Homework #9

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Due Date: December 1^{st}, 2016

1 Simpson 1/3 Rule For $1 + 2x + 3x^2$

$$\begin{aligned}\int_0^4 1 + 2x + 3x^2\,dx &\approx {}^h\!/_6(f(a) + 4\,f(a+h) + f(b)) \\ &= {}^4\!/_6(1 + 4(1 + 4 + 12) + (1 + 8 + 48)) \\ &= {}^2\!/_3(1 + 68 + 57) \\ &= 84\end{aligned}$$

2 Simpson's 3/8 Rule For x^3

$$\begin{aligned}\int_0^3 x^3\,dx &\approx {}^3\!/_8 h(f(a) + 3(a+h) + 3(a+2h) + f(b)) \\ &= {}^3\!/_8(0 + 1 + 8 + 27) \\ &= {}^3\!/_8 \cdot 36 \\ &= 13.5\end{aligned}$$

3 Trapezoidal, Richardson and Romberg Method of x^4

We know the equation for the trapezoid rule to be

$$I = \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n} + f(x_n)\right]$$

From this, we obtain

$$\begin{aligned}
I_4 &= \frac{4}{2}\left(f(0) + f(4)\right) = 512 \\
I_2 &= \frac{2}{2}\left(f(0) + 2\,f(2) + f(4)\right) = 288 \\
I_1 &= \frac{1}{2}\left(f(0) + 2\,f(2) + 2\,f(3) + f(4)\right) = 226
\end{aligned}$$

For Richardson, we calculate the terms as follows:

$$\begin{aligned}
I &\approx {}^4\!/\!_3 I(h_2) - {}^1\!/\!_3 I(h_1) \\
I_2 &= {}^4\!/\!_3(288) - {}^1\!/\!_3(512) = {}^{640}\!/\!_3 = 213.\bar{3} \\
I_1 &= {}^4\!/\!_3(226) - {}^1\!/\!_3(288) = {}^{616}\!/\!_3 = 205.\bar{3}
\end{aligned}$$

And for Romberg,

$$\begin{aligned}
I_{j,k} &= \frac{4^{k-1} I_{j+1,k-1} - I_{j,k-1}}{4^{k-1}} \\
I_1 &= \frac{4^2({}^{616}\!/\!_3) - {}^{640}\!/\!_3}{15} = \frac{1024}{5} = 204.8
\end{aligned}$$

We observe that the table looks like

h	**trapazoidal**	**Richardson**	**Romberg**
4	512		
2	288	$213.\bar{3}$	
1	226	$205.\bar{3}$	204.8

4 Derivation of Richardson Extrapolation

We recall that the value of integration I is equal to approximation + error, which we can eloquently write

$$I = I(h) + \mathcal{E}(h) \tag{1}$$

Where $\mathcal{E}$ represents our error and $I(h)$ represents our approximation. Supposing we have two different steps size (h_1 and h_2), we can rewrite equation (1) in the form

$$I = I(h_1) + \mathcal{E}(h_1) = I(h_2) + \mathcal{E}(h_2) \tag{2}$$

Now, let us expand $\mathcal{E}(h)$ from equation (1) and equation (2).

$$\mathcal{E} \approx -\frac{b-a}{2} h^2 \, \bar{f}'' \tag{3}$$

Where b and a are the upper and lower bounds, respectively, and $\bar{f}''$ is the average value of $\frac{d}{dx} f(x)$ (where $f(x)$ is the function we are integrating). Therefore, we can a ratio of the two errors

$$\frac{\mathcal{E}(h_1)}{\mathcal{E}(h_2)} \approx -\frac{\frac{b-a}{2} h_1^2 \, \bar{f}''}{\frac{b-a}{2} h_2^2 \, \bar{f}''} \approx \frac{h_1^2}{h_2^2}$$

Which can be algebraically manipulated to

$$\mathcal{E}(h_1) \approx \mathcal{E}(h_2) \left(\frac{h_1}{h_2} \right)^2 \tag{4}$$

Now we can substitute equation (4) back into equation (2) to obtain

$$I \approx I(h_1) + \mathcal{E}(h_2) \left(\frac{h_1}{h_2} \right)^2 \approx I(h_2) + \mathcal{E}(h_2)$$

Which can be solve for

$$\mathcal{E}(h_2) \approx \frac{I(h_2) - I(h_1)}{(h_1/h_2)^2 - 1}$$

Now that we have an estimate (hence the $\approx$) of the truncation error, We once plug this back into equation (2) to get

$$I \approx I(h_2) + \approx \frac{I(h_2) - I(h_1)}{(h_1/h_2)^2 - 1}.$$

Because we know the interval to be halved, we can safely assume $h_2 = h_1/2$. Thus, our equation reduces to

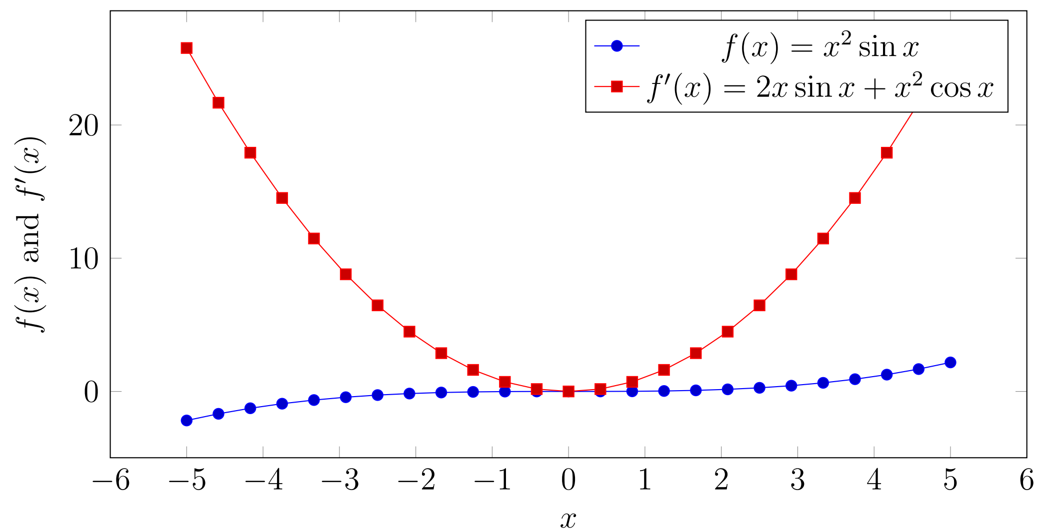
$$I \approx {}^4\!/_3\, I(h_2) - {}^1\!/_3\, I(h_1) \tag{5}$$

5 Derivative of $x^2 \sin x$

From the chain rule, we get

$$\frac{d}{dx} x^2 \sin x = 2x \sin x + x^2 \cos x$$

It can be plotted as follows



To compute the two derivatives for $h_1 = 0.2$ and $h_2 = 0.1$, we use the centered difference formulas. For h_1

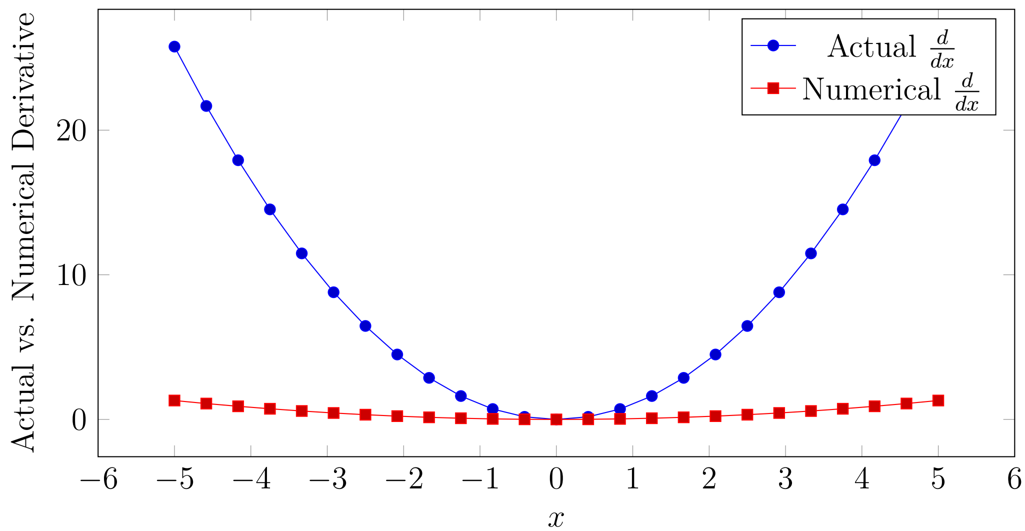
$$\begin{aligned}
\frac{d}{dx} &= \frac{f(x_{i+1}) - f(x_{i-1})}{2h} \\
&= \frac{f(x+h) - f(x-h)}{2h} \\
&= \frac{f(x + {}^1\!/_5) - f(x - {}^1\!/_5)}{{}^2\!/_5} \\
&= \frac{(x + {}^1\!/_5)^2 \sin(x + {}^1\!/_5) - ((x - {}^1\!/_5)^2 \sin(x - {}^1\!/_5))}{{}^2\!/_5}
\end{aligned}$$

Similarly, for h_2

$$\frac{(x + {}^1\!/_{10})^2 \sin(x + {}^1\!/_{10}) - ((x - {}^1\!/_{10})^2 \sin(x - {}^1\!/_{10}))}{{}^1\!/_5}$$

We combine these for obtain the Richardson extrapolation formula,

$$\begin{aligned} D &\approx {}^4\!/_3 D(h_2) - {}^1\!/_3 D(h_1) \quad (6) \\ &\approx {}^4\!/_3 \frac{(x + {}^1\!/_{10})^2 \sin(x + {}^1\!/_{10}) - ((x - {}^1\!/_{10})^2 \sin(x - {}^1\!/_{10}))}{{}^1\!/_5} \quad (7) \\ &- {}^1\!/_3 \frac{(x + {}^1\!/_5)^2 \sin(x + {}^1\!/_5) - ((x - {}^1\!/_5)^2 \sin(x - {}^1\!/_5))}{{}^2\!/_5} \quad (8) \end{aligned}$$



6 Central Difference of x^4

6.1 First Order

From the power rule, we notice that

$$\frac{d}{dx}x^4 = 4x^3 \implies f'(1) = 4$$

Using centered difference formula,

$$
\begin{aligned}
\frac{d}{dx} &= \frac{f(x_{i+1}) - f(x_{i-1})}{2h} \\
&= \frac{(1 + {}^1\!/\!_2)^4 - (1 - {}^1\!/\!_2)^4)}{(2{}^1\!/\!_2)} \\
&= 5
\end{aligned}
$$

We notice the error to be $|\frac{4-5}{4}| = |-.25| \implies \mathbf{25}\%$

6.2 Second Order

From the power rule, we notice that

$$\frac{d^2}{dx^2}x^4 = 12x^2 \implies f'(1) = 12.$$

Using centered difference formula,

$$
\begin{aligned}
\frac{d^2}{dx^2} &= \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2} \\
&= \frac{(1 + {}^1\!/\!_2)^4 - 2(1^4) + (1 - {}^1\!/\!_2)^4)}{({}^1\!/\!_2)^2} \\
&= {}^{25}\!/\!_2
\end{aligned}
$$

We notice the error to be $|\frac{12 - 25/2}{12}| = |-.041\bar{6}| \implies \mathbf{4.16}\%.$