One Factor Experiments

- The different values of the factor are called the **levels** of the factor and cal also be called **treatments**.
- The objects upon which measurements are made are called **experimental units**.
- The units assigned to a given treatment are called **replicas**.
- There are I **samples**, each from a different treatment.
- If particular treatments are chosen deliberately by the experimenter, rather than at random, then we say that it is a **fixed effects model**.
- We have I samples each from a different treatment.
- The treatment means are denoted $\mu_1, \ldots, m_I$.
- The **sample sizes** are denoted $J_1, \ldots, J_I$. The total number in all samples combined is denoted N.
- The **hypothesis** that we wish to test is

$$H_0 = \mu_1 = \ldots = \mu_1$$

versus

$$H_1 = \text{two or more of the } \mu_i \text{ are different}$$

- X_{ij} denotes the jth observation in the ith sample.
- The variation of the sample means around the sample grand mean is measured by a quantity called the **treatment sum of squares** ($SSTr$)

$$SSTr = \sum_{i=1}^{I} J_{i.}(\bar{X}_{i.} - \bar{X}_{..})^2$$
$$= \sum_{i=1}^{I} J_i \bar{X}_{i.}^2 - N\bar{X}_{..}^2$$

- The measure of the variation in the individual sample points around their respective sample means is called the **error sum of squares (SSE)**

$$SSE = \sum_{i=1}^{I}\sum_{j=1}^{J_i}(X_{ij} - \bar{X}_i)^2$$
$$= \sum_{i=1}^{I}\sum_{j=1}^{J_i} X_{ij}^2 - \sum_{i=1}^{I} J_i \bar{X}_i^2$$

The standard one-way ANOVE hypothesis test is valid *under the following conditions*

1. The treatment population must be normal.
2. The treatment populations must all have the same variance, denoted σ^2.

- The quantities $I-1$ and $N-I$ are the **degrees of freedom** for $SSTr$ and SSE, respectively.

$$MSTr = \frac{SSTr}{I-1} \qquad MSE = \frac{SSE}{N-I}$$

- It follows that

$$\mu_{MSTr} = \sigma^2 \text{ when } H_0 \text{ is true}$$
$$\mu_{MSTr} > \sigma^2 \text{ when } H_0 \text{ is false}$$
$$\mu_{MSE} = \sigma^2 \text{ whether or not } H_0 \text{ is true}$$

- When H_0 is true, $MSTr$ and MSE have the same mean. Therefore, when H_0 is true, we would expect their quotient to be near 1. The quotient is in fact the test statistic. The test statistic for testing $\mu_0 : \mu_1 = \ldots = \mu_i$ is

$$F = \frac{MSTr}{MSE}$$

When, F tends to be near 1. When H_0 is false, $MSTr$ is larger, and F is greater than 1.

- The analysis of variance identity is

$$SST = SSTr + SSE$$

The actual equation is as follows:

$$SST = \sum_{i=1}^{I}\sum_{j=1}^{J_i}(X_{ij} - \bar{X}_{..})^2$$
$$= \sum_{i=1}^{I}\sum_{j=1}^{J_i} X_{ij}^2 - N\bar{X}^2$$

- Null hypothesis (H_0) assumed to be true. It is the "status quo" so we believe it unless there is sufficient evidence to reject it.
- Alternative hypothesis (H_1 or H_α) hypothesis we want to establish, i.e. try to prove. It is "against current thinking or status" so strong evidence is needed to justify it is true.
- Decision rule (where α is the significant level):

$$p - \text{value} < \alpha \implies \text{reject } H_0$$
$$p - \text{value} \geq \alpha \implies \text{fail to reject } H_0$$

- When equal numbers of units are assigned to each treatment, the design is said to be **balanced**.

Two Factor Experiments

- Notation for two-way ANOVA:

I The number of levels of the row factor

J The number of levels of the column factor

$I \times J$ The number of treatment combinations

K The number of replicates for each treatment combination

X_{ijk} The sample value for the kth replicate corresponding to the treatment combination formed by the ith level of the row factor and jth level of the column factor.

- The two-way ANOVA model (μ is grand mean, α_i is the ith row effect, β_j is the jth column effect, and ϵ is the error).

$$X_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ijk}$$
$$\alpha_i = \bar{\mu}_{i.} - \mu$$
$$\beta_j = \bar{\mu}_{.j} - \mu$$
$$\gamma_{ij} = \mu_{ij} - \bar{\mu}_i - \bar{\mu}_j + \mu$$
$$\epsilon_{ijk} = X_{ijk} - \mu_{ij}$$

- To test hypothesis for two-way ANOVA, test for the presence of interaction effects,

$$H_{0,AB} : \gamma_{11} = \gamma_{12} = \cdots = \gamma_{IJ} = 0$$
$$H_{H,AB} : \text{at least one of the } \gamma_{ij} \text{ is nonzero}$$

Hypothesis for the main effects are tested **only if** the additive model holds *i.e., we fail to reject $H_{0,AB}$ for the first hypothesis test*. If interaction effects **does not exist**, continue to test the main effects.
Test the main effects of the row factor (A),

$$H_{0,A} : \alpha_1 = \alpha_2 = \cdots = \alpha_I = 0$$
$$H_{1,A} : \text{at least one of the } \alpha_i \text{ is nonzero}$$

The main effects of the column factor is the same.

Sum of Squares for Two-Way ANOVA

- Treatment sum of squares: SSTr (d.f. = I*J – 1)
- Row sum of squares: $SSA = JK\sum_{i=1}^{I}\hat{\alpha}_i^2$ (d.f. = I – 1)
- Column sum of squares: $SSB = IK\sum_{j=1}^{J}\hat{\beta}_j^2$ (d.f. = J – 1)
- Interaction sum of squares: $SSAB = K\sum_{i=1}^{I}\sum_{j=1}^{J}\hat{\gamma}_{ij}^2$ (d.f. = (I – 1)(J – 1))
- Error sum of squares: SSE (d.f. = I*J(K – 1))
- Total sum of squares: SST (d.f.= I*J*K – 1)

There is a different statistic for each hypothesis

$$F_{AB}^* = \frac{MSAB}{MSE} \qquad F_A^* = \frac{MSA}{MSE} \qquad \frac{MSB}{MSE}$$

and

$$MSA = \frac{SSA}{I-1} \qquad MSB = \frac{SSB}{J-1} \qquad MSAB + \frac{SSAB}{(I-1)(J-1)}$$

Steps for Analysis of Two-Way Experiments

For a set significance level α

1. Check (a) normality assumption and (b) equal variance assumption
2. Test if additive model holds:

$P_{AB}^* < \alpha \rightarrow reject\ H_{0,AB} \rightarrow STOP$

$P_{AB}^* > \alpha \rightarrow fail\ to\ reject\ H_{0,AB} \rightarrow continue\ with\ step\ 3\ and\ 4$

3. Test for row effects: $P_A^* < \alpha \rightarrow reject\ H_{0,A}$

$P_A^* > \alpha \rightarrow fail\ to\ reject\ H_{0,A}$

4. Test for column effects: $P_B^* < \alpha \rightarrow reject\ H_{0,B}$

$P_B^* > \alpha \rightarrow fail\ to\ reject\ H_{0,B}$

2^p Factorial Experiment

- 2^p factorial is a factorial experiment that has p factors each of which has 2
- levels – one level is designated "high" and the other is "low".

Hypothesis Testing Procedure for 2^3 Factorial Experiments

1) Test for 3-way interaction (ABC)
   - Reject $H_{0,ABC}$ → 3-way interaction is present → STOP
   - Fail to reject $H_{0,ABC}$ → 3-way interaction absent → continue to next step
2) Test for 2-way interactions (AB, AC, BC). For each of these,
   - Reject H_0 → DO NOT test component main effect
   - Fail to reject H_0 → component main effects can be tested
   - For example: AB interaction present → do not test A nor B main effects (may test C main effect as long as AC and BC interactions are both absent)
3) Test for main effects that are permitted by previous step

Bernoulli Distribution

The pmf is as follows

$$p(x) = P(X = x) = \begin{cases} p & x = 1 \\ 1-p & x = 0 \end{cases}$$

- Bernoulli's principle is as follows:
  - $X = 1$ if the experiment results in "success"
  - $X = 0$ if the experiment results in "failure"
- Notation $X\tilde{\ }Benoulli(p)$

- With $X\tilde{\ }Bernoulli(p)$, the mean and variance are as follows

$$\mu_x = p$$
$$\sigma_x^2 = p(1-p)$$

Binomial Distribution

With n trials, and same probability success rate p, and X the number of successes in the n trials

$$p(x) = P(X=x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 0, 1, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

- The mean and variance of a binomial random variable is

$$\mu_x = np$$
$$\sigma_x^2 = np(1-p)$$

- To estimate the success probability p we can computer the sample proportion $\bar{p}$

$$\bar{p} = \frac{\text{number of successes}}{\text{number of trials}} = \frac{X}{n}$$

- The uncertainty of $\bar{p}$

$$\sigma_{\bar{p}} = \sqrt{Var(\bar{p})} \approx \sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$$

Poisson Distribution

The Poisson distribution arises frequently in scientific work. One way to think of the Poisson distribution is as an approximation to the binomial distribution when n is large and p is small.

- with $\lambda = np$, and notation $X\tilde{\ }Poisson(\lambda)$

$$p(x) = P(X=x) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!} & x = 0, 1, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

- The mean and variance are defined as follows

$$\mu_x = \lambda \qquad \sigma_x^2 = \lambda$$

The Normal Distribution

If a continuous random variable x with mean μ and variance σ^2 has the following probability density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

The notation is $X\tilde{\ }N(\mu, \sigma^2)$

The standard unit equivalent

$$Z = \frac{X-\mu}{\sigma}$$

The Exponential Distribution

The notation is as follows $X\tilde{\ }Exp(\lambda)$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x = 0, 1, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

- The CDF of $X\tilde{\ }exp(\lambda)$ is

$$F(x) = \begin{cases} 0 & x \leq 0 \\ 1 - e^{-\lambda x} & x > 0 \end{cases}$$

- The mean of $X\tilde{\ }exp(\lambda)$ is $\mu_x = \frac{1}{\lambda}$
- The variance of $X\tilde{\ }exp(\lambda)$ is $\sigma_x^2 = \frac{1}{\lambda^2}$
- The exponential distribution is sometimes used to model the waiting time to an event. If events follow a Poisson process with rate parameter λ, and if T represents the waiting time from any starting point until the next event, then $T\tilde{\ }exp(\lambda)$.
- The lack of memory property is as follows: If $T\tilde{\ }exp(\lambda)$, and t and s are positive number, then

$$P(T > t + s | T > s) = P(T > t)$$

$$s_i^2 = \sigma_i^2 = \frac{1}{J_i - 1} \sum_{j=1}^{J_i} (X_{ij} - \bar{X_{i.}})^2$$

Median time is

$$\frac{\ln 2}{\lambda}$$

Source of Variation	**d.f.**	***SS***	***MS***	***F***	***p*-value**
Treatment Combination	$IJ-1$	$SSTr$	$MSTr$	$F_{Tr}{*} = MSTr/MSE$	p_{Tr}
Row factor (A)	$I-1$	SSA	MSA	$F_A{*} = MSA/MSE$	p_A
Column factor (B)	$J-1$	SSB	MSB	$F_B{*} = MSB/MSE$	p_B
Interaction (A*B)	$(I-1)(J-1)$	$SSAB$	$MSAB$	$F_{AB}{*} = MSAB/MSE$	p_{AB}
Error	$N-IJ$	SSE	MSE	-----	
Total	$N-1$	SST	-----	-----	