MATH 1215 Practice Final Key

This is a closed-book, closed-notes exam. The only items you are allowed to use are writing implements. If you are caught cheating, you will receive a zero grade for this exam. The max number of points per question is indicated in square brackets after each question. The sum of the max points for all the questions is 61, but note that the max exam score will be capped at 60 (i.e., there is 1 bonus point, but you can't score more than 100%). You have exactly 60 minutes to complete this exam. Keep your answers clear and concise while complete. Best of luck.

1. Calculate the arc length of $\ln(\cos x)$ on $[0, \frac{\pi}{4}]$. Simplify. [7]

$$\begin{aligned}
f(x) &= \ln(\cos x) \\
f'(x) &= \frac{1}{\cos x}(-\sin x) = -\tan x \\
f'(x)^2 + 1 &= \tan^2 x + 1 = \sec^2 x
\end{aligned}$$

$$\begin{aligned}
L &= \int_0^{\pi/4} \sqrt{1 + f'(x)^2}\, dx \\
&= \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx \\
&= \int_0^{\pi/4} \sec x\, dx \\
&= \ln|\sec x + \tan x|\Big|_0^{\pi/4} \\
&= \ln|\sec(\pi/4) + \tan(\pi/4)| - \ln|\sec(0) + \tan(0)| \\
&= \ln|\sqrt{2} + 1| - \ln|1 + 0| \\
&= \ln(\sqrt{2} + 1)
\end{aligned}$$

2. Calculate the following derivatives:

(a) $\frac{d^2}{dx^2}(2\log_7(x-3))$ [2]

$$\begin{aligned}
\frac{d}{dx} &= \frac{2}{(x-3)\ln 7} \\
\frac{d^2}{dx^2} &= \frac{2}{\ln 7} \times \frac{d}{dx}(\frac{1}{x-3}) \\
&= -\frac{2}{\ln 7 (x-3)^2}
\end{aligned}$$

(b) $\frac{d}{dx}(x^{\sec x})$ [4]

$$\begin{aligned}
x^{\sec x} &= e^{\ln x^{\sec x}} \\
&= e^{\sec x \ln x} \\
y &= e^{\sec x \ln x} \\
\frac{dy}{dx} &= e^{\sec x \ln x} \times \frac{d}{dx}(\sec x \ln x) \\
&= e^{\sec x \ln x} \times (\sec x \tan x (\ln x) + \sec x \frac{1}{x}) \\
&= x^{\sec x} \times (\sec x \tan x (\ln x) + \sec x \frac{1}{x})
\end{aligned}$$

3. Classify which function has a faster rate of growth via limit methods **or** growth classes. If the limits are comparable, identify the growth factor M.

(a) 1.00001^x and x^{40}. $1.00001^x \gg x^{40}$ [1]

(b) $\ln x^{18}$ and $\ln x$. Comparable, $M = 18$ [1]

(c) $x \ln x$, $\ln^3 x$, and e^x. $\ln^3 x \gg x \ln x \gg e^x$ [2]

4. Evaluate the following indefinite integrals, simplification is optional (i.e. $\ln(1), \frac{2}{4}, 5^4$ are all acceptable forms).

(a) $\int 18x^2 \ln x \; dx$ [4]

$$u = \ln x \qquad dv = 18x^2$$
$$du = \frac{1}{x}dx \qquad v = 6x^3$$
$$\begin{aligned}\int 18x^2 \ln x \; dx &= 6x^3 \ln x - \int 6x^2 dx \\ &= 6x^3 \ln x - 2x^3 + C\end{aligned}$$

(b) $\int \frac{2}{y^4\sqrt{y^2-25}} \, dy$. Assume θ to be in the first quadrant *at all times*[1]. [8]

$$\begin{aligned}y &= 5\sec\theta \\ \sqrt{y^2-25} &= \sqrt{25\sec^2\theta - 25} \\ &= 5\sqrt{\tan^2\theta} \\ &= 5|\tan\theta| \\ &= 5\tan\theta\end{aligned}$$

$$\begin{aligned}dy &= 5\sec\theta\tan\theta \; d\theta \\ \int \frac{2}{y^4\sqrt{y^2-25}} \, dy &= \int \frac{2}{5^4\sec^4\theta(5\tan\theta)} 5\sec\theta\tan\theta \; d\theta \\ &= \frac{2}{625}\int \frac{1}{\sec^3\theta} \, d\theta \\ &= \frac{2}{625}\int \cos^3\theta \; d\theta \\ &= \frac{2}{625}\int (1-\sin^2\theta)\cos\theta \; d\theta \qquad u = \sin\theta \\ &= \frac{2}{625}\int 1-u^2 \; du \\ &= \frac{2}{625}(u - \frac{1}{3}u^3) + C \\ &= \frac{2}{625}(\sin\theta - \frac{1}{3}\sin^3\theta) + C\end{aligned}$$

(c) $\int \frac{7}{y^2-9y-112} dy$ [5]

$$\begin{aligned}y^2 - 9y - 112 &= (y-16)(y+7) \\ \frac{7}{(y-16)(y+7)} &= \frac{A}{y-16} + \frac{B}{y+7} \\ 7 &= A(y+7) + B(y-16)\end{aligned}$$
$$A = \frac{7}{23} \qquad B = -\frac{7}{23}$$

$$\begin{aligned}\int \frac{7}{y^2-9y-112} &= \int \frac{A}{y-16} + \frac{B}{y+7} \, dy \\ &= \int \frac{\frac{7}{23}}{y-16} - \frac{\frac{7}{23}}{y+7} \, dy \\ &= \frac{7}{23}(\ln|y-16| - \ln|y+7|) + C\end{aligned}$$

5. State the convergence or divergence of the following sequences and series. State all preconditions. Unless specified, use any method.

[1] Reasoning: this ensures all trigonometric functions will always be positive.

(a) $\{-\frac{\sin n}{6n}\}$ [3]

$$\lim_{n\to\infty} -\frac{1}{6n} = \lim_{n\to\infty} \frac{1}{6n} = 0$$
$$-\frac{1}{6n} \ll -\frac{\sin n}{6n} \ll -\frac{1}{6n}$$

$\therefore$ By Squeeze Theorem, Convergent to 0.

(b) $\sum_{n=2}^{\infty} \frac{1}{1+n\ln n}$ via Limit Comparison Test. [5]

$$\begin{aligned} a_n &= \frac{1}{1+n\ln n} \\ \text{let } b_n &= \frac{1}{n\ln n} \end{aligned}$$

$$\begin{aligned} \lim_{n\to\infty} \frac{b_n}{a_n} &= \lim_{n\to\infty} (1 + \frac{1}{n\ln n}) \\ &= 1 \end{aligned}$$

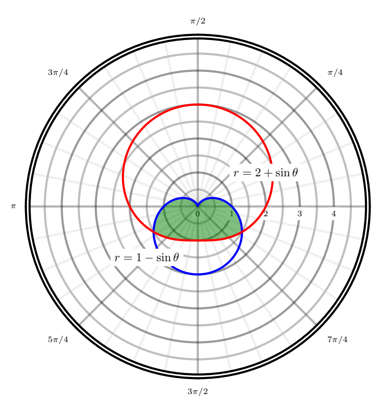
$$\begin{aligned} \int_2^{\infty} \frac{dx}{x\ln x} &= \int_2^{\infty} \frac{du}{u} \qquad u = \ln x \\ &= \infty \end{aligned}$$

$\therefore$ By Limit Comparison and Integral Test, Divergence.

(c) $\sum_{k=1}^{\infty} k\sin(\frac{1}{k})$ [6]

$$\begin{aligned} \lim_{k\to\infty} a_k &= \lim_{k\to\infty} k\sin(\frac{1}{k}) \\ &= \lim \frac{\sin(\frac{1}{k})}{\frac{1}{k}} \\ &= \lim \frac{-\frac{\cos(\frac{1}{k})}{k^2}}{-\frac{1}{k^2}} \\ &= \cos(\frac{1}{k}) \\ &= 1 \\ &\neq 0. \\ \therefore &\quad \text{Diverges by Divergence Test.} \end{aligned}$$

6. Set up the integral to find the area common to both $r = 2 + \sin\theta$ and $r = 1 - \sin\theta$ as shaded to the right. Do not simplify. **You do not need to evaluate the integral**. [5]



$$\int_0^{2\pi} \frac{1}{2}(2+\sin\theta)^2\, d\theta - \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2}\left[(2+\sin\theta)^2 - (1-\sin\theta)^2\right]$$

7. Set up the integral to find the pressure experienced by a circular plate with a radius of 2 that is submerged 6 meters below the water. This implies that the center of the plate is 8 meter below the water. Use ρ and g for pressure and acceleration. **You do not need to evaluate the integral**. [5]

$$2\rho g \int_{-2}^{2} (8-y)\sqrt{4-y^2}\, dy$$

8. Specify your answer as *True* or *False*. No counterexample is need.

(a) Given $\{a_n\}_{n=1}^{\infty}$, if a_n converges, then it it is bounded. Likewise, if a_n is bounded, then it is convergent. [1] **False.**

(b) If $\{a_n\}$ and $\{a_n\}$ are both divergent, then $\{a_n + b_n\}$ is divergent. [1] **False.**

(c) $\int_0^{\pi} \sec\theta \; d\theta$ is an improper integral. [1] **True.**