Chapter 12: Vectors and Vector Valued Functions

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Contents

12.1 Vectors in the Plane

Vectors, Equal Vectors, Scalars, Zero Vector

Vectors are quantities that have both **length** (or **magnitude**) and **direction**. Two vectors are **equal** if they have the same magnitude and direction. Quantities having magnitude but no direction are called **scalars**. One exception is the **zero** vector, denoted $\mathbf{0}$: It has length 0 and no direction.

Scalar Multiples and Parallel Vectors

Given a scalar c and a vector $\mathbf{u}$, the scalar multiple $c\mathbf{v}$ is a vector whose magnitude is $|c|$ multiplied by the magnitude of $\mathbf{v}$. If $c > 0$, then $c\mathbf{v}$ has the same direction as $\mathbf{v}$. If $c < 0$, then $c\mathbf{v}$ and $\mathbf{v}$ point in opposite directions. Two vectors are **parallel** if they are scalar multiples of each other.

Position Vectors and Vector Components

A vector $\mathbf{v}$ with its tail at the origin and head at the point (v_1, v_2) is called a **position vector** (or is said to be in **standard position**) and is written $\langle v_1, v_2\rangle$. The real numbers v_1 and v_2 are the x- and y-**components** of $\mathbf{v}$, respectively. The position vectors $\mathbf{u} = \langle v_1, v_2\rangle$ and $\mathbf{v} = \langle v_1, v_2\rangle$ are equal if and only if $u_1 = v_1$ and $u_2 = v_2$.

Magnitude of a Vector

Given the points $P(x_1, y_1)$ and $Q(x_2, y_2)$, the **magnitude**, or **length**, of $\vec{PQ} = \langle x_2 - x_1, y_2 - y_1\rangle$, denoted $|\vec{PQ}|$, is the distance between P and Q:

$$|\vec{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1}$$

The magnitude of the position vector $\mathbf{v} = \langle v_1, v_2\rangle$ is $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2}$

Vector Operations

Suppose c is a scalar, $\mathbf{u} = \langle u_1, u_2\rangle$ and $\mathbf{v} = \langle v_1, v_2\rangle$.

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2\rangle \qquad \text{Vector addition} \tag{2}$$

$$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2\rangle \qquad \text{Vector subtraction} \tag{3}$$

$$c\mathbf{u} = \langle cu_1, cu_2\rangle \qquad \text{Scalar multiplication} \tag{4}$$

Unit Vectors and Vectors of a Specified Length

A **unit vector** is any vector with length 1. Given a nonzero vector $\mathbf{v}, \pm\frac{\mathbf{v}}{|\mathbf{v}|}$ are unit vectors parallel to $\mathbf{v}$. For a scalar $c > 0$, the vectors $\pm\frac{c\mathbf{v}}{|v|}$ are vectors of length c parallel to $\mathbf{v}$.

Properties of Vector Operations

Suppose $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ are vectors and a and c are scalars. Then the following properties hold (for vectors in any number of dimensions).

$$\mathbf{u} + a = \mathbf{v} + \mathbf{u} \quad \text{Commutative property of addition} \tag{5}$$
$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}) \quad \text{Associative property of addition} \tag{6}$$
$$\mathbf{v} + \mathbf{0} = \mathbf{v} \quad \text{Additive identity} \tag{7}$$
$$\mathbf{v} + (-\mathbf{v}) = 0 \quad \text{Additive identity} \tag{8}$$
$$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v} \quad \text{Distributive property 1} \tag{9}$$
$$(a + c)\mathbf{v} = a\mathbf{v} + c\mathbf{v} \quad \text{Distributive property 2} \tag{10}$$
$$0\mathbf{v} = \mathbf{0} \quad \text{Multiplication by zero scalar} \tag{11}$$
$$c\mathbf{0} = \mathbf{0} \quad \text{Multiplication by zero vector} \tag{12}$$
$$1\mathbf{v} = \mathbf{v} \quad \text{Multiplicative identity} \tag{13}$$
$$a(c\mathbf{v}) = (ac)\mathbf{v} \quad \text{Associative property of scalar multiplication} \tag{14}$$

12.2 Vectors in Three Dimensions

Distance Formula in xyz-Space

The distance between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{15}$$

Spheres and Balls

A **sphere** centered at (a, b, c) with radius r is the set of points satisfying the equation

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2 \tag{16}$$

A ball centered at (a, b, c) with radius r is the set of points satisfying the inequality

$$(x - a)^2 + (y - b)^2 + (z - c)^2 \leq r^2 \tag{17}$$

Vector Operations in $\mathbb{R}$

Let c be a scalar, $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$.

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle \qquad \text{Vector addition} \tag{18}$$

$$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle \qquad \text{Vector subtraction} \tag{19}$$

$$c\mathbf{u} = \langle cu_1, cu_2, c_3 \rangle \tag{20}$$

Magnitude of a Vector

The **magnitude** (or **length**) of the vector $\vec{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$ is the distance from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$

$$|\vec{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{21}$$

12.3 Dot Product

Dot Product

Given two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in two or three dimensions, their **dot product** is

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta \tag{22}$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ with $0 \leq \theta \leq \pi$. If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then $\mathbf{u} \cdot \mathbf{v} = 0$, and θ is undefined.

Orthogonal Vectors

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if and only if $\mathbf{u} \cdot \mathbf{v} = 0$. The zero vector is orthogonal to all vectors. In two or three dimensions, two nonzero orthogonal vectors are perpendicular to each other.

Dot Product

Given two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$,

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \tag{23}$$

Properties of the Dot Product

Suppose $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors and let c be a scalar.

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} \qquad \text{Commutative property} \tag{24}$$

$$c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) \qquad \text{Associative property} \tag{25}$$

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} \tag{26}$$

(Orthogonal) Projection of u onto v

The **orthogonal projection of u onto v** , denoted $\text{proj}_{\mathbf{v}}\mathbf{u}$, where $\mathbf{v} \neq \mathbf{0}$, is

$$\text{proj}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}| \cos\theta \left(\frac{\mathbf{v}}{|\mathbf{v}|} \right) \tag{27}$$

The orthogonal projection may also be computed with the formulas

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \text{scal}_{\mathbf{v}}\mathbf{u}\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} \tag{28}$$

where the **scalar component of u in the direction of v** is

$$\text{scal}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}|\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} \tag{29}$$

Work

Let a constant force **F** be applied to an object, producing a displacement **d**. If the angle between **F** and **d** is θ, then the **work** done by the force is

$$W = |\mathbf{F}||\mathbf{d}|\cos\theta = \mathbf{F}\cdot\mathbf{d} \tag{30}$$

12.4 Cross Product

Given two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, the **cross product** $\mathbf{u} \times \mathbf{v}$ is a vector with magnitude

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta \tag{31}$$

where $0 \leq \theta \leq \pi$ is the angle between $\mathbf{u}$ and $\mathbf{v}$. The direction of $\mathbf{u} \times \mathbf{v}$ is given by the **right-hand rule**: When you put the vectors tail to tail and let the fingers of your right hand curl from $\mathbf{u}$ to $\mathbf{v}$ the direction of $\mathbf{u} \times \mathbf{v}$ is the direction of your thumb, orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. When $\mathbf{u} \times \mathbf{v} = 0$, the direction of $\mathbf{u} \times \mathbf{v}$ undefined.

Geometry of the Cross Product

Let $\mathbf{u}$ and $\mathbf{v}$ be two nonzero vectors in $\mathbb{R}^3$.

1. The vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel ($\theta = 0$ or $\theta = \pi$) if and only if $\mathbf{u} \times \mathbf{v} = 0$.

2. If $\mathbf{u}$ and $\mathbf{v}$ are two sides of a parallelogram, then the area of the parallelogram is

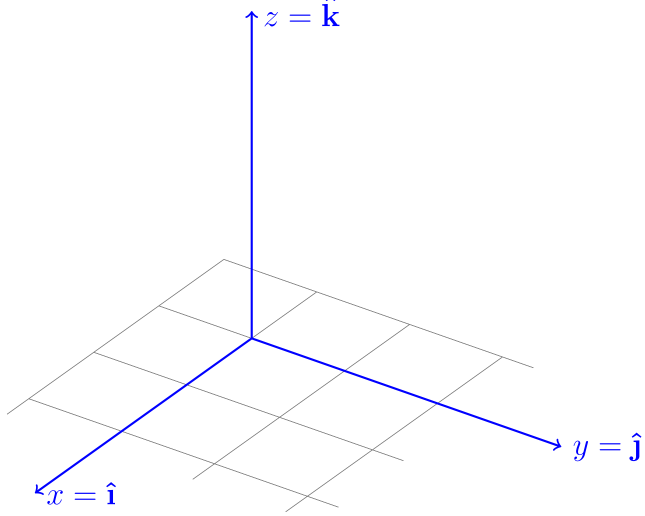
$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta \tag{32}$$

Properties of the Cross Product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be nonzero vectors in $\mathbb{R}^3$, and let a and b be scalars.

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= -(\mathbf{v} \times \mathbf{u}) && \text{Anticommutative property} && (33)\\
(a\mathbf{u}) \times (b\mathbf{v}) &= ab(\mathbf{u} \times \mathbf{v}) && \text{Associative property} && (34)\\
\mathbf{u} \times (\mathbf{v} + \mathbf{w}) &= (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w}) && \text{Distributive property} && (35)\\
(\mathbf{u} + \mathbf{v}) \times \mathbf{w} &= (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w}) && \text{Distributive property} && (36)
\end{aligned}$$

Cross Products of Coordinate Unit Vectors



$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = -(\hat{\mathbf{j}} \times \hat{\mathbf{i}}) = \hat{\mathbf{k}} \tag{37}$$

$$\hat{\mathbf{j}} \times \hat{\mathbf{k}} = -(\hat{\mathbf{k}} \times \hat{\mathbf{j}}) = \hat{\mathbf{i}} \tag{38}$$

$$\hat{\mathbf{k}} \times \hat{\mathbf{i}} = -(\hat{\mathbf{i}} \times \hat{\mathbf{k}}) = \hat{\mathbf{j}} \tag{39}$$

$$\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = 0 \tag{40}$$

Evaluating the Cross Product

Let $\mathbf{u} = u_1\hat{\mathbf{i}} + u_2\hat{\mathbf{j}} + u_3\hat{\mathbf{k}}$ and $\mathbf{v} = v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}} + v_3\hat{\mathbf{k}}$. Then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \hat{\mathbf{k}} \tag{41}$$

12.5 Lines and Curves in Space

Equation of a Line

An equation of the line passing through the point $P_0(x_0,\, y_0,\, z_0)$ in the direction of the vector $\mathbf{v} = \langle a,\, b,\, c\rangle$ is $\mathbf{r} = r_0 + t\mathbf{v}$, or

$$\langle x,\, y,\, z\rangle = \langle x_0,\, y_0,\, z_0\rangle + t\langle a,\, b,\, c\rangle, \quad \text{for} \quad -\infty < t < \infty \tag{42}$$

Equivalently, the parametric equations of the line are

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct, \quad \text{for} \quad -\infty < t < \infty \tag{43}$$

Limit of a Vector-Valued Function

A vector-valued function $\mathbf{r}$ approaches the limit $\mathbf{L}$ as t approaches a, written $\lim\limits_{t\to a} \mathbf{r}(t) = \mathbf{L}$, provided $\lim\limits_{t\to a} |\mathbf{r}(t) - \mathbf{L}| = 0$

12.6 Calculus of Vector-Valued Functions

Derivative and Tangent Vector

Let $\mathbf{r}(t) = f(t)\hat{\mathbf{\imath}} + g(t)\hat{\mathbf{\jmath}} + h(t)\hat{\mathbf{k}}$, where f, g, and h are differentiable functions on $(a\,b)$. Then $\mathbf{r}$ has a **derivative** (or is **differentiable**) on $(a\,b)$ and

$$\mathbf{r}'(t) = f'(i)\hat{\mathbf{\imath}} + g'(t)\hat{\mathbf{\jmath}} + h'(t)\hat{\mathbf{k}} \tag{44}$$

Provided $\mathbf{r}'(t) \neq \mathbf{0}, \mathbf{r}'(t)$ is a **tangent vector** (or velocity vector) at the point corresponding to $\mathbf{r}$.

Unit Tangent Vector

Let $\mathbf{r} = f(t)\hat{\mathbf{\imath}} + g(t)\hat{\mathbf{\jmath}} + h(t)\hat{\mathbf{k}}$ be a smooth parameterized curve, for $a \leq t \leq b$. The **unit tangent vector** for a particular value of t is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \tag{45}$$

Derivative Rules

Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector-valued functions and let f be a differentiable scalar-valued function, all at a point t. Let $\mathbf{c}$ be a constant vector. The following rules apply.

$$\frac{d}{dt}(\mathbf{c}) = \mathbf{0} \quad \text{Constant Rule} \tag{46}$$

$$\frac{d}{dt}(\mathbf{u}(t) + \mathbf{v}(t)) = \mathbf{u}'(t) + \mathbf{v}'(t) \quad \text{Sum Rule} \tag{47}$$

$$\frac{d}{dt}(f(t)\mathbf{u}(t)) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t) \quad \text{Product Rule} \tag{48}$$

$$\frac{d}{dt}(\mathbf{u}(f(t))) = \mathbf{u}'(f(t))f'(t) \quad \text{Chain Rule} \tag{49}$$

$$\frac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t) \quad \text{Dot Product Rule} \tag{50}$$

$$\frac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t) \quad \text{Cross Product Rule} \tag{51}$$

Indefinite Integral of a Vector-Valued Function

Let $\mathbf{r} = f\hat{\mathbf{i}} + g\hat{\mathbf{j}} + h\hat{\mathbf{k}}$ be a vector function and let $\mathbf{R} = F\hat{\mathbf{i}} + G\hat{\mathbf{j}} + H\hat{\mathbf{k}}$, where F, G, and H are antiderivatives of f, g, and h, respectively. The **indefinite integral** of $\mathbf{r}$ is

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C} \tag{52}$$

where $\mathbf{C}$ is an arbitrary constant vector.

Definite Integral of a Vector-Valued Function

Let $\mathbf{r}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$, where f, g, and h are integrable on the interval $[a,\, b]$.

$$\int \mathbf{r}(t)\,dt = \left[\int_a^b f(t)\,dt\right]\hat{\mathbf{i}} + \left[\int_a^b g(t)\,dt\right]\hat{\mathbf{j}} + \left[\int_a^b h(t)\,dt\right]\hat{\mathbf{k}} \tag{53}$$

12.7 Motion In Space

Position, Velocity, Speed, Acceleration

Let the position of an object moving in three-dimensional space be given by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, for $t \geq 0$. The **velocity** of the object is

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle \tag{54}$$

The **speed** of the object is the scalar function

$$|\mathbf{v}(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \tag{55}$$

The **acceleration** of the object is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

Motion with Constant $|\mathbf{r}|$

Let $\mathbf{r}$ describe a path on which $|\mathbf{r}|$ is constant (motion on a circle or sphere centered at the origin). Then, $\mathbf{r} \cdot \mathbf{v} = 0$, which means the position vector and the velocity vector are orthogonal at all times for which the functions are defined.

Two-Dimensional Motion in a Gravitational Field

Consider an object moving in a plane with horizontal x-axis and a vertical y-axis, subject only to the force of gravity. Given the initial velocity $\mathbf{v}(0) = \langle u_0,\, v_0\rangle$ and the initial position $\mathbf{r}(0) = \langle x_0,\, y_0\rangle$, the velocity of the object, for $t \geq 0$, is

$$\mathbf{v}(t) = \langle x'(t),\, y'(t)\rangle = \langle u_0,\, -gt + v_0\rangle \tag{56}$$

and the position is

$$\mathbf{r}(t) = \langle x(t),\, y(t)\rangle = \left\langle u_0\, t + x_0,\, -\frac{1}{2}gt^2 + v_0 t + y_0 \right\rangle \tag{57}$$

Two-Dimensional Motion

Assume an object traveling over horizontal ground, acted on only by the gravitational force, has an initial position $\langle x_0,\, y_0\rangle = \langle 0,\, 0\rangle$ and initial velocity

$\langle u_0, v_0 \rangle = \langle |\mathbf{v}_0| \cos\alpha, |\mathbf{v}_0| \sin\alpha \rangle$. The trajectory, which is a segment or a parabola, has the following properties.

$$\text{time of flight} = T = \frac{2|\mathbf{v}_0| \sin\alpha}{g} \tag{58}$$

$$\text{range} = \frac{|\mathbf{v}_0| \sin 2\alpha}{g} \tag{59}$$

$$\text{maximum height} = y\left(\frac{T}{2}\right) = \frac{(|\mathbf{v}_0| \sin\alpha)^2}{2g} \tag{60}$$

12.8 Length of Curves

Consider the parameterized curve $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, where f', g', and h' are continuous, and the curve is traversed once for $a \le t \le b$. The **arc length** of the curve between $(f(a), g(a), h(a))$ and $(f(b), g(b), h(b))$ is

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\, dt = \int_a^b |\mathbf{r}'(\mathbf{t})|\, dt \tag{61}$$

Arc Length of a Polar Curve

Let f have a continuous derivative on the interval $[\alpha, \beta]$. The **arc length** of the polar curve $r = f(\theta)$ on $[\alpha,\ \beta]$ is

$$L = \int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta. \tag{62}$$

Arc Length as a Function of a Parameter

Let $\mathbf{r}$(t) describe a smooth curve, for $t \ge a$. The arc length is given by

$$s(t) = \int_a^t |\mathbf{v}(u)|\, du, \tag{63}$$

where $|\mathbf{v}| = |\mathbf{r}'|$. Equivalently, $\frac{ds}{dt} = |\mathbf{v}t| > 0$. If $|\mathbf{v}(t)| = 1$, for all $t \ge a$, then the parameter t corresponds to arc length.

Arc Length as a Function of a Parameter

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12.9 Curvature and Normal Vectors

Curvature

Let $\mathbf{r}$ describe a smooth parameterized curve. If s denotes arc length and $\mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|}$ is the unit tangent vector, the **curvature** is $\kappa(s) = |\frac{d\mathbf{T}}{ds}|$

Curvature Formula

Let $\mathbf{r}(t)$ describes a smooth parameterized curve, where t is any parameter. If $\mathbf{v} = \mathbf{r}'$ is the velocity and $\mathbf{T}$ is the unit tangent vector, then the curvature is

$$\kappa(t) = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \left|\frac{\mathbf{T}'(t)}{\mathbf{r}'(t)}\right| \tag{65}$$

Alternative Curvature Formula

Let $\mathbf{r}$ be the position of an object moving on a smooth curve. The **curvature** at a point on the curve is

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}, \tag{66}$$

where $\mathbf{v} = \mathbf{r}'$ is the velocity and $\mathbf{a} = \mathbf{v}'$ is the acceleration.

Principal Unit Normal Vector

Let $\mathbf{r}$ describe a smooth parameterized curve. The **principal unit normal vector** at a point P on the curve at which $\kappa \neq 0$ is

$$\mathbf{N}(s) = \frac{{}^{d\mathbf{T}}\!/_{ds}}{|{}^{d\mathbf{T}}\!/_{ds}|} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} \tag{67}$$

In practice, we use the equivalent formula

$$\mathbf{N}(t) = \frac{{}^{d\mathbf{T}}\!/_{dt}}{|{}^{d\mathbf{T}}\!/_{dt}|} \tag{68}$$

evaluated at the value of t corresponding to P.

Properties of the Principal Unit Normal Vector

Let $\mathbf{r}$ describe a smooth parameterized curve with unit tangent vector $\mathbf{T}$ and principal unit normal vector $\mathbf{N}$.

1. $\mathbf{T}$ and $\mathbf{N}$ are orthogonal at all points of the curve; that is, $\mathbf{T}(t)\cdot\mathbf{N}(t) = 0$, at all points where $\mathbf{N}$ is defined.

2. The principal unit normal vector points to the inside of the curve—in the direction that the curve is turning.

Tangential and Normal Components of the Acceleration

The acceleration vector of an object moving in space along a smooth curve has the following representation in terms of its **tangential component** a_T (in the direction of $\mathbf{T}$) and its normal component a_N (in the direction of $\mathbf{N}$):

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T}, \tag{69}$$

where $a_N = \kappa|\mathbf{v}|^2 = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|}$ and $a_T = \frac{d^2s}{dt^2}$.

Unit Binormal Vector and Torsion

Let C be a smooth parameterized curve with unit tangent and principal unit normal vectors $\mathbf{T}$ and $\mathbf{N}$, respectively. Then, at each point of the curve at which the curvature is nonzero, the **unit binormal vector** is

$$\mathbf{B} = \mathbf{T}\times\mathbf{N} \tag{70}$$

and the **torsion** is

$$\tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N} \tag{71}$$

Formulas for Curves in Space

1. Position function: $\mathbf{r}(t) = \langle x(t),\, y(t),\, z(t)\rangle$

2. Velocity: $\mathbf{v} = \mathbf{r}'$

3. Acceleration: $\mathbf{a} = \mathbf{v}'$

4. Unit tangent vector: $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$

5. Principal unit normal vector: $\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$ (provided $d\mathbf{T}/dt \neq \mathbf{0}$)

6. Curvature: $\kappa = \frac{d\mathbf{T}}{ds} = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \left|\frac{\mathbf{T}'(t)}{\mathbf{r}'(t)}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$

7. Components of acceleration: $\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T}$, where $a_N = \kappa|\mathbf{v}|^2 = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|}$ and $a_T = \frac{d^2s}{dt^2} = \frac{\mathbf{v}\cdot\mathbf{a}}{|\mathbf{v}|}$

8. Unit binormal vector: $\mathbf{B} = \mathbf{B} \times \mathbf{N} = \frac{\mathbf{v}\times\mathbf{a}}{|\mathbf{v}\times\mathbf{a}|}$

9. Torsion $\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = \frac{(\mathbf{v}\times\mathbf{a})\cdot\mathbf{a}'}{|\mathbf{v}\times\mathbf{a}|^2} = \frac{(\mathbf{r}'\times\mathbf{r}'')\cdot\mathbf{r}'''}{(\mathbf{r}'\times\mathbf{r}'')^2}$