Chapter 11: Parametric and Polar Curves

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Contents

11.1 Parametric Equations

Forward or Positive Orientation

The direction in which a parametric curve is generated as the parameter increases is called the **forward**, or **positive**, **orientation** of the curve.

Derivative for Parametric Curves

Let $x = g(t)$ and $y = h(t)$, where g and h are differentiable on an interval $[a, b]$. Then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)} \tag{1}$$

provided $\frac{dx}{dt} \neq 0$.

11.2 Polar Coordinates

Introduction

- Up to now we have only studied in a Cartesian coordinate system.
  - A Cartesian coordinate system is just a plane described by Cartesian (or, algebraic) equations and points in a finite dimensions.
    - *One Dimension*: Lines.
    - *Two Dimensions*: x^2.
    - *Three, Four*: Upper-level Calculus and Physics.
- Let's define an alternative coordinate system — **polar coordinate.**
  - coordinates are constants on circles and rays.
  - Useful for navigation, position, and gravitation fields.

Defining Polar Coordinates

Pole The origin of the coordinate system.

Polar Axis Synonymous for the positive x-axis.

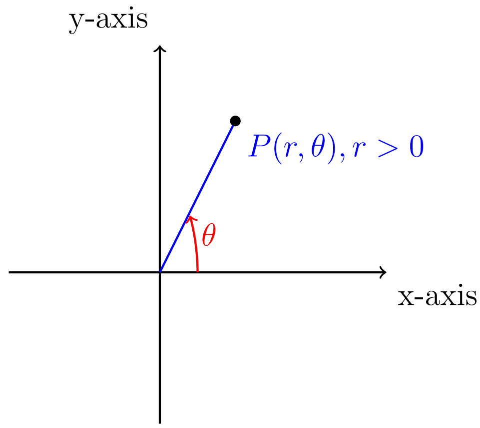
Polar Coordinates A polar coordinates P has the form (r, θ).

> **Radial Coordinate** The radial coordinate r describes the *signed*, or *directed*, distance from the origin to P.
>
> **Angular Coordinate** The angular coordinate θ describes an angle whose initial side is the positive x-axis and whose terminal side lies on the ray passing through the origin and P.

Notes

- **Positive angle measurements are measured *counterclockwise* from the origin.**
- Every point has multiple representations.
  - Angles are periodic, so multiples of 2π gives the same angle.

- Coordinates may be negative. So (r, θ) can be represented as $(-r, \theta + \pi)$ and $(-r, \theta - \pi)$



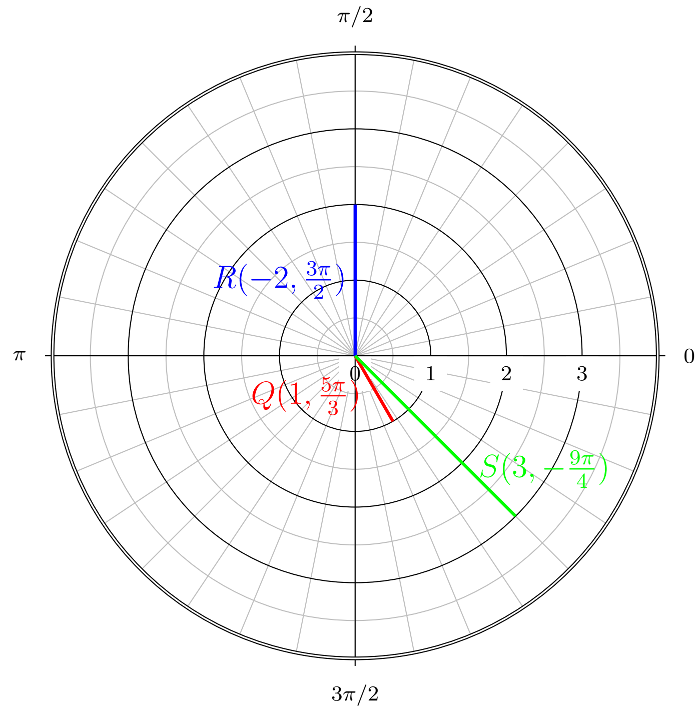
In summary,

- $(r, \theta + 2\pi)$ represents the same point as (r, θ)
- $P(r, \theta)$ and $P'(-r, \theta)$ are reflections through the origin.

Examples

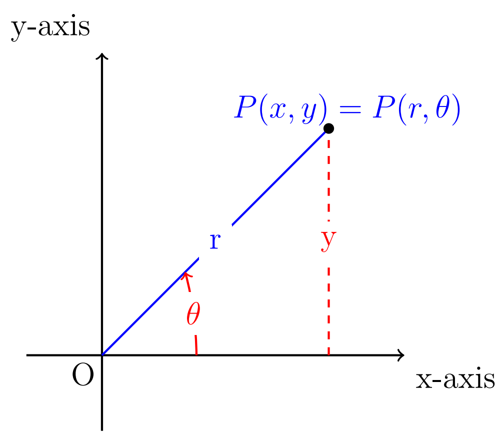
Graph the following points in polar coordinates:

- $Q(1, \frac{5\pi}{3})$
- $R(-2, \frac{3\pi}{2})$
- $S(3, -\frac{9\pi}{4})$
  - Now give two alternative representations.
  - $S'(3, \frac{1\pi}{4})$
  - $S''(-3, -\frac{5\pi}{4})$



Converting Between Cartesian and Polar Coordinates

- We sometimes need to convert between Cartesian and polar coordinates.
- Let's turn this problem into a right triangle.



A point with polar coordinates (r, θ) has Cartesian coordinates (x, y), where

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta \tag{2}$$

A point with Cartesian coordinates (x, y) has polar coordinates (r, θ), where

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x} \tag{3}$$

Examples

BE SURE TO GRAPH POINTS IN CARTESIAN FIRST.

Polar to Cartesian

Express the point with the following polar coordinates in Cartesian coordinates: $P(3, \frac{2\pi}{3})$

$$\begin{aligned} x &= r\cos\theta \\ &= 3\cos({}^{2\pi}\!/_{3}) \\ &= -3({}^{1}\!/_{2}) \\ &= -{}^{3}\!/_{2} \end{aligned} \qquad \begin{aligned} y &= r\sin\theta \\ &= 3\sin({}^{2\pi}\!/_{3}) \\ &= 3({}^{\sqrt{3}}\!/_{2}) \\ &= {}^{3\sqrt{3}}\!/_{2} \end{aligned}$$

Polar to Cartesian

Express the point with the following polar coordinates in Cartesian coordinates: $Q(e, -\frac{\pi}{4})$

$$\begin{aligned} x &= r\cos\theta \\ &= e\cos(-{}^{\pi}\!/_{4}) \\ &= e({}^{\sqrt{2}}\!/_{2}) \\ &= {}^{e\sqrt{2}}\!/_{2} \end{aligned} \qquad \begin{aligned} y &= r\sin\theta \\ &= e\sin(-{}^{\pi}\!/_{4}) \\ &= -e({}^{\sqrt{2}}\!/_{2}) \\ &= -{}^{e\sqrt{2}}\!/_{2} \end{aligned}$$

Cartesian To Polar

Express the point with the following Polar coordinates to Cartesian coordinates: $R(1, -1)$

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ &= \sqrt{1^2+(-1)^2} \\ &= \sqrt{2} \end{aligned} \qquad \begin{aligned} \tan\theta &= {}^{y}\!/_{x} \\ &= {}^{-1}\!/_{1} \\ &= -1 \\ \theta &= {}^{-\pi}\!/_{4} \text{ or } {}^{7\pi}\!/_{4}. \end{aligned}$$

Therefore, two possible solutions are: $(\sqrt{2}, {}^{-\pi}\!/_{4})$ or $(\sqrt{2}, {}^{7\pi}\!/_{4})$

Cartesian To Polar

Express the point with the following Polar coordinates to Cartesian coordinates: $S(1, \sqrt{3})$

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ &= \sqrt{(\sqrt{3})^2+(1)^2} \\ &= 2 \end{aligned} \qquad \begin{aligned} \tan\theta &= {}^{y}\!/_{x} \\ &= {}^{\sqrt{(3)}}\!/_{1} \\ \theta &= {}^{\pi}\!/_{3} \text{ or } {}^{4\pi}\!/_{3}. \end{aligned}$$

Therefore, two possible solutions are: $(2, {}^{\pi}\!/_{3})$ or $(2, {}^{4\pi}\!/_{3})$.

Basic Curves in Polar Coordinates

- A curve in polar coordinates is the set of **points** that satisfy an equation in r and θ.
- This makes graphing some things easier than others.
- Look at $r = 3$ is the set of all points that satisfy being away from the origin of 3 units.
  - This is because θ is not specified, it's arbitrary. Basically, θ is the function.
  - In general, $r = a$, $\forall a \in \mathbb{R}^+$ describes a circle.
- Taking the converse, let r be arbitrary.
  - If the r is arbitrary, and we specify the angle, what do you think we get?

– A line!

– Take $\sqrt{2}/2$.

Polar to Cartesian Graph Example

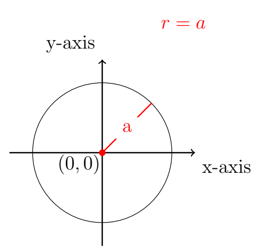
Convert the polar equation $r = 6\sin\theta$ to Cartesian coordinates and describe the corresponding graph.

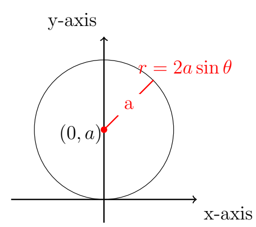
$$\begin{aligned} r^2 &= 6r\sin\theta && (4)\\ x^2+y^2 &= 6y && (5)\\ 0 &= x^2+y^2-6y && (6)\\ &= x^2+(y^2-6y+9)-9 && (7)\\ &= x^2+(y-3)^2-9 && (8) \end{aligned}$$

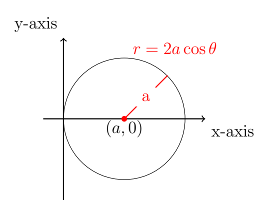
We recognize this to be the equation of a circle, centered at $(0,3)$ at 3. We can also generalize this.

Circle in Polar Coordinates

- The equation $r = a$ describes a circle of radius $|a|$ centered at $(0,0)$.
- The equation $r = 2a\sin\theta$ describes a circle of radius $|a|$ centered at $(0,a)$.
- The equation $r = 2a\cos\theta$ describes a circle of radius $|a|$ centered at $(a,0)$.



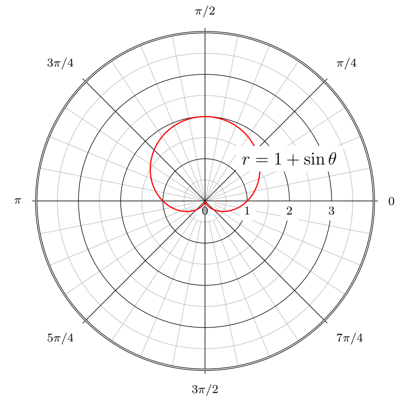




Graphing In Polar Coordinates

Graph the polar equation $r = f(\theta) = 1+\sin\theta$

θ	$r = 1 + sin\theta$
0	1
$\pi/6$	$3/2$
$\pi/2$	2
$5\pi/6$	$3/2$
π	1
$7\pi/6$	$1/2$
$3\pi/2$	0
$11\pi/6$	$1/2$
2π	1



The resulting curve is known as a **cardioid**.

Cartesian-to-Polar Method for Graphing $r = f(\theta)$

1. Graph $r = f(\theta)$ *as if* r *and* θ *were Cartesian coordinates* with θ on the horizontal axis and r on the vertical axis. Be sure to choose an interval in θ on which the entire polar curve is produced.

2. Use the Cartesian graph in Step 1 as a guide to sketch the points (r, θ) on the final *polar* curve.

Example

With the alternate graphing method, graph $r = 1 + \sin\theta$.

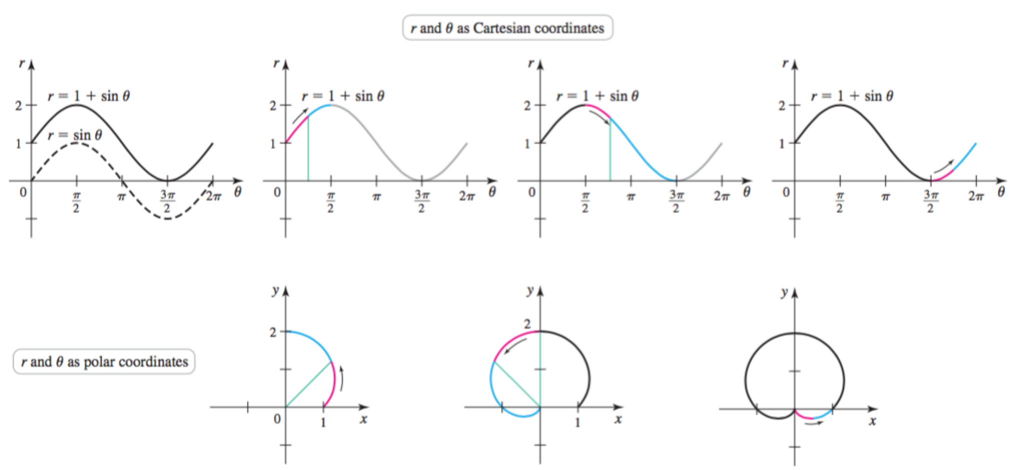


FIGURE 11.26

Symmetry In Polar Equations

Symmetry about the x-axis occurs if the point (r, θ) is on the graph whenever $(r, -\theta)$ is on the graph.

Symmetry about the y-axis occurs if the point (r, θ) is on the graph whenever $(r, \pi - \theta) = (-r, -\theta)$ is on the graph.

Symmetry about the origin occurs if the point $(r, \theta$ is on the graph whenever $(-r, \theta) = (r, \theta + \pi)$ is on the graph.

11.3 Calculus in Polar Coordinates

Slope of a Tangent Line

Let f be a differentiable function at θ_0. The slope of the line tangent to the curve $r = f(\theta)$ at the point $(f(\theta_0, \theta_0)$ is

$$\frac{dy}{dx} = \frac{f'(\theta_0)\sin\theta_0 + f(\theta_0)\cos\theta_0}{f'(\theta_0)\cos\theta_0 - f(\theta_0)} \tag{9}$$

Area of Regions in Polar Coordinates

Let R be the region bounded by the graphs of $r = f(\theta)$ and $r = g(\theta)$, between $\theta = \alpha$ and $\theta = \beta$, where f and g are continous and $f(\theta) \geq g(\theta) \geq 0$ on $[\alpha, \beta]$. The area of R is

$$\int_\alpha^\beta \frac{1}{2}(f(\theta)^2 - g(\theta)^2)\, d\theta \tag{10}$$