Homework #4

Analysis of Algorithms

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Question #1

```python
def sum_of_two_elements(S, x):
    S = sorted(S)
    i, j = 0, len(S) - 1
    while i <= j:
        if S[i] + S[j] == x:
            return True
        elif S[i] + S[j] < x:
            i += 1
        else:
            j -= 1

    return False
```

Table 1: Output for `sum_of_two_elements`

List	x	$n \lg n$ solution	n^2 solution
[219, 526, 87, 829, 59, 384, 849, 65, 463, 934]	449	True	True
[393, 983, 584, 160, 421, 652, 41, 719, 686, 181]	1338	True	True
[491, 154, 345, 508, 208, 50, 11, 183, 723, 994]	931	True	True
[798, 810, 263, 786, 177, 314, 211, 708, 300, 286]	1061	True	True
[262, 494, 533, 38, 105, 937, 383, 625, 733, 428]	645	True	True
[730, 371, 356, 591, 126, 416, 732, 84, 505, 165]	717	True	True
[469, 821, 559, 492, 479, 91, 621, 55, 550, 507]	605	True	True
[822, 407, 527, 858, 498, 360, 551, 532, 503, 274]	1054	True	True
[278, 265, 997, 628, 563, 536, 783, 817, 725, 124]	1380	True	True
[455, 745, 422, 274, 335, 781, 909, 867, 669, 681]	943	True	True
[209, 836, 717, 858, 446, 773, 507, 693, 907, 59]	268	True	True
[18, 639, 711, 738, 583, 69, 714, 503, 597, 280]	1452	True	True
[690, 867, 901, 558, 367, 927, 439, 590, 651, 447]	1459	True	True
[156, 129, 61, 363, 948, 347, 874, 914, 775, 73]	503	True	True
[781, 985, 385, 523, 753, 804, 740, 7, 155, 441]	596	True	True
[897, 799, 83, 402, 144, 820, 621, 22, 640, 660]	1420	True	True
[478, 364, 216, 907, 638, 576, 835, 487, 571, 883]	1051	True	True
[193, 116, 483, 17, 363, 276, 14, 534, 145, 636]	162	True	True
[626, 446, 185, 716, 514, 225, 953, 826, 758, 809]	1072	True	True
[268, 778, 934, 880, 347, 306, 90, 767, 626, 230]	1712	True	True
[170, 591, 380, 744, 868, 242, 736, 756, 45, 798]	624	False	False
[85, 354, 59, 185, 916, 42, 567, 532, 106, 285]	704	False	False
[938, 218, 584, 440, 574, 748, 450, 931, 955, 588]	909	False	False
[930, 337, 292, 491, 668, 486, 630, 320, 91, 797]	17	False	False
[277, 754, 116, 486, 75, 868, 788, 346, 326, 188]	847	False	False
[951, 37, 80, 72, 515, 45, 925, 533, 626, 767]	346	False	False
[529, 286, 483, 105, 283, 224, 461, 245, 447, 861]	482	False	False
[563, 577, 795, 911, 692, 462, 755, 311, 898, 268]	706	False	False
[798, 617, 27, 746, 149, 763, 546, 752, 692, 279]	816	False	False
[334, 339, 184, 870, 776, 716, 375, 752, 414, 453]	522	False	False
[683, 969, 687, 274, 870, 566, 139, 664, 699, 325]	17	False	False
[641, 411, 4, 735, 742, 205, 264, 548, 331, 617]	604	False	False
[876, 939, 653, 487, 148, 433, 807, 238, 848, 556]	872	False	False
[385, 21, 172, 598, 463, 721, 187, 670, 328, 917]	510	False	False
[3, 246, 181, 435, 937, 974, 817, 109, 841, 383]	518	False	False
[889, 542, 73, 455, 946, 307, 189, 988, 440, 349]	58	False	False
[383, 691, 490, 938, 233, 139, 818, 231, 825, 720]	170	False	False
[772, 543, 127, 538, 273, 508, 924, 793, 495, 150]	229	False	False
[311, 843, 307, 863, 305, 602, 696, 351, 910, 776]	42	False	False

Question #2

1. As follows, the inversions (i, j) are:
   a) (1, 5)
   b) (2, 5)

c) (3, 4)
d) (3, 5)
e) (4, 5)

$(A[i], A[j])$ are:

a) (2, 1)
b) (3, 5)
c) (8, 6)
d) (8, 1)
e) (6, 1)

2. For a reverse sorted array, there would $\binom{n}{2} = \frac{n(n-1)}{2}$ total inversions.
3. Divide the array recursively into half and count number of inversions in sub-arrays (an $\lg n$ algorithm). To count all inversions, it takes n steps. $n -$ operations $\times$ $\lg n -$ recursive steps. Combined, $\Theta(n \lg n)$ algorithm.

Question #3

We recall that the geometric series, and its derivatives, state that:

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad \text{and} \quad \sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2} \tag{1}$$

Next, we know that our sum is as follows:

$$\sum_{k=0}^{\infty} \frac{(k-1)}{2^k}$$

Breaking these up we have the following:

$$\sum_{k=0}^{\infty} k\left(\frac{1}{2}\right)^k - \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \tag{2}$$

Replacing Equation 2 with Equation 1, we get the following:

$$\begin{aligned}\sum_{k=0}^{\infty} k\left(\frac{1}{2}\right)^k - \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k &= \frac{1/2}{(1-1/2)^2} - \frac{1}{1-1/2} \\ &= 2-2 \\ &= 0\end{aligned}$$

Question #4

Because k^3 is monotonically increasing, we have the following bounds upper bound:

$$\begin{aligned}\sum_{k=1}^{n} k^3 &\leq \int_1^{n+1} x^3 \, dx \\ &= \left.\frac{x^4}{4}\right|_1^{n+1} \\ &= \frac{(n+1)^4 - 1}{4}\end{aligned}$$

For the lower bound,

$$\begin{aligned}\sum_{k=1}^{n} k^3 &\geq \int_0^{n} x^3 \, dx \\ &= \left.\frac{x^4}{4}\right|_0^{n} \\ &= \frac{(n)^4}{4}\end{aligned}$$

So, the bounds are as follows:

$$\frac{(n)^4}{4} \leq \sum_{k=1}^{n} k^3 \leq \frac{(n+1)^4 - 1}{4}$$

Question #5

We use integral approximations for all of the following.

Problem #5.1

$$\begin{aligned}\int_0^n x^r \, dx \quad &\leq \sum_{k=1}^{n} k^r \quad \leq \int_1^{n+1} x^r \, dx \\ \frac{n^{r+1}}{r+1} \quad &\leq \sum_{k=1}^{n} k^r \quad \leq \frac{(n+1)^{r+1} - 1}{r+1}\end{aligned}$$

Therefore, the bound is $\Theta\left(n^{r+1}\right)$.

Problem #5.2

For the following, we take s for 1,

$$\begin{aligned}
\int_0^n \lg^1 x \, dx \quad &\leq \sum_{k=1}^{n} \lg^1 k \quad \leq \int_1^{n+1} \lg^1 x \, dx \\
n\left(\lg(n) - 1\right) \quad &\leq \sum_{k=1}^{n} \lg^1 k \quad \leq (n+1)\left(\lg(n+1) - 1\right) + 1
\end{aligned}$$

Therefore, the bound is $\Theta\left(n \lg^1 n\right)$.

For the following, we take s for 2,

$$\begin{aligned}
\int_0^n \lg^2 x \, dx \quad &\leq \sum_{k=1}^{n} \lg^2 k \quad \leq \int_1^{n+1} \lg^2 x \, dx \\
n\left(\left(\lg(n) - 2\right)\lg(n) + 2\right) \quad &\leq \sum_{k=1}^{n} \lg^2 k \quad \leq (n+1)\left(\left(\lg(n+1) - 2\right)\lg(n+1) + 2\right) - 2
\end{aligned}$$

Therefore, the bound is $\Theta\left(n \lg^2 n\right)$.

For the following, we take s for 3,

$$\int_0^n \lg^3 x \, dx \quad \leq \sum_{k=1}^{n} \lg^3 k \quad \leq \int_1^{n+1} \lg^3 x \, dx$$

From this, we get,

$$\begin{aligned}
&n\left(\lg(n)\left(\left(\lg(n) - 3\right)\lg(n) + 6\right) - 6\right) \\
&\leq \sum_{k=1}^{n} \lg^3 k \\
&\leq (n+1)\left(\lg(n+1)\left(\left(\lg(n+1) - 3\right)\lg(n+1) + 6\right) - 6\right) + 6
\end{aligned}$$

Therefore, the bound is $\Theta\left(n \lg^3 n\right)$.

In general, the integral requires s integration by parts, multiplying $\lg n$ s times. So, for $\sum_{k=1}^{n} \lg^s k$, the bound would be $\Theta\left(n \lg^s n\right)$.

Problem #5.3

In general, we know that the lower-bound integral we have will have the form

$$\int_0^n x^r \lg^s x \, dx$$

And likewise for the upper bound. Using the same process as before (omitted due to tedious mathematics), we get the tight bound $\Theta\left(n^{r+1} \lg^s n\right)$.