Calculus III: Multivariable Calculus

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12 Vectors and Vector-Valued Functions

12.1 Vectors in the Plane

Vectors, Equal Vectors, Scalars, Zero Vector

Vectors are quantities that have both **length** (or **magnitude**) and **direction**. Two vectors are **equal** if they have the same magnitude and direction. Quantities having magnitude but no direction are called **scalars**. One exception is the **zero** vector, denoted **0**: It has length 0 and no direction.

Scalar Multiples and Parallel Vectors

Given a scalar c and a vector $\mathbf{u}$, the scalar multiple $c\mathbf{v}$ is a vector whose magnitude is $|c|$ multiplied by the magnitude of $\mathbf{v}$. If $c > 0$, then $c\mathbf{v}$ has the same direction as $\mathbf{v}$. If $c < 0$, then $c\mathbf{v}$ and $\mathbf{v}$ point in opposite directions. Two vectors are **parallel** if they are scalar multiples of each other.

Position Vectors and Vector Components

A vector $\mathbf{v}$ with its tail at the origin and head at the point (v_1, v_2) is called a **position vector** (or is said to be in **standard position**) and is written $\langle v_1, v_2\rangle$. The real numbers v_1 and v_2 are the x- and y-**components** of $\mathbf{v}$, respectively. The position vectors $\mathbf{u} = \langle v_1, v_2\rangle$ and $\mathbf{v} = \langle v_1, v_2\rangle$ are equal if and only if $u_1 = v_1$ and $u_2 = v_2$.

Magnitude of a Vector

Given the points $P(x_1, y_1)$ and $Q(x_2, y_2)$, the **magnitude**, or **length**, of $\vec{PQ} = \langle x_2 - x_1, y_2 - y_1\rangle$, denoted $|\vec{PQ}|$, is the distance between P and Q:

$$|\vec{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1}$$

The magnitude of the position vector $\mathbf{v} = \langle v_1, v_2\rangle$ is $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2}$

Vector Operations

Suppose c is a scalar, $\mathbf{u} = \langle u_1, u_2\rangle$ and $\mathbf{v} = \langle v_1, v_2\rangle$.

$$\mathbf{u}+\mathbf{v} = \langle u_1+v_1, u_2+v_2\rangle \qquad \text{Vector addition} \tag{2}$$
$$\mathbf{u}-\mathbf{v} = \langle u_1-v_1, u_2-v_2\rangle \qquad \text{Vector subtraction} \tag{3}$$
$$c\mathbf{u} = \langle cu_1, cu_2\rangle \qquad \text{Scalar multiplication} \tag{4}$$

Unit Vectors and Vectors of a Specified Length

A **unit vector** is any vector with length 1. Given a nonzero vector $\mathbf{v}, \pm\frac{\mathbf{v}}{|\mathbf{v}|}$ are unit vectors parallel to $\mathbf{v}$. For a scalar $c > 0$, the vectors $\pm\frac{c\mathbf{v}}{|v|}$ are vectors of length c parallel to $\mathbf{v}$.

Properties of Vector Operations

Suppose $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ are vectors and a and c are scalars. Then the following properties hold (for vectors in any number of dimensions).

$$\mathbf{u}+a = \mathbf{v}+\mathbf{u} \quad \text{Commutative property of addition} \tag{5}$$
$$(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w}) \quad \text{Associative property of addition} \tag{6}$$
$$\mathbf{v}+\mathbf{0} = \mathbf{v} \quad \text{Additive identity} \tag{7}$$
$$\mathbf{v}+(-\mathbf{v}) = 0 \quad \text{Additive identity} \tag{8}$$
$$c(\mathbf{u}+\mathbf{v}) = c\mathbf{u}+c\mathbf{v} \quad \text{Distributive property 1} \tag{9}$$
$$(a+c)\mathbf{v} = a\mathbf{v}+c\mathbf{v} \quad \text{Distributive property 2} \tag{10}$$
$$0\mathbf{v} = \mathbf{0} \quad \text{Multiplication by zero scalar} \tag{11}$$
$$c\mathbf{0} = \mathbf{0} \quad \text{Multiplication by zero vector} \tag{12}$$
$$1\mathbf{v} = \mathbf{v} \quad \text{Multiplicative identity} \tag{13}$$
$$a(c\mathbf{v}) = (ac)\mathbf{v} \quad \text{Associative property of scalar multiplication} \tag{14}$$

12.2 Vectors in Three Dimensions

Distance Formula in xyz-Space

The distance between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{15}$$

Spheres and Balls

A **sphere** centered at (a, b, c) with radius r is the set of points satisfying the equation

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2 \tag{16}$$

A ball centered at (a, b, c) with radius r is the set of points satisfying the inequality

$$(x - a)^2 + (y - b)^2 + (z - c)^2 \leq r^2 \tag{17}$$

Vector Operations in $\mathbb{R}$

Let c be a scalar, $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$.

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle \qquad \text{Vector addition} \tag{18}$$
$$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle \qquad \text{Vector subtraction} \tag{19}$$
$$c\mathbf{u} = \langle cu_1, cu_2, c_3 \rangle \tag{20}$$

Magnitude of a Vector

The **magnitude** (or **length**) of the vector $\vec{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$ is the distance from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$

$$|\vec{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{21}$$

12.3 Dot Product

Dot Product

Given two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in two or three dimensions, their **dot product** is

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta \tag{22}$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ with $0 \leq \theta \leq \pi$. If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then $\mathbf{u} \cdot \mathbf{v} = 0$, and θ is undefined.

Orthogonal Vectors

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if and only if $\mathbf{u} \cdot \mathbf{v} = 0$. The zero vector is orthogonal to all vectors. In two or three dimensions, two nonzero orthogonal vectors are perpendicular to each other.

Dot Product

Given two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$,

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \tag{23}$$

Properties of the Dot Product

Suppose $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors and let c be a scalar.

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= \mathbf{v} \cdot \mathbf{u} && \text{Commutative property} && (24)\\
c(\mathbf{u} \cdot \mathbf{v}) &= (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) && \text{Associative property} && (25)\\
\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} && && (26)
\end{aligned}$$

(Orthogonal) Projection of u onto v

The **orthogonal projection of u onto v** , denoted $\text{proj}_{\mathbf{v}}\mathbf{u}$, where $\mathbf{v} \neq \mathbf{0}$, is

$$\text{proj}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}| \cos\theta \left(\frac{\mathbf{v}}{|\mathbf{v}|} \right) \tag{27}$$

The orthogonal projection may also be computed with the formulas

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \text{scal}_{\mathbf{v}}\mathbf{u}\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} \tag{28}$$

where the **scalar component of u in the direction of v** is

$$\text{scal}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}|\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} \tag{29}$$

Work

Let a constant force $\mathbf{F}$ be applied to an object, producing a displacement $\mathbf{d}$. If the angle between $\mathbf{F}$ and $\mathbf{d}$ is θ, then the **work** done by the force is

$$W = |\mathbf{F}||\mathbf{d}|\cos\theta = \mathbf{F}\cdot\mathbf{d} \tag{30}$$

12.4 Cross Product

Given two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, the **cross product** $\mathbf{u} \times \mathbf{v}$ is a vector with magnitude

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta \tag{31}$$

where $0 \leq \theta \leq \pi$ is the angle between $\mathbf{u}$ and $\mathbf{v}$. The direction of $\mathbf{u} \times \mathbf{v}$ is given by the **right-hand rule**: When you put the vectors tail to tail and let the fingers of your right hand curl from $\mathbf{u}$ to $\mathbf{v}$ the direction of $\mathbf{u} \times \mathbf{v}$ is the direction of your thumb, orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. When $\mathbf{u} \times \mathbf{v} = 0$, the direction of $\mathbf{u} \times \mathbf{v}$ undefined.

Geometry of the Cross Product

Let $\mathbf{u}$ and $\mathbf{v}$ be two nonzero vectors in $\mathbb{R}^3$.

1. The vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel ($\theta = 0$ or $\theta = \pi$) if and only if $\mathbf{u} \times \mathbf{v} = 0$.
2. If $\mathbf{u}$ and $\mathbf{v}$ are two sides of a parallelogram, then the area of the parallelogram is

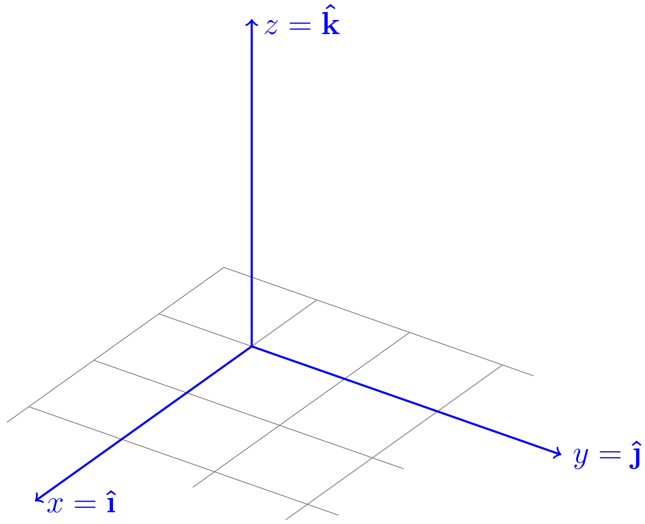
$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta \tag{32}$$

Properties of the Cross Product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be nonzero vectors in $\mathbb{R}^3$, and let a and b be scalars.

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= -(\mathbf{v} \times \mathbf{u}) && \text{Anticommutative property} && (33)\\
(a\mathbf{u}) \times (b\mathbf{v}) &= ab(\mathbf{u} \times \mathbf{v}) && \text{Associative property} && (34)\\
\mathbf{u} \times (\mathbf{v} + \mathbf{w}) &= (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w}) && \text{Distributive property} && (35)\\
(\mathbf{u} + \mathbf{v}) \times \mathbf{w} &= (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w}) && \text{Distributive property} && (36)
\end{aligned}$$

Cross Products of Coordinate Unit Vectors



$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = -(\hat{\mathbf{j}} \times \hat{\mathbf{i}}) = \hat{\mathbf{k}} \tag{37}$$

$$\hat{\mathbf{j}} \times \hat{\mathbf{k}} = -(\hat{\mathbf{k}} \times \hat{\mathbf{j}}) = \hat{\mathbf{i}} \tag{38}$$

$$\hat{\mathbf{k}} \times \hat{\mathbf{i}} = -(\hat{\mathbf{i}} \times \hat{\mathbf{k}}) = \hat{\mathbf{j}} \tag{39}$$

$$\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = 0 \tag{40}$$

Evaluating the Cross Product

Let $\mathbf{u} = u_1\hat{\mathbf{i}} + u_2\hat{\mathbf{j}} + u_3\hat{\mathbf{k}}$ and $\mathbf{v} = v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}} + v_3\hat{\mathbf{k}}$. Then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \hat{\mathbf{k}} \tag{41}$$

12.5 Lines and Curves in Space

Equation of a Line

An equation of the line passing through the point $P_0(x_0,\, y_0,\, z_0)$ in the direction of the vector $\mathbf{v} = \langle a,\, b,\, c\rangle$ is $\mathbf{r} = r_0 + t\mathbf{v}$, or

$$\langle x,\, y,\, z\rangle = \langle x_0,\, y_0,\, z_0\rangle + t\langle a,\, b,\, c\rangle, \quad \text{for} \quad -\infty < t < \infty \tag{42}$$

Equivalently, the parametric equations of the line are

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct, \quad \text{for} \quad -\infty < t < \infty \tag{43}$$

Limit of a Vector-Valued Function

A vector-valued function $\mathbf{r}$ approaches the limit $\mathbf{L}$ as t approaches a, written $\lim\limits_{t\to a} \mathbf{r}(t) = \mathbf{L}$, provided $\lim\limits_{t\to a} |\mathbf{r}(t) - \mathbf{L}| = 0$

12.6 Calculus of Vector-Valued Functions

Derivative and Tangent Vector

Let $\mathbf{r}(t) = f(t)\hat{\mathbf{\imath}} + g(t)\hat{\mathbf{\jmath}} + h(t)\hat{\mathbf{k}}$, where f, g, and h are differentiable functions on $(a\,b)$. Then $\mathbf{r}$ has a **derivative** (or is **differentiable**) on $(a\,b)$ and

$$\mathbf{r}'(t) = f'(i)\hat{\mathbf{\imath}} + g'(t)\hat{\mathbf{\jmath}} + h'(t)\hat{\mathbf{k}} \tag{44}$$

Provided $\mathbf{r}'(t) \neq \mathbf{0}, \mathbf{r}'(t)$ is a **tangent vector** (or velocity vector) at the point corresponding to $\mathbf{r}$.

Unit Tangent Vector

Let $\mathbf{r} = f(t)\hat{\mathbf{\imath}} + g(t)\hat{\mathbf{\jmath}} + h(t)\hat{\mathbf{k}}$ be a smooth parameterized curve, for $a \leq t \leq b$. The **unit tangent vector** for a particular value of t is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \tag{45}$$

Derivative Rules

Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector-valued functions and let f be a differentiable scalar-valued function, all at a point t. Let $\mathbf{c}$ be a constant vector. The following rules apply.

$$\frac{d}{dt}(\mathbf{c}) = \mathbf{0} \quad \text{Constant Rule} \tag{46}$$

$$\frac{d}{dt}(\mathbf{u}(t) + \mathbf{v}(t)) = \mathbf{u}'(t) + \mathbf{v}'(t) \quad \text{Sum Rule} \tag{47}$$

$$\frac{d}{dt}(f(t)\mathbf{u}(t)) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t) \quad \text{Product Rule} \tag{48}$$

$$\frac{d}{dt}(\mathbf{u}(f(t))) = \mathbf{u}'(f(t))f'(t) \quad \text{Chain Rule} \tag{49}$$

$$\frac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t) \quad \text{Dot Product Rule} \tag{50}$$

$$\frac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t) \quad \text{Cross Product Rule} \tag{51}$$

Indefinite Integral of a Vector-Valued Function

Let $\mathbf{r} = f\hat{\mathbf{i}} + g\hat{\mathbf{j}} + h\hat{\mathbf{k}}$ be a vector function and let $\mathbf{R} = F\hat{\mathbf{i}} + G\hat{\mathbf{j}} + H\hat{\mathbf{k}}$, where F, G, and H are antiderivatives of f, g, and h, respectively. The **indefinite integral** of $\mathbf{r}$ is

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C} \tag{52}$$

where $\mathbf{C}$ is an arbitrary constant vector.

Definite Integral of a Vector-Valued Function

Let $\mathbf{r}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$, where f, g, and h are integrable on the interval $[a, b]$.

$$\int \mathbf{r}(t)\,dt = \left[\int_a^b f(t)\,dt\right]\hat{\mathbf{i}} + \left[\int_a^b g(t)\,dt\right]\hat{\mathbf{j}} + \left[\int_a^b h(t)\,dt\right]\hat{\mathbf{k}} \tag{53}$$

12.7 Motion In Space

Position, Velocity, Speed, Acceleration

Let the position of an object moving in three-dimensional space be given by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, for $t \geq 0$. The **velocity** of the object is

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle \tag{54}$$

The **speed** of the object is the scalar function

$$|\mathbf{v}(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \tag{55}$$

The **acceleration** of the object is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

Motion with Constant $|\mathbf{r}|$

Let $\mathbf{r}$ describe a path on which $|\mathbf{r}|$ is constant (motion on a circle or sphere centered at the origin). Then, $\mathbf{r} \cdot \mathbf{v} = 0$, which means the position vector and the velocity vector are orthogonal at all times for which the functions are defined.

Two-Dimensional Motion in a Gravitational Field

Consider an object moving in a plane with horizontal x-axis and a vertical y-axis, subject only to the force of gravity. Given the initial velocity $\mathbf{v}(0) = \langle u_0,\, v_0\rangle$ and the initial position $\mathbf{r}(0) = \langle x_0,\, y_0\rangle$, the velocity of the object, for $t \geq 0$, is

$$\mathbf{v}(t) = \langle x'(t),\, y'(t)\rangle = \langle u_0,\, -gt + v_0\rangle \tag{56}$$

and the position is

$$\mathbf{r}(t) = \langle x(t),\, y(t)\rangle = \left\langle u_0\, t + x_0,\, -\frac{1}{2}gt^2 + v_0 t + y_0 \right\rangle \tag{57}$$

Two-Dimensional Motion

Assume an object traveling over horizontal ground, acted on only by the gravitational force, has an initial position $\langle x_0,\, y_0\rangle = \langle 0,\, 0\rangle$ and initial velocity

$\langle u_0, v_0 \rangle = \langle |\mathbf{v}_0| \cos\alpha, |\mathbf{v}_0| \sin\alpha \rangle$. The trajectory, which is a segment or a parabola, has the following properties.

$$\text{time of flight} = T = \frac{2|\mathbf{v}_0| \sin\alpha}{g} \tag{58}$$

$$\text{range} = \frac{|\mathbf{v}_0| \sin 2\alpha}{g} \tag{59}$$

$$\text{maximum height} = y\left(\frac{T}{2}\right) = \frac{(|\mathbf{v}_0| \sin\alpha)^2}{2g} \tag{60}$$

12.8 Length of Curves

Consider the parameterized curve $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, where f', g', and h' are continuous, and the curve is traversed once for $a \leq t \leq b$. The **arc length** of the curve between $(f(a), g(a), h(a))$ and $(f(b), g(b), h(b))$ is

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\, dt = \int_a^b |\mathbf{r}'(\mathbf{t})|\, dt \tag{61}$$

Arc Length of a Polar Curve

Let f have a continuous derivative on the interval $[\alpha, \beta]$. The **arc length** of the polar curve $r = f(\theta)$ on $[\alpha,\, \beta]$ is

$$L = \int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta. \tag{62}$$

Arc Length as a Function of a Parameter

Let $\mathbf{r}$(t) describe a smooth curve, for $t \geq a$. The arc length is given by

$$s(t) = \int_a^t |\mathbf{v}(u)|\, du, \tag{63}$$

where $|\mathbf{v}| = |\mathbf{r}'|$. Equivalently, $\frac{ds}{dt} = |\mathbf{v}t| > 0$. If $|\mathbf{v}(t)| = 1$, for all $t \geq a$, then the parameter t corresponds to arc length.

Arc Length as a Function of a Parameter

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$$s(t) = \int_a^t |\mathbf{v}(u)|\, du, \tag{64}$$

where $|\mathbf{v}| = |\mathbf{r}'|$. Equivalently, $\frac{ds}{dt} = \mathbf{v}(t) > 0$. If $|\mathbf{v}(\mathbf{t})| = 1$, for all $t \geq a$, then the parameter t corresponds to arc length.

12.9 Curvature and Normal Vectors

Curvature

Let $\mathbf{r}$ describe a smooth parameterized curve. If s denotes arc length and $\mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|}$ is the unit tangent vector, the **curvature** is $\kappa(s) = |\frac{d\mathbf{T}}{ds}|$

Curvature Formula

Let $\mathbf{r}(t)$ describes a smooth parameterized curve, where t is any parameter. If $\mathbf{v} = \mathbf{r}'$ is the velocity and $\mathbf{T}$ is the unit tangent vector, then the curvature is

$$\kappa(t) = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \left|\frac{\mathbf{T}'(t)}{\mathbf{r}'(t)}\right| \tag{65}$$

Alternative Curvature Formula

Let $\mathbf{r}$ be the position of an object moving on a smooth curve. The **curvature** at a point on the curve is

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}, \tag{66}$$

where $\mathbf{v} = \mathbf{r}'$ is the velocity and $\mathbf{a} = \mathbf{v}'$ is the acceleration.

Principal Unit Normal Vector

Let $\mathbf{r}$ describe a smooth parameterized curve. The **principal unit normal vector** at a point P on the curve at which $\kappa \neq 0$ is

$$\mathbf{N}(s) = \frac{{}^{d\mathbf{T}}/_{ds}}{|{}^{d\mathbf{T}}/_{ds}|} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} \tag{67}$$

In practice, we use the equivalent formula

$$\mathbf{N}(t) = \frac{{}^{d\mathbf{T}}/_{dt}}{|{}^{d\mathbf{T}}/_{dt}|} \tag{68}$$

evaluated at the value of t corresponding to P.

Properties of the Principal Unit Normal Vector

Let $\mathbf{r}$ describe a smooth parameterized curve with unit tangent vector $\mathbf{T}$ and principal unit normal vector $\mathbf{N}$.

1. $\mathbf{T}$ and $\mathbf{N}$ are orthogonal at all points of the curve; that is, $\mathbf{T}(t)\cdot\mathbf{N}(t) = 0$, at all points where $\mathbf{N}$ is defined.
2. The principal unit normal vector points to the inside of the curve—in the direction that the curve is turning.

Tangential and Normal Components of the Acceleration

The acceleration vector of an object moving in space along a smooth curve has the following representation in terms of its **tangential component** a_T (in the direction of $\mathbf{T}$) and its normal component a_N (in the direction of $\mathbf{N}$):

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T}, \tag{69}$$

where $a_N = \kappa|\mathbf{v}|^2 = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|}$ and $a_T = \frac{d^2s}{dt^2}$.

Unit Binormal Vector and Torsion

Let C be a smooth parameterized curve with unit tangent and principal unit normal vectors $\mathbf{T}$ and $\mathbf{N}$, respectively. Then, at each point of the curve at which the curvature is nonzero, the **unit binormal vector** is

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} \tag{70}$$

and the **torsion** is

$$\tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N} \tag{71}$$

Formulas for Curves in Space

1. Position function: $\quad \mathbf{r}(t) = \langle x(t),\, y(t),\, z(t)\rangle$
2. Velocity: $\quad \mathbf{v} = \mathbf{r}'$
3. Acceleration: $\quad \mathbf{a} = \mathbf{v}'$

4. Unit tangent vector: $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$

5. Principal unit normal vector: $\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$ (provided $d\mathbf{T}/dt \neq \mathbf{0}$)

6. Curvature: $\kappa = \frac{d\mathbf{T}}{ds} = \frac{1}{|\mathbf{v}|}|\frac{d\mathbf{T}}{dt}| = |\frac{\mathbf{T}'(t)}{\mathbf{r}'(t)}| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$

7. Components of acceleration: $\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T}$, where $a_N = \kappa|\mathbf{v}|^2 = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|}$ and $a_T = \frac{d^2s}{dt^2} = \frac{\mathbf{v}\cdot\mathbf{a}}{|\mathbf{v}|}$

8. Unit binormal vector: $\mathbf{B} = \mathbf{B} \times \mathbf{N} = \frac{\mathbf{v}\times\mathbf{a}}{|\mathbf{v}\times\mathbf{a}|}$

9. Torsion $\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = \frac{(\mathbf{v}\times\mathbf{a})\cdot\mathbf{a}'}{|\mathbf{v}\times\mathbf{a}|^2} = \frac{(\mathbf{r}'\times\mathbf{r}'')\cdot\mathbf{r}'''}{(\mathbf{r}'\times\mathbf{r}'')^2}$

13 Functions of Several Variables

13.1 Planes and Surfaces

Plane in $\mathbb{R}^3$

Given a fixed point P_0 and a nonzero **normal vector n**, the set of points P in $\mathbb{R}^3$ for which $\vec{P_0P}$ is orthogonal to **n** is called a **plane**.

General Equation of a Plane in $\mathbb{R}^3$

The plane passing through the point $P_0(x_0, y_0, z_0)$ with a nonzero normal vector $\mathbf{n} = \langle a, b, c\rangle$ is described by the equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad \text{or} \quad ax + by + cz = d \tag{72}$$

where $d = ax_0 + by_0 + cz_0$.

Parallel and Orthogonal Planes

Two distinct planes are **parallel** if their respective normal vectors are parallel (that is, the normal vectors are scalar multiples of each other). Two planes are **orthogonal** if their respective normal vectors are orthogonal (that is, the dot product of the normal vectors is zero).

Cylinder

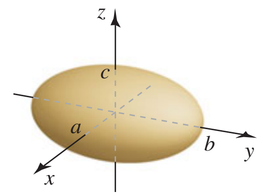
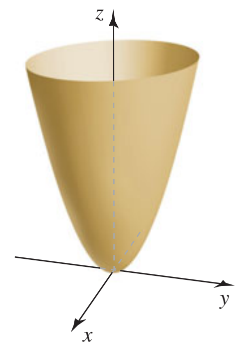
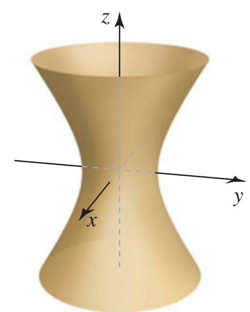
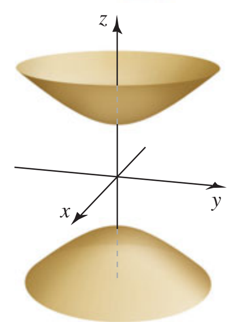
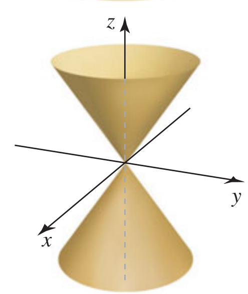
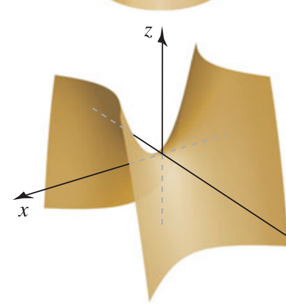
Given a curve C in a plane P and a line ℓ not in P, a **cylinder** is the surface consisting of all lines parallel to ℓ that pass through C.

Trace

A **trace** of a surface is the set of points at which the surface intersects a plane that is parallel to one of the coordinate planes. The trace in the coordinate planes are called **xy-trace**, the **xz-trace**, and the **yz-trace**.

Quadratic Surfaces

Name	Standard Equation	Features
Ellipsoid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	All traces are ellipses.
Elliptic paraboloid	$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$	Traces with $z=z_0>0$ are ellipses. Traces with $x=x_0$ or $y=y_0$ are parabolas.
Hyperboloid of one sheet	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	Traces with $z=z_0$ are ellipses for all z_0. Traces with $x=x_0$ or $y=y_0$ are hyperbolas.
Hyperboloid of two sheets	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	Traces with $z=z_0$ with $\lvert z_0\rvert>\lvert c\rvert$ are ellipses. Traces with $x=x_0$ and $y=y_0$ are hyperbolas.
Elliptic cone	$-\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$	Traces with $z=z_0\neq 0$ are ellipses. Traces with $x=x_0$ or $y=y_0$ are hyperbolas or intersecting lines.
Hyperbolic paraboloid	$z=\frac{x^2}{a^2}-\frac{y^2}{b^2}$	Traces with $z=z_0\neq 0$ are hyperbolas. Traces with $x=x_0$ or $y=y_0$ are parabolas.

Name	Standard Equation	Features	Graph
Ellipsoid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	All traces are ellipses.	
Elliptic paraboloid	$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$	Traces with $z = z_0 > 0$ are ellipses. Traces with $x = x_0$ or $y = y_0$ are parabolas.	
Hyperboloid of one sheet	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	Traces with $z = z_0$ are ellipses for all z_0. Traces with $x = x_0$ or $y = y_0$ are hyperbolas.	
Hyperboloid of two sheets	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	Traces with $z = z_0$ with $\lvert z_0 \rvert > \lvert c \rvert$ are ellipses. Traces with $x = x_0$ and $y = y_0$ are hyperbolas.	
Elliptic cone	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$	Traces with $z = z_0 \neq 0$ are ellipses. Traces with $x = x_0$ or $y = y_0$ are hyperbolas or intersecting lines.	
Hyperbolic paraboloid	$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$	Traces with $z = z_0 \neq 0$ are hyperbolas. Traces with $x = x_0$ or $y = y_0$ are parabolas.	

13.2 Graphs and Level Curves

Function, Domain, and Range with Two Independent Variables

A **function** $z = f(x, y)$ assigns to each point (x, y) in a set D in $\mathbb{R}^2$ a unique real number z in a subset of $\mathbb{R}$. The set D is the **domain** of f. The **range** of f is the set of real numbers z that are assumed as the points (x, y) vary over the domain.

Function, Domain, and Range with n Independent Variables

The **function** $y = f(x_1, x_2, \ldots, x_n)$ assigns a unique real number y to each point $(x_1, x_2, \ldots, x_n)$ in a set D in $\mathbb{R}^n$. The set D is the **domain** of f. The **range** is the set of real numbers y that are assumed as the points $(x_1, x_2, \ldots, x_n)$ vary over the domain.

13.3 Limits and Continuity

Limit of a Function of Two Variables

The function f has the **limit** L as $P(x, y)$ approaches $P_0(a, b)$ written

$$\lim (x, y) \to (a, b) f(x, y) = \lim P \to P_0 f(x, y) = L \tag{73}$$

if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon \tag{74}$$

whenever (x, y) is in the domain of f and

$$0 < |PP_0| = \sqrt{(x-a)^2 + (y-b)^2} < \delta \tag{75}$$

Limits of Constant and Linear Functions

Let a, b, and c be real numbers.

1. Constant functions $f(x, y) = c : \lim\limits_{(x, y) \to (a, b)} c = c$
2. Linear function $f(x, y) = x : \lim\limits_{(x, y) \to (a, b)} x = a$
3. Linear function $f(x, y) = y : \lim\limits_{(x, y) \to (a, b)} y = b$

Limit Laws and Functions of Two Variables

Let L and M be real numbers and suppose that $\lim\limits_{(x, y) \to (a, b)} f(x, y) = L$ and $\lim\limits_{(x, y) \to (a, b)} g(x, y) = M$. Assume c is a constant, and $\forall m, n \in \mathbb{Z}$.

1 Sum $\lim\limits_{(x, y) \to (a, b)} (f(x, y) + g(x, y)) = L + M$

2 Difference $\lim\limits_{(x, y) \to (a, b)} (f(x, y) - g(x, y)) = L - M$

3 Constant multiple $\lim\limits_{(x, y) \to (a, b)} cf(x, y) = cL$

4 Product $\lim\limits_{(x, y) \to (a, b)} f(x, y) \cdot g(x, y) = L \cdot M$

5 Quotient $\lim\limits_{(x, y) \to (a, b)} \left[\frac{f(x, y)}{g(x, y)}\right] = \frac{L}{M}$

6 Power $\lim_{(x,\,y)\to(a,\,b)} (f(x,\,y))^n = L^n$

7 $^m\!/_n$ Power If m and n have no common factors and $n \neq 0$, then $\lim_{(x,\,y)\to(a,\,b)} [f(x,\,y)]^{m/n} = L^{m/n}$, where we assume $L > 0$ if n is even.

Interior and Boundary Points

Let R be a region in $\mathbb{R}^2$. An **interior point** P of R lies entirely within R, which means it is possible to find a disk centered at P that contains only points of R.
A **boundary point** Q of R lies on the edge of R in the sense that *every* disk centered at Q contains at least one point in R and at least one point not in R.

Open and Closed Sets

A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points.

Two-Path Test for Nonexistence of Limits

If $f(x,\,y)$ approaches two different values as $(x,\,y)$ approaches $(a,\,b)$ along two different paths in the domain of f, then $\lim_{(x,\,y)\to(a,\,b)} f(x,\,y)$ does not exist.

Continuity

A function f is continuous at the point $(a,\,b)$ provided

1. f is defined at $(a,\,b)$.
2. $\lim_{(x,\,y)\to(a,\,b)} f(x,\,y)$ exists.
3. $\lim_{(x,\,y)\to(a,\,b)} f(x,\,y) = f(a,\,b)$.

Continuity of Composite Functions

If $u = g(x,\,y)$ is continuous at $(a,\,b)$ and $z = f(u)$ is continuous at $g(a,\,b)$, then the composite function $z = f(g(x,\,y))$ is continuous at $(a,\,b)$.

13.4 Partial Derivatives

The **partial derivative of f with respect to x at the point (a, b)** is

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}. \tag{76}$$

The **partial derivative of f with respect to y at the point (a, b)** is

$$f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}. \tag{77}$$

provided these limits exists.

Equality of Mixed Partial Derivatives

Assume that f is defined on an open set D of $\mathbb{R}^2$, and f_{xy} and f_{yx} are continuous throughout D. Then $f_{xy} = f_{yx}$ at all points of D.

Differentiability

The function $z = f(x,\, y)$ is **differentiable at** $(a,\, b)$ provided $f_x(a,\, b)$ and $f_y(a,\, b)$ exist and the change $\Delta z = f(a+\Delta x,\, b+\Delta y) - f(a,\, b)$ equals

$$\Delta z = f_x(a,\, b)\Delta x + f_y(a,\, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y, \tag{78}$$

where for fixed a and b, ε_1 and ε_2 are functions that depend only on Δx and Δy, with $(\varepsilon_1, \varepsilon_2) \to (0,\, 0)$ as $(\Delta x,\, \Delta y) \to (0,\, 0)$. A function is **differentiable** on an open set R if it is differentiable at every point on R.

Conditions for Differentiability

Suppose the function f has partial derivatives f_x and f_y defined on an open set containing $(a,\, b)$, with f_x and f_y continuous $(a,\, b)$. Then f is differentiable at $(a,\, b)$.

Differentiability Implies Continuity

If a function f is differentiable at $(a,\, b)$, then it is continuous at $(a,\, b)$

13.5 The Chain Rule

Chain Rule (One Independent Variable)

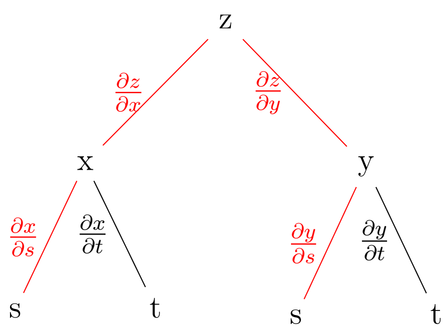
Let z be a differentiable function of x and y on its domain, where x and y are differentiable functions of t on an interval I. Then

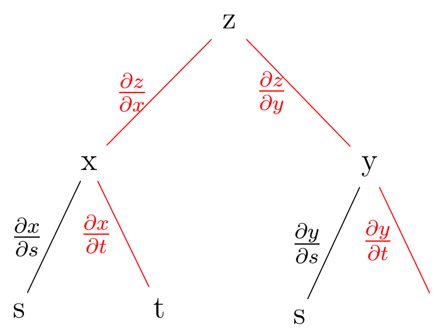
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}. \tag{79}$$

Chain Rule (Two Independent Variables)

Let z be a differentiable function of x and y, where x and y are differentiable functions of s and t. Then

$$\frac{dz}{ds} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \tag{80}$$





Implicit Differentiation

Let F be differentiable on its domain and suppose that $F(x, y) = 0$ defines y as a differentiable function of x. Provided $F_y \neq 0$.

$$\frac{dy}{dx} = -\frac{F_x}{F_y} \tag{81}$$

13.6 Directional Derivatives and the Gradient

Directional Derivative

Let f be a differentiable at (a, b) and let $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ be a unit vector in the xy-plane. The **directional derivatives of** f **at** (a, b) in the direction of $\mathbf{u}$ is

$$D_u f(a, b) = \lim_{h \to 0} \frac{f(a + h\cos\theta, b + h\sin\theta) - f(a, b)}{h} \tag{82}$$

provided the limit exits.

Directional Derivative

Let f be differentiable on (a, b) and let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vecor in the x, y-plane. The **directional derivative of** f **at a** (a, b) **in the direction of** u is

$$D_u f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle \cdot \langle u_1, u_2 \rangle \tag{83}$$

Gradient (Two Dimensions)

Let f be differentiable at the point (x, y). The **gradient** of f at (x, y) is the vector-valued function

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = f_x(x, y)\hat{\mathbf{i}} + f_y(x, y)\hat{\mathbf{j}} \tag{84}$$

Directions of Change

Let f be differentiable at (a, b) with $\nabla f(a, b) \neq \mathbf{0}$

1. f has its maximum rate of increase at (a, b) in the direction of the gradient $\nabla f(a, b)$. The rate of increase in this direction is $|\nabla f(a, b)|$

2. f has its maximum rate of decrease at (a, b) in the direction of $-\nabla f(a, b)$. The rate of decrease in this direction is $-|\nabla f(a, b)|$.

3. The directional derivative is zero in any direction orthogonal to $\nabla f(a, b)$.

The Gradient and Level Curves

Given a function f differentiable at $(a,\, b)$, the line tangent to the level curve of f at $(a,\, b)$ is orthogonal to the gradient $\nabla f(a,\, b)$, provided $\nabla f(a,\, b) \neq \mathbf{0}$.

Gradient and Directional Derivative in Three Dimensions

Let f be differentiable at the point $(x,\, y,\, z)$. The **gradient** of f at $(x,\, y,\, z)$ is the vector-valued function

$$\begin{aligned} \nabla f(x,\, y,\, z) &= \langle f_x(x,\, y,\, z),\, f_y(x,\, y,\, z),\, f_z(x,\, y,\, z)\rangle && (85) \\ &= f_x(x,\, y,\, z)\hat{\mathbf{i}} + f_y(x,\, y,\, z)\hat{\mathbf{j}} + f_z(x,\, y,\, z)\hat{\mathbf{k}} && (86) \end{aligned}$$

The **directional derivative** of f in the direction of the unit vector $\mathbf{u} = \langle u_1,\, u_2,\, u_3\rangle$ at the point $(a,\, b,\, c)$ is $D_u f(a,\, b,\, c) = \nabla f(a,\, b,\, c) \cdot \mathbf{u}$

13.7 Tangent Planes and Linear Approximation

Let F be differentiable at the point $P_0(a,\, b,\, c)$ with $\nabla F(a,\, b,\, c) \neq \mathbf{0}$. The plane tangent to the surface $F(x,\, y,\, z) = 0$ at P_0, called the **tangent plane**, is the plane passing through P_0 orthogonal $\nabla F(a,\, b,\, c)$. An equation of the tangent plane is

$$F_x(a,\, b,\, c)(x - a) + F_y(a,\, b,\, c)(y - b) + F_z(a,\, b,\, c)(z - c) = 0. \tag{87}$$

Tangent Plane for $z = f(x,\, y)$

Let f be differentiable at the point $(a,\, b)$. An equation of the plane tangent to the surface $z = f(x,\, y)$ at the point $(a,\, b,\, f(a,\, b))$ is

$$z = f_x(a,\, b)(x - a) + f_y(a,\, b)(y - b) + f(a,\, b) \tag{88}$$

Linear Approximation

Let f be differentiable at $(a,\, b)$. The linear approximation to the surface $z = f(x,\, y)$ at the point $(a,\, b,\, f(a,\, b))$ is the tangent plane at that point, given by the equation

$$L(x,\, y) = f_x(a,\, b)(x - a) + f_y(a,\, b)(y - b) + f(a,\, b) \tag{89}$$

The Differential dz

Let f be differentiable at the point $(a,\, b)$. The change in $z = f(x,\, y)$ as the independent variables change from $(a,\, b)$ to $(a + dx,\, b + dy)$ is denoted by the differential dz:

$$\Delta z \approx dz = f_x(a,\, b)dx + f_y(a,\, b)dy \tag{90}$$

13.8 Maximum/Minimum Problems

Local Maximum/Minimum Values

A function f has a **local maximum value** at (a, b) if $f(x, y) \leq f(a, b)$ for all (x, y) in the domain of f in some open disk centered at (a, b). A function f has a **local minimum value** at (a, b) if $f(x, y) \geq f(a, b)$ for all (x, y) in the domain of f in some open disk centered at (a, b). Local maximum and local minimum values are also called **local extreme values** or **local extrema**.

Derivatives and Local Maximum/Minimum Values

If f has a local maximum or minimum value at (a, b) and the partial derivatives f_x and f_y exist at (a, b) then $f_x(a, b) = f_y(a, b) = 0$.

Critical Point

An interior point $(a,\ b)$ in the domain of f is a **critical point** of f is either

1. $f_x(a,\ b) = f_y(a,\ b) = 0$, or
2. one (or both) of f_x or f_y does not exist at $(a,\ b)$

Saddle Point

A function f has a **saddle point** at a critical point $(a,\ b)$ if, in every open disk centered at $(a,\ b)$, there are points $(x,\ y)$ for which $f(x,\ y) > f(a,\ b)$ and points for which $f(x,\ y) < f(a,\ b)$

Second Derivative Test

Suppose that the second partial derivative of f are continuous throughout an open disk centered at the point $(a,\ b)$, where $f_x(a,\ b) = f_y(a,\ b) = 0$. Let $D(x,\ y) = f_{xx}(x,\ y)f_{yy}(x,\ y) - (f_{xy}(x,\ y))^2$.

1. If $D(a,\ b) > 0$ and $f_{xx}(a,\ b) < 0$, then f has a local maximum value at $(a,\ b)$.
2. If $D(a,\ b) > 0$ and $f_{xx}(a,\ b) > 0$, then f has a local minimum value at $(a,\ b)$.

3. If $D(a, b) < 0$, then f has a saddle point at (a, b).

4. If $D(a, b) = 0$, then the test is inconclusive.

Absolute Maximum/Minimum Values

If $f(x, y) \leq f(a, b)$ for all (x, y) in the domain of f, then f has an **absolute maximum value** at (a, b). If $f(x, y) \geq f(a, b)$ for all (x, y) in the domain of f, then f has an **absolute minimum value** at (a, b).

Finding Absolute Maximum/Minimum Values on Closed, Bounded Sets

Let f be continuous on a closed bounded set R in $\mathbb{R}^2$. To find the absolute maximum and minimum values of f on R:

1. Determine the values of f at all critical points in R.

2. Find the maximum and minimum values of f on the boundary of R.

3. The greatest function values found in Step 1 and 2 is the absolute maximum value of f on R, and the least function value found in Steps 1 and 2 is the absolute minimum value of f on R.

13.9 Lagrange Multipliers

Parallel Gradients (Ball Park Theorem)

Let f be a differentiable function in a region of $\mathbb{R}^2$ that contains the smooth curve C given by $g(x, y) = 0$. Assume that f has a local extreme value (relative to values of f on C) at a point $P(a, b)$ on C. Then $\nabla f(a, b)$ is orthogonal to the line tangent to C at P. Assuming $\nabla g(a, b) \neq 0$, it follows that there is a real number λ (called a **Lagrange multiplier**) such that $\nabla f(a, b) = \lambda \nabla g(a, b)$.

Method of Lagrange Multipliers in Two Variables

Let the objective function f and the constraint function g be differentiable on a region of $\mathbb{R}^2$ with $\lambda g(x, y) \neq 0$ on the curve $g(x, y) = 0$. To locate the maximum and minimum values of f subject to the constraint $g(x, y) = 0$, carry out the following steps.

1. Find the values of x, y and λ (if they exist) that satisfy the equations

$$\nabla f(x,\, y) = \lambda \nabla g(x,\, y) \quad \text{and} \quad g(x,\, y) = 0 \tag{91}$$

2. Among the values $(x,\, y)$ found in Step 1, select the largest and smallest corresponding function values, which are the maximum and minimum values of f subject to the constraint.

Method of Lagrange Multipliers in Three Variables

Let f and g be differentiable on a region of $\mathbb{R}^3$ with $\nabla g(x,\, y,\, z) neq 0$ on the surface $g(x,\, y,\, z) = 0$. To locate the maximum and minimum values of f subject to the constraint $g(x,\, y,\, z) = 0$, carry out the following steps.

1. Find the values of $x,\, y,\, z$ and λ that satisfy the equations

$$\nabla f(x,\, y,\, z) = \lambda \nabla g(x,\, y,\, z) \quad \text{and} \quad g(x,\, y,\, z) = 0 \tag{92}$$

2. Among the points $(x,\, y,\, z)$ found in Step 1, select the largest and smallest corresponding values of the objective function. These values are the maximum and minimum values of f subject to the constraint.

14 Multiple Integration

14.1 Double Integrals over Rectangular Regions

Volumes and Double Integrals

A function f defined on a rectangular region R in the xy-plane is **integratabtle** on R if $\lim\limits_{\Delta\to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$ exists fora ll partitions of R and for all choices of (x_k^*, y_k^*) within those partitions. The limit is the **double integral of f over R**, which we write

$$\iint\limits_R f(x, y)\, dA = \lim_{\Delta\to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k \tag{93}$$

If f is nonnegative on R, then the double integral equals the volume of the solid bounded by $z = f(x, y)$ and the xy-plane over R.

Double Integrals on Rectangular Regions

Let f be continuous on the rectangular region $R = \{(x, y) : a \leq x \leq b,\ c \leq y \leq d\}$. The double integral of f over R may be evaluated by either of two iterated integrals:

$$\iint\limits_R f(x, y)\, dA = \int_c^d \int_a^b dx\, dy = \int_a^b \int_c^d dy\, dx \tag{94}$$

Average Value of a Function over a Plane Region

The **average value** of an integrable function f over a region R is

$$\bar{f} = \frac{1}{\text{area of } R} \iint\limits_R f(x, y)\, dA \tag{95}$$

14.2 Double Integrals over General Regions

Let R be a region bounded below and above by the graphs of the continuous functions $y = g(x)$ and $y = h(x)$, respectively, and by the lines $x = a$ and $x = b$. If f is continuous on R, then

$$\iint\limits_R f(x, y)\, dA = \int_a^b \int_{g(x)}^{h(x)} f(x, y)\, dy\, dx \tag{96}$$

Let R be a region bounded on the left and right by the graphs of the continuous functions $x = g(y)$ and $x = h(y)$, respectively, and the lines $y = c$ and $y = d$. If f is continuous on R, then

$$\iint\limits_R f(x,\, y)\, dA = \int_c^d \int_{g(y)}^{h(y)} f(x,\, y)\, dx\, dy \tag{97}$$

Areas of Regions by Double Integrals

Let R be a region in the xy-plane. Then

$$\text{area of } R = \iint\limits_R \tag{98}$$

14.3 Double Integrals in Polar Coordinates

Double Integrals over Polar Rectangular Region

Let f be continuous on the region in the xy-plane $R = \{(r, \theta) : 0 \le a \le r \le b, \alpha \le \theta \le \beta\}$, where $\beta - \alpha \le 2\pi$. Then

$$\iint_R f(r,\theta)\, dA = \int_\alpha^\beta \int_a^b f(r,\theta)\ r\, dr\, d\theta \tag{99}$$

Double Integrals over More General Polar Regions

Let f be continuous on the region in the xy-plane

$$R = \{(r,\theta) : 0 \le g(\theta) \le r \le h(\theta), \alpha \le \theta \le \beta\} \tag{100}$$

where $0 < \beta - \alpha \le 2\pi$. Then

$$\iint_R f(r,\theta)\, dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r,\theta) r\, dr\, d\theta. \tag{101}$$

Area of Polar Regions

The area of the region $R = \{(r,\theta) : 0 \le g(\theta) \le r \le h(\theta), \alpha \le \theta \le \beta\}$, where $0 < \beta - \alpha \le 2\pi$, is

$$A = \iint_R dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} r\, dr\, d\theta \tag{102}$$

14.4 Triple Integrals

Let f be continuous over the region

$$D = \{(x,\, y,\, z) : a \le x \le b, g(x) \le y \le h(x),\, G(x,\, y) \le z \le H(x,\, y)\} \tag{103}$$

where g, h, G, and H are continuous functions. Then f is integrable over D and the triple integral is evaluated as the iterated integral

$$\iiint\limits_D f(x,\, y,\, z)\, dV = \int_a^b \int_{g(x)}^{h(x)} \int_{G(x,y)}^{H(x,y)} f(x,\, y,\, z)\, dz\, dy\, dx. \tag{104}$$

Average Value of a Function of Three Variables

If f is continuous on a region D of $\mathbb{R}^3$, then the average value of over D is

$$\bar{f} = \frac{1}{\text{volume }(D)} \iiint\limits_D f(x,\, y,\, z)\, dV \tag{105}$$

14.5 Triple Integrals in Cylindrical and Spherical Coordinates

Transformations Between Cylindrical and Rectangular Coordinates

Rectangular $\to$ Cylindrical	Cylindrical $\to$ Rectangular
$r^2 = x^2 + y^2$	$x = r\cos\theta$
$\tan\theta = {}^{y}\!/\!{}_{x}$	$y = r\sin\theta$
$z = z$	$z = z$

Triple Integrals in Cylindrical Coordinates

Let f be continuous over the region

$$D = \{(r,\, \theta,\, z) : g(\theta) \le r \le h(\theta),\, \alpha \le \theta \le \beta,\, G(x,\, y) \le z \le H(x,\, y).\} \tag{106}$$

Then f is integrable over D and the triple integral of f over D in cylindrical coordinates is

$$\iiint\limits_D f(r,\, \theta,\, z)\, dV = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} \int_{G(r\cos\theta,\, r\sin\theta)}^{H(r\cos\theta,\, r\sin\theta)} f(r,\, \theta,\, z)\, dz\; r\, dr\, d\theta \tag{107}$$

Tranformations Between Spherical and Rectangular Coordinates

Rectangular $\to$ Spherical	Spherical $\to$ Rectangular
$\rho^2 = x^2 + y^2 + z^2$	$x = \rho\sin\varphi\cos\theta$
Use trigonometry to find φ and θ	$y = \rho\sin\varphi\sin\theta$
	$z = \rho\cos\varphi$

Triple Integrals in Spherical Coordinates

Let f be continuous over the region

$$D = \{(\rho,\, \varphi,\, \theta) : g(\varphi,\, \theta) \le \rho \le h(\varphi,\, \theta),\, a \le \varphi b,\, \alpha \le \theta \le \beta\} \tag{108}$$

Then f is integrable over D, and the triple integral of f over D in spherical coordinates is

$$\iiint\limits_D f(\rho,\, \varphi,\, \theta)dV = \int_\alpha^\beta \int_a^b \int_{g(\varphi,\theta)}^{h(\varphi,\theta)} f(\rho,\, \varphi,\, \theta)\rho^2 \sin\varphi \; d\rho\, d\varphi\, d\theta \tag{109}$$

14.6 Integrals for Mass Calculations

Center of Mass in One Dimension

Let ρ be an integrable density function on the interval $[a, b]$ (which represents a thin rod or wire). The **center of mass** is location at the point $\bar{x} = \frac{M}{m}$, where the **total moment** M and mass m are

$$M = \int_a^b x\rho(x)\,dx \quad \text{and} \quad m = \int_a^b \rho(x)\,dx \tag{110}$$

Center of Mass in Two Dimensions

Let ρ be integrable density function defined over a closed bounded region R in $\mathbb{R}^2$. The coordinates of the center of mass of the object represented by R are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint\limits_R x\rho(x, y)\,dA \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = \frac{1}{m} = \iint\limits_R y\rho(x, y)\,dA \tag{111}$$

Center of Mass in Three Dimensions

Let ρ be integrable density function on a closed bounded region D in $\mathbb{R}^3$. The coordinates of the center of mass of the region are

$$\bar{x} = \frac{M_{yz}}{m} = \frac{1}{m}\iiint\limits_R x\rho(x, y, z)\,dV \tag{112}$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{1}{m}\iiint\limits_R y\rho(x, y, z)\,dV \tag{113}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{1}{m}\iiint\limits_R z\rho(x, y, z)\,dV \tag{114}$$

where $m = \iiint_D \rho(x, y, z)\,dV$ is the mass, and M_{yz}, M_{xz} and M_{xy} are the moments with respect to the coordinate planes.

15 Vector Calculus

15.1 Vector Fields

Vector Fields in Two Dimensions

Let f and g be defined on a region R of $\mathbb{R}^2$. A **vector field** in $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point in R a vector $\langle f(x,y),\, g(x,y)\rangle$. The vector field is written as

$$\mathbf{F}(x,y) = \langle f(x,y),\, g(x,y)\rangle \quad \text{or} \quad \mathbf{F}(x,y) = f(x,y)\hat{\mathbf{\imath}} + g(x,y)\hat{\mathbf{\jmath}} \tag{115}$$

A vector field $\mathbf{F} = \langle f, g\rangle$ is continuous or differentiable on a region R of $\mathbb{R}^2$ if f and g are continuous or differentiable on R, respectively.

Radial Vector Fields in $\mathbb{R}^2$

Let $\mathbf{r} = \langle x,\, y\rangle$. A vector field of the form $\mathbf{F} = f(x,\, y)\mathbf{r}$, where f is a scalar-valued function, is a **radial vector field**. Of specific interest are the radial vector field

$$\mathbf{F}(x,\, y) = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x,\, y\rangle}{|\mathbf{r}|^p} \tag{116}$$

where p is a real number. At every point (except the origin), the vectors of this field are directed outward from the origin with the magnitude of $|\mathbf{F}| = \frac{1}{|\mathbf{r}|^{p-1}}$.

Vector Fields in Three Dimensions

Let $f,\ g,$ and h be defined on a region D of $\mathbb{R}^3$. A **vector field** in $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point in D a vector $\langle f(x,\, y,\, z),\, g(x,\, y,\, z),\, h(x,\, y,\, z)\rangle$. The vector field is written as

$$\mathbf{F}(x,\, y,\, z) = \langle f(x,\, y,\, z),\, g(x,\, y,\, z),\, h(x,\, y,\, z)\rangle \quad \text{or} \tag{117}$$

$$\mathbf{F}(x,\, y,\, z) = f(x,\, y,\, z)\hat{\mathbf{\imath}} + g(x,\, y,\, z)\hat{\mathbf{\jmath}} + h(x,\, y,\, z)\hat{\mathbf{k}} \tag{118}$$

A vector field $\mathbf{F} = \langle f,\, g,\, h\rangle$ is continuous or differentiable on a region D of $\mathbb{R}^3$ if $f,\ g,\ h$ are continuous or differentiable on R, respectively. Of particular importance are the **radial vector fields**

$$\mathbf{F}(x,\, y,\, z) = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x,\, y,\, z\rangle}{|\mathbf{r}|^p} \tag{119}$$

where p is a real number.

Gradient Fields and Potential Functions

Let $z = \varphi(x,\, y)$ and $w = \varphi(x,\, y,\, z)$ be differentiable functions on regions of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. The vector field $\mathbf{F} = \nabla\varphi$ is **gradient field**, and the function φ is a **potential function** for $\mathbf{F}$.

15.2 Line Integrals

Scalar Line Integral in the Plane, Arc Length Parameter

Suppose the scalar-valued function f is defined on the smooth curve C : $\mathbf{r}(s) = \langle x(s),\, y(s)\rangle$, parameterized by the arc length s. The **line integral of** f **over** C is

$$\int\limits_C f(x(s),\, y(s))\, ds = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x(s_k^*),\, y(s_k^*))\Delta s_k, \tag{120}$$

provided this limit exists over all partitions of C. When the limit exists, f is said to be **integrable** on C.

Evaluating Scalar Line Integrals in $\mathbb{R}^2$

Let f be continuous on a region containing a smooth curve C : $\mathbf{r}(t) = \langle x(t),\, y(t)\rangle$, for $a \leq t \leq b$. Then

$$\int\limits_C f\, ds = \int_a^b f(x(t),\, y(t))|\mathbf{r}'(t)|\, dt \tag{121}$$

$$= \int_a^b f(x(t),\, y(t))\sqrt{x'(t)^2 + y'(t)^2}\, dt \tag{122}$$

Evaluating the Line Integral $\int\limits_C f\, ds$

1. Find a parametric description of C in the form $\mathbf{r}(t) = \langle x(t),\, y(t)\rangle$, for $a \leq t \leq b$
2. Computer $|\mathbf{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$
3. Make substitutions for x and y in the integrand and evaluate an ordinary integral

$$\int\limits_C f\, ds = \int_a^b f(x(t),\, y(t))|\mathbf{r}'(t)|\, dt \tag{123}$$

Evaluating Scalar Line Integrals in $\mathbb{R}^3$

Let f be continuous on a region containing a smooth curve $C : \mathbf{r}(t) = \langle x(t),\, y(t),\, z(t)\rangle$, for $a \le t \le b$. Then

$$\int f\,ds = \int_a^b f(x(t),\, y(t),\, z(t))|\mathbf{r}'(t)|\,dt \tag{124}$$

$$= \int_a^b f(x(t),\, y(t),\, z(t))\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt \tag{125}$$

Line Integral of a Vector Field

Let $\mathbf{F}$ be a vector field that is continuous on a region containing a smooth oriented curve C parameterized by arc length. Let $\mathbf{T}$ be the unit tangent vector at each point of C consistent with the orientation. The line integral of $\mathbf{F}$ over C is $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$.

Different Forms of Line Integrals of Vector Fields

The line integral $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ may be expressed in the following forms, where $\mathbf{F} = \langle f,\, g,\, h\rangle$, for $a \le t \le b$:

$$\int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_a^b (f\,x'(t),\, g\,y'(t),\, h\,z'(t))\,dt \tag{126}$$

$$= \int_C f\,dx + g\,dy + h\,dz \tag{127}$$

$$= \int_C \mathbf{F}\cdot d\mathbf{r} \tag{128}$$

For line integrals in the plane, we let $\mathbf{F} = \langle f,\, g\rangle$ and assume C is parameterized in the form $\mathbf{r}(t) = \langle x(t),\, y(t)\rangle$, for $a \le t \le b$. Then

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_a^b (f\,x'(t) + g\,y'(t))\,dt = \int_C f\,dx + g\,dy = \int_C \mathbf{F}\cdot d\mathbf{r} \tag{129}$$

Work Done in a Force Field

Let $\mathbf{F}$ be a continuous force field in a region D of $\mathbb{R}^3$ and let $C : \mathbf{r}(t) = \langle x(t),\, y(t),\, z(t)\rangle$, for $a \le t \le b$, be a smooth curve in D with a unit tangent vector $\mathbf{T}$ consistent with the orientation. The work done in moving an object C in the positive direction is

$$W = \int\limits_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F} \cdot \mathbf{r}'(\mathbf{t})\, dt \tag{130}$$

Circulation

Let $\mathbf{F}$ be a continuous vector field on a region D of $\mathbb{R}^3$ and let C be a closed smooth oriented curve in D. The **circulation** of $\mathbf{F}$ on C is $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$, where $\mathbf{T}$ is the unit vector tangent to C consistent with the orientation.

Flux

Let $F = \langle f, g\rangle$ be continuous vector field on a region R of $\mathbb{R}^2$. Let $C : \mathbf{r}(t) = \langle x(t),\, y(t)\rangle$, for $a \le t \le b$, be a smooth oriented curve in R that does not intersect itself. The **flux** of the vector field across C is

$$\int\limits_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_a^b (f\, y'(t) - g\, x'(t))\, dt, \tag{131}$$

where $n = \mathbf{T} \times \hat{\mathbf{k}}$ is the unit normal vector and $\mathbf{T}$ is the unit tangent vector consistent with the orientation. If C is a closed curve with counterclockwise orientation, $\mathbf{n}$ is the outward normal vector and the flux integral gives the **outward flux** across C.

15.3 Conservative Vector Fields

Simple and Closed Curves

Suppose a curve C (in $\mathbb{R}^2$ and $\mathbb{R}^3$) is described parametrically by $\mathbf{r}(t)$, where $a \leq t \leq b$. Then C is a **simple curve** if $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ for all t_1 and t_2, with $a < t_1 < t_2 < b$; that is, C never intersects itself between its endpoints. The curve C is **closed** if $\mathbf{r}(a) = \mathbf{r}(b)$; that is, the initial and terminal points of C are the same.

Connected and Simply Connected Regions

An open region R in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$) is **connected** if it possible to connect any two points of R by a continuous curve lying in R. An open region R is **simply connected** if every closed simple curve in R can be deformed and contracted to a point in R.

Conservative Vector Field

A vector field F is said to be **conservative** on a region (in $\mathbb{R}^2$ or $\mathbb{R}^3$) if there exists a scalar function φ such that $\mathbf{F} = \nabla\varphi$ on that region.

Test for Conservative Vector Fields

Let $\mathbf{F} = \langle f,\, g,\, h\rangle$ be a vector field defined on a connected and simply connected region D of $\mathbb{R}^3$, where f, g, and h have continuous first partial derivatives on D. Then $\mathbf{F}$ is a conservative vector field on D (there is a potential function φ such that $\mathbf{F} = \nabla\varphi$) if and only if

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \quad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \quad \text{and} \quad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y} \tag{132}$$

For vector fields in $\mathbb{R}^2$, we have the single condition $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$.

Finding Potential Functions in $\mathbb{R}^3$

Suppose $\mathbf{F} = \langle f,\, g,\, h\rangle$ is a conservative vector field. To find φ such that $\mathbf{F} = \nabla\varphi$, take the following steps:

1. Integral $\varphi_x = f$ with respect to x to obtain φ, which includes an arbitrary function $c(y,\, z)$.

2. Compute φ_y and equate it to g to obtain an expression for $c_y(y, z)$.

3. Integrate $c_y(y, z)$ with respect to y to obtain $c(y, z)$, including an arbitrary function $d(z)$.

4. Compute φ_z and equate it to h to get $d(z)$.

Beginning the procedure with $\varphi_y = g$ or $\varphi_z = h$ maybe be easier in some cases.

Fundamental Theorem for Line Integrals

Let $\mathbf{F}$ be a continuous vector field on an open connected region R in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$). There exists a potential function φ with $\mathbf{F} = \nabla\varphi$ (which means that $\mathbf{F}$ is conservative) if and only if

$$\int\limits_C \mathbf{F} \cdot \mathbf{T}\, ds = \int\limits_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A) \tag{133}$$

for all points A and B in R and all smooth oriented curves C from A to B.

Line Integrals on Closed Curves

Let R in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$) be an open region. Then $\mathbf{F}$ is a conservative vector field on R if and only if $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all simple closed smooth oriented curves C in R.

15.4 Green's Theorem

Let C be a simple closed smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g\rangle$, where f and g have continuous first partial derivatives in R. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA. \tag{134}$$

Two-Dimensional Curl

The **two-dimensional curl** of the vector field $\mathbf{F} = \langle f, g\rangle$ is $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}$. If the curl is zero throughout a region, the vector field is said to be **irrotational** on that region.

Area of a Plane Region by Line Integrals

Under the conditions of Green's Theorem, the area of a region R enclosed by a curve C is

$$\oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C (x\,dy - y\,dx) \tag{135}$$

Green's Thoerem, Flux Form

Let C be a simple closed smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g\rangle$, where f and g have continuous first partial derivatives in R. Then

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C f\,dy - g\,dx = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) \tag{136}$$

where $\mathbf{n}$ is the outward unit normal vector on the curve.

Two-Dimensional Divergence

The **two-dimensional divergence** of the vector field $\mathbf{F} = \langle f, g\rangle$ is $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$. If the divergence is zero thoughout a region, the vector field is said to be **source free** on that region.

15.5 Divergence and Curl

Divergence of a Vector Field

The **divergence** of a vector field $\mathbf{F} = \langle f,\, g,\, h\rangle$ that is differentiable on a region of $\mathbb{R}^3$ is

$$\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \tag{137}$$

If $\nabla\cdot\mathbf{F} = 0$, the vector field is **source free.**

Divergence of Radial Vector Fields

For a real number p, the divergence of the radial vector field

$$\mathbf{F} = \frac{\mathbf{r}}{|r|^p} = \frac{\langle x,\, y,\, z\rangle}{(x^2+y^2+z^2)^{\frac{p}{2}}} \text{ is } \nabla\cdot\mathbf{F} = \frac{3-p}{|\mathbf{r}|^p} \tag{138}$$

Curl of a Vector Field

The **curl** of a vector field $\mathbf{F} = \langle f,\, g,\, h\rangle$ that is differentiable on a region of $\mathbb{R}^3$ is

$$\nabla\times\mathbf{F} = \operatorname{curl}\mathbf{F} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\hat{\mathbf{k}} \tag{139}$$

If $\nabla\times\mathbf{F} = \mathbf{0}$, the vector field is **irrotational**.

Curl of a Conservative Vector Field

The **general rotation vector field** is $\mathbf{F} = \mathbf{a}\times\mathbf{r}$, where the nonzero constant vector $\mathbf{a} = \langle a_1,\, a_2,\, a_3\rangle$ is the axis of ration and $\mathbf{r} = \langle x,\, y,\, z\rangle$. For all nonzero choices of $\mathbf{a}$, $|\nabla\times\mathbf{F}| = 2|\mathbf{a}|$ and $\nabla\cdot\mathbf{F} = 0$. The constant angular speed of the vector field is

$$\omega = |\mathbf{a}| = \frac{1}{2}|\nabla\times\mathbf{F}| \tag{140}$$

Curl of a Conservative Vector Field

Suppose that $\mathbf{F}$ is a conservative vector field on an open region D of $\mathbb{R}^3$. Let $\mathbf{F} = \nabla\varphi$, where φ is a potential function with continuous second partial derivatives on D. Then $\nabla \times \mathbf{F} = \nabla \times \nabla\varphi = \mathbf{0}$; that is, the curl of the gradient is the zero vector and $\mathbf{F}$ is irrotational.

Divergence of the Curl

Suppose that $\mathbf{F} = \langle f,\, g,\, h\rangle$, where $f,\ g,$ and h have continuous second partial derivatives. Then $\nabla \cdot (\nabla \times \mathbf{F}) = 0$: The divergence of the curl is zero.

Product Rule for the Divergence

Let u be a scalar-valued functino that is differentiable on a region D and let $\mathbf{F}$ be a vector field that is differentiable on D. Then

$$\nabla \cdot (u\mathbf{F}) = \nabla u \cdot \mathbf{F} + u(\nabla \cdot \mathbf{F}) \tag{141}$$

Properties of a Conservative Vector Field

Let $\mathbf{F}$ be a conservative vector field whose components have continuous second partial derivatives on an open connected region D in $\mathbb{R}^3$. Then $\mathbf{F}$ has the following equivalent properties.

1. There exists a potential function φ such that $\mathbf{F} = \nabla\varphi$
2. $\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$ for all points A and B in D and all smooth oriented curves C from A and B.
3. $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all simple smooth closed oriented curves C in D.
4. $\nabla \times \mathbf{F} = \mathbf{0}$ at all points of D.

15.6 Surface Integrals

Surface Integrals of Scalar-Valued Functions on Parameterized Surface

Let f be a continuous function on a smooth surface S given parametrically by $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$, where $R = \{(u, v) : a \le u \le b, c \le v \le d\}$. Assume also that the tangent vectors $\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u} = \langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\rangle$, and $\mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v} = \langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\rangle$ are continuous on R and the normal vectors $\mathbf{n} = \mathbf{t}_u \times \mathbf{t}_v$ is nonzero on R. Then the **surface integral** of the scalar-valued function f over S is

$$\iint_S f(x, y, z)\, dS = \iint_R f(x(u, v), y(u, v), z(u, v))|\mathbf{t}_u \times \mathbf{t}_v|\, dA \tag{142}$$

If $f(x, y, z) = 1$, the integral equals the surface area of S.

Evaluation of Surface Integrals of Scalar-Valued Functions on Explicitly Defined Surfaces

Let f be a continuous function on a smooth surfaces S given by $z = g(x, y)$, for (x, y) in a region R. The surface integral of f over S is

$$\iint_S f(x, y, z)\, dS = \iint_R f(x, y, g(x, y))\sqrt{z_x^2 + z_y^2 + 1}\, dA \tag{143}$$

If $f(x, y, z) = 1$, the surface integral equals the area of the surface.

Surface Integral of a Vector Field

Suppose $\mathbf{F} = \langle f, g, h\rangle$ is a continuous vector field on a region of $\mathbb{R}^3$ containing a smooth oriented surface S. If S is defined parametrically as $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$, for (u, v) is a region R, then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \mathbf{F} \cdot (\mathbf{t}_u \times \mathbf{t}_v)\, dA \tag{144}$$

where $\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u} = \langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\rangle$ and $\mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v} = \langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\rangle$ are continuous on R, the normal vector $\mathbf{n} = \mathbf{t}_u \times \mathbf{t}_v$ is nonzero on R, and the direction of $\mathbf{n}$ is consistent with the orientation of S. If S is defined in the form $z = g(z, y)$, for (x, y) in a region R, then

$$\iint\limits_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint\limits_R (-f z_x - g z_y + h) \, dA \tag{145}$$

15.7 Stokes' Theorem

Let S be a smooth oriented surface in $\mathbb{R}^2$ with a smooth closed boundary C whose orientation is consistent with that of S. Assume that $\mathbf{F} = \langle f,\, g,\, h\rangle$ is a vector field whose components have continuous first partial derivatives on S. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS \tag{146}$$

where $\mathbf{n}$ is the unit vector normal to S determined by the orientation of S.

Curl $\mathbf{F} = \mathbf{0}$ Implies $\mathbf{F}$ is Conservative

Suppose that $\nabla \times \mathbf{F} = \mathbf{0}$ throughout an open simply connected region D of $\mathbb{R}^3$. Then $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all closed simple smooth curves C in D and $\mathbf{F}$ is a conservative vector field on D.

15.8 Divergence Theorem

Let $\mathbf{F}$ be a vector field whose components have continuous first partial derivatives in a connected and simply connected region D in $\mathbb{R}^3$ enclosed by a smooth oriented surface S. Then

$$\iint\limits_{S} \mathbf{F} \cdot \mathbf{n}\, dS = \iiint\limits_{D} \nabla \cdot \mathbf{F}\, dV \tag{147}$$

where $\mathbf{n}$ is the unit outward normal vector on S.

Divergence Theorem for Hollow Regions

Suppose the vector field $\mathbf{F}$ satisfies the conditions of the Divergence Theorem on a region D bounded by two smooth oriented surfaces S_1 and S_2, where S_1 lies within S_2. Let S be the entire boundary of $D(S = S_1 \cup S_2)$ and let $\mathbf{n_1}$ and $\mathbf{n}_2$ be the outward unit normal vectors for S_1 and S_2, respectively. Then

$$\iiint\limits_{D} \nabla \cdot \mathbf{F}\, dV = \iint \mathbf{F} \cdot \mathbf{n} dS = \iint\limits_{S_2} \mathbf{F} \cdot \mathbf{n_2}\, dS = \iint\limits_{S_1} \mathbf{F} \cdot \mathbf{n_1}\, dS \tag{148}$$