Calculus I: Single-Variable Calculus

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Contents

1 Functions

1.1 Review of Functions

A **function** is a rule that assigns to each value x in a set D a unique value denoted $f(x)$. The set D is the **domain** of the function. The **range** is the set of all values of $f(x)$ produced as x varies over the domain.

Vertical Line Test

A graph represents a function if and only if it passes the **vertical line test**. Every vertical line intersects the graph at most once. A graph that fails this test does not represent a function.

Composite Functions

Given two functions f and g, the composite functions $f \circ g$ is defined by $(f \circ g)(x) = f(g(x))$. It is evaluated in two steps: $y = f(u)$, where $u = g(x)$. The domain of $f \circ g$ consists of all x in the domain of g such that $u = g(x)$ is in the domain of f.

Symmetry in Functions

An **even function** has the property that $f(-x) = f(x)$, for all x in the domain. The graph of an even function symmetric about the y-axis. Polynomials consisting of only even powers of the variable (of the form x^{2n}, where n is a nonnegative integer) are even functions.
An **odd function** f has the property that $f(-x) = -f(x)$, for all x in the domain. The graph of an odd function is symmetric about the origin. Polynomials consisting of only odd powers of the variable (of the form x^{2n+1}, where n is a nonnegative integer) are odd functions.

1.2 Review of Functions

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1.3 Representing Functions

Some brief families of functions can include

Polynomials are functions of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where the **coefficients** $a_0, a_1, \ldots, a_n$ are real numbers with $a_n \neq 0$ and the nonnegative integer n is the **degree** of the polynomial. The domain of any polynomial is the set of all real numbers. An nth-degree polynomial can have as many as n real **zeros** or **roots** — values of x.

Rational Functions are ratios of the form $f(x) = p(x)/q(x)$, where p and q are polynomials. Because division by zero is prohibited, the domain of a rational function is the set of all real numbers except those for which the denominator is zero.

Algebriac Functions are constructed using the operations of algebra: addition, subtraction, multiplication, division, and roots. Examples of algebraic functions are $f(x) = \sqrt{2x^3 + 4}$ and $f(x) = x^{1/4}(x^3 + 2)$. In general, if an even root (square root, fourth root, and so forth) appears, then the domain does not contain points at which the quantity under the root is negative (and perhaps other points).

Exponential Functions have the form $f(x) = b^x$, where the base $b \neq 1$ is a positive real number. Closely associated with exponential functions are logarithmic functions of the form $f(x) = \log_b x$, where $b > 0$ and $b \neq q$. An exponential function has a domain consisting of all real numbers. Logarithmic functions are defined for positive, real numbers. The most important function is the **natural exponential function** $f(x) = e^x$, with base $b = e$, where $e \approx 2.71828\ldots$ is one of the fundamental constants of mathematics. Associated with the natural exponential function is the **natural logarithmic function** $f(x) = \ln x$, which also has the base $b = e$.

Trigonometric Functions are $\sin x$, $\cos x$, $\tan x$, $\sec x$, and $\cos x$; they are fundamental to mathematics and many areas of application. Also important are their relatives, the **inverse trigonometric functions**.

Transcendental Functions Trigonometric, exponential, and logarithmic functions are few examples of a large family called transcendental functions.

Transformations

Given the real numbers a, b, c, and d and the function f, the graph of $y = cf(a(x - b)) + d$ is obtained from the graph of $y = f(x)$ in the following steps.

$$
\begin{aligned}
y = f(x) &\xrightarrow{\text{horizontal scaling by a factor of } |a|} y = f(ax) \\
&\xrightarrow{\text{horizontal shift by } b \text{ units}} y = f(a(x-b)) \\
&\xrightarrow{\text{vertical scaling by a factor of } |c|} y = cf(a(x-b)) \\
&\xrightarrow{\text{horizontal scaling by a factor of |a|}} y = cf(a(x-b)) + d
\end{aligned}
$$

1.4 Inverse, Exponential, and Logarithmic Functions

The Natural Exponential Function

The **natural exponential function** is $f(x) = e^x$, which as the base $e = 2.718281828459\ldots$.

Inverse Function

Given a function f, its inverse (if it exists) is a function f^{-1} such that whenever $y = f(x)$, then $f^{-1}(y) = x$.

One-to-One Functions and the Horizontal Line Test

A function f is **one-to-one** on a domain D if each value of $f(x)$ corresponds to exactly one value of x in D. More precisely, f is one-to-one on D is $y(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ for x_1 and x_2 in D. The **horizontal line test** says that every horizontal line intersects the graph of a one-to-one function at most once.

Existence of Inverse Functions

Let f be one-to-one function on a domain D with a range R. Then f has a unique inverse f^{-1} with domain R and range D such that

$$f^{-1}(f(x)) = x \qquad \text{and} \qquad f(f^{-1}(y)) = y,$$

where x is in D and y is in R.

Finding an Inverse Function

Suppose f is one-to-one on an interval I. To find f^{-1}:

- Solve $y = f(x)$ for x. If necessary, choose the function that corresponds to I.
- Interchange x and y and write $y = f^{-1}(x)$.

Logarithmic Function Base b

For any base $b > 0$, with $b \neq 1$, the **logarithmic function base** b, denoted $y = \log_b x$, is the inverse of the exponential function $y = b^x$. The inverse of the natural exponential function with base $b = e$ is the **natural logarithm function**, denoted $y = \ln x$.

Inverse Relations For Exponential and Logarithmic Functions

For any base $b > 0$, with $b \neq 1$, the following inverse relations hold:

- $b^{\log_b x} = x$, for $x > 0$
- $\log_b b^x = x$, for any real values of x

Change-of-Base Rules

Let b be a positive real number with $b \neq 1$. Then

$$b^x = e^{x \ln b}, \text{for all } x \qquad \text{and} \qquad \log_b x = \frac{\ln x}{\ln b}, \text{for } x > 0$$

More generally, if c is a positive real number with $c \neq 1$, then

$$b^x = c^{x \log_c b}, \text{for all } x \qquad \text{and} \qquad \log_b x = \frac{\log_c x}{\log_c b}, \text{for } x > 0$$

1.5 Trigonometric Functions and Their Inverses

Let $P(x, y)$ be a point on a circle of radius r associated with the angle θ. Then

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x} \tag{1}$$
$$\cot\theta = \frac{x}{y} \qquad \sec\theta = \frac{r}{x} \qquad \csc\theta = \frac{r}{y} \tag{2}$$
$$\tag{3}$$

Trigonometric Identities

Reciprocal Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} \tag{4}$$
$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \tag{5}$$

Pythagorean

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \cot^2\theta = \csc^2\theta \qquad \tan^2\theta + 1 = \sec^2\theta \tag{6}$$

Double- and Half-Angle Formulas

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \cos 2\theta = \sin^2\theta - \sin^2\theta \tag{7}$$
$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2} \tag{8}$$

Period of Trigonometric Function

The function $\sin\theta$, $\cos\theta$, $\sec\theta$, and $\csc\theta$ have a period of 2π

$$\sin(\theta + 2\pi) = \sin\theta \qquad \cos(\theta + 2\pi) = \cos\theta \tag{9}$$
$$\sec(\theta + 2\pi) = \sec\theta \qquad \csc(\theta + 2\pi) = \csc\theta \tag{10}$$

for all θ in the domain.
The functions $\tan\theta$ and $\cot\theta$ have a period of π:

$$\tan(\theta + \pi) = \tan\theta \qquad \cot(\theta + \pi) = \cot\theta \tag{11}$$

for all θ in the domain.

Inverse Sine and Cosine

$y = \sin^{-1} x$ is the value of y such that $x = \sin y$, where $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. $y = \cos^{-1} x$ is the value of y such that $x = \cos y$, where $0 \leq y \leq \pi$. The domain of both $\sin^{-1} x$ and $\cos^{-1} x$ is $\{x : -1 \leq x \leq 1\}$.

Other Inverse Trigonometric Functions

- $\tan^{-1} x$ is the value of y such that $x = \tan y$, where $-\frac{\pi}{2} < \frac{\pi}{2}$.
- $\cot^{-1} x$ is the value of y such that $x = \tan y$, where $0 < y < \pi$.

The domain of both $\tan^{-1} x$ and $\cot^{-1} x$ is $\{x : -\infty < x < \infty\}$

- $\sec^{-1} x$ is the value of y such that $x = \sec y$, where $0 < y < \pi$ with $y \neq \frac{\pi}{2}$
- $\tan^{-1} x$ is the value of y such that $x = \tan y$, where $-\frac{\pi}{2} < \frac{\pi}{2}$.

2 Limits

2.2 Definitions of Limits

Limits of a Function (Preliminary)

Suppose the function f is defined for all x near a except possibly at a. If $f(x)$ is arbitrarily close to L (as close to L as we like) for all x sufficiently close (but not equal) to a, we write

$$\lim_{x \to a} f(x) = L$$

and say the limit of $f(x)$ as x approaches a equals L.

One-Sided Limits

1 Right-sided limits Suppose f is defined for all x near a with $x > a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x > a$, we write

$$\lim_{x \to a^+} f(x) = L \tag{12}$$

and say the limit of $f(x)$ as x approaches a from the right equals L.

2 Left-sided limits Suppose f is defined for all x near a with $x < a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x < a$, we write

$$\lim_{x \to a^-} f(x) = L \tag{13}$$

and say the limit of $f(x)$ as x approaches a from the left equals L.

Relationship Between One-Sided and Two-Sided Limits

Assume f is defined for all x near a except possibly at a. Then $\lim\limits_{x \to a} f(x) = L$ if and only if $\lim\limits_{x \to a^+} = \lim\limits_{x \to a^-} = L$.

2.3 Techniques For Computing Limits

Limits of Linear Functions

Let a, b, and m be real numbers. For Linear functions $f(x) = mx + b$,

$$\lim_{x \to a} f(x) = f(a) = ma + b \tag{14}$$

Limit Laws

Assume $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ exist. The following properties hold, where c is a real number, and $m > 0$ and $n > 0$ are integers.

1 Sum $\lim\limits_{x \to a} [f(x) + g(x)] = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$

2 Difference $\lim\limits_{x \to a} [f(x) - g(x)] = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x)$

3 Constant Multiple $\lim\limits_{x \to a} [c\, f(x)] = c \lim\limits_{x \to a}$

4 Product $\lim\limits_{x \to a} [f(x)g(x)] = \left[\lim\limits_{x \to a} f(x)\right] \left[\lim\limits_{x \to a} g(x)\right]$

5 Quotient $\lim\limits_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided $\lim\limits_{x \to a} g(x) \neq 0$

6 Power $\lim\limits_{x \to a} [f(x)]^n = \left[\lim\limits_{x \to a} f(x)\right]^n$

7 Fractional Power $\lim\limits_{x \to a} [f(x)]^{n/m} = \left[\lim\limits_{x \to a} f(x)\right]^{n/m}$, provided $f(x) \geq 0$, for x near a, if m is even and n/m is reduced to lowest terms.

Limits of Polynomial and Rational Functions

Assume p and q are polynomials and a is a constant

- Polynomial functions: $\lim\limits_{x\to a} p(x) = p(a)$
- Rational functions: $\lim\limits_{x\to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$, provided $q(a) \neq 0$

Limit Laws For One-Sided Limits

Laws 1–6 hold with $\lim\limits_{x\to a}$ replaced by $\lim\limits_{x\to a^+}$ or $\lim\limits_{x\to a^-}$. Law 7 is modified as follows, assume $m > 0$ and $n > 0$ are integers.

7 Fractional Power

- $\lim\limits_{x\to a^+} [f(x)]^{n/m}$, provided $f(x) \geq 0$, for x near a with $x > a$, if m is even and n/m is reduced to lowest terms
- $\lim\limits_{x\to a^-} [f(x)]^{n/m}$, provided $f(x) \geq 0$, for x near a with $x < a$, if m is even and n/m is reduced to lowest terms

The Squeeze Theorem

Assume the function f, g, and h satisfy $f(x) \leq g(x) \leq h(x)$, for all values of x near a, except possibly at a. If $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} h(x) = L$, then $\lim\limits_{x\to a} g(x) = L$.

2.4 Infinite Limits

Suppose f is defined for all x near a. If $f(x)$ grows arbitrarily large for all x sufficiently close (but not equal) to a, we write

$$\lim_{x\to a} f(x) = \infty \tag{15}$$

We say the limit of $f(x)$ as x approaches a is infinity.
If $f(x)$ is negative and grows arbitrarily large in magnitude for all x sufficiently close (but not equal) to a, we write

$$\lim_{x\to a} f(x) = -\infty \tag{16}$$

In this case, we say the limit of $f(x)$ as x approaches a is negative infinity. In both cases, *the limit does not exist.*

One-Sided Infinite Limits

Suppose f is defined for all x near a with $x > a$. If $f(x)$ becomes arbitrarily large for all x sufficiently close to a with $x > a$, we write $\lim_{x\to a^+} f(x) = \infty$. The one-sided infinite limit $\lim_{x\to a^+} = -\infty$, $\lim_{x\to a^-} f(x) = \infty$, and $\lim_{x\to a^-} = -\infty$ are defined analogously.

Vertical Asymptote

If $\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a^+} = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$, the line $x = a$ is called a **vertical asymptote** of f.

2.5 Limits at Infinity

If $f(x)$ becomes arbitrarily close to a finite number L for all sufficiently large and positive x, then we write

$$\lim_{x\to\infty} f(x) = L \tag{17}$$

We say the limit of $f(x)$ as x approaches infinity is L. In this case the line $y = L$ is a **horizontal asymptote** of f. The limit at negative infinity, $\lim_{x\to-\infty} f(x) = M$, is defined analogously. When the limit exists, the horizontal asymptote is $y = M$.

Infinite Limits at Infinity

If $f(x)$ becomes arbitrarily large as x becomes arbitrary large, then we write

$$\lim_{x\to\infty} f(x) = \infty \tag{18}$$

The limits $\lim_{x\to\infty} = -\infty$, $\lim_{x\to-\infty} = \infty$, and $\lim_{x\to-\infty} = -\infty$ are defined similarly.

Limit of Infinity at Powers and Polynomials

Let n be a positive integer and let p be the polynomial $p(x) = a_n x^m + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$, where $a_n \neq 0$.

1. $\lim_{x\to\pm\infty} x^n = \infty$ when n is even.
2. $\lim_{x\to\infty} x^n = \infty$ and $\lim_{x\to-\infty} x^n = -\infty$ when n is odd.

3. $\lim_{x\to\pm\infty} \frac{1}{x^n} = \lim_{x\to\pm\infty} x^{-n} = 0$

4. $\lim_{x\to\pm\infty} p(x) = \infty or -\infty$, depending on the degree of the polynomial or the leading coefficient of a_n.

End Behavior and Asymptotes of Rational Functions

Suppose $f(x) = \frac{p(x)}{q(x)}$ is a rational function, where

$$p(x) = a_m x^m + a_{m-1} + \cdots + a_2 x^2 + a_1 x + a_0 \tag{19}$$

and

$$q(x) = b_m x^m + b_{m-1} + \cdots + b_2 x^2 + b_1 x + b_0 \tag{20}$$

with $a_m \neq 0$ and $b_n \neq 0$.

1. If $m < n$ then $\lim_{x\to\pm\infty} f(x) = 0$, and $y = 0$ is a horizontal asymptote of f.
2. If $m = n$, then $\lim_{x\to\pm\infty} f(x) = a_m/b_n$, and $y = a_m/b_n$ is a horizontal asymptote.
3. If $m > n$, then $\lim_{x\to\pm\infty} f(x) = \infty$ or $-\infty$, and f has no horizontal asymptote.
4. If $m = n + 1$, then $\lim_{x\to\pm\infty} f(x) = \infty$ or $-\infty$, f has no horizontal asymptote, but f has a slant asymptote.
5. Assuming that $f(x)$ is in reduced form (p and q share no common factors), vertical asymptotes occur at the zeros of q.

End Behavior of e^x, e^{-x}, and $\ln x$

The end behavior for e^x and e^{-x} on $(-\infty, \infty)$ and $\ln x$ on $(0, \infty)$ is given by the following limits:

$$\lim_{x\to\infty} e^x = \infty \quad \text{and} \quad \lim_{x\to-\infty} e^x = 0 \tag{21}$$

$$\lim_{x\to\infty} e^{-x} = \infty \quad \text{and} \quad \lim_{x\to-\infty} e^x = 0 \tag{22}$$

$$\lim_{x\to 0^+} \ln x = -\infty \quad \text{and} \quad \lim_{x\to\infty} \ln x = \infty \tag{23}$$