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Chapter 8: Integration Techniques

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8.1 Basic Approaches

$$\int k\,dx = kx + C \tag{1}$$

$$\int k^p\,dx = \frac{k^{p+1}}{p+1} + C, p \in \mathbf{R} \wedge \neq -1 \tag{2}$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C \tag{3}$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C \tag{4}$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C \tag{5}$$

$$\int \csc^2 ax\,dx = -\frac{1}{a}\cot ax + C \tag{6}$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax + C \tag{7}$$

$$\int \csc ax \cot ax\,dx = -\frac{1}{a}\csc ax \cot ax + C \tag{8}$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C \tag{9}$$

$$\int \frac{dx}{x} = \ln|x| + C \tag{10}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C \tag{11}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C \tag{12}$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C \tag{13}$$

8.2 Integration By Parts

Suppose that u and v are differentiable functions. Then

$$\int u\,dv = uv - \int v\,du \tag{14}$$

Integration by Parts for Definite Integrals

Let u and v be differentiable. Then

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)\,dx \tag{15}$$

Integral of $\ln x$

$$\int \ln x\,dx = x\ln x - x + C \tag{16}$$

8.3 Trigonometric Integrals

$\int \sin^m x \cos^n x \; dx$ **Strategy**

m **is odd,** n **real** Split off $\sin x$, rewrite the resulting even power of $\sin x$ in terms of $\cos x$, and then use $u = \cos x$

n **odd,** m **real** Split off $\cos x$, rewrite the resulting even power of $\cos x$ in terms of $\sin x$, and then use $u = \sin x$.

m **and** n **both even, nonnegative** Use half-angle identities to transform the integrand into a polynomial in $\cos 2x$, and apply the preceding strategies once again to powers of $\cos 2x$ greater than 1.

Reduction Formulas

Assume n is a positive integer.

$$\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \; dx \tag{17}$$

$$\int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \; dx \tag{18}$$

$$\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \; dx, n \neq 1 \tag{19}$$

$$\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \; dx, n \neq 1 \tag{20}$$

Integrals of $\tan x, \cot x, \sec x,$ and $\csc x$

$$\int \tan x \, dx = -\ln|\cos x| + C = \ln|\sec x| + C \tag{21}$$

$$\int \cot x \, dx = \ln|\sin x| + C \tag{22}$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C \tag{23}$$

$$\int \csc x \, dx = -\ln|\csc x + \cot x| + C \tag{24}$$

$\int \tan^m x \sec^n x \, dx$ **Strategy**

n **even** Split off $\sec^2 x$, rewrite the remaining even power of $\sec x$ in terms of $\tan x$, and use $u = \tan x$.

m **odd** Split off $\sec x \tan x$, rewrite the remaining even power of $\tan x$ in terms of $\sec x$, and use $u = \sec x$.

m **even and** n **odd** Rewrite the even power of $\tan x$ in terms of $\sec x$ to produce a polynomial in $\sec x$; apply reduction formula 4 to each term.

8.4 Trigonometric Substitutions

The Integral Contains...	Corresponding Substitution	Useful Identity
$a^2 - x^2$	$x = a\sin\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \forall \lvert x\rvert \le a$	$a^2 - a^2\sin^2\theta = a^2\cos^2\theta$
$a^2 + x^2$	$x = a\tan\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$a^2 + a^2\tan^2\theta = a^2\sec^2\theta$
$x^2 - a^2$	$x = a\sec\theta,$ $\begin{cases} 0 \le \theta < \frac{\pi}{2}, \forall x \ge a \\ \frac{\pi}{2} < \theta \le \pi, \forall x \le -a \end{cases}$	$a^2\sec^2\theta - a^2 = a^2\tan^2\theta$

8.5 Partial Fractions

Partial Fractions with Simple Linear Factors

Suppose $f(x) = p(x) > q(x)$, where p and q are polynomials with no common factors and with the degree of p less than the degree of q. Assume that q is the product of simple linear factors. The partial fraction decomposition is obtained as follows.

Step 1. Factor the denominator q in the form $(x-r_1)(x-r_2)\cdots(x-r_n)$, where $r_1, \ldots, r_n$ are real numbers.

Step 2. Partial fraction decomposition Form the partial fraction decomposition by writing

$$\frac{p(x)}{q(x)} = \frac{A_1}{x-r_1} + \frac{A_2}{x-r_2} + \cdots + \frac{A_n}{x-r_n} \tag{25}$$

Step 3. Clear denominators Multiply both sides of the equation in Step 2 by $q(x) = (x-r_1)(x-r_2)\cdots(x-r_n)$, which produces conditions for $A_1, \ldots, A_n$.

Step 4. Solve for coefficients Equate like powers of x in Step 3 to solve for the undertmined coefficients $A_1, \ldots, A_n$

Partial Fractions For Repeated Linear Factors

Suppose the repeated linear factor $(x-r)^m$ appears in the denominator of a proper rational function in reduced form. The partial fraction decomposition has a partial fraction for each power of $(x-r)$ up to and including the mth power; that is, the partial fraction decomposition contains the sum

$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_m}{(x-r)^m} \tag{26}$$

where $A_1, \ldots, A_m$ are constants to be determined.

Partial Fractions with Simple Irreducible Quadratic Factors

Suppose a simple irreducible factor $ax^2 + bx + c$ appears in the denominator of a proper rational function in reduced form. The partial fraction decomposition contains a term of the form

$$\frac{Ax + B}{ax^2 + bx + c} \tag{27}$$

where A and B are unknown coefficients to be determined.

Partial Fraction Decomposition

Let $f(x) = p(x)/q(x)$ be a proper rational function in reduced form. Assume the denominator q has been factored completely over the real numbers and m is a positive integer.

Simple linear factor A factor $x-r$ in the denominator requires the partial fraction $\frac{A}{x-r}$.

Repeated linear factor A factor $(x-r)^m$ with $m > 1$ in the denominator requires the partial fractions.

$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_m}{(x-r)^m} \tag{28}$$

Simple irreducible quadratic factor An irreducible factor $ax^2 + bx + c$ in the denominator requires the partial fraction

$$\frac{Ax + B}{ax^2 + bx + c} \tag{29}$$

Repeated irreducible quadratic factor An irreducible factor $(ax^2+bx+c)^m$ with $m > 1$ in the denominator requires the partial fractions

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_mx + B_m}{(ax^2 + bx + c)^m} \tag{30}$$

8.8 Improper Integrals

Improper Integrals over Infinite Intervals

1. If f is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx \tag{31}$$

provided the limit exists.

2. If f is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx \tag{32}$$

provided the limit exists.

3. If f is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{a\to-\infty} \int_a^c f(x)\,dx + \lim_{b\to\infty} \int_c^b f(x)\,dx \tag{33}$$

provided both limits exist, where c is any real number.

In each case, if the limit exists, the improper integral is said to **converge**, if it does not exist, the improper integral is said to **diverge**.

Improper Integrals with an Unbounded Integrand

1. Suppose f is continuous on $(a, b]$ with $\lim_{x\to a^+} f(x) = \pm\infty$. Then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx \tag{34}$$

provided the limit exists.

2. Suppose f is continuous on $[a, b)$ with $\lim_{x\to b^-} f(x) = \pm\infty$. Then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx \tag{35}$$

provided the limit exists.

3. Suppose f is continuous on $[a, b]$ except at the interior point p where f is unbounded. Then

$$\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx, \tag{36}$$

provided the improper integrals on the right side exist.

In each case, if the limit exists, the improper integrals is said to **converge**, if it does not exists, the improper integral is said to **diverge**.