Numerical Methods Crib Sheet
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Integration Equations

Note that h usually refers to $(b-a)$. Also, Trapezoidal needs 2 points, Simpson's $1/3$ uses 3, Simpson's $3/8$ uses 4 and Boole's 5. Also note that for something like $[0,4]$,
$h=4, n=1, h=2, n=2, h=1, n=4$

`Trapezoidal`	$I = \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n} + f(x_n)\right]$
`Richardson`	$I \approx 4/3 I(h_2) - 1/3 I(h_1)$
	$\approx 4/3 I(\text{current cell}) - 1/3 I(\text{previous cell})$
`Romberg`	$I_{j,k} = \frac{4^{k-1} I_{j+1,k-1} - I_{j,k-1}}{4^{k-1}-1}$
	$I_{j,k} = \frac{4^{\text{cell your on - 1}} I_{\text{cell your on}} - I_{\text{cell one up}}}{4^{k-1}-1}$
`Simpson 1/3`	$h/6(f(a) + 4f(a+h) + f(b))$
`Simpson 3/8`	$3\,h/8(f(a+h) + 3f(a+h) + 3f(a+2h) + f(b))$

Differentiation Equations

Forward Finite-Divided

$$\frac{d}{dx} = \frac{f(x+h) - f(x)}{h} \quad O(h)$$

Backward Finite-Divided

$$\frac{d}{dx} = \frac{f(x) - f(x-h)}{h} \quad O(h)$$

Central Difference

$$\frac{d}{dx} = \frac{f(x+h) - f(x-h)}{2h} \quad O(h^2)$$

$$\frac{d^2}{dx^2} = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \quad O(h^4)$$

Richardson Extrapolation

$$I = I(h) + \mathcal{E}(h)$$

$$I = I(h_1) + \mathcal{E}(h_1) = I(h_2) + \mathcal{E}(h_2)$$

$$\mathcal{E} \approx -\frac{b-a}{2} h^2 \bar{f}''$$

$$\frac{\mathcal{E}(h_1)}{\mathcal{E}(h_2)} \approx -\frac{\frac{b-a}{2} h_1^2 \bar{f}''}{\frac{b-a}{2} h_2^2 \bar{f}''} \approx \frac{h_1^2}{h_2^2}$$

$$\mathcal{E}(h_1) \approx \mathcal{E}(h_2) \left(\frac{h_1}{h_2}\right)^2$$

$$I \approx I(h_1) + \mathcal{E}(h_2) \left(\frac{h_1}{h_2}\right)^2 \approx I(h_2) + \mathcal{E}(h_2)$$

$$\mathcal{E}(h_2) \approx \frac{I(h_2) - I(h_1)}{(h_1/h_2)^2 - 1}$$

$$I \approx I(h_2) + \approx \frac{I(h_2) - I(h_1)}{(h_1/h_2)^2 - 1}$$

$$I \approx 4/3\, I(h_2) - 1/3\, I(h_1)$$

Differential Equations

Midpoint Method

Note that $y(a) = b \implies x_i = a, y_i = b$.

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_1 + h/2, y_i + k_1 h/2)$$

$$y_{i+1} = y_i + k_2 \cdot h$$

Heun Method

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + h, y_i + k_1 h)$$

$$y_{i+1} = y_i + \frac{h \cdot (k_1 + k_2)}{2}$$

RK-3

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + h/2, y_i + k_1 h/2)$$

$$k_3 = f(x_i + h, y_i + (-k_1 + 2k_2)h)$$

$$y_{i+1} = y_i + h \cdot (k_i + 4k_2 + k_3)/6$$

RK-4

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_1 + h/2, y_i + k_1 h/2)$$

$$k_3 = f(x_i + h/2, y_i + k_2 h/2)$$

$$k_4 = f(x_i + h, y_i + k_3 h)$$

$$y_{i+i} = y_i + h(k_1 + 2k_2 + 2k_3 + k_3)/6$$