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Chapter 7: Logarithmic and Exponential Functions

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Contents

7.1 Inverse Function

Derivative of the Inverse Function

Let f be differentiable and have an inverse on an interval I. If x_0 is a point of I at which $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}, \text{ where } y_0 = f(x_0) \tag{1}$$

7.2 The Natural Logarithmic and Exponential Functions

The Natural Logarithm

The **natural logarithm** of a number $x > 0$, denoted x, is defined

$$\ln x = \int_1^x \frac{dt}{t} \tag{2}$$

Properties of the Natural Logarithm

1. The domain and range of $\ln x$ are $(0, \infty)$ and $(-\infty, \infty)$, respectively.
2. $\ln(xy) = \ln x + \ln y, \forall x, y \in \mathbf{R}^+$
3. $\ln(x/y) = \ln x - \ln y, \forall x, y \in \mathbf{R}^+$
4. $\ln x^p = p \ln x, \forall x \in \mathbf{Q}^+$
5. $\frac{d}{dx}(\ln |x|) = \frac{1}{x}, \forall x \in \mathbf{R} \wedge x \neq 0$
6. $\frac{d}{dx}(\ln |u(x)|) = \frac{u'(x)}{u(x)}$
7. $\int \frac{dx}{x} = \ln |x| + C$

The Number e

The number e is the real number that satisfies

$$\ln e = \int_1^e \frac{dt}{t} = 1 \tag{3}$$

The Exponential Function

$\forall x, y \in \mathbf{R}$

$$y = e^x \Leftrightarrow x = \ln y \tag{4}$$

Properties of e

- $e^{x+y} = e^x e^y$
- $e^{x-y} = e^x / e^y$
- $(e^x)^y = e^{xy}, \forall y \in \mathbf{Q}$
- $\ln(e^x) = x, \forall x \in \mathbf{R}$
- $e^{\ln\ x} = x, \forall x \in \mathbf{R}^+$

Exponential Functions with General Bases

Let $b \in \mathbf{R}^+ \wedge b \neq 1$. $\forall x \in \mathbf{R}^+$,

$$b^x = e^{x \ln b} \tag{5}$$

Derivative and Integral of the Exponential Function

$\forall x \in \mathbf{R}$,

$$\frac{d}{dx}(e^{u(x)}) = e^{u(x)} u'(x) \tag{6}$$

$$\int e^x \, dx = e^x + C \tag{7}$$

7.3 Logarithmic and Exponential Functions with Other Bases

Logarithmic Function Base b

For any base $b > 0$, with $b \neq 1$, the **logarithmic function base** b, denoted $\log_b x$, is the inverse of the exponential function b^x.

Inverse Relations for Exponential and Logarithmic Functions

For any base $b > 0$, with $b \neq 1$, the following inverse relation holds.

- $b^{\log_b x} = x$, $\forall x \in \mathbf{R}^+$
- $\log_b b^x = x$, $\forall x$

Derivative of b^x

If $b > 0 \land b \neq 1, \forall x$,

$$\frac{d}{dx}(b^x) = b^x \ln b \tag{8}$$

Indefinite Integral of b^x

For $b > 0 \land b \neq 1$,

$$\int b^x dx = \frac{1}{\ln b} b^x + C \tag{9}$$

General Power Rule

$\forall p, x \in \mathbf{R}^+$,

$$\frac{d}{dx}(x^p) = px^{p-1} \tag{10}$$

Furthermore, if u is a positive differentiable function on its domain, then

$$\frac{d}{dx}(u(x)^p) = p(u(x))^{p-1} \times u'(x) \tag{11}$$

Derivative of $\log_b x$

If $b > 1$,

$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b} \quad \forall x \in \mathbf{R}^+ \wedge x \neq 0 \tag{12}$$

$$\frac{d}{dx}(\log_b |x|) = \frac{1}{x \ln b} \quad \forall x \in \mathbf{R}^+ \wedge x \neq 0 \tag{13}$$

7.5 Inverse Trigonometric Functions

Derivative of Inverse Trigonometric Functions

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ $\quad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$, $\{x \in \mathbf{R} \mid -1 < x < 1\}$
$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ $\quad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$, $\{x \in \mathbf{R} \mid -\infty < x < \infty\}$
$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$ $\quad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$, $\{x \in \mathbf{R} \mid |x| > 1\}$

Integrals Involving Inverse Trigonometric Functions

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C \tag{14}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C \tag{15}$$

$$\int \frac{dx}{x\sqrt{a^2 + x^2}} = \frac{1}{a} \sec^{-1} \left|\frac{x}{a}\right| + C \tag{16}$$

7.6 L'Hôpital's Rule and Growth Rates of Functions

Indeterminate forms $1^\infty, 0^0, \infty^0$

Assume $\lim_{x\to a} f(x)^{g(x)}$ has the indeterminate form 1^∞, 0^0, or ∞^0.

1. Evaluate $L = \lim_{x\to a} g(x)\ln f(x)$. This limit can be put in the form $0/0$ or ∞/∞, both of which are handled by l'Hôpital's rule.
2. Then $\lim_{x\to a} f(x)^{g(x)} = e^L$

Growth Rates of Functions (as $x \to \infty$**)**

Suppose f and g are functions with $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$. Then f **grows faster than g** as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0 \quad \text{or, quantitatively, if} \quad \lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty \tag{17}$$

The functions f and g have **comparable growth rates** if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = M$$

where $M \in \mathbb{R}^+$.

Ranking Growth Rates as $x \to \infty$

Let $f << g$ mean that g grows faster than f as $f \to \infty$. With positive real numbers p, q, r, s and $b > 1$,

$$\ln^q x << x^p << x^p \ln^r x << x^{p+s} << b^x << x^x \tag{18}$$

7.7 Hyperbolic Functions

Hyperbolic Functions

Hyperbolic Cosine

$$\cosh x = \frac{e^x + e^{-x}}{2} \tag{19}$$

Hyperbolic Sine

$$\sinh x = \frac{e^x - e^{-x}}{2} \tag{20}$$

Hyperbolic Tangent

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \tag{21}$$

Hyperbolic Cotangent

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} \tag{22}$$

Hyperbolic Secant

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \tag{23}$$

Hyperbolic Cosecant

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} \tag{24}$$

Hyperbolic Identities

$\cosh^2 x - \sinh^2 x = 1$	$\cosh(-x) = \cosh x$
$1 - \tanh^2 x = \operatorname{sech}^2 x$	$\sinh(-x) = -\sinh x$
$\coth^2 x - 1 = \operatorname{csch}^2 x$	$\tanh(-x) = -\tanh x$
$\coth(x+y) = \cosh x \cosh y + \sinh x \sinh y$	
$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$	
$\cosh 2x = \cosh^x + \sinh^2$	$\sinh 2x = 2 \sinh x \cosh x$
$\cosh^2 x = \frac{\cosh 2x + 1}{2}$	$\sinh^2 x = \frac{\cosh 2x - 1}{2}$

Derivatives and Integral Formulas

1. $\frac{d}{dx}(\cosh x) = \sinh x \quad \Rightarrow \quad \int \sinh x \; dx = \cosh x + C$
2. $\frac{d}{dx}(\sinh x) = \cosh x \quad \Rightarrow \quad \int \cosh x \; dx = \sinh x + C$
3. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \quad \Rightarrow \quad \int \operatorname{sech}^2 x \; dx = \tanh x + C$
4. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x \quad \Rightarrow \quad \int \operatorname{csch}^2 x \; dx = -\coth x + C$
5. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x \quad \Rightarrow \quad \int \operatorname{sech} x \tanh x \; dx = -\operatorname{sech} x + C$
6. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x \quad \Rightarrow \quad \int \operatorname{csch} x \coth x \; dx = -\operatorname{csch} x + C$

Integrals of Hyperbolic Functions

1. $\int \tanh x \, dx = \ln \cosh x + C$
2. $\int \coth x \, dx = \ln |\sinh x| + C$
3. $\int \operatorname{sech} x \, dx = \tan^{-1} |\sinh x| + C$
4. $\int \operatorname{csch} x \, dx = \ln |\tanh(x/2)| + C$

Inverses of the Hyperbolic Functions Expressed as Logarithms

$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}) \ (x \geq 1)$ $\quad \operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x} \ (0 < x \leq 1)$
$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$ $\quad \operatorname{csch}^{-1} x = \sinh^{-1} \frac{1}{x} \ (x \neq 0)$
$\tanh^{-1} x = \frac{1}{2} \ln(\frac{1+x}{1-x}) \ (|x| < 1)$ $\quad \coth^{-1} x = \tanh^{-1} \frac{1}{x} \ (|x| > 1)$

Derivatives of the Inverse Hyperbolic Functions

$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}} \ (x > 1)$ $\quad \frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$
$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2} \ (|x| < 1)$ $\quad \frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2} \ (|x| > 1)$
$\frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}} \ (0 < x < 1)$ $\quad \frac{d}{dx}(\operatorname{csch}^{-1} x) = -\frac{1}{|x|\sqrt{1+x^2}} \ (x \neq 0)$

Integral Formulas

1. $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1} \frac{x}{a} + C,$ for $x > a$
2. $\int \frac{dx}{x^2+a^2} = \sinh^{-1} \frac{x}{a} + C,$ for all x

3. $\int \frac{dx}{a^2-x^2} = \frac{1}{a}\tanh^{-1}\frac{x}{a} + C,\ \text{for}|x| < a = \frac{1}{a}\coth^{-1}\frac{x}{a} + C, \text{for } |x| > a$

4. $\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\frac{x}{a} + C,\ \text{for } 0 < x < a$

5. $\int \frac{dx}{x\sqrt{a^2+x^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\frac{|x|}{a} + C,\ \text{for } x \neq 0$