Chapter #1

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Problem #1

Suppose P is a level and M is a level set. The statement that P is a **bottom level** of M means if x is a level in M then P cannot be above x.

Class Solution

P is a bottom level of the level set M if P is a level in M so that if x is a level in M then P is not above x.

Problem #2

Theorem 1. *If M is a level set having exactly one level, M **does** a top level.*

Proof. Suppose that $\exists P \in M$, where P is a level and M is a level set. Suppose, also, that P is the only level in M. By our definition of **top level**, we state that the **top level** of M means P is a level in M and if x is a level in M then x is not above P. But we only have one level in M, that is P.

Therefore, x must be P, and because Axiom 1 (If P is a level then P is not above P) agrees with our definition x is not above P. □

Class Solution

Yes, because if level set M contains a single level P any level x within M would be P so according to Axiom 1 satisfies the definition of a top level of level set M.

Problem #3

With respect to all these definitions, can infer that P is a bottom level in set M. One can infer that level P is a bottom level.

Problem #4

No.

Proof. Suppose not. That is, suppose there exists levels P, Q, x in the level set M, where P is a different name for Q. Because they are both representations of each other, we will write Q to represent both P and Q.

Rewriting Axiom 2 in this notation, it states as follows:

> If each of Q and Q is a level and Q is above Q then there exists a level x such that Q is above x and x is above Q.

This argument is no longer valid, because it builds on the premise that Q is above Q (by Axiom 1, this not valid). This is a violation of Axiom 1. $\square$

Problem #5

If P and Q are levels then P is above Q and Q is above P, then (by Axiom 3) this implies that P is transitive above P, which by Axiom 1 is a contradiction.

Class Solution (Osman)

No, if P and Q are levels such that P is above Q and Q is above P, then by Axiom 4 the P is above P which violates Axiom 1.

Problem #6

No.

Proof. Suppose not. That is, suppose that there exists two, different levels P, Q in the level set M. By Axiom 3, P is above Q or Q is above P, but not both (because that would imply P is above P). By our definition of top level,

P and Q must be a top level if $\exists x \in M$, where x is not above P and Q, but by Axiom 2 $\exists x \in M$ such that P is above x and x is above Q. Because the statements x is above and P, Q and P is above x and x is above Q cannot both hold, this is a contradiction. □

Class Solution (Matthew Kovar)

Proof. If P and Q are both top levels in a level set M, then there is no level x in the level set M above P or Q (by Definition 1), which means that P cannot be above Q and Q cannot be above P. However, this violates Axiom 3 which states that P must be above Q or Q must be above P. Therefore, there cannot be more than one top level in a level set. □

Problem #7

The following answer is wrong. An empty level set is simply an empty set, not an empty level set.

(a) A level set that satisfies the property of having no top level is the empty set, $\varnothing$.

(b) The empty set $\varnothing$ also satisfies this property.

(c) Assume $\exists P \in M$, where P is an arbitrary level and M to be the level set of all levels. Therefore, there are is no top level (if $Q \in M$, if Q is said to be the top level, $\exists x \in M$, where x is above Q), but P such that if $x \in M$ then P is above x.

Class Solution (Brett Sears, Illya Starikov, Matthew Healy)

(a) (Brett Sears) Let M be a level set containing some level P. A level x is included in M if x is above P. Suppose T is the top level of M. By Axiom 5, we know there is also a level U above T, T is above P or P. By Axiom 4, U is above P and included in M. Therefore there is no level that satisfies the definition of of a top level, and this is a contradiction.

(a) (Illya Starikov) Suppose M to be the level set of all levels. By our definition of top level, there exists a top level in M if there is a level $x, P \in M$, then x is not above P (where P is to be the candidate top level). However, for every level P, there is a level x above it. $\therefore$ M has no top level.

(b) (Matthew Healy) Suppose M to be the level set of all levels. For any bottom level b which we have defined a level such that for any level $x \in M$, b is not above x. By Axiom 5, there exists level x and y, such that $y > b > x$. Since this causes b to be above another level $x \in M$, b no longer satisfies the definition of a bottom level.

(c) (Brett Sears) Let M be the level set that includes every level P is above, but does not include P. Let T be a potential top level for M. By Axiom 2, there is a level x such that P is above x, x is above T. Since P is above x, x must be included in the level set M. However, since x is above T, the definition of a top level is not satisfied. So M does not have a definitive top level.

Problem #8

Class Solution

Let level set S_1 have a top level Q. Under the hypothesis of Axiom 6, suppose level set S_2 has a bottom level P. By Axiom 2, there exists a level x such that P is above x and x is above Q. Since x must be in S_1 or S_2 under the hypothesis of Axiom 6 and since Q is the top level of S_1, then x must be in S_2. Since bottom level P in S_2 is above level x in S_2, Kyle Foster's definition of a bottom level is violated and level P cannot be a bottom level. Therefore, if S_1 has a top level, S_2 cannot have a bottom level.

Problem #9

The level P is not the top level of the level set M means ... *there is more than one level in M.*

Class Solution (Terry Maxwell)

The level P is not the top level of the level set M means ... *the level P may or may not be included in the level set M and if P is included in M then there must exists a level x in level set M and x must be above P.*

Theorem #1

Supposing A is above B, B is above C, C is above D. M is the set of all level. x such that A is above x and x is above B or C is above x and x is above D. B is above y and y is above C.

Problem #10

Suppose P and Q are levels. To say that P is **below** Q is to say that Q is above P.

Conjecture 1. *If P is a level then P is not below P.*

Proof. Suppose there is a level P such that P is below P. By the definition of below this says P is above P. This contradicts Axiom 1. Thus there is no level which is below itself. $\square$

The Bare Denial of Above

The bare denial of P is above Q is P *is below* Q *or* P *is* Q.