

Chapter 14: Multiple Integration

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Contents

14.1 Double Integrals over Rectangular Regions	2
14.2 Double Integrals over General Regions	3
14.3 Double Integrals in Polar Coordinates	4
14.4 Triple Integrals	5
14.5 Triple Integrals in Cylindrical and Spherical Coordinates . . .	6
14.6 Integrals for Mass Calculations	8

14.1 Double Integrals over Rectangular Regions

Volumes and Double Integrals

A function f defined on a rectangular region R in the xy -plane is **integratable** on R if $\lim_{\Delta \rightarrow 0} \sum_{k=1}^n f(x_k^*, y_k^*) \Delta A_k$ exists for all partitions of R and for all choices of (x_k^*, y_k^*) within those partitions. The limit is the **double integral of f over R** , which we write

$$\iint_R f(x, y) dA = \lim_{\Delta \rightarrow 0} \sum_{k=1}^n f(x_k^*, y_k^*) \Delta A_k \quad (1)$$

If f is nonnegative on R , then the double integral equals the volume of the solid bounded by $z = f(x, y)$ and the xy -plane over R .

Double Integrals on Rectangular Regions

Let f be continuous on the rectangular region $R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$. The double integral of f over R may be evaluated by either of two iterated integrals:

$$\iint_R f(x, y) dA = \int_c^d \int_a^b f(x, y) dx dy = \int_a^b \int_c^d f(x, y) dy dx \quad (2)$$

Average Value of a Function over a Plane Region

The **average value** of an integrable function f over a region R is

$$\bar{f} = \frac{1}{\text{area of } R} \iint_R f(x, y) dA \quad (3)$$

14.2 Double Integrals over General Regions

Let R be a region bounded below and above by the graphs of the continuous functions $y = g(x)$ and $y = h(x)$, respectively, and by the lines $x = a$ and $x = b$. If f is continuous on R , then

$$\iint_R f(x, y) dA = \int_a^b \int_{g(x)}^{h(x)} f(x, y) dy dx \quad (4)$$

Let R be a region bounded on the left and right by the graphs of the continuous functions $x = g(y)$ and $x = h(y)$, respectively, and the lines $y = c$ and $y = d$. If f is continuous on R , then

$$\iint_R f(x, y) dA = \int_c^d \int_{g(y)}^{h(y)} f(x, y) dx dy \quad (5)$$

Areas of Regions by Double Integrals

Let R be a region in the xy -plane. Then

$$\text{area of } R = \iint_R 1 dA \quad (6)$$

14.3 Double Integrals in Polar Coordinates

Double Integrals over Polar Rectangular Region

Let f be continuous on the region in the xy -plane $R = \{(r, \theta) : 0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta\}$, where $\beta - \alpha \leq 2\pi$. Then

$$\iint_R f(r, \theta) dA = \int_{\alpha}^{\beta} \int_a^b f(r, \theta) r dr d\theta \quad (7)$$

Double Integrals over More General Polar Regions

Let f be continuous on the region in the xy -plane

$$R = \{(r, \theta) : 0 \leq g(\theta) \leq r \leq h(\theta), \alpha \leq \theta \leq \beta\} \quad (8)$$

where $0 < \beta - \alpha \leq 2\pi$. Then

$$\iint_R f(r, \theta) dA = \int_{\alpha}^{\beta} \int_{g(\theta)}^{h(\theta)} f(r, \theta) r dr d\theta. \quad (9)$$

Area of Polar Regions

The area of the region $R = \{(r, \theta) : 0 \leq g(\theta) \leq r \leq h(\theta), \alpha \leq \theta \leq \beta\}$, where $0 < \beta - \alpha \leq 2\pi$, is

$$A = \iint_R dA = \int_{\alpha}^{\beta} \int_{g(\theta)}^{h(\theta)} r dr d\theta \quad (10)$$

14.4 Triple Integrals

Let f be continuous over the region

$$D = \{(x, y, z) : a \leq x \leq b, g(x) \leq y \leq h(x), G(x, y) \leq z \leq H(x, y)\} \quad (11)$$

where g, h, G , and H are continuous functions. Then f is integrable over D and the triple integral is evaluated as the iterated integral

$$\iiint_D f(x, y, z) dV = \int_a^b \int_{g(x)}^{h(x)} \int_{G(x, y)}^{H(x, y)} f(x, y, z) dz dy dx. \quad (12)$$

Average Value of a Function of Three Variables

If f is continuous on a region D of \mathbb{R}^3 , then the average value of f over D is

$$\bar{f} = \frac{1}{\text{volume}(D)} \iiint_D f(x, y, z) dV \quad (13)$$

14.5 Triple Integrals in Cylindrical and Spherical Coordinates

Transformations Between Cylindrical and Rectangular Coordinates

Rectangular \rightarrow Cylindrical Cylindrical \rightarrow Rectangular

$$r^2 = x^2 + y^2$$

$$\tan \theta = y/x$$

$$z = z$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$

Triple Integrals in Cylindrical Coordinates

Let f be continuous over the region

$$D = \{(r, \theta, z) : g(\theta) \leq r \leq h(\theta), \alpha \leq \theta \leq \beta, G(x, y) \leq z \leq H(x, y)\} \quad (14)$$

Then f is integrable over D and the triple integral of f over D in cylindrical coordinates is

$$\iiint_D f(r, \theta, z) dV = \int_{\alpha}^{\beta} \int_{g(\theta)}^{h(\theta)} \int_{G(r \cos \theta, r \sin \theta)}^{H(r \cos \theta, r \sin \theta)} f(r, \theta, z) dz r dr d\theta \quad (15)$$

Transformations Between Spherical and Rectangular Coordinates

Rectangular \rightarrow Spherical Spherical \rightarrow Rectangular

$$\rho^2 = x^2 + y^2 + z^2$$

Use trigonometry to find φ and θ

$$x = \rho \sin \varphi \cos \theta$$

$$y = \rho \sin \varphi \sin \theta$$

$$z = \rho \cos \varphi$$

Triple Integrals in Spherical Coordinates

Let f be continuous over the region

$$D = \{(\rho, \varphi, \theta) : g(\varphi, \theta) \leq \rho \leq h(\varphi, \theta), a \leq \varphi \leq b, \alpha \leq \theta \leq \beta\} \quad (16)$$

Then f is integrable over D , and the triple integral of f over D in spherical coordinates is

$$\iiint_D f(\rho, \varphi, \theta) dV = \int_{\alpha}^{\beta} \int_a^b \int_{g(\varphi, \theta)}^{h(\varphi, \theta)} f(\rho, \varphi, \theta) \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta \quad (17)$$

14.6 Integrals for Mass Calculations

Center of Mass in One Dimension

Let ρ be an integrable density function on the interval $[a, b]$ (which represents a thin rod or wire). The **center of mass** is location at the point $\bar{x} = \frac{M}{m}$, where the **total moment** M and mass m are

$$M = \int_a^b x\rho(x) dx \quad \text{and} \quad m = \int_a^b \rho(x) dx \quad (18)$$

Center of Mass in Two Dimensions

Let ρ be integrable density function defined over a closed bounded region R in \mathbb{R}^2 . The coordinates of the center of mass of the object represented by R are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m} \iint_R x\rho(x, y) dA \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = \frac{1}{m} \iint_R y\rho(x, y) dA \quad (19)$$

Center of Mass in Three Dimensions

Let ρ be integrable density function on a closed bounded region D in \mathbb{R}^3 . The coordinates of the center of mass of the region are

$$\bar{x} = \frac{M_{yz}}{m} = \frac{1}{m} \iiint_R x\rho(x, y, z) dV \quad (20)$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{1}{m} \iiint_R y\rho(x, y, z) dV \quad (21)$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{1}{m} \iiint_R z\rho(x, y, z) dV \quad (22)$$

where $m = \iiint_D \rho(x, y, z) dV$ is the mass, and M_{yz} , M_{xz} and M_{xy} are the moments with respect to the coordinate planes.