elliptic paraboloid	$z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$	Traces with $z = z_0 > 0$ are ellipses. Traces with $x = x_0$ or $y = y_0$ are parabolas.	
Hyperboloid of one sheet	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$	Traces with $z = z_0$ are ellipses for all z_0. Traces with $x = x_0$ or $y = y_0$ are hyperbolas.	
Hyperboloid of two sheets	$-\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$	Traces with $z = z_0$ with $\lvert z_0 \rvert > \lvert c \rvert$ are ellipses. Traces with $x = x_0$ and $y = y_0$ are hyperbolas.	
Elliptic cone	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z^2}{c^2}$	Traces with $z = z_0 \neq 0$ are ellipses. Traces with $x = x_0$ or $y = y_0$ are hyperbolas or intersecting lines.	
Hyperbolic paraboloid	$z = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$	Traces with $z = z_0 \neq 0$ are hyperbolas. Traces with $x = x_0$ or $y = y_0$ are parabolas.	

13.2 Graphs and Level Curves

Function, Domain, and Range with Two Independent Variables

A **function** $z = f(x, y)$ assigns to each point (x, y) in a set D in $\mathbb{R}^2$ a unique real number z in a subset of $\mathbb{R}$. The set D is the **domain** of f. The **range** of f is the set of real numbers z that are assumed as the points (x, y) vary over the domain.

Function, Domain, and Range with n Independent Variables

The **function** $y = f(x_1, x_2, \ldots, x_n)$ assigns a unique real number y to each point $(x_1, x_2, \ldots, x_n)$ in a set D in $\mathbb{R}^n$. The set D is the **domain** of f. The **range** is the set of real numbers y that are assumed as the points $(x_1, x_2, \ldots, x_n)$ vary over the domain.

13.3 Limits and Continuity

Limit of a Function of Two Variables

The function f has the **limit** L as $P(x, y)$ approaches $P_0(a, b)$ written

$$\lim (x, y) \to (a, b) f(x, y) = \lim P \to P_0 f(x, y) = L \qquad (2)$$

if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon \qquad (3)$$

whenever (x, y) is in the domain of f and

$$0 < |PP_0| = \sqrt{(x-a)^2 + (y-b)^2} < \delta \qquad (4)$$

Limits of Constant and Linear Functions

Let a, b, and c be real numbers.

1. Constant functions $f(x, y) = c : \lim_{(x,y)\to(a,b)} c = c$
2. Linear function $f(x, y) = x : \lim_{(x,y)\to(a,b)} x = a$
3. Linear function $f(x, y) = y : \lim_{(x,y)\to(a,b)} y = b$

Limit Laws and Functions of Two Variables

Let L and M be real numbers and suppose that $\lim_{(x,y)\to(a,b)} f(x, y) = L$ and $\lim_{(x,y)\to(a,b)} g(x, y) = M$. Assume c is a constant, and $\forall m, n \in \mathbb{Z}$.

1 Sum $\lim_{(x,y)\to(a,b)} (f(x, y) + g(x, y)) = L + M$

2 Difference $\lim_{(x,y)\to(a,b)} (f(x, y) - g(x, y)) = L - M$

3 Constant multiple $\lim_{(x,y)\to(a,b)} cf(x, y) = cL$

4 Product $\lim_{(x,y)\to(a,b)} f(x, y) \cdot g(x, y) = L \cdot M$

5 Quotient $\lim_{(x,y)\to(a,b)} \left[\frac{f(x,y)}{g(x,y)}\right] = \frac{L}{M}$

6 Power $\lim_{(x,\,y)\to(a,\,b)} (f(x,\,y))^n = L^n$

7 $^m/_n$ **Power** If m and n have no common factors and $n \neq 0$, then $\lim_{(x,\,y)\to(a,\,b)} [f(x,\,y)]^{m/n} = L^{m/n}$, where we assume $L > 0$ if n is even.

Interior and Boundary Points

Let R be a region in $\mathbb{R}^2$. An **interior point** P of R lies entirely within R, which means it is possible to find a disk centered at P that contains only points of R.
A **boundary point** Q of R lies on the edge of R in the sense that *every* disk centered at Q contains at least one point in R and at least one point not in R.

Open and Closed Sets

A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points.

Two-Path Test for Nonexistence of Limits

If $f(x,\,y)$ approaches two different values as $(x,\,y)$ approaches $(a,\,b)$ along two different paths in the domain of f, then $\lim_{(x,\,y)\to(a,\,b)} f(x,\,y)$ does not exist.

Continuity

A function f is continuous at the point $(a,\,b)$ provided

1. f is defined at $(a,\,b)$.
2. $\lim_{(x,\,y)\to(a,\,b)} f(x,\,y)$ exists.
3. $\lim_{(x,\,y)\to(a,\,b)} f(x,\,y) = f(a,\,b)$.

Continuity of Composite Functions

If $u = g(x,\,y)$ is continuous at $(a,\,b)$ and $z = f(u)$ is continuous at $g(a,\,b)$, then the composite function $z = f(g(x,\,y))$ is continuous at $(a,\,b)$.

13.4 Partial Derivatives

The **partial derivative of f with respect to x at the point (a, b)** is

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}. \tag{5}$$

The **partial derivative of f with respect to y at the point (a, b)** is

$$f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}. \tag{6}$$

provided these limits exists.

Equality of Mixed Partial Derivatives

Assume that f is defined on an open set D of $\mathbb{R}^2$, and f_{xy} and f_{yx} are continuous throughout D. Then $f_{xy} = f_{yx}$ at all points of D.

Differentiability

The function $z = f(x,\, y)$ is **differentiable at** $(a,\, b)$ provided $f_x(a,\, b)$ and $f_y(a,\, b)$ exist and the change $\Delta z = f(a + \Delta x,\, b + \Delta y) - f(a,\, b)$ equals

$$\Delta z = f_x(a,\, b)\Delta x + f_y(a,\, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y, \tag{7}$$

where for fixed a and b, ε_1 and ε_2 are functions that depend only on Δx and Δy, with $(\varepsilon_1, \varepsilon_2) \to (0,\, 0)$ as $(\Delta x,\, \Delta y) \to (0,\, 0)$. A function is **differentiable** on an open set R if it is differentiable at every point on R.

Conditions for Differentiability

Suppose the function f has partial derivatives f_x and f_y defined on an open set containing $(a,\, b)$, with f_x and f_y continuous $(a,\, b)$. Then f is differentiable at $(a,\, b)$.

Differentiability Implies Continuity

If a function f is differentiable at $(a,\, b)$, then it is continuous at $(a,\, b)$

13.5 The Chain Rule

Chain Rule (One Independent Variable)

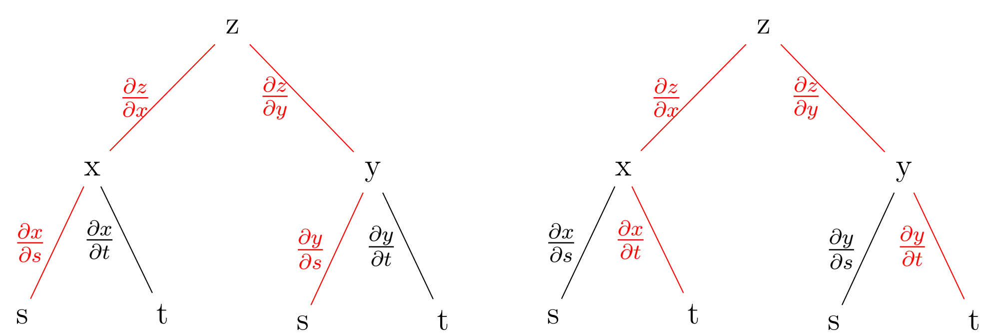
Let z be a differentiable function of x and y on its domain, where x and y are differentiable functions of t on an interval I. Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}. \tag{8}$$

Chain Rule (Two Independent Variables)

Let z be a differentiable function of x and y, where x and y are differentiable functions of s and t. Then

$$\frac{dz}{ds} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \tag{9}$$



Implicit Differentiation

Let F be differentiable on its domain and suppose that $F(x, y) = 0$ defines y as a differentiable function of x. Provided $F_y \neq 0$.

$$\frac{dy}{dx} = -\frac{F_x}{F_y} \tag{10}$$

13.6 Directional Derivatives and the Gradient

Directional Derivative

Let f be a differentiable at $(a,\, b)$ and let $\mathbf{u} = \langle \cos\theta,\, \sin\theta \rangle$ be a unit vector in the xy-plane. The **directional derivatives of** f **at** $(a,\, b)$ in the direction of $\mathbf{u}$ is

$$D_u f(a,\, b) = \lim_{h\to 0} \frac{f(a + h\cos\theta,\, b + h\sin\theta) - f(a,\, b)}{h} \tag{11}$$

provided the limit exits.

Directional Derivative

Let f be differentiable on $(a,\, b)$ and let $\mathbf{u} = \langle u_1,\, u_2 \rangle$ be a unit vecor in the $x,\, y$-plane. The **directional derivative of** f **at a** $(a,\, b)$ **in the direction of** u is

$$D_u f(a,\, b) = \langle f_x(a,\, b),\, f_y(a,\, b) \rangle \cdot \langle u_1,\, u_2 \rangle \tag{12}$$

Gradient (Two Dimensions)

Let f be differentiable at the point $(x,\, y)$. The **gradient** of f at $(x,\, y)$ is the vector-valued function

$$\nabla f(x,\, y) = \langle f_x(x,\, y),\, f_y(x,\, y) \rangle = f_x(x,\, y)\hat{\mathbf{i}} + f_y(x,\, y)\hat{\mathbf{j}} \tag{13}$$

Directions of Change

Let f be differentiable at $(a,\, b)$ with $\nabla f(a,\, b) \neq \mathbf{0}$

1. f has its maximum rate of increase at $(a,\, b)$ in the direction of the gradient $\nabla f(a,\, b)$. The rate of increase in this direction is $|\nabla f(a,\, b)|$
2. f has its maximum rate of decrease at $(a,\, b)$ in the direction of $-\nabla f(a,\, b)$. The rate of decrease in this direction is $-|\nabla f(a,\, b)|$.
3. The directional derivative is zero in any direction orthogonal to $\nabla f(a,\, b)$.

The Gradient and Level Curves

Given a function f differentiable at $(a,\, b)$, the line tangent to the level curve of f at $(a,\, b)$ is orthogonal to the gradient $\nabla f(a,\, b)$, provided $\nabla f(a,\, b) \neq \mathbf{0}$.

Gradient and Directional Derivative in Three Dimensions

Let f be differentiable at the point $(x,\, y,\, z)$. The **gradient** of f at $(x,\, y,\, z)$ is the vector-valued function

$$\begin{aligned}
\nabla f(x,\, y,\, z) &= \langle f_x(x,\, y,\, z),\, f_y(x,\, y,\, z),\, f_z(x,\, y,\, z)\rangle && (14)\\
&= f_x(x,\, y,\, z)\hat{\mathbf{i}} + f_y(x,\, y,\, z)\hat{\mathbf{j}} + f_z(x,\, y,\, z)\hat{\mathbf{k}} && (15)
\end{aligned}$$

The **directional derivative** of f in the direction of the unit vector $\mathbf{u} = \langle u_1,\, u_2,\, u_3\rangle$ at the point $(a,\, b,\, c)$ is $D_u f(a,\, b,\, c) = \nabla f(a,\, b,\, c) \cdot \mathbf{u}$

13.7 Tangent Planes and Linear Approximation

Let F be differentiable at the point $P_0(a,\, b,\, c)$ with $\nabla F(a,\, b,\, c) \neq \mathbf{0}$. The plane tangent to the surface $F(x,\, y,\, z) = 0$ at P_0, called the **tangent plane**, is the plane passing through P_0 orthogonal $\nabla F(a,\, b,\, c)$. An equation of the tangent plane is

$$F_x(a,\, b,\, c)(x-a) + F_y(a,\, b,\, c)(y-b) + F_z(a,\, b,\, c)(z-c) = 0. \tag{16}$$

Tangent Plane for $z = f(x,\, y)$

Let f be differentiable at the point $(a,\, b)$. An equation of the plane tangent to the surface $z = f(x,\, y)$ at the point $(a,\, b,\, f(a,\, b))$ is

$$z = f_x(a,\, b)(x-a) + f_y(a,\, b)(y-b) + f(a,\, b) \tag{17}$$

Linear Approximation

Let f be differentiable at $(a,\, b)$. The linear approximation to the surface $z = f(x,\, y)$ at the point $(a,\, b,\, f(a,\, b))$ is the tangent plane at that point, given by the equation

$$L(x,\, y) = f_x(a,\, b)(x-a) + f_y(a,\, b)(y-b) + f(a,\, b) \tag{18}$$

The Differential dz

Let f be differentiable at the point $(a,\, b)$. The change in $z = f(x,\, y)$ as the independent variables change from $(a,\, b)$ to $(a + dx,\, b + dy)$ is denoted by the differential dz:

$$\Delta z \approx dz = f_x(a,\, b)dx + f_y(a,\, b)dy \tag{19}$$

13.8 Maximum/Minimum Problems

Local Maximum/Minimum Values

A function f has a **local maximum value** at (a, b) if $f(x, y) \leq f(a, b)$ for all (x, y) in the domain of f in some open disk centered at (a, b). A function f has a **local minimum value** at (a, b) if $f(x, y) \geq f(a, b)$ for all (x, y) in the domain of f in some open disk centered at (a, b). Local maximum and local minimum values are also called **local extreme values** or **local extrema**.

Derivatives and Local Maximum/Minimum Values

If f has a local maximum or minimum value at (a, b) and the partial derivatives f_x and f_y exist at (a, b) then $f_x(a, b) = f_y(a, b) = 0$.

Critical Point

An interior point $(a,\ b)$ in the domain of f is a **critical point** of f is either

1. $f_x(a,\ b) = f_y(a,\ b) = 0$, or
2. one (or both) of f_x or f_y does not exist at $(a,\ b)$

Saddle Point

A function f has a **saddle point** at a critical point $(a,\ b)$ if, in every open disk centered at $(a,\ b)$, there are points $(x,\ y)$ for which $f(x,\ y) > f(a,\ b)$ and points for which $f(x,\ y) < f(a,\ b)$

Second Derivative Test

Suppose that the second partial derivative of f are continuous throughout an open disk centered at the point $(a,\ b)$, where $f_x(a,\ b) = f_y(a,\ b) = 0$. Let $D(x,\ y) = f_{xx}(x,\ y)f_{yy}(x,\ y) - (f_{xy}(x,\ y))^2$.

1. If $D(a,\ b) > 0$ and $f_{xx}(a,\ b) < 0$, then f has a local maximum value at $(a,\ b)$.
2. If $D(a,\ b) > 0$ and $f_{xx}(a,\ b) > 0$, then f has a local minimum value at $(a,\ b)$.

3. If $D(a, b) < 0$, then f has a saddle point at (a, b).

4. If $D(a, b) = 0$, then the test is inconclusive.

Absolute Maximum/Minimum Values

If $f(x, y) \leq f(a, b)$ for all (x, y) in the domain of f, then f has an **absolute maximum value** at (a, b). If $f(x, y) \geq f(a, b)$ for all (x, y) in the domain of f, then f has an **absolute minimum value** at (a, b).

Finding Absolute Maximum/Minimum Values on Closed, Bounded Sets

Let f be continuous on a closed bounded set R in $\mathbb{R}^2$. To find the absolute maximum and minimum values of f on R:

1. Determine the values of f at all critical points in R.

2. Find the maximum and minimum values of f on the boundary of R.

3. The greatest function values found in Step 1 and 2 is the absolute maximum value of f on R, and the least function value found in Steps 1 and 2 is the absolute minimum value of f on R.

13.9 Lagrange Multipliers

Parallel Gradients (Ball Park Theorem)

Let f be a differentiable function in a region of $\mathbb{R}^2$ that contains the smooth curve C given by $g(x,y) = 0$. Assume that f has a local extreme value (relative to values of f on C) at a point $P(a,b)$ on C. Then $\nabla f(a,b)$ is orthogonal to the line tangent to C at P. Assuming $\nabla g(a,b) \neq 0$, it follows that there is a real number λ (called a **Lagrange multiplier**) such that $\nabla f(a,b) = \lambda \nabla g(a,b)$.

Method of Lagrange Multipliers in Two Variables

Let the objective function f and the constraint function g be differentiable on a region of $\mathbb{R}^2$ with $\lambda g(x,y) \neq 0$ on the curve $g(x,y) = 0$. To locate the maximum and minimum values of f subject to the constraint $g(x,y) = 0$, carry out the following steps.

1. Find the values of x, y and λ (if they exist) that satisfy the equations

$$\nabla f(x,\, y) = \lambda \nabla g(x,\, y) \quad \text{and} \quad g(x,\, y) = 0 \tag{20}$$

2. Among the values $(x,\, y)$ found in Step 1, select the largest and smallest corresponding function values, which are the maximum and minimum values of f subject to the constraint.

Method of Lagrange Multipliers in Three Variables

Let f and g be differentiable on a region of $\mathbb{R}^3$ with $\nabla g(x,\, y,\, z) neq 0$ on the surface $g(x,\, y,\, z) = 0$. To locate the maximum and minimum values of f subject to the constraint $g(x,\, y,\, z) = 0$, carry out the following steps.

1. Find the values of $x,\, y,\, z$ and λ that satisfy the equations

$$\nabla f(x,\, y,\, z) = \lambda \nabla g(x,\, y,\, z) \quad \text{and} \quad g(x,\, y,\, z) = 0 \tag{21}$$

2. Among the points $(x,\, y,\, z)$ found in Step 1, select the largest and smallest corresponding values of the objective function. These values are the maximum and minimum values of f subject to the constraint.