

Chapter 10: Power Series

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10.1 Approximating Functions with Polynomials

Taylor Polynomials

Let f be a function with $f', f'', \dots, f^{(n)}$ defined at a . The **n th-order Taylor polynomial** for f with its **center** at a , denoted p_n , has the property that it matches f in value, slope, and all derivatives up to the n th derivative at a ; that is,

$$p_n(a) = f(a), p'_n(a) = f'(a), \dots, p_n^{(n)}(a) = f^{(n)}(a). \quad (1)$$

The n th-order Taylor polynomial centered at a is

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \quad (2)$$

More compactly, $p_n(x) = \sum_{k=0}^n c_k(x-a)^k$, where the **coefficients** are

$$c_k = \frac{f^{(k)}(a)}{k!}, \quad \text{for } k = 0, 1, 2, \dots, n \quad (3)$$

Remainder in a Taylor Polynomial

Let p_n be the Taylor polynomial of order n for f . The **remainder** in using p_n to approximate f at the point x is

$$R_n(x) = f(x) - p_n(x) \quad (4)$$

Taylor's Theorem

Let f have continuous derivatives up to $f^{(n+1)}$ on an open interval I containing a . For all x in I ,

$$f(x) = p_n(x) + R_n(x), \quad (5)$$

where p_n is the n th-order Taylor polynomial for f centered at a , and the remainder is

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{(n+1)}, \quad (6)$$

for some point c between x and a .

Estimate of the Remainder

Let n be a fixed positive integer. Suppose there exists a number M such that $|f^{(n+1)}(c)| \leq M$, for all c between a and x inclusive. The remainder in the n th-order Taylor polynomial for f centered at a satisfies

$$|R_n(x)| = |f(x) - p_n(x)| \leq M \frac{|x - a|^{n+1}}{(n+1)!} \quad (7)$$

10.2 Properties of Power Series

Power Series

A **power series** has the general form

$$\sum_{k=0}^{\infty} c_k(x-a)^k \quad (8)$$

where a and c_k are real numbers, and x is a variable. The c_k 's are the **coefficients** of the power series and a is the **center** of the power series. The set of values of x for which the series converges is its **interval of convergence**. The **radius of convergence** of the power series, denoted R , is the distance from the center of the series to the boundary of the interval of convergence.

Convergence of Power Series

A power series $\sum_{k=0}^{\infty} c_k(x-a)^k$ centered at a converges in one of three ways:

1. The series converges absolutely for all x , in which case the interval of convergence is $(-\infty, \infty)$ and the radius of convergence is $R = \infty$.
2. There is a real number $R > 0$ such that the series converges absolutely for $|x-a| < R$ and diverges for $|x-a| > R$, in which case the radius of convergence is R .
3. The series converges only at a , in which case the radius of convergence is $R = 0$.

Combining Power Series

Suppose the power series $\sum c_k x^k$ and $\sum d_k x^k$ converge absolutely to $f(x)$ and $g(x)$, respectively, on an interval I .

1. **Sum and difference:** The power series $\sum (c_k \pm d_k)x^k$ converges absolutely to $f(x) \pm g(x)$ on I .
2. **Multiplication by a power:** The power series $x^m \sum c_k x^k = \sum c_k x^{k+m}$ converges absolutely to $x^m f(x)$ on I , provided m is an integer such that $k+m \geq 0$ for all terms of the series.

3. **Composition:** If $h(x) = bx^m$, where m is a positive integer and b is a real number, the power series $\sum c_k(h(x))^k$ converges absolutely to the composite function $f(h(x))$, for all x such that $h(x)$ is in I .

Differentiating and Integrating Power Series

Let the function f be defined by the power series $\sum c_k(x - a)^k$ on its interval of convergence I .

- f is a continuous function on I .
- The power series may be differentiated or integrated term by term, and the resulting power series converges to $f'(x)$ or $\int f(x) dx + C$, respectively, at all points in the interior of I , where C is an arbitrary constant.

10.3 Taylor Series

Taylor/Maclaurin Series for a Function

Suppose the function f has derivatives of all orders on an interval containing the point a . The **Taylor series for f centered at a** is

$$f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f^{(3)}(a)}{3!}(x - a)^3 + \cdots \quad (9)$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x - a)^k. \quad (10)$$

A Taylor series centered at 0 is called a **Maclaurin series**.

Binomial Coefficients

$$\forall p, k \in \mathbb{R} \wedge k \geq 1$$

$$\binom{p}{k} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}. \quad (11)$$

With the special case of $\binom{p}{0} = 1$.

Binomial Series

$\forall p \in \mathbb{R} \wedge p \neq 0$, the Taylor series for $f(x) = (1+x)^p$ centered at 0 is the **binomial series**

$$\sum_{k=0}^{\infty} x^k = \sum_{k=0}^{\infty} \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!} x^k \quad (12)$$

$$= 1 + px + \frac{p(p-1)(p-2)}{3!} x^3 + \cdots \quad (13)$$

The series converges for $|x| < 1$ (and possibly at the endpoints, depending on p). If p is a nonnegative integer, the series terminates and results in a polynomial of degree p .

Convergence of Taylor Series

Let f have derivatives of all orders on an open interval I containing a . The Taylor series for f centered at a converges to f , for all x in I , if and only if $\lim_{n \rightarrow \infty} R_n(x) = 0$, for all x in I , where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1} \tag{14}$$

is the remainder at x (with c between x and a).

Taylor Series Functions

Table 10.5

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^k + \cdots = \sum_{k=0}^{\infty} x^k, \quad \text{for } |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^k x^k + \cdots = \sum_{k=0}^{\infty} (-1)^k x^k, \quad \text{for } |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \text{for } |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad \text{for } |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \quad \text{for } |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{k+1} x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}, \quad \text{for } -1 < x \leq 1$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{x^k}{k}, \quad \text{for } -1 \leq x < 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^k x^{2k+1}}{2k+1} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}, \quad \text{for } |x| \leq 1$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}, \quad \text{for } |x| < \infty$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \quad \text{for } |x| < \infty$$

$$(1+x)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k, \quad \text{for } |x| < 1 \text{ and } \binom{p}{k} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}, \binom{p}{0} = 1$$