Theorem 1. *Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the non-negative numbers by the recurrence*

$$T(n) = a\,T(n/b) + f(n) \tag{1}$$

where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then $T(n)$ has the following asymptotic bounds:

1. *If $f(n) = \mathcal{O}\left(n^{\log_b a - \epsilon}\right)$ for some constant $\epsilon > 0$, then $T(n) = \Theta\left(n^{\log_b a}\right)$.*
2. *If $f(n) = \Theta\left(n^{\log_b a}\right)$, then $T(n) = \Theta\left(n^{\log_b a} \lg n\right)$.*
3. *If $f(n) = \Omega\left(n^{\log_b a + \epsilon}\right)$ for some constant $\epsilon > 0$, and if*

$$a\,f(n/b) < c\,f(n)$$

for some constant $c < 1$ and sufficiently large n, then $T(n) = \Theta\left(f(n)\right)$.

Question #1

Problem #1.1

From Equation 1, we have the following:

$$a = 2 \qquad b = 4 \qquad f(n) = 1$$

From this,

$$\begin{aligned} f(n) &= \mathcal{O}\left(n^{\log_b a - \epsilon}\right) \\ 1 &= \mathcal{O}\left(n^{\log_4 2 - \epsilon}\right) \\ &= \mathcal{O}\left(n^{1/2 - \epsilon}\right) \\ &= \mathcal{O}\left(n^{1/2 - 1/2}\right) \\ &= \mathcal{O}\left(1\right) \end{aligned}$$

Therefore, the tight asymptotic bound for the recurrence is

$$\begin{aligned} T(n) &\in \Theta\left(n^{\log_b a}\right) \\ &\in \Theta\left(n^{\log_4 2}\right) \\ &\in \Theta\left(n^{1/2}\right) \\ &\in \Theta\left(\sqrt{n}\right) \end{aligned}$$

Therefore, the bound is $\Theta\left(\sqrt{n}\right)$.

Problem #1.2

From Equation 1, we have the following:

$$a = 2 \qquad b = 4 \qquad f(n) = \sqrt{n}$$

From this,

$$\sqrt{n} = \Theta\left(n^{\log_4 2}\right)$$

Therefore, the tight asymptotic bound for the the recurrence is

$$T(n) \in \Theta\left(\sqrt{n}\lg n\right)$$

Problem #1.3

From Equation 1, we have the following:

$$a = 2 \qquad b = 4 \qquad f(n) = n$$

From this,

$$n = \Omega\left(n^{\log_4 2 + {}^1\!/_2}\right)$$

Therefore, the tight asymptotic bound for the the recurrence is

$$T(n) \in \Theta\left(n\right)$$

Problem #1.4

From Equation 1, we have the following:

$$a = 2 \qquad b = 4 \qquad f(n) = n^2$$

From this,

$$n^2 = \Omega\left(n^{\log_4 2 + {}^3\!/_2}\right)$$

Therefore, the tight asymptotic bound for the recurrence is

$$T(n) \in \Theta\left(n^2\right)$$

Question #2

Recall Strassen's algorithm belongs to the complexity class $\Theta\left(n^{\lg 7}\right)$. By using the Case 1 of the Master Theorem, we must find a that solves the inequality $\log_4 a < \lg 7$. The integer that solves the equation $\log_4 a = \lg 7$ is 49; therefore,

$$a = 48$$

Question #3

Because the process is identical to Problem , the work is omitted; only answers are provided.

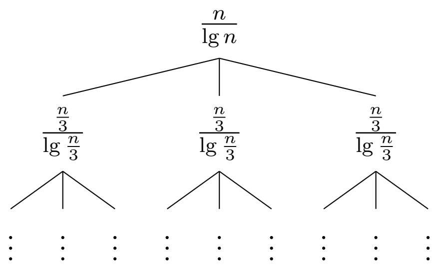
1. Master Theorem (*Case #3*), $T(n) \in \Theta\left(n^4\right)$
2. Master Theorem (*Case #3*), $T(n) \in \Theta\left(n\right)$
3. Master Theorem (*Case #2*), $T(n) \in \Theta\left(n^2 \lg n\right)$
4. Master Theorem (*Case #3*), $T(n) \in \Theta\left(n^2\right)$
5. Master Theorem (*Case #1*), $T(n) \in \Theta\left(n^{\log_2 7}\right)$
6. Master Theorem (*Case #2*), $T(n) \in \Theta\left(\sqrt{n} \lg n\right)$
7. Because $(n-2) \in \mathcal{O}\left(n\right)$, we can write an equivalent recurrence relation,

   $$T(n) = T(n/1) + n^2$$

   Because $\log_1 1$ is undefined, we cannot use the Master Theorem; but we do not need it. We have n^2 work to do n times. This implies that $T(n) \in \Theta\left(n^3\right)$.

Question #4

1. Master Theorem (*Case #1*), $T(n) \in \Theta\left(n^{\log_3 4}\right)$
2. Using a recursion tree,



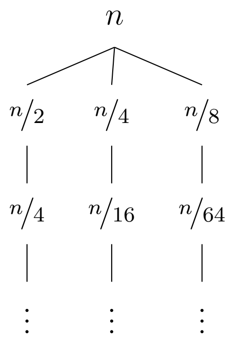
We see that the $T(n) \in \Theta(n \lg \lg n)$.

3. Master Theorem (*Case #3*), $T(n) \in \Theta(n^{2.5})$.

4. We augment the equation to be $3T(n/3) + n/2$, where we can use Master Theorem (*Case #2*), $T(n) \in \Theta(n \lg n)$.

5. With similar reasoning as in Step #2, we use a recursion tree to get the results $T(n) \in \Theta(n \lg \lg n)$

6. We solve so via the substitution method. We use a recursion tree to get our initial guess.



Guessing $T(n) \in \Theta(n)$, we get the following:

$$\begin{aligned} T(n) &= T(n/2) + T(n/4)T(n/8) + n \\ &= c\,n/2 + c\,n/4 + c\,n/8 + n \\ &= \frac{7}{8}n + n \\ &\le c\,n \end{aligned}$$

We see this inequality holds for $c \geq 8$. Therefore, $T(n) \in \Theta(n)$.

7. Using a recursion tree,

We see that $T(n) \in \Theta\left(n \lg \lg n\right)$.

8. Using a recursion tree,

$$\begin{array}{c} \lg n \\ | \\ \lg\left(n-1\right) \\ | \\ \lg\left(n-2\right) \\ | \\ \vdots \end{array}$$

We see that $T(n) \in \Theta\left(n \lg n\right)$.

9. Using a recursion tree,

$$\begin{array}{c} \frac{1}{\lg n} \\ | \\ \frac{1}{\lg(n-2)} \\ | \\ \frac{1}{\lg(n-4)} \\ | \\ \vdots \end{array}$$

We see that $T(n) \in \Theta\left(\lg \lg n\right)$.

10. We guess

$$T(n) \in \Theta\left(n \lg \lg n\right)$$

$$\begin{aligned} T(n) &= \sqrt{n}\, T\left(\sqrt{n}\right) + n \\ &= \sqrt{n}\, T\left(c\sqrt{n} \lg \lg \sqrt{n}\right) + n \\ &= c\, n \lg \lg n - c\, n + n \\ &\leq c\, n \lg \lg n \end{aligned}$$

We see this inequality for $c \in \mathbb{R}^{+}$. Therefore $T(n) \in \Theta\left(n \lg \lg n\right)$.

Question #5

Options c and e.

Question #6

1. We choose a root among the vertices, call it v_0. If we choose the kth smallest element, the smaller subtree will have $i-1$ vertices and the larger will have $n-i$ vertices. Summing over all possibilities, we get the following form for b_n:

$$\begin{aligned} b_n &= \sum_{k=1}^{n} b_{k-1}b_{n-k} \\ &= \sum_{k=0}^{n-1} b_k b_k b_{n-k-1} \end{aligned}$$

2.

$$\begin{aligned} B(x) &= \sum_{n=0}^{\infty} b_n x^n \\ &= 1 + \sum_{n=1}^{\infty} b_n x^n \\ &= 1 + \sum_{n=1}^{\infty}\sum_{k=0}^{n-1} b_k b_{n-k-1} x^{n-k-1} \\ &= 1 + x\sum_{n=1}^{\infty}\sum_{k=0}^{n-1} b_k x^k b_{n-k-1} x^{n-k-1} \\ &= 1 + x\sum_{n=0}^{\infty}\sum_{k=0}^{n} b_k x^k b_{n-k} x^{n-k} \\ &= 1 + x\,B(x)^2 \end{aligned}$$

3. We use the Taylor expansion of $\sqrt{1-4x}$, we have the following

$$
\begin{aligned}
B(x) &= \frac{1}{2x}\left(1 - \sum_{n=0}^{\infty} \frac{1}{1-2n}\binom{2n}{n}x^n\right) \\
&= -\frac{1}{2x}\sum_{n=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{1-2n}\binom{2n}{n}x^n \\
&= \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{2n-1}\binom{2n}{n}x^{n-1} \\
&= \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{2n+1}\binom{2n+2}{n+1}x^n
\end{aligned}
$$

Therefore $b_n = \frac{1}{n+1}\binom{2n}{n}$.

4.

$$
\begin{aligned}
b_n &= \frac{1}{n+1}\frac{(2n)!}{n!n!} \\
&\approx \frac{1}{n+1}\frac{\sqrt{4\,\pi n}(2n/e)^{2n}}{2\,\pi n(n/e)^{2n}} \\
&= \frac{1}{n+1}\frac{4^n}{\sqrt{\pi n}} \\
&= \left(\frac{1}{n} + \left(\frac{1}{n+1} - \frac{1}{n}\right)\right)\frac{4^n}{\sqrt{\pi n}} \\
&= \left(\frac{1}{n} - \frac{1}{n^2+n}\right)\frac{4^n}{\sqrt{\pi n}} \\
&= \frac{1}{n}\left(1 - \frac{1}{n+1}\right)\frac{4^n}{\sqrt{\pi n}} \\
&= \frac{4^n}{\sqrt{\pi}n^{3/2}}(1 + O(1/n))
\end{aligned}
$$