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Chapter 9: Sequences and Infinite Series

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9.1 An Overview

Sequence

A **sequence** $\{a_n\}$ is an ordered list of numbers of the form

$$\{a_1, a_2, a_3, \dots, a_n, \dots\} \tag{1}$$

A sequence may be generated by a **recurrence relations** of the form $a_{n+1} = f(a_n)$, for $n = 1, 2, 3, \dots$, where a_1 is given. A sequence may also be defined with an **explicit form** of the form $a_n = f(n)$, for $n = 1, 2, 3, \dots$.

Limit of a Sequence

If the terms of a sequence $\{a_n\}$ approach a unique number L as n increases, then we say $\lim\limits_{n\to\infty} = L$ exists, and the sequence **converges** to L. If the terms of the sequence do not approach a single number as n increases, the sequence has no limits, and the sequence **diverges**.

Infinite Series

Given a set of numbers $\{a_1, a_2, a_3, \dots\}$, the sum

$$a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k \tag{2}$$

is called an **infinity series**. Its **sequence of partial sums** $\{S_n\}$ has the terms

$$\begin{aligned}
S_1 &= a_1 && (3)\\
S_2 &= a_1 + a_2 && (4)\\
S_3 &= a_2 + a_2 + a_3 && (5)\\
&\vdots && (6)\\
S_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k, \quad n = 1, 2, 3, \dots && (7)
\end{aligned}$$

If the sequence of partial sums $\{S_n\}$ has a limit L, the infinite series **converges** to that limits, and we write

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n = L \tag{8}$$

If the sequence of partial sums diverges, the infinite series also **diverges**.

9.2 Sequences

Limits of Sequences from Limits of Functions

Suppose f is a function such that $f(n) = a_n$ for all positive integers n. If $\lim_{x\to\infty} = L$, then the limits of the sequences $\{a_n\}$ is also L.

Properties of Limits of Sequences

Assume that the sequence $\{a_n\}$ and $\{b_n\}$ have limits A and B, respectively. Then,

1. $\lim_{n\to\infty}(a_n \pm b_n) = A \pm B$
2. $\lim_{n\to\infty} ca_n = cA$, where c is a real number
3. $\lim_{n\to\infty} a_n b_n = AB$
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$, provided $B \neq 0$.

Geometric Sequences

Let r be a real number. Then,

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \Leftrightarrow |r| < 1 \\ 1 & \Leftrightarrow r = 1 \\ \text{does not exist} & \Leftrightarrow r \leq -1 \vee r > 1 \end{cases} \tag{9}$$

Squeeze Theorem for Sequences

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences with $a_n \leq b_n \leq c_n$ for all n greater than some index N. If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, the $\lim_{n\to\infty} b_n = L$.

Bounded Monotonic Sequences

A bounded monotonic sequence converges.

Growth Rates of Sequences

The following sequences are ordered according to increasing growth rates as $n \to \infty$; that is, if $\{a_n\}$ appears before $\{b_n\}$ in the list, then $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\lim_{n\to\infty} \frac{b_n}{a_n} = \infty$

$$\{\ln^q n\} \ll \{n^p\} \ll \{n^p \ln^r n\} \ll \{n^{p+s}\} \ll \{b^n\} \ll \{n!\} \ll \{n^n\} \tag{10}$$

The ordering applies for $p, q, r, s, b \in \mathbb{R}^+ \wedge b > 1$.

Limit of a Sequence

The sequence $\{a_n\}$ converges to L provided the terms of a_n can be made arbitrarily close to L by taking n sufficiently large. More precisely, $\{a_n\}$ has the unique limit L if given any tolerance $\epsilon > 0$, it is possible to find a positive integer N (depending only on ϵ) such that

$$|a_n - L| < \epsilon \quad \text{whenever } n > N \tag{11}$$

if the **limit of a sequence** is L, we say the sequence **converges** to L, written

$$\lim_{n\to\infty} a_n = L \tag{12}$$

A sequence that does not converge is said to **diverge.**

9.3 Infinite Series

Geometric Series

$$\sum_{k=0}^{n-1} ar^k = S_n = a\frac{1-r^n}{1-r} \tag{13}$$

Geometric Series

Let $a \neq 0$ and r be real numbers. If $|r| < 1$, then $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$. If $|r| \geq 1$, then the series diverges.

9.4 The Divergence and Integral Tests

Divergence Test

If $\sum a_k$ converges, then $\lim_{k\to\infty} a_k = 0$. Equivalently, if $\lim_{k\to\infty} a_k \neq 0$, then the series diverges. However, this cannot be used to prove convergence. If $\lim_{k\to\infty} a_k = 0$, the test is inconclusive.

Harmonic Series

The harmonic series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ diverges, even though the terms of the series approach zero.

Integral Test

Suppose f is a continuous, positive, decreasing function, for $x \geq 1$, and let $a_k = f(k)$, for $k = 1, 2, 3, \ldots$. Then

$$\sum_{k=1}^{\infty} a_k \quad \text{and} \quad \int_1^{\infty} f(x)\, dx \tag{14}$$

either both converge or both diverge. In the case of convergence, the value of the integral is *not*, in general, equal to the value of the series.

Convergence of the p-Series

The p-Series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges, for $p > 1$, and diverges for $p \leq 1$.

Estimating Series with Positive Terms

Let f be continuous, positive, decreasing function, for $x \geq 1$, and let $a_k = f(k)$, for $k = 1, 2, 3, \ldots$. Let $S = \sum_{k=1}^{\infty} a_k$ be a convergence series and let $S_n = \sum_{k=1}^{n}$ be the sum of of the first n terms of the series. The remainder $R_n = S - S_n$ satisfies

$$R_n \leq \int_n^{\infty} f(x)\, dx \tag{15}$$

Furthermore, the exact value of the series is bounded as follows:

$$S_n + \int_{n+1}^{\infty} f(x)\, dx \leq \sum_{k=1}^{\infty} a_k \leq S_n + \int_n^{\infty} f(x)\, dx. \tag{16}$$

Properties of Convergent Series

1. Suppose $\sum a_k$ converges to A and let c be a real number. The series $\sum ca_k$ converges and $\sum ca_k = c\sum a_k = cA$

2. Suppose $\sum a_k$ converges to A and $\sum b_k$ converges to B. The series $\sum(a_k \pm b_k)$ converges and $\sum(a_k \pm b_k) = \sum a_k \pm \sum b_k = A \pm B$

3. *Whether* a series converges does not depend on a fininte number of terms added to or removed form the series. Specifically, if M is a positive integer, then $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=M}^{\infty} a_k$ both converge or both diverge. However, the *value* of a convergent series does change if nonzero terms are added or deleted.

9.5 The Ratio, Root, and Comparison Tests

Useful Identities

$$\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k = \lim_{k\to\infty}\left(\frac{k+1}{k}\right)^k = e \tag{17}$$

$$\lim_{k\to\infty} k^{\frac{1}{k}} = 1 \tag{18}$$

The Ratio Test

Let $\sum a_k$ be an infinite series with positive terms and let $r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$.

1. If $0 \le r < 1$, the series converges.
2. If $r > 1$ (including $r = \infty$), the series diverges.
3. If $r = 1$, the test is inconclusive.

The Root Test

Let $\sum a_k$ be an infinite series with nonegative terms and let $\rho = \lim_{k\to\infty} \sqrt[k]{a_k}$.

1. If $0 \le \rho < 1$, the series converges.
2. If $\rho > 1$ (including $\rho = \infty$), the series diverges.
3. If $\rho = 1$, test is inconclusive.

The Comparison Test

Let $\sum a_k$ and $\sum b_k$ be a series with postivie terms.

1. If $0 < a_k \le b_k$ and $\sum b_k$ converges, then $\sum a_k$ converges.
2. If $0 < b_k \le a_k$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

The Limit Comparison Test

Let $\sum a_k$ and $\sum b_k$ be series with positive terms and let

$$\lim_{k \to \infty} \frac{a_k}{b_k} = L \tag{19}$$

- If $0 < L < \infty$ (that is, L is a finite, positive number), then $\sum a_k$ and $\sum b_k$ either both converge or both diverge.
- If $L = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges.
- If $L = \infty$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

Guidelines

- Begin with the Divergence Test. If you show that $\lim\limits_{k \to \infty} a_k \neq 0$, then the series diverges and your work is finished.
- Is the series a special series? Recall the convergence properties for the following series:
  - Geometric series: $\sum ar^k$ converges for $|r| < 1$ and diverges for $|r| \geq 1$ $(a \neq 0)$.
  - p-series: $\sum \frac{1}{k^p}$ converges for $p > 1$, and diverges for $p \leq 1$.
  - Check also for telescoping series.
- If the general kth term of the series looks like a function you can integrate, then try the Integral Test.
- If the general kth term of the series involves $k!, k^k,$ or a^k, where a is a constant, the Ratio Test is advisable. Series with k in an exponent may yield to the Root Test.
- If the general kth term of the series is a rational function of k (or a root of a rational function), use the Comparison or the Limit Comparison Test. Use the families of series given in Step 2 as comparison series.

9.6 Alternating Series

The Alternating Series Test

The alternating series $\sum(-1)^{k+1}a_k$ converges provided

1. the terms of the series are non-increasing in magnitude ($0 < a_{k+1} \leq a_k$, for k greater than some index N) and
2. $\lim\limits_{k\to\infty} = 0$

Alternating Harmonic Series

The alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ converges (even though the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ diverges).

Remainder in Alternating Series

Let $R_n = |S - S_n|$ be the remainder in approximating the value of a convergent alternating series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ by the sum of its first n terms. Then $R_n \leq a_{n+1}$. In other words, the remainder is less than or equal to the magnitude of the first neglected term.

Absolute and Conditional Convergence

Assume the infinite series $\sum a_k$ converges. The series $\sum a_k$ **converges absolutely** if the series $\sum |a_k|$ converges. Otherwise, the series $\sum a_k$ **converges conditionally**.

Absolute Convergence Implies Convergence

If $\sum |a_k|$ converges, then $\sum a_k$ converges (absolute convergence implies convergence). If $\sum a_k$ diverges, then $\sum |a_k|$ diverges.

Series or Test	Form of Series	Condition for Convergence	Condition for Divergence	Comments
Geometric Series	$\sum_{k=0}^{\infty} ar^k, a \neq 0$	$\|r\| < 1$	$\|r\| \geq 1$	If $\|r\| < 1$, then $\sum a_{k=1}^{\infty} ar^k = \frac{a}{1-r}$
Divergence Test	$\sum_{k=1}^{\infty} a_k$	Does Not Apply	$\lim_{k\to\infty a_k \neq 0}$	Cannot be used to prove convergence.
Integral Test	$\sum a_{k=1}^{\infty} a_k$, where $a_k = f(k)$ and f is continuous, positive, and decreasing.	$\int_1^{\infty} f(x)dx < \infty$	$\int_1^{\infty} f(x)dx$ does not exist	The value of the integral is not the value of the series.
p-Series	$\sum a_{k=1}^{\infty} \frac{1}{k^p}$	$p > 1$	$p \leq 1$	Useful for comparison tests.
Ratio Test	$\sum a_{k=1}^{\infty}$ where $a_k > 0$	$\lim_k \to \infty \frac{a_{k+1}}{a_k} < 1$	$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} > 1$	Inconclusive if $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 1$
Root Test	$\sum a_{k=1}^{\infty}$ where $a_k \geq 0$	$\lim_{k\to\infty} \sqrt[k]{a_k} < 1$	$\lim_{k\to\infty} \sqrt[k]{a_k} > 1$	Inconclusive if $\lim_{k\to\infty} \sqrt[k]{a_k} = 1$
Comparison Test	$\sum a_{k=1}^{\infty}$ where $a_k > 0$	$0 < a_k \leq b_k$ and $\sum_{k=1}^{\infty} b_k$ converges	$0 < b_k \leq a_k$ and $\sum a_{k=1}^{\infty} b_k$ diverges	$\sum a_{k=1}^{\infty} a_k$ is given; you supply $\sum a_{k=1}^{\infty} b_k$
Limit Comparison Test	$\sum a_{k=1}^{\infty} a_k$ where $a_k > 0, b_k > 0$	$0 \leq \lim_{k\to\infty} \frac{a_k}{b_k} < \infty$ and $\sum_{k=1}^{\infty} b_k$ converges	$\lim_{k\to\infty} \frac{a_k}{b_k} > 0$ and $\sum a_{k=1}^{\infty} b_k$ diverges.	$\sum a_{k=1}^{\infty} a_k$ is given; you supply $\sum a_{k=1}^{\infty} b_k$
Alternating Series Test	$\sum_{k=1}^{\infty} (-1)^k a_k$, where $a_k > 0, 0 < a_{k+1} \leq a_k$	$\lim_{k\to\infty} a_k = 0$	$\lim_{k\to\infty} a_k \neq 0$	Remainder R_n satisfies $R_n \leq a_{n+1}$
Absolute Convergence	$\sum a_{k=1}^{\infty} a_k, a_k$ arbitrary	$\sum_{k=1}^{\infty} \|a_k\|$ converges.	Applies to arbitrary series	