Homework #5

Analysis of Algorithms

Illya Starikov

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Question #1

Theorem 1. *If $L \geq 2$, then every binary tree with L leaves contains a subtree having between $\frac{L}{3}$ and $\frac{2L}{3}$ leaves, inclusive.*

Proof. Suppose not. That is, suppose that there exists a binary tree with more than 1 leaves with all subtrees containing leaves in the range $\frac{L}{3} > n > \frac{2L}{3}$. The root contains L leaves, and all bottommost leaves contain 1 leaves.

This implies that two subtrees with size less than $\frac{L}{3}$ (because our hypothesis states all leaves must in the range $\frac{L}{3} > n > \frac{2L}{3}$) combined to make a tree of size greater than $\frac{2L}{3}$. This is a contradiction

$$\frac{L}{3} + \frac{L}{3} \not> \frac{2L}{3}$$

Because we have reached a contradiction, our hypothesis does not hold. Therefore, binary tree with L leaves contains a subtree having between $\frac{L}{3}$ and $\frac{2L}{3}$ leaves, inclusive. □

Question #2

Theorem 2. *For every binary tree T, let us associate a "weight" $w(q) = 2^{-depth(q)}$. For all leaves $q \in T$,*

$$\sum_q w(q) \leq 1$$

Proof. We prove so by induction.

(Base Case) For the root,

$$\sum_{x \in T} w(x) = 2^{-0} = 1$$

(Inductive Hypothesis) Suppose for any tree of n nodes, $\sum_{x \in T}^{n} w(q) \leq 1$.

(Inductive Step) Suppose the binary tree T_0 to have $n-1$ nodes. We must show that $\sum_{x \in T_0}^{n-1} w(x) \leq 1$. Suppose we take the left subtree T_L or the right subtree T_R. By the inductive hypothesis, the sum of the weights are less than or equal to 1; however, since the depth of a node in T_0 is one greater than the depth of T_L or T_R, the respective weights of ever leaf in T_0 is halved. Thus,

$$\sum_{x \in T_L} w(x) \leq 1/2 \qquad \sum_{x \in T_R} w(x) \leq 1/2$$

Because both of these are bounded by $1/2$, their sum forms T_0, which is bounded by 1.

By the principle of mathematical induction, $\sum_q w(q) \leq 1$ for any tree. $\square$

Question #3

Theorem 3. *Any planar graph can be colored with six or fewer colors.*

Proof. We prove so by induction. We take n to be an arbitrary number of nodes in an arbitrary graph G.

(Base Case) For n vertices, where $1 \leq n \leq 6$, we can color the graph G with 6 or less colors (for every vertices maps to a single, unique color).

(Inductive Hypothesis) Suppose for $n-1$ vertices, where $n > 1$, the graph G can be colored with 6 or less colors.

(Inductive Step) We now prove that G can be colored with n vertices can be colored with 6 or fewer colors. Recall all connected, simple planar graph contains a maximum of 5 degrees. Suppose that the any vertex $v \in G$ has this max degree 5.

We remove this vertex, name it v_0, with all incident edges. We now have less than n vertices, and by our inductive hypothesis, this graph now can be colored with 6 or less colors. Adding this vertex back, we see that there are 5 incident vertexes, hence 5 colors. We use the 6$^{\text{th}}$ color for v_0.

Hence, the inductive step holds. By the principle of mathematical induction, any planar graph can be colored with six or fewer colors. $\square$

Question #4

Theorem 4. *Any point x in a graph is a cut-point iff there exists two vertices in the graph, a and b, such that every path between a and b has to pass through x.*

Proof. Assume G to be a connected graph, and $\exists$ vertices $a, b \in G$, where every path $a \longleftrightarrow b$ path passes through x. Also assume there exists no path $a \longleftrightarrow b \in G - \{x\}$. Therefore, $G - \{x\}$ is disconnected, for a and b are in two different components of $G - \{x\}$. This proves that x is a cut-point. $\square$

Question #5

Euler Path A path that uses every edge of a graph exactly once.

Hamiltonian Cycle A path through a graph that starts and ends at the same vertex and includes every other vertex exactly once.

Yes, all **cyclic graphs** meet both these conditions.