

Chapter 8: Integration Techniques

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June 30, 2025

Contents

8.1	Basic Approaches	2
8.2	Integration By Parts	3
8.3	Trigonometric Integrals	4
8.4	Trigonometric Substitutions	6
8.5	Partial Fractions	7
8.8	Improper Integrals	9

8.1 Basic Approaches

$$\int k \, dx = kx + C \quad (1)$$

$$\int k^p \, dx = \frac{k^{p+1}}{p+1} + C, p \in \mathbf{R} \wedge \neq -1 \quad (2)$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax + C \quad (3)$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax + C \quad (4)$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax + C \quad (5)$$

$$\int \csc^2 ax \, dx = -\frac{1}{a} \cot ax + C \quad (6)$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax + C \quad (7)$$

$$\int \csc ax \cot ax \, dx = -\frac{1}{a} \csc ax \cot ax + C \quad (8)$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + C \quad (9)$$

$$\int \frac{dx}{x} = \ln |x| + C \quad (10)$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C \quad (11)$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C \quad (12)$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \left| \frac{x}{a} \right| + C \quad (13)$$

8.2 Integration By Parts

Suppose that u and v are differentiable functions. Then

$$\int u \, dv = uv - \int v \, du \quad (14)$$

Integration by Parts for Definite Integrals

Let u and v be differentiable. Then

$$\int_a^b u(x)v'(x) \, dx = u(x)v(x) \Big|_a^b - \int_a^b v(x)u'(x) \, dx \quad (15)$$

Integral of $\ln x$

$$\int \ln x \, dx = x \ln x - x + C \quad (16)$$

8.3 Trigonometric Integrals

$\int \sin^m x \cos^n x \, dx$ **Strategy**

m is odd, n real Split off $\sin x$, rewrite the resulting even power of $\sin x$ in terms of $\cos x$, and then use $u = \cos x$

n odd, m real Split off $\cos x$, rewrite the resulting even power of $\cos x$ in terms of $\sin x$, and then use $u = \sin x$.

m and n both even, nonnegative Use half-angle identities to transform the integrand into a polynomial in $\cos 2x$, and apply the preceding strategies once again to powers of $\cos 2x$ greater than 1.

Reduction Formulas

Assume n is a positive integer.

$$\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx \quad (17)$$

$$\int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx \quad (18)$$

$$\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx, n \neq 1 \quad (19)$$

$$\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx, n \neq 1 \quad (20)$$

Integrals of $\tan x$, $\cot x$, $\sec x$, and $\csc x$

$$\int \tan x \, dx = -\ln |\cos x| + C = \ln |\sec x| + C \quad (21)$$

$$\int \cot x \, dx = \ln |\sin x| + C \quad (22)$$

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C \quad (23)$$

$$\int \csc x \, dx = -\ln |\csc x + \cot x| + C \quad (24)$$

$\int \tan^m x \sec^n x \, dx$ **Strategy**

n **even** Split off $\sec^2 x$, rewrite the remaining even power of $\sec x$ in terms of $\tan x$, and use $u = \tan x$.

m **odd** Split off $\sec x \tan x$, rewrite the remaining even power of $\tan x$ in terms of $\sec x$, and use $u = \sec x$.

m **even and** n **odd** Rewrite the even power of $\tan x$ in terms of $\sec x$ to produce a polynomial in $\sec x$; apply reduction formula 4 to each term.

8.4 Trigonometric Substitutions

The Integral Contains...	Corresponding Substitution	Useful Identity
$a^2 - x^2$	$x = a \sin \theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \forall x \leq a$	$a^2 - a^2 \sin^2 \theta = a^2 \cos^2 \theta$
$a^2 + x^2$	$x = a \tan \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$a^2 + a^2 \tan^2 \theta = a^2 \sec^2 \theta$
$x^2 - a^2$	$x = a \sec \theta,$	$a^2 \sec^2 \theta - a^2 = a^2 \tan^2 \theta$
	$\begin{cases} 0 \leq \theta < \frac{\pi}{2}, \forall x \geq a \\ \frac{\pi}{2} < \theta \leq \pi, \forall x \leq -a \end{cases}$	

8.5 Partial Fractions

Partial Fractions with Simple Linear Factors

Suppose $f(x) = p(x) > q(x)$, where p and q are polynomials with no common factors and with the degree of p less than the degree of q . Assume that q is the product of simple linear factors. The partial fraction decomposition is obtained as follows.

Step 1. Factor the denominator q in the form $(x - r_1)(x - r_2) \cdots (x - r_n)$, where r_1, \dots, r_n are real numbers.

Step 2. Partial fraction decomposition Form the partial fraction decomposition by writing

$$\frac{p(x)}{q(x)} = \frac{A_1}{x - r_1} + \frac{A_2}{x - r_2} + \cdots + \frac{A_n}{x - r_n} \quad (25)$$

Step 3. Clear denominators Multiply both sides of the equation in Step 2 by $q(x) = (x - r_1)(x - r_2) \cdots (x - r_n)$, which produces conditions for A_1, \dots, A_n .

Step 4. Solve for coefficients Equate like powers of x in Step 3 to solve for the undetermined coefficients A_1, \dots, A_n .

Partial Fractions For Repeated Linear Factors

Suppose the repeated linear factor $(x - r)^m$ appears in the denominator of a proper rational function in reduced form. The partial fraction decomposition has a partial fraction for each power of $(x - r)$ up to and including the m th power; that is, the partial fraction decomposition contains the sum

$$\frac{A_1}{(x - r)} + \frac{A_2}{(x - r)^2} + \frac{A_3}{(x - r)^3} + \cdots + \frac{A_m}{(x - r)^m} \quad (26)$$

where A_1, \dots, A_m are constants to be determined.

Partial Fractions with Simple Irreducible Quadratic Factors

Suppose a simple irreducible factor $ax^2 + bx + c$ appears in the denominator of a proper rational function in reduced form. The partial fraction decomposition contains a term of the form

$$\frac{Ax + B}{ax^2 + bx + c} \quad (27)$$

where A and B are unknown coefficients to be determined.

Partial Fraction Decomposition

Let $f(x) = p(x)/q(x)$ be a proper rational function in reduced form. Assume the denominator q has been factored completely over the real numbers and m is a positive integer.

Simple linear factor A factor $x - r$ in the denominator requires the partial fraction $\frac{A}{x-r}$.

Repeated linear factor A factor $(x - r)^m$ with $m > 1$ in the denominator requires the partial fractions.

$$\frac{A_1}{(x - r)} + \frac{A_2}{(x - r)^2} + \frac{A_3}{(x - r)^3} + \cdots + \frac{A_m}{(x - r)^m} \quad (28)$$

Simple irreducible quadratic factor An irreducible factor $ax^2 + bx + c$ in the denominator requires the partial fraction

$$\frac{Ax + B}{ax^2 + bx + c} \quad (29)$$

Repeated irreducible quadratic factor An irreducible factor $(ax^2 + bx + c)^m$ with $m > 1$ in the denominator requires the partial fractions

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_mx + B_m}{(ax^2 + bx + c)^m} \quad (30)$$

8.8 Improper Integrals

Improper Integrals over Infinite Intervals

1. If f is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx \quad (31)$$

provided the limit exists.

2. If f is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx \quad (32)$$

provided the limit exists.

3. If f is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x) dx = \lim_{a \rightarrow -\infty} \int_a^c f(x) dx + \lim_{b \rightarrow \infty} \int_c^b f(x) dx \quad (33)$$

provided both limits exist, where c is any real number.

In each case, if the limit exists, the improper integral is said to **converge**, if it does not exist, the improper integral is said to **diverge**.

Improper Integrals with an Unbounded Integrand

1. Suppose f is continuous on $(a, b]$ with $\lim_{x \rightarrow a^+} f(x) = \pm \infty$. Then

$$\int_a^b f(x) dx = \lim_{c \rightarrow a^+} \int_c^b f(x) dx \quad (34)$$

provided the limit exists.

2. Suppose f is continuous on $[a, b)$ with $\lim_{x \rightarrow b^-} f(x) = \pm \infty$. Then

$$\int_a^b f(x) dx = \lim_{c \rightarrow b^-} \int_a^c f(x) dx \quad (35)$$

provided the limit exists.

3. Suppose f is continuous on $[a, b]$ except at the interior point p where f is unbounded. Then

$$\int_a^b f(x) dx = \int_a^p f(x) dx + \int_p^b f(x) dx, \quad (36)$$

provided the improper integrals on the right side exist.

In each case, if the limit exists, the improper integrals is said to **converge**, if it does not exists, the improper integral is said to **diverge**.