Chapter 15: Vector Calculus

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June 30, 2025

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15.1 Vector Fields

Vector Fields in Two Dimensions

Let f and g be defined on a region R of $\mathbb{R}^2$. A **vector field** in $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point in R a vector $\langle f(x,y),\, g(x,y)\rangle$. The vector field is written as

$$\mathbf{F}(x,y) = \langle f(x,y),\, g(x,y)\rangle \quad \text{or} \quad \mathbf{F}(x,y) = f(x,y)\hat{\mathbf{\imath}} + g(x,y)\hat{\mathbf{\jmath}} \tag{1}$$

A vector field $\mathbf{F} = \langle f, g\rangle$ is continuous or differentiable on a region R of $\mathbb{R}^2$ if f and g are continuous or differentiable on R, respectively.

Radial Vector Fields in $\mathbb{R}^2$

Let $\mathbf{r} = \langle x,\, y\rangle$. A vector field of the form $\mathbf{F} = f(x,\, y)\mathbf{r}$, where f is a scalar-valued function, is a **radial vector field**. Of specific interest are the radial vector field

$$\mathbf{F}(x,\, y) = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x,\, y\rangle}{|\mathbf{r}|^p} \tag{2}$$

where p is a real number. At every point (except the origin), the vectors of this field are directed outward from the origin with the magnitude of $|\mathbf{F}| = \frac{1}{|\mathbf{r}|^{p-1}}$.

Vector Fields in Three Dimensions

Let $f,\ g,$ and h be defined on a region D of $\mathbb{R}^3$. A **vector field** in $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point in D a vector $\langle f(x,\, y,\, z),\, g(x,\, y,\, z),\, h(x,\, y,\, z)\rangle$. The vector field is written as

$$\mathbf{F}(x,\, y,\, z) = \langle f(x,\, y,\, z),\, g(x,\, y,\, z),\, h(x,\, y,\, z)\rangle \quad \text{or} \tag{3}$$

$$\mathbf{F}(x,\, y,\, z) = f(x,\, y,\, z)\hat{\mathbf{\imath}} + g(x,\, y,\, z)\hat{\mathbf{\jmath}} + h(x,\, y,\, z)\hat{\mathbf{k}} \tag{4}$$

A vector field $\mathbf{F} = \langle f,\, g,\, h\rangle$ is continuous or differentiable on a region D of $\mathbb{R}^3$ if $f,\ g,\ h$ are continuous or differentiable on R, respectively. Of particular importance are the **radial vector fields**

$$\mathbf{F}(x,\, y,\, z) = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x,\, y,\, z\rangle}{|\mathbf{r}|^p} \tag{5}$$

where p is a real number.

Gradient Fields and Potential Functions

Let $z = \varphi(x, y)$ and $w = \varphi(x, y, z)$ be differentiable functions on regions of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. The vector field $\mathbf{F} = \nabla\varphi$ is **gradient field**, and the function φ is a **potential function** for $\mathbf{F}$.

15.2 Line Integrals

Scalar Line Integral in the Plane, Arc Length Parameter

Suppose the scalar-valued function f is defined on the smooth curve C : $\mathbf{r}(s) = \langle x(s),\, y(s)\rangle$, parameterized by the arc length s. The **line integral of f over C** is

$$\int\limits_C f(x(s),\, y(s))\, ds = \lim_{\Delta\to 0} \sum_{k=1}^{n} f(x(s_k^*),\, y(s_k^*))\Delta s_k, \tag{6}$$

provided this limit exists over all partitions of C. When the limit exists, f is said to be **integrable** on C.

Evaluating Scalar Line Integrals in $\mathbb{R}^2$

Let f be continuous on a region containing a smooth curve C : $\mathbf{r}(t) = \langle x(t),\, y(t)\rangle$, for $a \le t \le b$. Then

$$\begin{aligned} \int\limits_C f\, ds &= \int_a^b f(x(t),\, y(t))|\mathbf{r}'(t)|\, dt && (7) \\ &= \int_a^b f(x(t),\, y(t))\sqrt{x'(t)^2 + y'(t)^2}\, dt && (8) \end{aligned}$$

Evaluating the Line Integral $\int\limits_C f\, ds$

1. Find a parametric description of C in the form $\mathbf{r}(t) = \langle x(t),\, y(t)\rangle$, for $a \le t \le b$
2. Computer $|\mathbf{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$
3. Make substitutions for x and y in the integrand and evaluate an ordinary integral

$$\int\limits_C f\, ds = \int_a^b f(x(t),\, y(t))|\mathbf{r}'(t)|\, dt \tag{9}$$

Evaluating Scalar Line Integrals in $\mathbb{R}^3$

Let f be continuous on a region containing a smooth curve $C : \mathbf{r}(t) = \langle x(t),\, y(t),\, z(t)\rangle$, for $a \le t \le b$. Then

$$\int f\,ds = \int_a^b f(x(t),\, y(t),\, z(t))|\mathbf{r}'(t)|\,dt \tag{10}$$

$$= \int_a^b f(x(t),\, y(t),\, z(t))\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt \tag{11}$$

Line Integral of a Vector Field

Let $\mathbf{F}$ be a vector field that is continuous on a region containing a smooth oriented curve C parameterized by arc length. Let $\mathbf{T}$ be the unit tangent vector at each point of C consistent with the orientation. The line integral of $\mathbf{F}$ over C is $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$.

Different Forms of Line Integrals of Vector Fields

The line integral $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ may be expressed in the following forms, where $\mathbf{F} = \langle f,\, g,\, h\rangle$, for $a \le t \le b$:

$$\int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_a^b (f\,x'(t),\, g\,y'(t),\, h\,z'(t))\,dt \tag{12}$$

$$= \int_C f\,dx + g\,dy + h\,dz \tag{13}$$

$$= \int_C \mathbf{F}\cdot d\mathbf{r} \tag{14}$$

For line integrals in the plane, we let $\mathbf{F} = \langle f,\, g\rangle$ and assume C is parameterized in the form $\mathbf{r}(t) = \langle x(t),\, y(t)\rangle$, for $a \le t \le b$. Then

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_a^b (f\,x'(t) + g\,y'(t))\,dt = \int_C f\,dx + g\,dy = \int_C \mathbf{F}\cdot d\mathbf{r} \tag{15}$$

Work Done in a Force Field

Let $\mathbf{F}$ be a continuous force field in a region D of $\mathbb{R}^3$ and let $C : \mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, for $a \le t \le b$, be a smooth curve in D with a unit tangent vector $\mathbf{T}$ consistent with the orientation. The work done in moving an object C in the positive direction is

$$W = \int\limits_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F} \cdot \mathbf{r}'(\mathbf{t})\, dt \tag{16}$$

Circulation

Let $\mathbf{F}$ be a continuous vector field on a region D of $\mathbb{R}^3$ and let C be a closed smooth oriented curve in D. The **circulation** of $\mathbf{F}$ on C is $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$, where $\mathbf{T}$ is the unit vector tangent to C consistent with the orientation.

Flux

Let $F = \langle f, g\rangle$ be continuous vector field on a region R of $\mathbb{R}^2$. Let $C : \mathbf{r}(t) = \langle x(t), y(t)\rangle$, for $a \le t \le b$, be a smooth oriented curve in R that does not intersect itself. The **flux** of the vector field across C is

$$\int\limits_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_a^b (f\, y'(t) - g\, x'(t))\, dt, \tag{17}$$

where $n = \mathbf{T} \times \hat{\mathbf{k}}$ is the unit normal vector and $\mathbf{T}$ is the unit tangent vector consistent with the orientation. If C is a closed curve with counterclockwise orientation, $\mathbf{n}$ is the outward normal vector and the flux integral gives the **outward flux** across C.

15.3 Conservative Vector Fields

Simple and Closed Curves

Suppose a curve C (in $\mathbb{R}^2$ and $\mathbb{R}^3$) is described parametrically by $\mathbf{r}(t)$, where $a \leq t \leq b$. Then C is a **simple curve** if $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ for all t_1 and t_2, with $a < t_1 < t_2 < b$; that is, C never intersects itself between its endpoints. The curve C is **closed** if $\mathbf{r}(a) = \mathbf{r}(b)$; that is, the initial and terminal points of C are the same.

Connected and Simply Connected Regions

An open region R in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$) is **connected** if it possible to connect any two points of R by a continuous curve lying in R. An open region R is **simply connected** if every closed simple curve in R can be deformed and contracted to a point in R.

Conservative Vector Field

A vector field F is said to be **conservative** on a region (in $\mathbb{R}^2$ or $\mathbb{R}^3$) if there exists a scalar function φ such that $\mathbf{F} = \nabla\varphi$ on that region.

Test for Conservative Vector Fields

Let $\mathbf{F} = \langle f,\, g,\, h\rangle$ be a vector field defined on a connected and simply connected region D of $\mathbb{R}^3$, where f, g, and h have continuous first partial derivatives on D. Then $\mathbf{F}$ is a conservative vector field on D (there is a potential function φ such that $\mathbf{F} = \nabla\varphi$) if and only if

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \quad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \quad \text{and} \quad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y} \tag{18}$$

For vector fields in $\mathbb{R}^2$, we have the single condition $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$.

Finding Potential Functions in $\mathbb{R}^3$

Suppose $\mathbf{F} = \langle f,\, g,\, h\rangle$ is a conservative vector field. To find φ such that $\mathbf{F} = \nabla\varphi$, take the following steps:

1. Integral $\varphi_x = f$ with respect to x to obtain φ, which includes an arbitrary function $c(y,\, z)$.

2. Compute φ_y and equate it to g to obtain an expression for $c_y(y,\, z)$.

3. Integrate $c_y(y,\, z)$ with respect to y to obtain $c(y,\, z)$, including an arbitrary function $d(z)$.

4. Compute φ_z and equate it to h to get $d(z)$.

Beginning the procedure with $\varphi_y = g$ or $\varphi_z = h$ maybe be easier in some cases.

Fundamental Theorem for Line Integrals

Let $\mathbf{F}$ be a continuous vector field on an open connected region R in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$). There exists a potential function φ with $\mathbf{F} = \nabla\varphi$ (which means that $\mathbf{F}$ is conservative) if and only if

$$\int\limits_C \mathbf{F} \cdot \mathbf{T}\, ds = \int\limits_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A) \tag{19}$$

for all points A and B in R and all smooth oriented curves C from A to B.

Line Integrals on Closed Curves

Let R in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$) be an open region. Then $\mathbf{F}$ is a conservative vector field on R if and only if $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all simple closed smooth oriented curves C in R.

15.4 Green's Theorem

Let C be a simple closed smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g \rangle$, where f and g have continuous first partial derivatives in R. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA. \tag{20}$$

Two-Dimensional Curl

The **two-dimensional curl** of the vector field $\mathbf{F} = \langle f, g \rangle$ is $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}$. If the curl is zero throughout a region, the vector field is said to be **irrotational** on that region.

Area of a Plane Region by Line Integrals

Under the conditions of Green's Theorem, the area of a region R enclosed by a curve C is

$$\oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C (x\,dy - y\,dx) \tag{21}$$

Green's Thoerem, Flux Form

Let C be a simple closed smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g \rangle$, where f and g have continuous first partial derivatives in R. Then

$$\oint_C \mathbf{F} \cdot \mathbf{n}\,ds = \oint_C f\,dy - g\,dx = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) \tag{22}$$

where $\mathbf{n}$ is the outward unit normal vector on the curve.

Two-Dimensional Divergence

The **two-dimensional divergence** of the vector field $\mathbf{F} = \langle f, g \rangle$ is $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$. If the divergence is zero thoughout a region, the vector field is said to be **source free** on that region.

15.5 Divergence and Curl

Divergence of a Vector Field

The **divergence** of a vector field $\mathbf{F} = \langle f, g, h \rangle$ that is differentiable on a region of $\mathbb{R}^3$ is

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \tag{23}$$

If $\nabla \cdot \mathbf{F} = 0$, the vector field is **source free.**

Divergence of Radial Vector Fields

For a real number p, the divergence of the radial vector field

$$\mathbf{F} = \frac{\mathbf{r}}{|r|^p} = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{\frac{p}{2}}} \text{ is } \nabla \cdot \mathbf{F} = \frac{3 - p}{|\mathbf{r}|^p} \tag{24}$$

Curl of a Vector Field

The **curl** of a vector field $\mathbf{F} = \langle f, g, h \rangle$ that is differentiable on a region of $\mathbb{R}^3$ is

$$\nabla \times \mathbf{F} = \operatorname{curl} \mathbf{F} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) \hat{\mathbf{i}} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) \hat{\mathbf{j}} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \hat{\mathbf{k}} \tag{25}$$

If $\nabla \times \mathbf{F} = \mathbf{0}$, the vector field is **irrotational**.

Curl of a Conservative Vector Field

The **general rotation vector field** is $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, where the nonzero constant vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is the axis of ration and $\mathbf{r} = \langle x, y, z \rangle$. For all nonzero choices of $\mathbf{a}$, $|\nabla \times \mathbf{F}| = 2|\mathbf{a}|$ and $\nabla \cdot \mathbf{F} = 0$. The constant angular speed of the vector field is

$$\omega = |\mathbf{a}| = \frac{1}{2} |\nabla \times \mathbf{F}| \tag{26}$$

Curl of a Conservative Vector Field

Suppose that $\mathbf{F}$ is a conservative vector field on an open region D of $\mathbb{R}^3$. Let $\mathbf{F} = \nabla\varphi$, where φ is a potential function with continuous second partial derivatives on D. Then $\nabla \times \mathbf{F} = \nabla \times \nabla\varphi = \mathbf{0}$; that is, the curl of the gradient is the zero vector and $\mathbf{F}$ is irrotational.

Divergence of the Curl

Suppose that $\mathbf{F} = \langle f,\, g,\, h\rangle$, where $f,\ g,$ and h have continuous second partial derivatives. Then $\nabla \cdot (\nabla \times \mathbf{F}) = 0$: The divergence of the curl is zero.

Product Rule for the Divergence

Let u be a scalar-valued functino that is differentiable on a region D and let $\mathbf{F}$ be a vector field that is differentiable on D. Then

$$\nabla \cdot (u\mathbf{F}) = \nabla u \cdot \mathbf{F} + u(\nabla \cdot \mathbf{F}) \tag{27}$$

Properties of a Conservative Vector Field

Let $\mathbf{F}$ be a conservative vector field whose components have continuous second partial derivatives on an open connected region D in $\mathbb{R}^3$. Then $\mathbf{F}$ has the following equivalent properties.

1. There exists a potential function φ such that $\mathbf{F} = \nabla\varphi$
2. $\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$ for all points A and B in D and all smooth oriented curves C from A and B.
3. $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all simple smooth closed oriented curves C in D.
4. $\nabla \times \mathbf{F} = \mathbf{0}$ at all points of D.

15.6 Surface Integrals

Surface Integrals of Scalar-Valued Functions on Parameterized Surface

Let f be a continuous function on a smooth surface S given parametrically by $\mathbf{r}(u,\, v) = \langle x(u,\, v),\, y(u,\, v),\, z(u,\, v)\rangle$, where $R = \{(u,\, v) : a \le u \le b, c \le v \le d\}$. Assume also that the tangent vectors $\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u} = \langle \frac{\partial x}{\partial u},\, \frac{\partial y}{\partial u},\, \frac{\partial z}{\partial u}\rangle$, and $\mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v} = \langle \frac{\partial x}{\partial v},\, \frac{\partial y}{\partial v},\, \frac{\partial z}{\partial v}\rangle$ are continuous on R and the normal vectors $\mathbf{n} = \mathbf{t}_u \times \mathbf{t}_v$ is nonzero on R. Then the **surface integral** of the scalar-valued function f over S is

$$\iint_S f(x,\, y,\, z)\, dS = \iint_R f(x(u,\, v),\, y(u,\, v),\, z(u,\, v))|\mathbf{t}_u \times \mathbf{t}_v|\, dA \tag{28}$$

If $f(x,\, y,\, z) = 1$, the integral equals the surface area of S.

Evaluation of Surface Integrals of Scalar-Valued Functions on Explicitly Defined Surfaces

Let f be a continuous function on a smooth surfaces S given by $z = g(x,\, y)$, for $(x,\, y)$ in a region R. The surface integral of f over S is

$$\iint_S f(x,\, y,\, z)\, dS = \iint_R f(x,\, y,\, g(x,\, y))\sqrt{z_x^2 + z_y^2 + 1}\, dA \tag{29}$$

If $f(x,\, y,\, z) = 1$, the surface integral equals the area of the surface.

Surface Integral of a Vector Field

Suppose $\mathbf{F} = \langle f,\, g,\, h\rangle$ is a continuous vector field on a region of $\mathbb{R}^3$ containing a smooth oriented surface S. If S is defined parametrically as $\mathbf{r}(u,\, v) = \langle x(u,\, v),\, y(u,\, v),\, z(u,\, v)\rangle$, for $(u,\, v)$ is a region R, then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \mathbf{F} \cdot (\mathbf{t}_u \times \mathbf{t}_v)\, dA \tag{30}$$

where $\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u} = \langle \frac{\partial x}{\partial u},\, \frac{\partial y}{\partial u},\, \frac{\partial z}{\partial u}\rangle$ and $\mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v} = \langle \frac{\partial x}{\partial v},\, \frac{\partial y}{\partial v},\, \frac{\partial z}{\partial v}\rangle$ are continuous on R, the normal vector $\mathbf{n} = \mathbf{t}_u \times \mathbf{t}_v$ is nonzero on R, and the direction of $\mathbf{n}$ is consistent with the orientation of S. If S is defined in the form $z = g(z,\, y)$, for $(x,\, y)$ in a region R, then

$$\iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint\limits_R (-f z_x - g z_y + h)\, dA \tag{31}$$

15.7 Stokes' Theorem

Let S be a smooth oriented surface in $\mathbb{R}^2$ with a smooth closed boundary C whose orientation is consistent with that of S. Assume that $\mathbf{F} = \langle f,\, g,\, h\rangle$ is a vector field whose components have continuous first partial derivatives on S. Then

$$\oint\limits_C \mathbf{F} \cdot d\mathbf{r} = \iint\limits_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS \tag{32}$$

where $\mathbf{n}$ is the unit vector normal to S determined by the orientation of S.

Curl $\mathbf{F} = \mathbf{0}$ Implies $\mathbf{F}$ is Conservative

Suppose that $\nabla \times \mathbf{F} = \mathbf{0}$ throughout an open simply connected region D of $\mathbb{R}^3$. Then $\oint\limits_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all closed simple smooth curves C in D and $\mathbf{F}$ is a conservative vector field on D.

15.8 Divergence Theorem

Let $\mathbf{F}$ be a vector field whose components have continuous first partial derivatives in a connected and simply connected region D in $\mathbb{R}^3$ enclosed by a smooth oriented surface S. Then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_D \nabla \cdot \mathbf{F}\, dV \tag{33}$$

where $\mathbf{n}$ is the unit outward normal vector on S.

Divergence Theorem for Hollow Regions

Suppose the vector field $\mathbf{F}$ satisfies the conditions of the Divergence Theorem on a region D bounded by two smooth oriented surfaces S_1 and S_2, where S_1 lies within S_2. Let S be the entire boundary of $D(S = S_1 \cup S_2)$ and let $\mathbf{n_1}$ and $\mathbf{n}_2$ be the outward unit normal vectors for S_1 and S_2, respectively. Then

$$\iiint_D \nabla \cdot \mathbf{F}\, dV = \iint \mathbf{F} \cdot \mathbf{n} dS = \iint_{S_2} \mathbf{F} \cdot \mathbf{n_2}\, dS = \iint_{S_1} \mathbf{F} \cdot \mathbf{n_1}\, dS \tag{34}$$