

Chapter 11: Parametric and Polar Curves

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11.1 Parametric Equations

Forward or Positive Orientation

The direction in which a parametric curve is generated as the parameter increases is called the **forward**, or **positive, orientation** of the curve.

Derivative for Parametric Curves

Let $x = g(t)$ and $y = h(t)$, where g and h are differentiable on an interval $[a, b]$. Then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)} \quad (1)$$

provided $\frac{dx}{dt} \neq 0$.

11.2 Polar Coordinates

Introduction

- Up to now we have only studied in a Cartesian coordinate system.
 - A Cartesian coordinate system is just a plane described by Cartesian (or, algebraic) equations and points in a finite dimensions.
 - * *One Dimension*: Lines.
 - * *Two Dimensions*: x^2 .
 - * *Three, Four*: Upper-level Calculus and Physics.
- Let's define an alternative coordinate system — **polar coordinate**.
 - coordinates are constants on circles and rays.
 - Useful for navigation, position, and gravitation fields.

Defining Polar Coordinates

Pole The origin of the coordinate system.

Polar Axis Synonymous for the positive x -axis.

Polar Coordinates A polar coordinates P has the form (r, θ) .

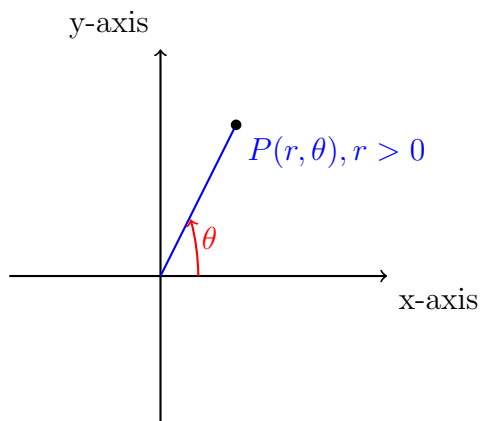
Radial Coordinate The radial coordinate r describes the *signed*, or *directed*, distance from the origin to P .

Angular Coordinate The angular coordinate θ describes an angle whose initial side is the positive x -axis and whose terminal side lies on the ray passing through the origin and P .

Notes

- **Positive angle measurements are measured *counterclockwise* from the origin.**
- Every point has multiple representations.
 - Angles are periodic, so multiples of 2π gives the same angle.

- Coordinates may be negative. So (r, θ) can be represented as $(-r, \theta + \pi)$ and $(-r, \theta - \pi)$



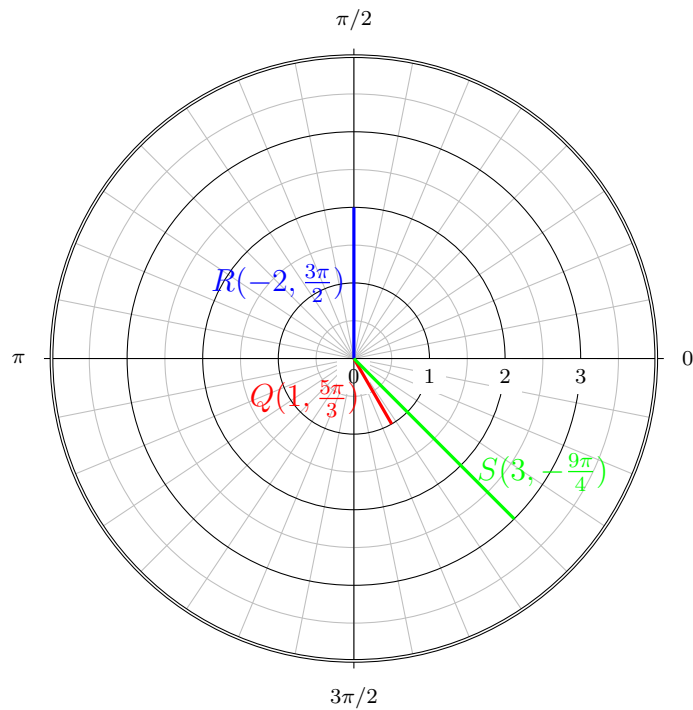
In summary,

- $(r, \theta + 2\pi)$ represents the same point as (r, θ)
- $P(r, \theta)$ and $P'(-r, \theta)$ are reflections through the origin.

Examples

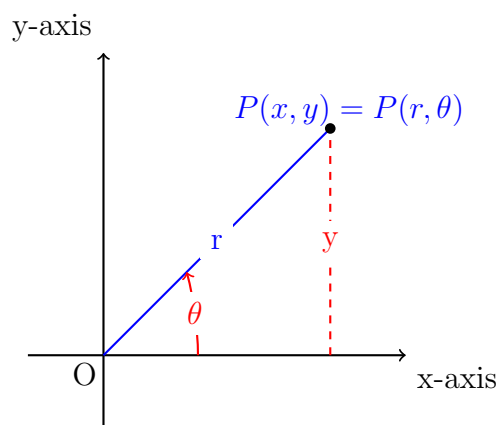
Graph the following points in polar coordinates:

- $Q(1, \frac{5\pi}{3})$
- $R(-2, \frac{3\pi}{2})$
- $S(3, -\frac{9\pi}{4})$
 - Now give two alternative representations.
 - $S'(3, \frac{1\pi}{4})$
 - $S''(-3, -\frac{5\pi}{4})$



Converting Between Cartesian and Polar Coordinates

- We sometimes need to convert between Cartesian and polar coordinates.
- Let's turn this problem into a right triangle.



A point with polar coordinates (r, θ) has Cartesian coordinates (x, y) , where

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta \quad (2)$$

A point with Cartesian coordinates (x, y) has polar coordinates (r, θ) , where

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan \theta = \frac{y}{x} \quad (3)$$

Examples

BE SURE TO GRAPH POINTS IN CARTESIAN FIRST.

Polar to Cartesian

Express the point with the following polar coordinates in Cartesian coordinates: $P(3, \frac{2\pi}{3})$

$$\begin{aligned} x &= r \cos \theta & y &= r \sin \theta \\ &= 3 \cos(2\pi/3) & &= 3 \sin(2\pi/3) \\ &= -3(1/2) & &= 3(\sqrt{3}/2) \\ &= -3/2 & &= 3\sqrt{3}/2 \end{aligned}$$

Polar to Cartesian

Express the point with the following polar coordinates in Cartesian coordinates: $Q(e, -\frac{\pi}{4})$

$$\begin{aligned} x &= r \cos \theta & y &= r \sin \theta \\ &= e \cos(-\pi/4) & &= e \sin(-\pi/4) \\ &= e(\sqrt{2}/2) & &= -e(\sqrt{2}/2) \\ &= e\sqrt{2}/2 & &= -e\sqrt{2}/2 \end{aligned}$$

Cartesian To Polar

Express the point with the following Polar coordinates to Cartesian coordinates: $R(1, -1)$

$$\begin{array}{ll}
r &= \sqrt{x^2 + y^2} \\
&= \sqrt{1^2 + (-1)^2} \\
&= \sqrt{2} \\
\tan \theta &= y/x \\
&= -1/1 \\
&= -1 \\
\theta &= -\pi/4 \text{ or } 7\pi/4.
\end{array}$$

Therefore, two possible solutions are: $(\sqrt{2}, -\pi/4)$ or $(\sqrt{2}, 7\pi/4)$

Cartesian To Polar

Express the point with the following Polar coordinates to Cartesian coordinates: $S(1, \sqrt{3})$

$$\begin{array}{ll}
r &= \sqrt{x^2 + y^2} \\
&= \sqrt{(\sqrt{3})^2 + (1)^2} \\
&= 2 \\
\tan \theta &= y/x \\
&= \sqrt{3}/1 \\
\theta &= \pi/3 \text{ or } 4\pi/3.
\end{array}$$

Therefore, two possible solutions are: $(2, \pi/3)$ or $(2, 4\pi/3)$.

Basic Curves in Polar Coordinates

- A curve in polar coordinates is the set of **points** that satisfy an equation in r and θ .
- This makes graphing some things easier than others.
- Look at $r = 3$ is the set of all points that satisfy being away from the origin of 3 units.
 - This is because θ is not specified, it's arbitrary. Basically, θ is the function.
 - In general, $r = a, \forall a \in \mathbb{R}^+$ describes a circle.
- Taking the converse, let r be arbitrary.
 - If the r is arbitrary, and we specify the angle, what do you think we get?

- A line!
- Take $\sqrt{2}/2$.

Polar to Cartesian Graph Example

Convert the polar equation $r = 6 \sin \theta$ to Cartesian coordinates and describe the corresponding graph.

$$r^2 = 6r \sin \theta \quad (4)$$

$$x^2 + y^2 = 6y \quad (5)$$

$$0 = x^2 + y^2 - 6y \quad (6)$$

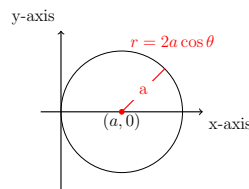
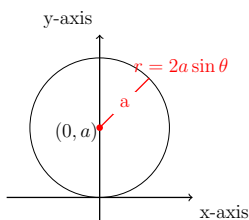
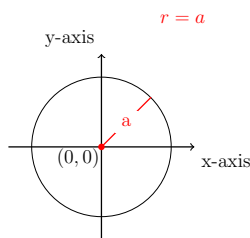
$$= x^2 + (y^2 - 6y + 9) - 9 \quad (7)$$

$$= x^2 + (y - 3)^2 - 9 \quad (8)$$

We recognize this to be the equation of a circle, centered at $(0, 3)$ at 3. We can also generalize this.

Circle in Polar Coordinates

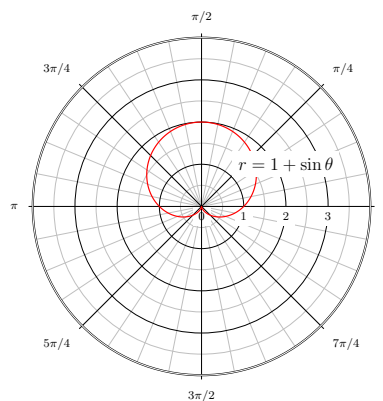
- The equation $r = a$ describes a circle of radius $|a|$ centered at $(0, 0)$.
- The equation $r = 2a \sin \theta$ describes a circle of radius $|a|$ centered at $(0, a)$.
- The equation $r = 2a \cos \theta$ describes a circle of radius $|a|$ centered at $(a, 0)$.



Graphing In Polar Coordinates

Graph the polar equation $r = f(\theta) = 1 + \sin \theta$

θ	$r = 1 + \sin \theta$
0	1
$\pi/6$	$3/2$
$\pi/2$	2
$5\pi/6$	$3/2$
π	1
$7\pi/6$	$1/2$
$3\pi/2$	0
$11\pi/6$	$1/2$
2π	1



The resulting curve is known as a **cardioid**.

Cartesian-to-Polar Method for Graphing $r = f(\theta)$

1. Graph $r = f(\theta)$ as if r and θ were Cartesian coordinates with θ on the horizontal axis and r on the vertical axis. Be sure to choose an interval in θ on which the entire polar curve is produced.
2. Use the Cartesian graph in Step 1 as a guide to sketch the points (r, θ) on the final *polar* curve.

Example

With the alternate graphing method, graph $r = 1 + \sin \theta$.

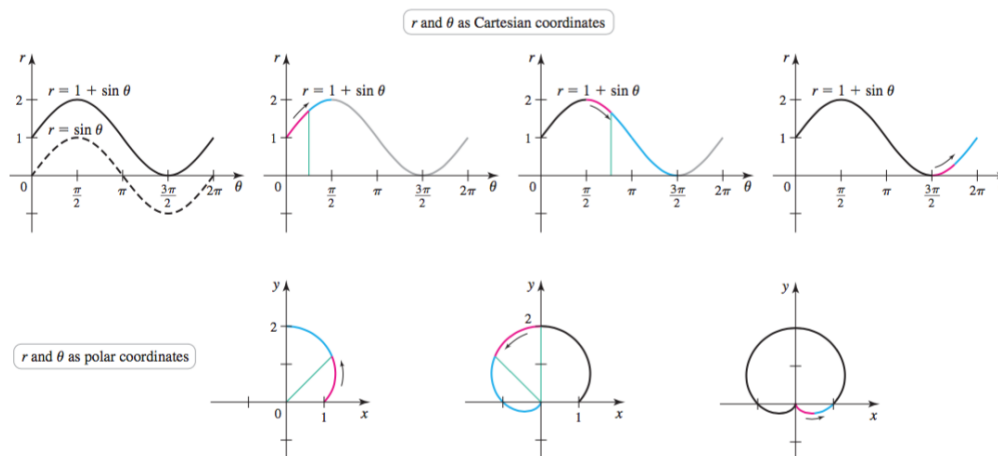


FIGURE 11.26

Symmetry In Polar Equations

Symmetry about the x-axis occurs if the point (r, θ) is on the graph whenever $(r, -\theta)$ is on the graph.

Symmetry about the y-axis occurs if the point (r, θ) is on the graph whenever $(r, \pi - \theta) = (-r, -\theta)$ is on the graph.

Symmetry about the origin occurs if the point (r, θ) is on the graph whenever $(-r, \theta) = (r, \theta + \pi)$ is on the graph.

11.3 Calculus in Polar Coordinates

Slope of a Tangent Line

Let f be a differentiable function at θ_0 . The slope of the line tangent to the curve $r = f(\theta)$ at the point $(f(\theta_0), \theta_0)$ is

$$\frac{dy}{dx} = \frac{f'(\theta_0) \sin \theta_0 + f(\theta_0) \cos \theta_0}{f'(\theta_0) \cos \theta_0 - f(\theta_0)} \quad (9)$$

Area of Regions in Polar Coordinates

Let R be the region bounded by the graphs of $r = f(\theta)$ and $r = g(\theta)$, between $\theta = \alpha$ and $\theta = \beta$, where f and g are continuous and $f(\theta) \geq g(\theta) \geq 0$ on $[\alpha, \beta]$. The area of R is

$$\int_{\alpha}^{\beta} \frac{1}{2} (f(\theta)^2 - g(\theta)^2) d\theta \quad (10)$$