

# Homework #9

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## 1 Simpson 1/3 Rule For $1 + 2x + 3x^2$

$$\begin{aligned}\int_0^4 1 + 2x + 3x^2 dx &\approx h/6(f(a) + 4f(a+h) + f(b)) \\ &= 4/6(1 + 4(1 + 4 + 12) + (1 + 8 + 48)) \\ &= 2/3(1 + 68 + 57) \\ &= 84\end{aligned}$$

## 2 Simpson's 3/8 Rule For $x^3$

$$\begin{aligned}\int_0^3 x^3 dx &\approx 3/8h(f(a) + 3f(a+h) + 3f(a+2h) + f(b)) \\ &= 3/8(0 + 1 + 8 + 27) \\ &= 3/8 \cdot 36 \\ &= 13.5\end{aligned}$$

## 3 Trapezoidal, Richardson and Romberg Method of $x^4$

We know the equation for the trapezoid rule to be

$$I = \frac{h}{2} \left[ f(x_0) + 2 \sum_{i=1}^n f(x_i) \right]$$

From this, we obtain

$$\begin{aligned} I_4 &= \frac{4}{2} (f(0) + f(4)) = 512 \\ I_2 &= \frac{2}{2} (f(0) + 2f(2) + f(4)) = 288 \\ I_1 &= \frac{1}{2} (f(0) + 2f(2) + 2f(3) + f(4)) = 226 \end{aligned}$$

For Richardson, we calculate the terms as follows:

$$\begin{aligned} I &\approx 4/3 I(h_2) - 1/3 I(h_1) \\ I_2 &= 4/3(288) - 1/3(512) = 640/3 = 213.\bar{3} \\ I_1 &= 4/3(226) - 1/3(288) = 616/3 = 205.\bar{3} \end{aligned}$$

And for Romberg,

$$\begin{aligned} I_{j,k} &= \frac{4^{k-1} I_{j+1,k-1} - I_{j,k-1}}{4^{k-1}} \\ I_1 &= \frac{4^2(616/3) - 640/3}{15} = \frac{1024}{5} = 204.8 \end{aligned}$$

We observe that the table looks like

| <b>h</b> | <b>trapazoidal</b> | <b>Richardson</b> | <b>Romberg</b> |
|----------|--------------------|-------------------|----------------|
| 4        | 512                |                   |                |
| 2        | 288                | 213. $\bar{3}$    |                |
| 1        | 226                | 205. $\bar{3}$    | 204.8          |

## 4 Derivation of Richardson Extrapolation

We recall that the value of integration  $I$  is equal to approximation + error, which we can eloquently write

$$I = I(h) + \mathcal{E}(h) \quad (1)$$

Where  $\mathcal{E}$  represents our error and  $I(h)$  represents our approximation. Supposing we have two different steps size ( $h_1$  and  $h_2$ ), we can rewrite equation (1) in the form

$$I = I(h_1) + \mathcal{E}(h_1) = I(h_2) + \mathcal{E}(h_2) \quad (2)$$

Now, let us expand  $\mathcal{E}(h)$  from equation (1) and equation (2).

$$\mathcal{E} \approx -\frac{b-a}{2}h^2 \bar{f}'' \quad (3)$$

Where  $b$  and  $a$  are the upper and lower bounds, respectively, and  $\bar{f}''$  is the average value of  $\frac{d}{dx}f(x)$  (where  $f(x)$  is the function we are integrating). Therefore, we can a ratio of the two errors

$$\frac{\mathcal{E}(h_1)}{\mathcal{E}(h_2)} \approx -\frac{\frac{b-a}{2}h_1^2 \bar{f}''}{\frac{b-a}{2}h_2^2 \bar{f}''} \approx \frac{h_1^2}{h_2^2}$$

Which can be algebraically manipulated to

$$\mathcal{E}(h_1) \approx \mathcal{E}(h_2) \left( \frac{h_1}{h_2} \right)^2 \quad (4)$$

Now we can substitute equation (4) back into equation (2) to obtain

$$I \approx I(h_1) + \mathcal{E}(h_2) \left( \frac{h_1}{h_2} \right)^2 \approx I(h_2) + \mathcal{E}(h_2)$$

Which can be solve for

$$\mathcal{E}(h_2) \approx \frac{I(h_2) - I(h_1)}{(h_1/h_2)^2 - 1}$$

Now that we have an estimate (hence the  $\approx$ ) of the truncation error, We once plug this back into equation (2) to get

$$I \approx I(h_2) + \frac{I(h_2) - I(h_1)}{(h_1/h_2)^2 - 1}.$$

Because we know the interval to be halved, we can safely assume  $h_2 = h_1/2$ . Thus, our equation reduces to

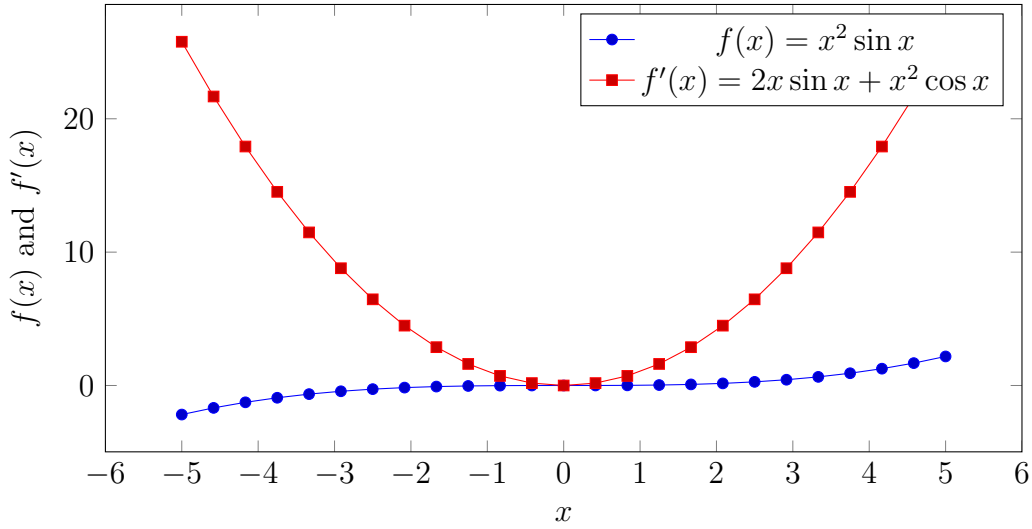
$$I \approx \frac{4}{3} I(h_2) - \frac{1}{3} I(h_1) \quad (5)$$

## 5 Derivative of $x^2 \sin x$

From the chain rule, we get

$$\frac{d}{dx} x^2 \sin x = 2x \sin x + x^2 \cos x$$

It can be plotted as follows



To compute the two derivatives for  $h_1 = 0.2$  and  $h_2 = 0.1$ , we use the centered difference formulas. For  $h_1$

$$\begin{aligned} \frac{d}{dx} &= \frac{f(x_{i+1}) - f(x_{i-1}))}{2h} \\ &= \frac{f(x+h) - f(x-h)}{2h} \\ &= \frac{f(x + 1/5) - f(x - 1/5)}{2/5} \\ &= \frac{(x + 1/5)^2 \sin(x + 1/5) - ((x - 1/5)^2 \sin(x - 1/5))}{2/5} \end{aligned}$$

Similarly, for  $h_2$

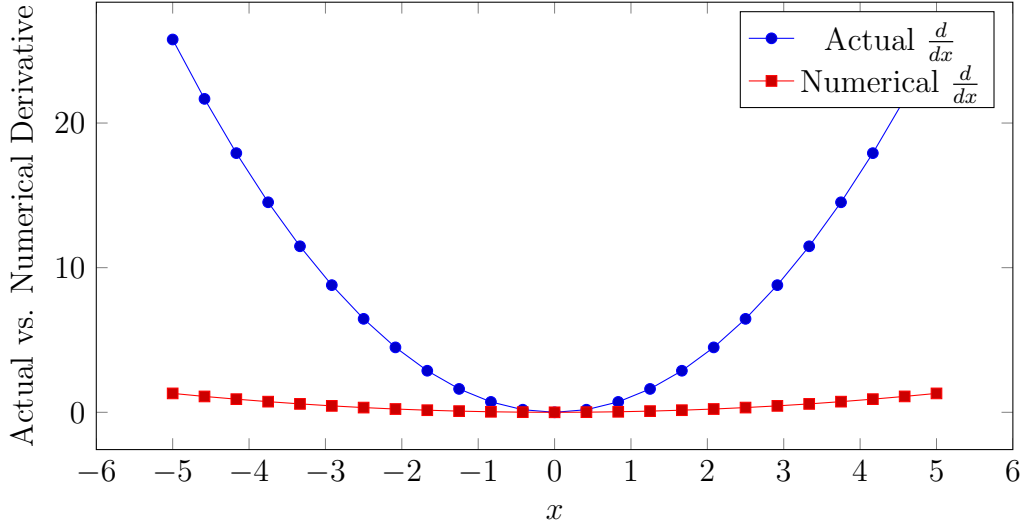
$$\frac{(x + 1/10)^2 \sin(x + 1/10) - ((x - 1/10)^2 \sin(x - 1/10))}{1/5}$$

We combine these for obtain the Richardson extrapolation formula,

$$D \approx 4/3 D(h_2) - 1/3 D(h_1) \quad (6)$$

$$\approx 4/3 \frac{(x + 1/10)^2 \sin(x + 1/10) - ((x - 1/10)^2 \sin(x - 1/10))}{1/5} \quad (7)$$

$$- 1/3 \frac{(x + 1/5)^2 \sin(x + 1/5) - ((x - 1/5)^2 \sin(x - 1/5))}{2/5} \quad (8)$$



## 6 Central Difference of $x^4$

### 6.1 First Order

From the power rule, we notice that

$$\frac{d}{dx} x^4 = 4x^3 \implies f'(1) = 4$$

Using centered difference formula,

$$\begin{aligned}
\frac{d}{dx} &= \frac{f(x_{i+1}) - f(x_{i-1}))}{2h} \\
&= \frac{(1 + 1/2)^4 - (1 - 1/2)^4}{(2^{1/2})} \\
&= 5
\end{aligned}$$

We notice the error to be  $|\frac{4-5}{4}| = |- .25| \implies \mathbf{25\%}$

## 6.2 Second Order

From the power rule, we notice that

$$\frac{d^2}{dx^2} x^4 = 12x^2 \implies f'(1) = 12.$$

Using centered difference formula,

$$\begin{aligned}
\frac{d^2}{dx^2} &= \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1}))}{h^2} \\
&= \frac{(1 + 1/2)^4 - 2(1^4) + (1 - 1/2)^4}{(1/2)^2} \\
&= 25/2
\end{aligned}$$

We notice the error to be  $|\frac{12-25/2}{12}| = |- .041\bar{6}| \implies \mathbf{4.16\%}$ .