Chapter #1

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Problem #1

Suppose P is a level and M is a level set. The statement that P is a **bottom level** of M means if x is a level in M then P cannot be above x.

Class Solution

P is a bottom level of the level set M if P is a level in M so that if x is a level in M then P is not above x.

Problem #2

Theorem 1. *If M is a level set having exactly one level, M **does** a top level.*

Proof. Suppose that $\exists P \in M$, where P is a level and M is a level set. Suppose, also, that P is the only level in M. By our definition of **top level**, we state that the **top level** of M means P is a level in M and if x is a level in M then x is not above P. But we only have one level in M, that is P.

Therefore, x must be P, and because Axiom 1 (If P is a level then P is not above P) agrees with our definition x is not above P. $\square$

Class Solution

Yes, because if level set M contains a single level P any level x within M would be P so according to Axiom 1 satisfies the definition of a top level of level set M.

Problem #3

With respect to all these definitions, can infer that P is a bottom level in set M. One can infer that level P is a bottom level.

Problem #4

No.

Proof. Suppose not. That is, suppose there exists levels P, Q, x in the level set M, where P is a different name for Q. Because they are both representations of each other, we will write Q to represent both P and Q.

Rewriting Axiom 2 in this notation, it states as follows:

> If each of Q and Q is a level and Q is above Q then there exists a level x such that Q is above x and x is above Q.

This argument is no longer valid, because it builds on the premise that Q is above Q (by Axiom 1, this not valid). This is a violation of Axiom 1. □

Problem #5

If P and Q are levels then P is above Q and Q is above P, then (by Axiom 3) this implies that P is transitive above P, which by Axiom 1 is a contradiction.

Class Solution (Osman)

No, if P and Q are levels such that P is above Q and Q is above P, then by Axiom 4 the P is above P which violates Axiom 1.

Problem #6

No.

Proof. Suppose not. That is, suppose that there exists two, different levels P, Q in the level set M. By Axiom 3, P is above Q or Q is above P, but not both (because that would imply P is above P). By our definition of top level,

P and Q must be a top level if $\exists x \in M$, where x is not above P and Q, but by Axiom 2 $\exists x \in M$ such that P is above x and x is above Q. Because the statements x is above and P, Q and P is above x and x is above Q cannot both hold, this is a contradiction. $\square$

Class Solution (Matthew Kovar)

Proof. If P and Q are both top levels in a level set M, then there is no level x in the level set M above P or Q (by Definition 1), which means that P cannot be above Q and Q cannot be above P. However, this violates Axiom 3 which states that P must be above Q or Q must be above P. Therefore, there cannot be more than one top level in a level set. $\square$

Problem #7

The following answer is wrong. An empty level set is simply an empty set, not an empty level set.

(a) A level set that satisfies the property of having no top level is the empty set, $\varnothing$.

(b) The empty set $\varnothing$ also satisfies this property.

(c) Assume $\exists P \in M$, where P is an arbitrary level and M to be the level set of all levels. Therefore, there are is no top level (if $Q \in M$, if Q is said to be the top level, $\exists x \in M$, where x is above Q), but P such that if $x \in M$ then P is above x.

Class Solution (Brett Sears, Illya Starikov, Matthew Healy)

(a) (Brett Sears) Let M be a level set containing some level P. A level x is included in M if x is above P. Suppose T is the top level of M. By Axiom 5, we know there is also a level U above T, T is above P or P. By Axiom 4, U is above P and included in M. Therefore there is no level that satisfies the definition of of a top level, and this is a contradiction.

(a) (Illya Starikov) Suppose M to be the level set of all levels. By our definition of top level, there exists a top level in M if there is a level $x, P \in M$, then x is not above P (where P is to be the candidate top level). However, for every level P, there is a level x above it. $\therefore M$ has no top level.

(b) (Matthew Healy) Suppose M to be the level set of all levels. For any bottom level b which we have defined a level such that for any level $x \in M$, b is not above x. By Axiom 5, there exists level x and y, such that $y > b > x$. Since this causes b to be above another level $x \in M$, b no longer satisfies the definition of a bottom level.

(c) (Brett Sears) Let M be the level set that includes every level P is above, but does not include P. Let T be a potential top level for M. By Axiom 2, there is a level x such that P is above x, x is above T. Since P is above x, x must be included in the level set M. However, since x is above T, the definition of a top level is not satisfied. So M does not have a definitive top level.

Problem #8

Class Solution

Let level set S_1 have a top level Q. Under the hypothesis of Axiom 6, suppose level set S_2 has a bottom level P. By Axiom 2, there exists a level x such that P is above x and x is above Q. Since x must be in S_1 or S_2 under the hypothesis of Axiom 6 and since Q is the top level of S_1, then x must be in S_2. Since bottom level P in S_2 is above level x in S_2, Kyle Foster's definition of a bottom level is violated and level P cannot be a bottom level. Therefore, if S_1 has a top level, S_2 cannot have a bottom level.

Problem #9

The level P is not the top level of the level set M means ... *there is more than one level in M.*

Class Solution (Terry Maxwell)

The level P is not the top level of the level set M means ... *the level P may or may not be included in the level set M and if P is included in M then there must exists a level x in level set M and x must be above P.*

Theorem #1

Theorem 2. *If M is a level set and a level B is above every level of M then M has a top level or there is a bottom level of the level set of all the levels that are above every level in M.*

Proof. Let C be the level set of all levels above every level above M, and let D be the level set of all levels not in C.

Any level x must be in either level set C or level set D. Assume there is a level y in C such that y is not above a level z in D. If y is z, then this would imply y is in both level set C and D, which is a contradiction. If y is not above z and not z, then, z is above y. Since y is above every level in M, then, since the hypothesis of Axiom 4 is satisfied, z would be above every level in M, which implies z is in both set C and set D, contradicting sets C and D's definitions. Since every level in level set C is above every level in level set D, the hypothesis of Axiom 6 is satisfied. Therefore, through the conclusion of Axiom 6, level set D has a top level or level set C has a bottom level.

But does this satisfy the original hypothesis of Theorem 1?

Yes, since level set D is also a level set, and a level B is above every level of M and, by our definition of C, B is in set C.

Claim: Level set D and level set M must both have a top level or both not have a top level.

Bare denial of the claim: Suppose level set D does not have a top level and level set M does, or level set D does have a top level and level set M does not.

Case #1 D does not have a top level and M does, and D does have a top level and level set M does not. A level set, by our definition of top level, cannot have both a top level and no top level at the same time.

Case #2 D does have a top level T and M does not.

By our definition of level set D, level set M does not contain levels that are above any levels in level set D. Yet, level set D also contains all of the levels within level set M, since the level set M is not included in level set C. So, if D does have a top level, then, because

1. M does not have any levels that are above the levels in D.
2. D contains all levels of M.
3. D does not contain any levels that are above all levels in M.

We have already shown $C = \{\text{all levels above every level in } M\}$, $D = \{\text{every level not in } C\}$, and Axiom 6 to be satisfied. If T is below $\forall x \in C$ then $\forall x \in M$ not above T and $T \notin C \implies T \in M$ or $Q \in M$ above T.

This must be true because their is no level Q in M that can satisfy being above T because it must then be a member of D and violate the definition of top level held by T, forcing Q to be in C. Therefore $T \in M$ and thus the top level of M.

Case #3 M does have a top level T and D does not. In this instance, see Case 2. If M has a top level, then D must include that top level T and no levels above T.

Therefore, since our bare denial has been proven false, our claim must be true! □

Problem #10

Suppose P and Q are levels. To say that P is **below** Q is to say that Q is above P.

Conjecture 1. *If P is a level then P is not below P.*

Proof. Suppose there is a level P such that P is below P. By the definition of below this says P is above P. This contradicts Axiom 1. Thus there is no level which is below itself. □

The Bare Denial of Above

The bare denial of P is above Q is P *is below* Q *or* P *is* Q.

Problem #11

Class Solution

A level P is an accumulation level of the level set M if a segment s contains P then s also contains a level of M distinct from P.

Problem #12

Class Solution (Terry Maxwell)

A level P is not an accumulation level of level set M if P is contained in a segment s, and there are no levels in s that are contained in level set M other than P.

Problem #13

Class Solution

1. (Daniel Welker) M is the level set containing only the level A. By Axiom 5, there must exist levels x and y such that y is above A and x is below A. These levels x and y may not be in the level set M because we have defined the level set M to contain only the level A. The levels x and y may form a segment $s(x, y)$ which contains the level A in level set M. Since there are no distinct levels of M in $s(x, y)$ other than A, the bare denial of the definition of an accumulation level is satisfied so that A is not an accumulation level of M.

2. (Daniel Welker) Suppose B to be above A such that $y > B > A > x$. Then B would be in segment $s(x, y)$. The segment s contains levels A and B, but only A is a member of level set M. Then $s(x, y)$ is not an accumulation level of M because it does not contain a level of M distinct from A.

Problem #14

Class Solution (Elisabeth Warner)

By Axiom 5, there exists levels x and y such that y is above A and A is above x. There also exists levels z and w such that z is above B and B is above w. However, because M is the level set that only contains A and B, levels x, y, z, and w are not in M.

In order for B (or A) to be an accumulation level of M, every segment s containing B (or A) must also contain a level of M distinct from B (or A).

Take A to be above B If A is above B and y is above A, then y is above B (Axiom 4). Take segment $s(w,\ y)$. B is above w but below y, so B is in the segment. A is above B (and thus above w) and y is above A, thus A is also in s. B is a possible accumulation level of M because segment $s(w,\ y)$ contains A (a level in M distinct from B).

However, take segment $s(w,\ A)$. As B is below A, B is included in s, but the level A is not included in the segment $s(w,\ A)$, by the definition of a segment. There is not a level in $s(w,\ A)$ of M distinct from B, therefore B is not an accumulation level of M.

Nevertheless, A could be an accumulation level of M. Take segment $s(B, y)$. A is above B and y is above A, so A is in segment s. However, B is not in segment s(endpoints not in segments), therefore there $s(B,\ y)$ does not contain B(a level in M distinct from A), thus A is not an accumulation level of M.

Suppose y is an accumulation of M. By Axiom 5, there exist levels j and k, where j is above y and y is above k. Take segment $s(A,\ j)$. Level y is in the segment due to the fact that y is above A but below j. Segment s does not contain either A or B, thus it does not contain a level in M distinct from y. Therefore, y is not an accumulation level of M.

Take B to be above A B is above A and z is above B, thus z is above A. Take segment $s(x,\ z)$. A is above x and z is above A, thus A is in the segment. B is above A but below z, so B is also in segment s. Segment $s(x,\ z)$ contains a level in M distinct from A, thus A is an accumulation level of M.

Take segment $s(x, B)$. A is in s, as A is below B, but B is not in s. Segment $s(x, B)$ does not contain a level in M distinct from A, thus A is not an accumulation level of M.

Testing B: take segment $s(A, z)$. B is in s (B is above A but below z), but A is not in s (endpoints of a segment). Therefore, segment $s(A, z)$ does not contain A (a level in M distinct from B), thus B is not an accumulation level of M.

There is not any level that satisfies the qualifications to be an accumulation level of level set M.

Problem #15

Class Solution (Brett Sears)

Yep.

Suppose there were no level x such that x is an accumulation level of s. In other words, suppose it were possible to define a segment $q(z, y)$, such that x is included in q with no other levels also in s distinct from x.

Let x be a level between (A, B).

If z is between (A, B) and y is above B. Then, there must be a level x between z and B (Axiom 2). Then, there must also be another level R between z and x. Since R and x are both between z and y and both included within (A, B), (z, y) must contain a level distinct from x that is also included in (A, B).

If z is between A and B and y is between A and B. Then, there must be a level x between z and y (Axiom 2). Then, there must also be another level P between z and x (Axiom 2). Since B is above x is above z, and x is above z is above A, B is above x is above A (Axiom 4), which means x is between A and B and thus included on (A, B). A similar argument can be made for level P. However, P and x are both included in (z, y), so (z, y) must contain a level distinct from x.

If z is below A and y is between A and B. Then, there must be a level x between y and A (Axiom 2). Likewise, there must also be another level R between x and $y(x, y)$.

If z is below A and y is below A. Suppose x is included in (z, y). Then, x is between z and y; in other words, y is above x is above z. However, x

is also included in (A, B). x must be above A since x is between A and B. So, x must be above A must be above y. However, y is above x, leading to a contradiction.

If z is above B and y is above B. Suppose x is included in (z, y). Then, x is between z and y; in other words, y is above x is above z. However, x is also included in (A, B). x must be above A since x is between A and B. So, x is above z is above B. However, B is above x, and by Axiom 4 we would have x is above x, leading to a contradiction.

If z is A and y is B. Then any level between z and y is also included in (A, B).

All possible segments have been exhausted.