# Locally Minimax Optimal and Dimension-Agnostic Discrete Argmin Inference

Ilmun Kim

Department of Mathematical Sciences
KAIST

# Joint work with

Aaditya Ramdas
(Carnegie Mellon University)

# Outline of this talk

(1) Problem Setup

(2) The Proposed Methods

(3) Theoretical Properties

(4) Empirical Results

(5) Summary

# Problem Setup: Discrete Argmin Inference

- Suppose $X_1, \ldots, X_{2n} \overset{\text{i.i.d.}}{\sim} P$ in $\mathbb{R}^d$ with mean $\mu = (\mu_1, \ldots, \mu_d)^\top$ and let

$$\Theta = \Theta(P) := \arg\min_{k \in [d]} \mu_k$$

# Problem Setup: Discrete Argmin Inference

- Suppose $X_1, \ldots, X_{2n} \overset{\text{i.i.d.}}{\sim} P$ in $\mathbb{R}^d$ with mean $\mu = (\mu_1, \ldots, \mu_d)^\top$ and let

$$\Theta = \Theta(P) := \arg\min_{k \in [d]} \mu_k$$

i.e., the set of all coordinates whose mean equals the *smallest* in $\mu$

# Problem Setup: Discrete Argmin Inference

- Suppose $X_1, \ldots, X_{2n} \overset{\text{i.i.d.}}{\sim} P$ in $\mathbb{R}^d$ with mean $\mu = (\mu_1, \ldots, \mu_d)^\top$ and let

$$\Theta = \Theta(P) := \arg\min_{k \in [d]} \mu_k$$

i.e., the set of all coordinates whose mean equals the *smallest* in $\mu$

- We would like to form a **confidence set** $\widehat{\Theta}$ for $\Theta$

# Problem Setup: Discrete Argmin Inference

- Suppose $X_1, \ldots, X_{2n} \overset{\text{i.i.d.}}{\sim} P$ in $\mathbb{R}^d$ with mean $\mu = (\mu_1, \ldots, \mu_d)^\top$ and let

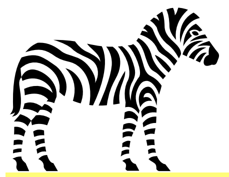
$$\Theta = \Theta(P) := \arg\min_{k \in [d]} \mu_k$$

i.e., the set of all coordinates whose mean equals the *smallest* in $\mu$

- We would like to form a **confidence set** $\widehat{\Theta}$ for $\Theta$

**Each $r \in \Theta$** is included in $\widehat{\Theta}$ with high probability:

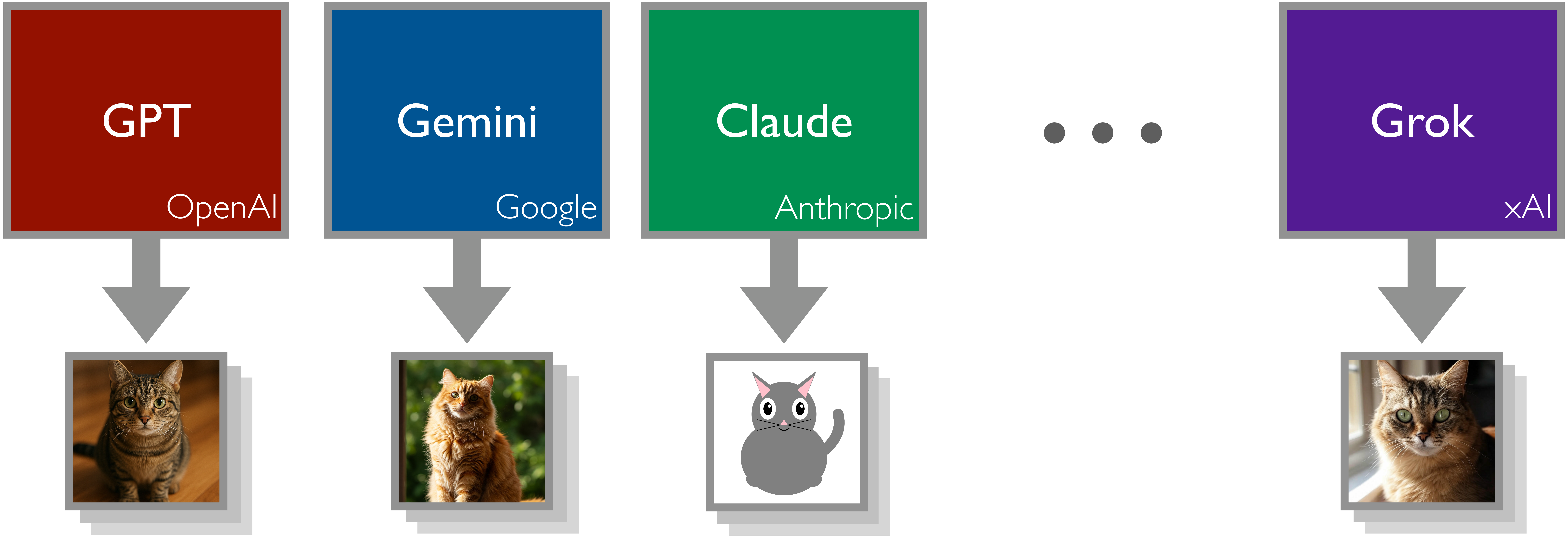
$$\inf_{P \in \mathscr{P}} \inf_{r \in \Theta(P)} P\left(r \in \widehat{\Theta}\right) \geq 1 - \alpha$$



$\mathscr{P}$: class of distributions
$\alpha$: target error level

# Modern Application

**Prompt:** "Generate cat images"



# Modern Application

**Prompt:** "Generate cat images"

| GPT | Gemini | Claude | ... | Grok |
|---|---|---|---|---|
| OpenAI | Google | Anthropic | | xAI |

- The best model minimizes the **population risk**

# Modern Application

**Prompt:** "Generate cat images"

| GPT | Gemini | Claude | … | Grok |
|---|---|---|---|---|
| OpenAI | Google | Anthropic | | xAI |

- The best model minimizes the **population risk**
- But **the population risk is unknown** → **the empirical risk**

# Modern Application

**Prompt:** "Generate cat images"

| GPT | Gemini | Claude | … | Grok |
|---|---|---|---|---|
| OpenAI | Google | Anthropic | | xAI |

- The best model minimizes the **population risk**
- But **the population risk is unknown** → **the empirical risk**
- We must **account for statistical uncertainty** to determine which models are plausibly **optimal with statistical confidence**

# Related Work

- **Classical work:** Bechhofer (1954), Gupta (1956, 1965), Futschik and Pflug (1995) etc

  → rely on parametric models, independence between coordinates, absence of ties

# Related Work

- **Classical work:** Bechhofer (1954), Gupta (1956, 1965), Futschik and Pflug (1995) etc

  → rely on parametric models, independence between coordinates, absence of ties

- **Hansen et al. (2011):** propose a sequential procedure for MCS with uniform coverage

  → widely cited (2550+), produce wide sets; extremely slow to run

# Related Work

- **Classical work:** Bechhofer (1954), Gupta (1956, 1965), Futschik and Pflug (1995) etc

  → rely on parametric models, independence between coordinates, absence of ties

- **Hansen et al. (2011):** propose a sequential procedure for MCS with uniform coverage

  → widely cited (2550+), produce wide sets; extremely slow to run

- **Mogstad et al. (2024):** introduce a bootstrap approach for rank inference (can be tweaked)

  → efficiency not analyzed; results limited to fixed dimensional settings

# Related Work

- **Classical work:** Bechhofer (1954), Gupta (1956, 1965), Futschik and Pflug (1995) etc

  → rely on parametric models, independence between coordinates, absence of ties

- **Hansen et al. (2011):** propose a sequential procedure for MCS with uniform coverage

  → widely cited (2550+), produce wide sets; extremely slow to run

- **Mogstad et al. (2024):** introduce a bootstrap approach for rank inference (can be tweaked)

  → efficiency not analyzed; results limited to fixed dimensional settings

- **Zhang et al. (2024):** introduce a cross-validation + privacy approach for argmin inference

  → require careful tuning and fall short of minimax optimality

# We aim to develop a method for argmin inference with

1. **Dimension-agnostic validity:** works in both **low**- and **high**-dimensional settings

# We aim to develop a method for argmin inference with

1. **Dimension-agnostic validity:** works in both **low**- and **high**-dimensional settings
2. **Powerful inference:** attains **local minimax rates** across diverse regimes

# We aim to develop a method for argmin inference with

1. **Dimension-agnostic validity:** works in both **low**- and **high**-dimensional settings
2. **Powerful inference:** attains **local minimax rates** across diverse regimes
3. **Robustness to complex data:** handles **ties** and strong **dependence**

# We aim to develop a method for argmin inference with

1. **Dimension-agnostic validity:** works in both **low**- and **high**-dimensional settings
2. **Powerful inference:** attains **local minimax rates** across diverse regimes
3. **Robustness to complex data:** handles **ties** and strong **dependence**
4. **Model-agnostic applicability:** requires **no parametric** assumptions

# We aim to develop a method for argmin inference with

1. **Dimension-agnostic validity:** works in both **low**- and **high**-dimensional settings
2. **Powerful inference:** attains **local minimax rates** across diverse regimes
3. **Robustness to complex data:** handles **ties** and strong **dependence**
4. **Model-agnostic applicability:** requires **no parametric** assumptions
5. **Tuning-free implementation:** requires **no** (non-trivial) **tuning** parameters

# We aim to develop a method for argmin inference with

1. **Dimension-agnostic validity:** works in both **low**- and **high**-dimensional settings
2. **Powerful inference:** attains **local minimax rates** across diverse regimes
3. **Robustness to complex data:** handles **ties** and strong **dependence**
4. **Model-agnostic applicability:** requires **no parametric** assumptions
5. **Tuning-free implementation:** requires **no** (non-trivial) **tuning** parameters

→ Sample Splitting + Studentization (Kim and Ramdas, 2024)

# Dimension-Agnostic Argmin Inference

# Formal (Primal) Goal.

- We seek a **dimension-agnostic confidence set** $\widehat{\Theta}$ for $\Theta$ such that

$$\boxed{\liminf_{n\to\infty} \inf_{P\in\mathscr{P}} \inf_{r\in\Theta(P)} P\left(r \in \widehat{\Theta}\right) \geq 1-\alpha, \quad \text{regardless of the sequence } (d_n)_{n=1}^{\infty}}$$

# Formal (Primal) Goal.

- We seek a **dimension-agnostic confidence set** $\widehat{\Theta}$ for $\Theta$ such that

$$\boxed{\liminf_{n\to\infty} \inf_{P\in\mathscr{P}} \inf_{r\in\Theta(P)} P\left(r \in \widehat{\Theta}\right) \geq 1-\alpha, \quad \text{regardless of the sequence } (d_n)_{n=1}^{\infty}}$$

- We also seek a **confidence set** $\widehat{\Theta}$ for $\Theta$ such that its **expected cardinality** is small and **ideally optimal**

# Formal (Dual) Goal.

- Given some fixed $r \in [d]$, consider the **null** and **alternative** hypotheses:

$$\boxed{H_0 : r \in \Theta \quad \text{versus} \quad H_1 : r \notin \Theta}$$

# Formal (Dual) Goal.

- Given some fixed $r \in [d]$, consider the **null** and **alternative** hypotheses:

$$\boxed{H_0 : r \in \Theta \quad \text{versus} \quad H_1 : r \notin \Theta}$$

- Let $\psi_r : \{X_1, \ldots, X_{2n}\} \to \{0,1\}$ denote a test function that rejects $H_0$ if $\psi_r = 1$

# Formal (Dual) Goal.

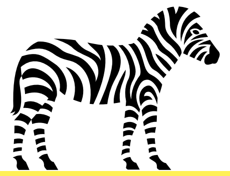
- Given some fixed $r \in [d]$, consider the **null** and **alternative** hypotheses:

$$\boxed{H_0 : r \in \Theta \quad \text{versus} \quad H_1 : r \notin \Theta}$$

- Let $\psi_r : \{X_1, \ldots, X_{2n}\} \to \{0,1\}$ denote a test function that rejects $H_0$ if $\psi_r = 1$

- We seek a **dimension-agnostic test** $\psi_r$ such that

$$\limsup_{n\to\infty} \sup_{P\in\mathscr{P}_{0,r}} P\left(\psi_r = 1\right) \le \alpha, \quad \text{regardless of the sequence } (d_n)_{n=1}^{\infty}$$



$\mathscr{P}_{0,r} \subseteq \mathscr{P}$: a class of distributions s.t. $H_0 : r \in \Theta$

# Formal (Dual) Goal.

- Given some fixed $r \in [d]$, consider the **null** and **alternative** hypotheses:

$$\boxed{H_0 : r \in \Theta \quad \text{versus} \quad H_1 : r \notin \Theta}$$

- Let $\psi_r : \{X_1, \ldots, X_{2n}\} \to \{0,1\}$ denote a test function that rejects $H_0$ if $\psi_r = 1$

- We seek a **dimension-agnostic test** $\psi_r$ such that

$$\limsup_{n\to\infty} \sup_{P \in \mathscr{P}_{0,r}} P\big(\psi_r = 1\big) \le \alpha, \quad \text{regardless of the sequence } (d_n)_{n=1}^{\infty}$$

- We can construct a **DA confidence set** using **DA tests**

$\mathscr{P}_{0,r} \subseteq \mathscr{P}$: a class of distributions s.t. $H_0 : r \in \Theta$

# DA tests yield a DA confidence set via duality

- Suppose that each $\psi_k$ satisfies the **dimension-agnostic (DA)** property:

$$\limsup_{n\to\infty} \sup_{P\in\mathscr{P}_{0,k}} P\left(\psi_k = 1\right) \le \alpha, \quad \text{regardless of the sequence } (d_n)_{n=1}^{\infty}$$

# DA tests yield a DA confidence set via duality

- Suppose that each $\psi_k$ satisfies the **dimension-agnostic (DA)** property:

$$\limsup_{n\to\infty} \sup_{P\in\mathscr{P}_{0,k}} P\big(\psi_k = 1\big) \le \alpha, \quad \text{regardless of the sequence } (d_n)_{n=1}^{\infty}$$

- We run $d$ such DA tests $\psi_1, \ldots, \psi_d$ and let

$$\widehat{\Theta} = \big\{k \in [d] : \psi_k = 0\big\}$$

# DA tests yield a DA confidence set via duality

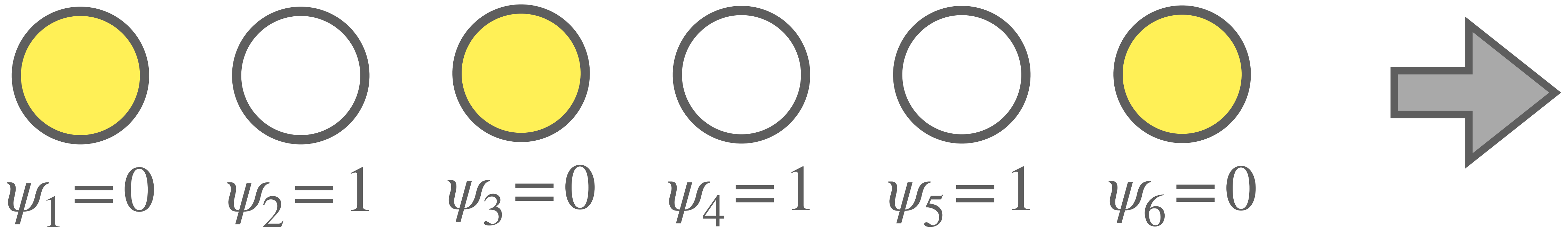
- Suppose that each $\psi_k$ satisfies the **dimension-agnostic (DA)** property:

$$\limsup_{n\to\infty} \sup_{P\in\mathscr{P}_{0,k}} P\big(\psi_k = 1\big) \le \alpha, \quad \text{regardless of the sequence } (d_n)_{n=1}^{\infty}$$

- We run $d$ such DA tests $\psi_1, \ldots, \psi_d$ and let

$$\boxed{\widehat{\Theta} = \big\{k \in [d] : \psi_k = 0\big\}}$$

For example,




$$\widehat{\Theta} = \{1,3,6\}$$

# DA tests yield a DA confidence set via duality

- Suppose that each $\psi_k$ satisfies the **dimension-agnostic (DA)** property:

$$\boxed{\limsup_{n\to\infty} \sup_{P\in\mathscr{P}_{0,k}} P\big(\psi_k = 1\big) \le \alpha, \quad \text{regardless of the sequence } (d_n)_{n=1}^{\infty}}$$

- We run $d$ such DA tests $\psi_1, \ldots, \psi_d$ and let

$$\boxed{\widehat{\Theta} = \big\{k \in [d] : \psi_k = 0\big\}}$$

which results in the DA confidence set

$$\boxed{\liminf_{n\to\infty} \inf_{P\in\mathscr{P}} \inf_{r\in\Theta(P)} P\big(r \in \widehat{\Theta}\big) \ge 1-\alpha, \quad \text{regardless of the sequence } (d_n)_{n=1}^{\infty}}$$

# Procedures

# Reformulation

- Recall **our (dual) goal** is to test

$$\boxed{H_0 : r \in \Theta \quad \text{vs.} \quad H_1 : r \notin \Theta}$$

# Reformulation

- Recall **our (dual) goal** is to test

$$\boxed{H_0 : r \in \Theta \quad \text{vs.} \quad H_1 : r \notin \Theta}$$

- We can **reformulate** this as

$$\boxed{H_0 : \mu_r - \mu_s \leq 0 \quad \text{vs.} \quad H_1 : \mu_r - \mu_s > 0}$$

where $s = \arg \min_{k \in [d] \setminus \{r\}} \mu_k$

# Reformulation

- Recall **our (dual) goal** is to test

$$\boxed{H_0 : r \in \Theta \quad \text{vs.} \quad H_1 : r \notin \Theta}$$

- We can **reformulate** this as

$$\boxed{H_0 : \mu_r - \mu_s \leq 0 \quad \text{vs.} \quad H_1 : \mu_r - \mu_s > 0}$$

where $s = \arg \min_{k \in [d] \setminus \{r\}} \mu_k$

i.e., **Target** $\mu_r$ is **greater** than the **smallest** mean $\mu_s$?

# Reformulation

- Recall **our (dual) goal** is to test

$$\boxed{H_0 : r \in \Theta \quad \text{vs.} \quad H_1 : r \notin \Theta}$$

- We can **reformulate** this as

$$\boxed{H_0 : \mu_r - \mu_s \leq 0 \quad \text{vs.} \quad H_1 : \mu_r - \mu_s > 0}$$

where $s = \arg\min_{k \in [d] \setminus \{r\}} \mu_k$

i.e., Target $\mu_r$ is **greater** than the **smallest** mean $\mu_s$?

- When $s$ is **known**, use a **one-sided t-test**

$$\frac{\bar{X}_r - \bar{X}_s}{\hat{\sigma}_{r,s}} > z_{1-\alpha}$$

# Reformulation

- Recall **our (dual) goal** is to test

$$H_0 : r \in \Theta \quad \text{vs.} \quad H_1 : r \notin \Theta$$

- We can **reformulate** this as

$$H_0 : \mu_r - \mu_s \leq 0 \quad \text{vs.} \quad H_1 : \mu_r - \mu_s > 0$$

where $s = \arg \min_{k \in [d] \setminus \{r\}} \mu_k$

- When $s$ is **known**, use a **one-sided t-test**

$$\frac{\bar{X}_r - \bar{X}_s}{\hat{\sigma}_{r,s}} > z_{1-\alpha}$$

- But $s$ is **unknown:** one idea (Mogstad et al. 2024)

$$\max_{k \in [d] \setminus r} \frac{\bar{X}_r - \bar{X}_k}{\hat{\sigma}_{r,k}} > c_{1-\alpha}$$

i.e., **Target** $\mu_r$ is **greater** than the **smallest** mean $\mu_s$?

# Reformulation

- Recall **our (dual) goal** is to test

$$\boxed{H_0 : r \in \Theta \quad \text{vs.} \quad H_1 : r \notin \Theta}$$

- We can **reformulate** this as

$$\boxed{H_0 : \mu_r - \mu_s \leq 0 \quad \text{vs.} \quad H_1 : \mu_r - \mu_s > 0}$$

where $s = \arg \min_{k \in [d] \setminus \{r\}} \mu_k$

i.e., **Target** $\mu_r$ is **greater** than the **smallest** mean $\mu_s$?

- When $s$ is **known**, use a **one-sided t-test**

$$\frac{\bar{X}_r - \bar{X}_s}{\hat{\sigma}_{r,s}} > z_{1-\alpha}$$

- But $s$ is **unknown:** one idea (Mogstad et al. 2024)

$$\max_{k \in [d] \setminus r} \frac{\bar{X}_r - \bar{X}_k}{\hat{\sigma}_{r,k}} > c_{1-\alpha}$$

- Hard to calibrate $c_{1-\alpha}$ in **high-dimensions**

# Dimension-Agnostic Argmin Test

💡**Idea:** use $\mathscr{D}_1$ to estimate $s$ and use $\mathscr{D}_2$ for inference

## Step 1 (Sample Splitting)

- Split the data into two

$$\mathscr{D}_1 \quad \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{n-1} \\ X_n \end{bmatrix} \qquad \mathscr{D}_2 \quad \begin{bmatrix} X_{n+1} \\ X_{n+2} \\ \vdots \\ X_{2n-1} \\ X_{2n} \end{bmatrix}$$

# Dimension-Agnostic Argmin Test

$s = \arg\min_{k \in [d]\setminus\{r\}} \mu_k$

💡**Idea:** use $\mathscr{D}_1$ to estimate $s$ and use $\mathscr{D}_2$ for inference

## Step I (Sample Splitting)

- Split the data into two

$$\mathscr{D}_1 \quad \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{n-1} \\ X_n \end{bmatrix} \qquad \mathscr{D}_2 \quad \begin{bmatrix} X_{n+1} \\ X_{n+2} \\ \vdots \\ X_{2n-1} \\ X_{2n} \end{bmatrix}$$

## Step II (Model Selection)

- Based on $\mathscr{D}_1$, compute

**1. Plug-in version**

$$\hat{s} = \arg\max_{k \in [d]\setminus\{r\}} \bar{X}_r^{(2)} - \bar{X}_k^{(2)}$$

**2. Noise-adjusted version**

$$\hat{s} = \arg\max_{k \in [d]\setminus\{r\}} \frac{\bar{X}_r^{(2)} - \bar{X}_k^{(2)}}{\hat{\sigma}_{r,k}^{(2)} \vee \kappa}$$

# Dimension-Agnostic Argmin Test

💡**Idea:** use $\mathscr{D}_1$ to estimate $s$ and use $\mathscr{D}_2$ for inference

## Step I (Sample Splitting)

- Split the data into two

$$\mathscr{D}_1 \quad \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{n-1} \\ X_n \end{bmatrix} \qquad \mathscr{D}_2 \quad \begin{bmatrix} X_{n+1} \\ X_{n+2} \\ \vdots \\ X_{2n-1} \\ X_{2n} \end{bmatrix}$$

## Step II (Model Selection)

- Based on $\mathscr{D}_1$, compute

**1. Plug-in version**

$$\hat{s} = \arg \max_{k \in [d] \setminus \{r\}} \overline{X}_r^{(2)} - \overline{X}_k^{(2)}$$

**2. Noise-adjusted version**

$$\hat{s} = \arg \max_{k \in [d] \setminus \{r\}} \frac{\overline{X}_r^{(2)} - \overline{X}_k^{(2)}}{\hat{\sigma}_{r,k}^{(2)} \vee \kappa}$$

# Dimension-Agnostic Argmin Test

💡**Idea:** use $\mathscr{D}_1$ to estimate $s$ and use $\mathscr{D}_2$ for inference

**Step III** (Student's t-statistic)

- Given $\hat{s} \in [d] \backslash \{r\}$, compute a one-sided t-statistic

$$T = \frac{\overline{X}_r^{(1)} - \overline{X}_{\hat{s}}^{(1)}}{\hat{\sigma}_{r,\hat{s}}^{(1)}}$$

based on $\mathscr{D}_2$

# Dimension-Agnostic Argmin Test

💡**Idea:** use $\mathcal{D}_1$ to estimate $s$ and use $\mathcal{D}_2$ for inference

## Step III (Student's t-statistic)

- Given $\widehat{s} \in [d] \backslash \{r\}$, compute a one-sided t-statistic

$$T = \frac{\overline{X}_r^{(1)} - \overline{X}_{\widehat{s}}^{(1)}}{\widehat{\sigma}_{r,\widehat{s}}^{(1)}}$$

based on $\mathcal{D}_2$

## Step IV (Decision)

- **Reject** the null if

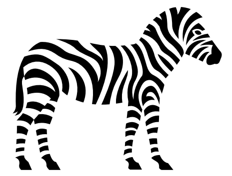
$$T > z_{1-\alpha}$$

- **Accept** the null o.w.

$\Phi(z_{1-\alpha}) = 1 - \alpha$

# Theoretical Properties

1. Asymptotic Validity
2. Power Analysis



$X = (X^{(1)}, \ldots, X^{(d)})$

# Asymptotic Validity

**Assumption (Truncated Moment Condition)**

Let $W_k := (X^{(r)} - \mu_r) - (X^{(k)} - \mu_k)$ and

# Asymptotic Validity

$X = (X^{(1)}, \ldots, X^{(d)})$

**Assumption (Truncated Moment Condition)**

Let $W_k := (X^{(r)} - \mu_r) - (X^{(k)} - \mu_k)$ and

$$M_k := \sup_{P \in \mathscr{P}_{0,r}} \mathbb{E}_P \left[ \frac{W_k^2}{\mathbb{E}_P[W_k^2]} \min\left\{ 1, \frac{|W_k|}{n^{1/2}(\mathbb{E}_P[W_k^2])^{1/2}} \right\} \right]$$

# Asymptotic Validity

$X = (X^{(1)}, \ldots, X^{(d)})$

**Assumption (Truncated Moment Condition)**

Let $W_k := (X^{(r)} - \mu_r) - (X^{(k)} - \mu_k)$ and

$$M_k := \sup_{P \in \mathscr{P}_{0,r}} \mathbb{E}_P \left[ \frac{W_k^2}{\mathbb{E}_P[W_k^2]} \min\left\{ 1, \frac{|W_k|}{n^{1/2}(\mathbb{E}_P[W_k^2])^{1/2}} \right\} \right]$$

Assume that $\max_{k \in [d] \setminus \{r\}} M_k = o(1)$

# Asymptotic Validity

$X = (X^{(1)}, \ldots, X^{(d)})$

**Assumption (Truncated Moment Condition)**

Let $W_k := (X^{(r)} - \mu_r) - (X^{(k)} - \mu_k)$ and

$$M_k := \sup_{P \in \mathscr{P}_{0,r}} \mathbb{E}_P \left[ \frac{W_k^2}{\mathbb{E}_P[W_k^2]} \min\left\{ 1, \frac{|W_k|}{n^{1/2}(\mathbb{E}_P[W_k^2])^{1/2}} \right\} \right]$$

Assume that $\max_{k \in [d] \setminus \{r\}} M_k = o(1)$

- **Equivalent** to Lindeberg's condition (weaker than Lyapunov)

# Asymptotic Validity

$X = (X^{(1)}, \ldots, X^{(d)})$

## Assumption (Truncated Moment Condition)

Let $W_k := (X^{(r)} - \mu_r) - (X^{(k)} - \mu_k)$ and

$$M_k := \sup_{P \in \mathscr{P}_{0,r}} \mathbb{E}_P \left[ \frac{W_k^2}{\mathbb{E}_P[W_k^2]} \min\left\{ 1, \frac{|W_k|}{n^{1/2}(\mathbb{E}_P[W_k^2])^{1/2}} \right\} \right]$$

Assume that $\max_{k \in [d] \setminus \{r\}} M_k = o(1)$

- **Equivalent** to Lindeberg's condition (weaker than Lyapunov)
- If $X \sim N(\mu, \Sigma)$, it only requires $\mathrm{Var}[W_k] > 0$

# Asymptotic Validity

$X = (X^{(1)}, \ldots, X^{(d)})$

## Assumption (Truncated Moment Condition)

Let $W_k := (X^{(r)} - \mu_r) - (X^{(k)} - \mu_k)$ and

$$M_k := \sup_{P \in \mathscr{P}_{0,r}} \mathbb{E}_P \left[ \frac{W_k^2}{\mathbb{E}_P[W_k^2]} \min\left\{ 1, \frac{|W_k|}{n^{1/2}(\mathbb{E}_P[W_k^2])^{1/2}} \right\} \right]$$

Assume that $\max_{k \in [d] \setminus \{r\}} M_k = o(1)$

- **Equivalent** to Lindeberg's condition (weaker than Lyapunov)
- If $X \sim N(\mu, \Sigma)$, it only requires $\mathrm{Var}[W_k] > 0$
- **Arbitrary dependence** among the components except $X^{(r)}$

# Asymptotic Validity

$$M_k := \sup_{P \in \mathscr{P}_{0,r}} \mathbb{E}_P\left[\frac{W_k^2}{\mathbb{E}_P[W_k^2]} \min\left\{1, \frac{|W_k|}{n^{1/2}(\mathbb{E}_P[W_k^2])^{1/2}}\right\}\right]$$

Assume that $\max_{k \in [d] \setminus \{r\}} M_k = o(1)$

**Theorem** Kim and Ramdas (2025)

Let $\mathscr{P}_{0,r}$ be the class of null distributions with $H_0 : r \in \Theta$ that satisfy the **truncated moment condition.** Then the **DA test** is asymptotically **valid as**

$$\limsup_{n \to \infty} \sup_{P \in \mathscr{P}_{0,r}} P\big(\psi_r = 1\big) \le \alpha, \quad \text{regardless of the sequence } (d_n)_{n=1}^{\infty}$$

# Asymptotic Validity

$$M_k := \sup_{P \in \mathscr{P}_{0,r}} \mathbb{E}_P \left[ \frac{W_k^2}{\mathbb{E}_P[W_k^2]} \min\left\{ 1, \frac{|W_k|}{n^{1/2}(\mathbb{E}_P[W_k^2])^{1/2}} \right\} \right]$$

Assume that $\max_{k \in [d] \setminus \{r\}} M_k = o(1)$

**Theorem** Kim and Ramdas (2025)

Let $\mathscr{P}_{0,r}$ be the class of null distributions with $H_0 : r \in \Theta$ that satisfy the **truncated moment condition.** Then the **DA test** is asymptotically **valid as**

$$\limsup_{n \to \infty} \sup_{P \in \mathscr{P}_{0,r}} P\big(\psi_r = 1\big) \le \alpha, \quad \text{regardless of the sequence } (d_n)_{n=1}^{\infty}$$

It holds for both

**1. Plug-in version**

$$\widehat{s} = \arg\max_{k \in [d] \setminus \{r\}} \overline{X}_r^{(2)} - \overline{X}_k^{(2)}$$

**2. Noise-adjusted version**

$$\widehat{s} = \arg\max_{k \in [d] \setminus \{r\}} \frac{\overline{X}_r^{(2)} - \overline{X}_k^{(2)}}{\widehat{\sigma}_{r,k}^{(2)} \vee \kappa}$$

# Theoretical Properties

1. Asymptotic Validity
2. Power Analysis

# Confusion Set

- The problem **difficulty** depends on the cardinality of a **confusion set** $\mathbb{C}_r$

$$\mathbb{C}_r := \left\{ k \in [d] \backslash \{r\} : \frac{\mu_r - \mu_\star}{2} \leq \mu_k - \mu_\star \leq C_n \sqrt{\frac{\log(d)}{n}} \right\}$$

# Confusion Set

- The problem **difficulty** depends on the cardinality of a **confusion set** $\mathbb{C}_r$

$$\mathbb{C}_r := \left\{ k \in [d] \backslash \{r\} : \frac{\mu_r - \mu_\star}{2} \leq \mu_k - \mu_\star \leq C_n \sqrt{\frac{\log(d)}{n}} \right\}$$

$\mu_\star = \min_{i \in [d]} \mu_i$

$C_n$ : any increasing sequence

# Confusion Set

$\mu_\star = \min_{i \in [d]} \mu_i$

- The problem **difficulty** depends on the cardinality of a **confusion set** $\mathbb{C}_r$

$$\mathbb{C}_r := \left\{ k \in [d] \backslash \{r\} : \frac{\mu_r - \mu_\star}{2} \le \underbrace{\mu_k - \mu_\star \le C_n \sqrt{\frac{\log(d)}{n}}}_{\mathscr{A}} \right\}$$

$C_n$ : any increasing sequence

- **Intuition for $\mathscr{A}^c$**

When $\mu_k$ is far from $\mu_\star$, it is unlikely that $\hat{s} = k$

$\therefore$ **Well-separated set**

# Confusion Set

$\mu_\star = \min_{i \in [d]} \mu_i$

- The problem **difficulty** depends on the cardinality of a **confusion set** $\mathbb{C}_r$

$$\mathbb{C}_r := \left\{ k \in [d] \backslash \{r\} : \underbrace{\frac{\mu_r - \mu_\star}{2} \le \mu_k - \mu_\star}_{\mathscr{B}} \le C_n \sqrt{\frac{\log(d)}{n}} \right\}$$

$\mathscr{A}$: $\mu_k - \mu_\star \le C_n \sqrt{\frac{\log(d)}{n}}$

$C_n$ : any increasing sequence

- **Intuition for $\mathscr{A}^c$**

When $\mu_k$ is far from $\mu_\star$, it is unlikely that $\widehat{s} = k$

$\therefore$ **Well-separated set**

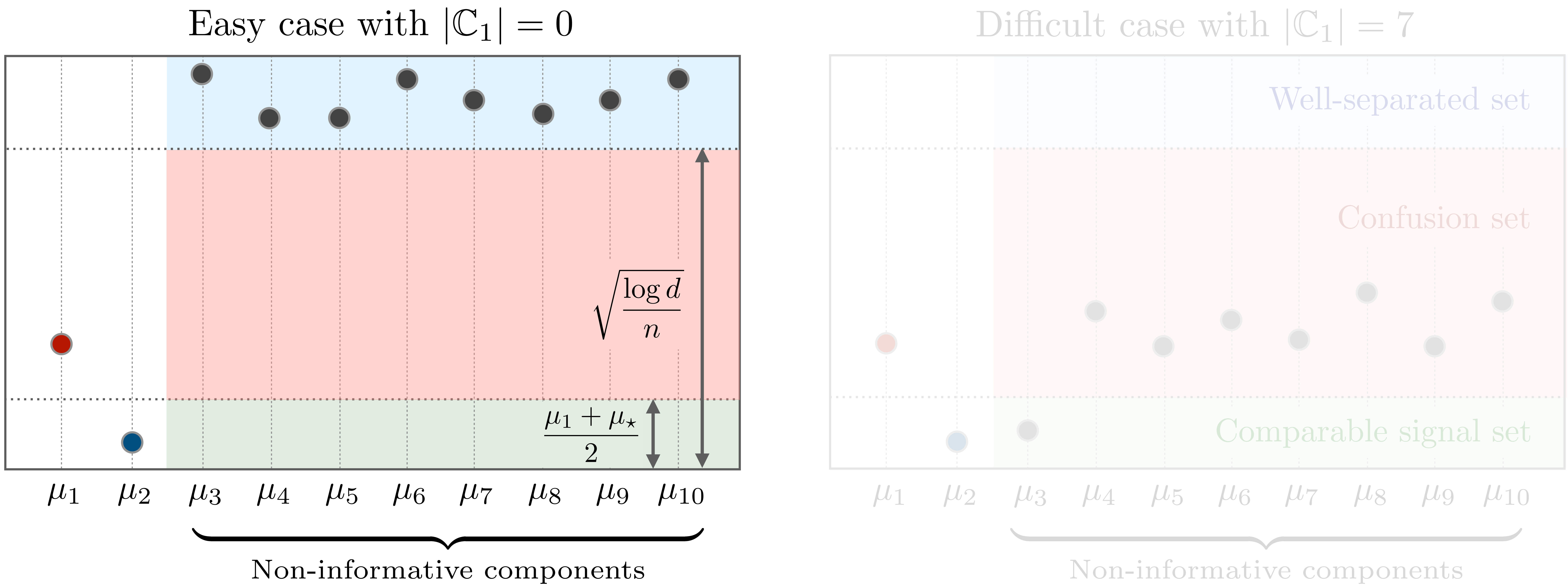
- **Intuition for $\mathscr{B}^c$**

$$\frac{\mu_r - \mu_\star}{2} > \mu_k - \mu_\star \iff \mu_r - \mu_k > \frac{\mu_r - \mu_\star}{2}$$

$\therefore$ **Comparable signal**

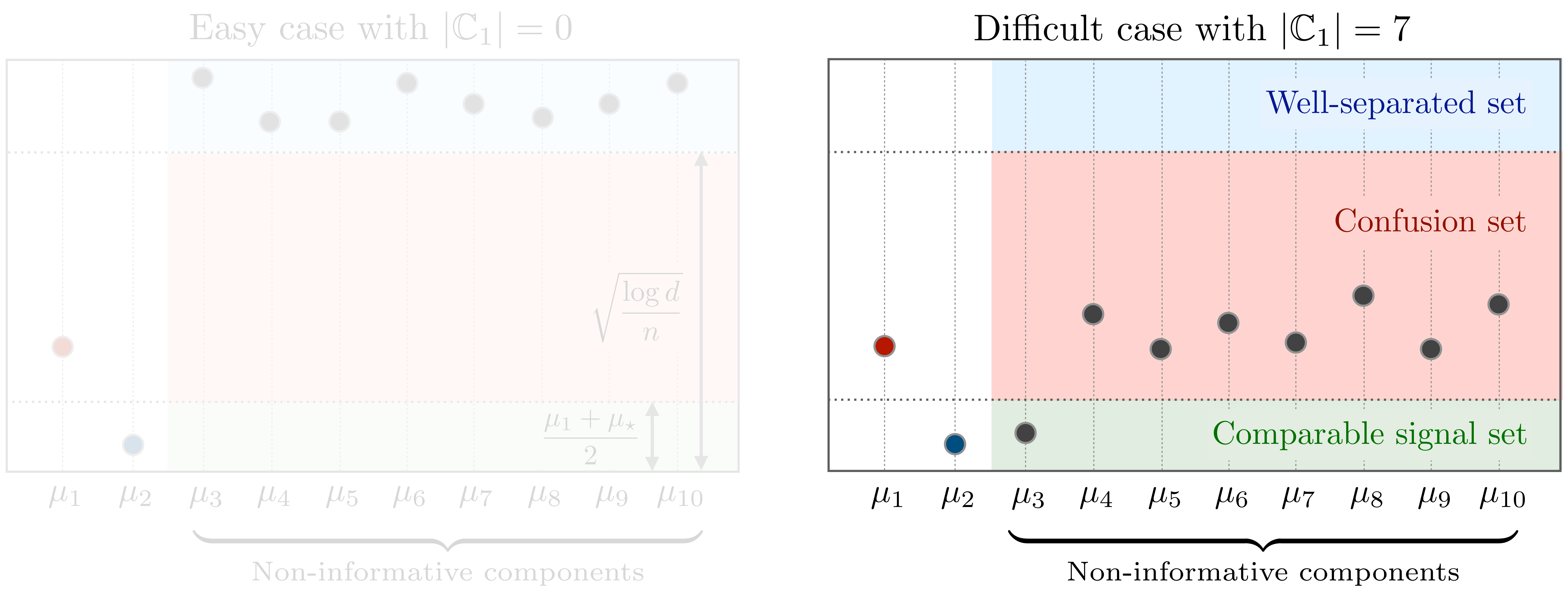
# Confusion Set

- Set $\mu_r = \mu_1$ and $\mu_\star = \mu_2$



# Confusion Set

- Set $\mu_r = \mu_1$ and $\mu_\star = \mu_2$



# Power Analysis

- Let $\mathscr{P}$ be a collection of **sub-Gaussian** distributions with $\sigma^2$

# Power Analysis

- Let $\mathscr{P}$ be a collection of **sub-Gaussian** distributions with $\sigma^2$

- Define a class of **alternative** distributions

$$\mathscr{P}_{1,r}(\varepsilon;\tau) := \left\{P \in \mathscr{P} : \mu_r - \mu_\star \geq \varepsilon \text{ and } |\mathbb{C}_r| = \tau\right\}$$

# Power Analysis

- Let $\mathscr{P}$ be a collection of **sub-Gaussian** distributions with $\sigma^2$

- Define a class of **alternative** distributions

$$\mathscr{P}_{1,r}(\varepsilon;\tau) := \left\{P \in \mathscr{P} : \mu_r - \mu_\star \geq \varepsilon \text{ and } |\mathbb{C}_r| = \tau\right\}$$

- The **critical radius** $\varepsilon^\star$ is defined as

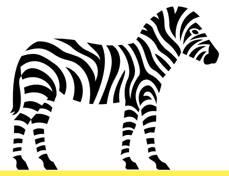
$$\varepsilon^\star = \varepsilon^\star(\tau) = \sqrt{\frac{1 \vee \log(\tau)}{n}}$$

# Power Analysis

**Theorem** Kim and Ramdas (2025)

For any $\tau$, suppose that $\varepsilon \gg \varepsilon^\star$. Then the asymptotic uniform power of the **DA test** is equal to one:

$$\lim_{n\to\infty} \inf_{P\in\mathscr{P}_{1,r}(\varepsilon;\tau)} P\big(\psi_r = 1\big) = 1$$



$$\varepsilon^\star = \sqrt{\frac{1 \vee \log(\tau)}{n}}$$

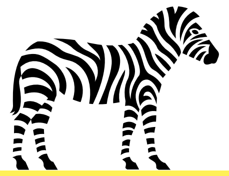
# Power Analysis

**Theorem** Kim and Ramdas (2025)

For any $\tau$, suppose that $\varepsilon \gg \varepsilon^\star$. Then the asymptotic uniform power of the **DA test** is equal to one:

$$\lim_{n \to \infty} \inf_{P \in \mathscr{P}_{1,r}(\varepsilon;\tau)} P\big(\psi_r = 1\big) = 1$$

- Adaptivity to **unknown** $|\mathbb{C}_r|$
- The rate changes from $1/\sqrt{n}$-rate to $\sqrt{\log(d)/n}$



$\varepsilon^\star = \sqrt{\frac{1 \vee \log(\tau)}{n}}$

# Power Analysis

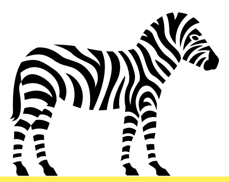
**Theorem** Kim and Ramdas (2025)

For any $\tau$, suppose that $\varepsilon \gg \varepsilon^\star$. Then the asymptotic uniform power of the **DA test** is equal to one:

$$\lim_{n\to\infty} \inf_{P \in \mathscr{P}_{1,r}(\varepsilon;\tau)} P\left(\psi_r = 1\right) = 1$$

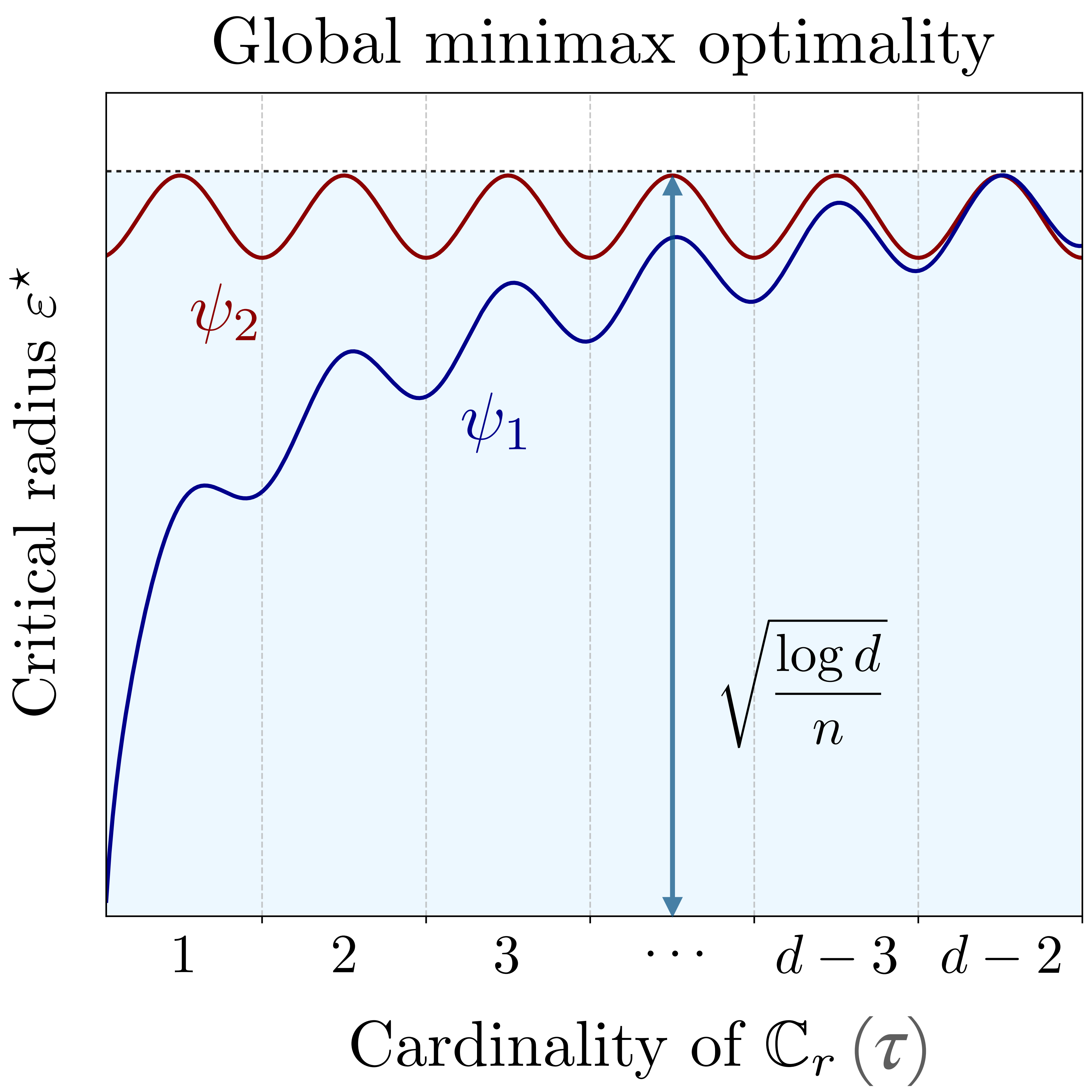
- Adaptivity to **unknown** $|\mathbb{C}_r|$
- The rate changes from $1/\sqrt{n}$-rate to $\sqrt{\log(d)/n}$

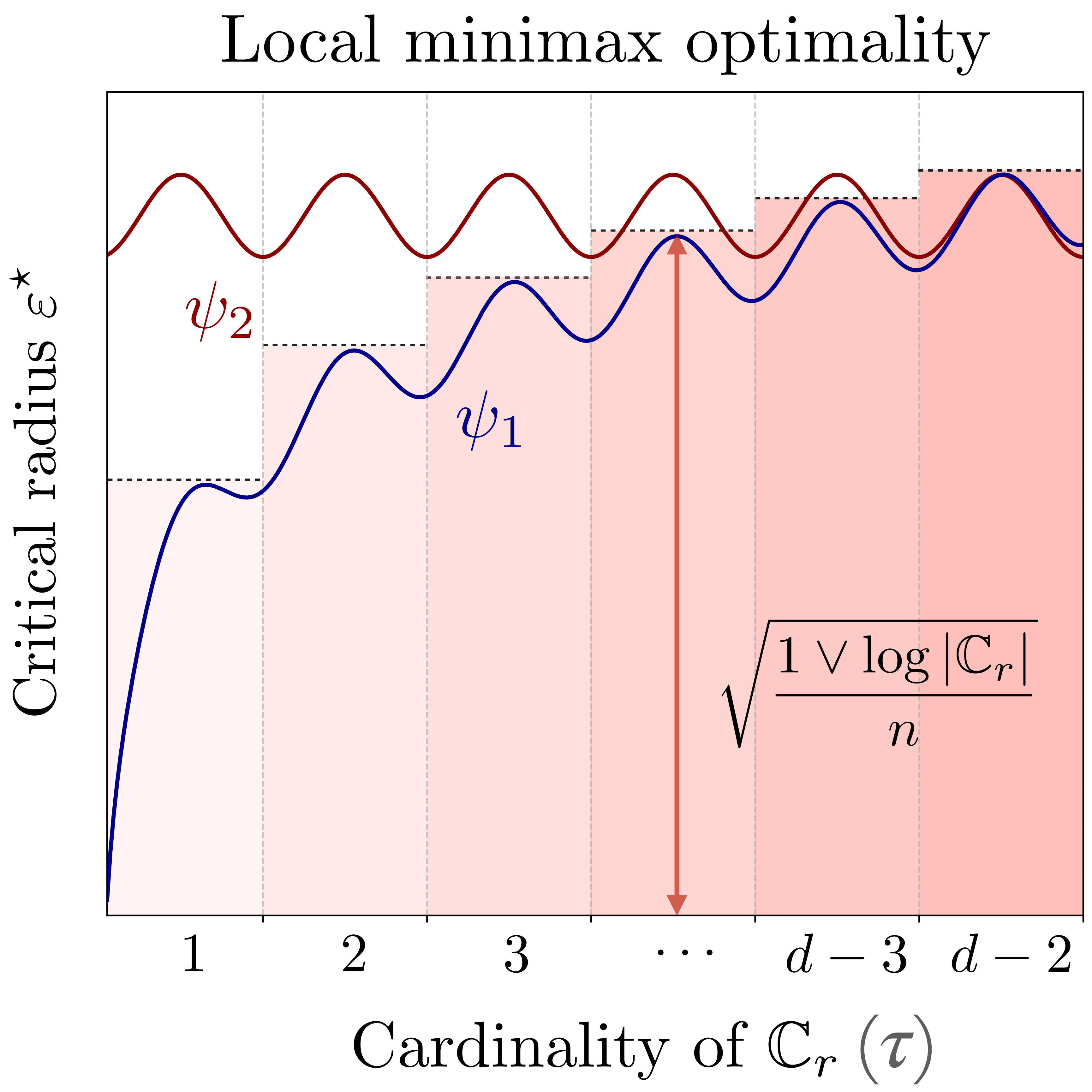
**Question.** Can we further **improve** the **separation rate**?



$$\varepsilon^\star = \sqrt{\frac{1 \vee \log(\tau)}{n}}$$

# Local Minimax Optimality



# Local Minimax Optimality

**Theorem** Kim and Ramdas (2025)

Let $\Psi_\alpha$ be the set of all asymptotic level-$\alpha$ tests over $\mathscr{P}_{0,r}$

$$\Psi_\alpha := \left\{ \psi : \limsup_{n\to\infty} \sup_{P\in\mathscr{P}_{0,r}} P(\psi = 1) \leq \alpha \right\}$$

# Local Minimax Optimality

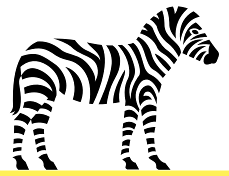
**Theorem** Kim and Ramdas (2025)

Let $\Psi_\alpha$ be the set of all asymptotic level-$\alpha$ tests over $\mathscr{P}_{0,r}$

$$\Psi_\alpha := \left\{ \psi : \limsup_{n\to\infty} \sup_{P\in\mathscr{P}_{0,r}} P(\psi = 1) \le \alpha \right\}$$

If $\varepsilon \ll \varepsilon^\star$, then the **asymptotic type II error** is at least $\beta$:

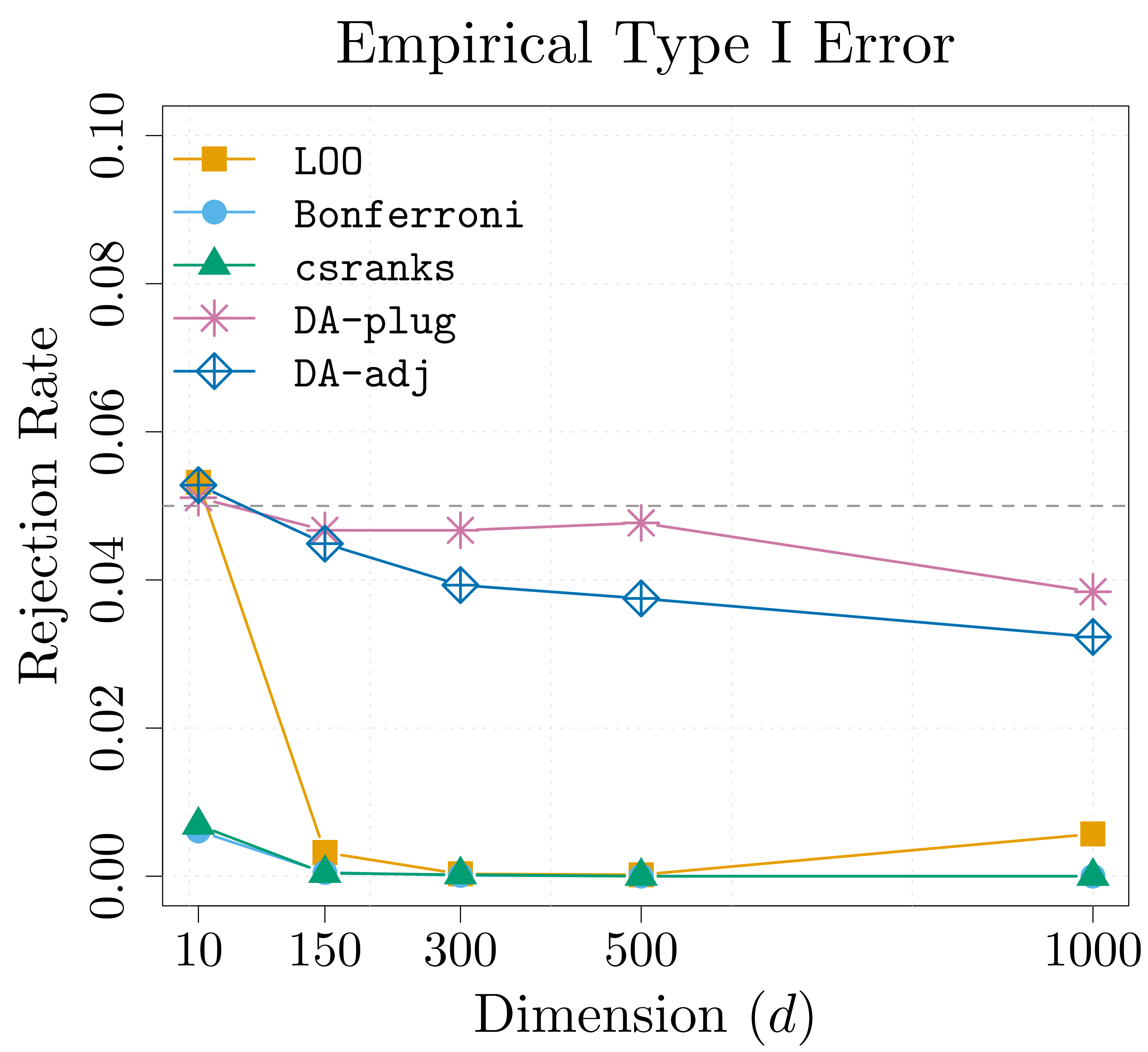
$$\liminf_{n\to\infty} \inf_{\psi\in\Psi_\alpha} \sup_{P\in\mathscr{P}_{1,r}(\varepsilon;\tau)} P(\psi = 0) \ge \beta$$

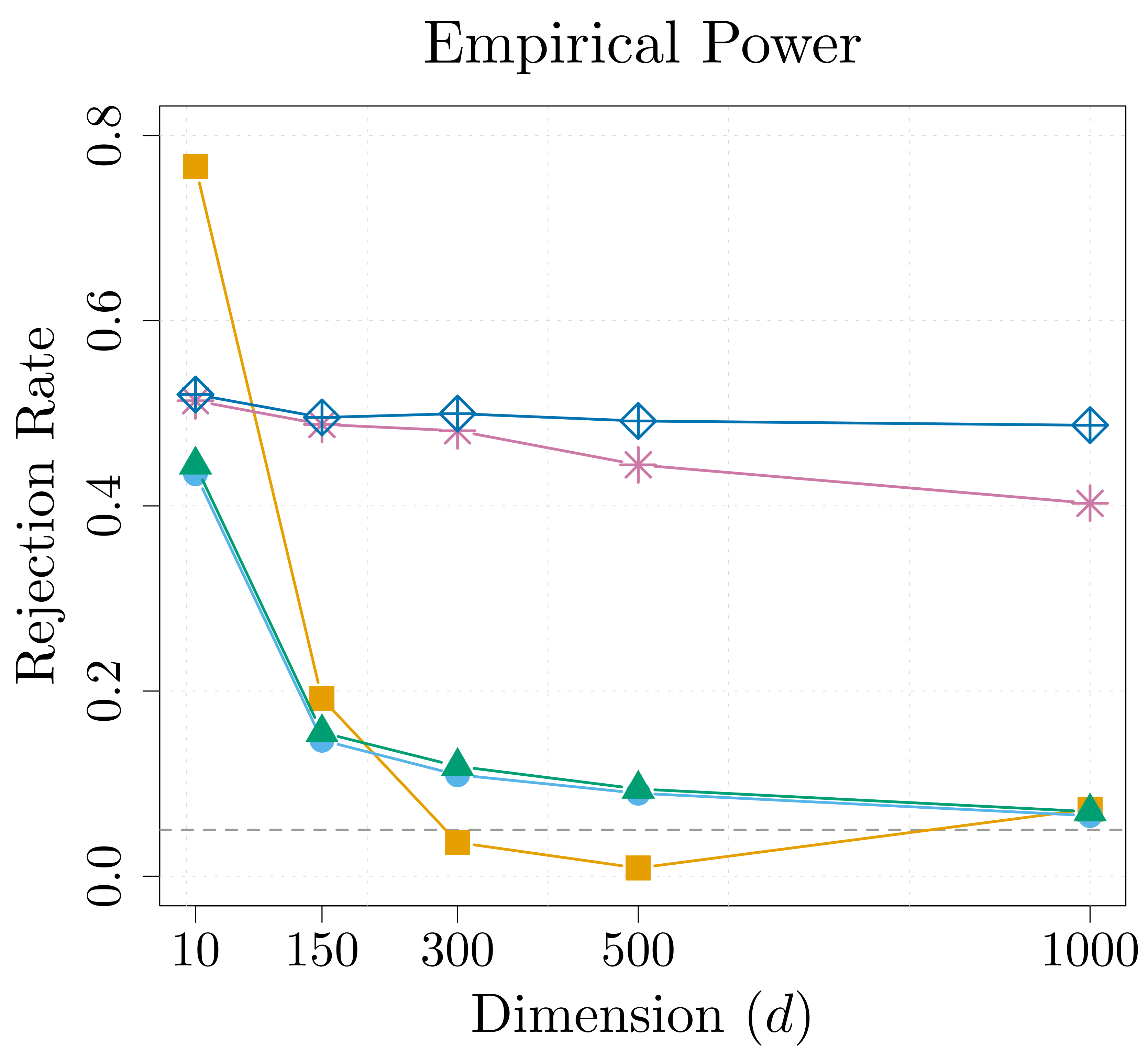


$$\varepsilon^\star = \sqrt{\frac{1 \vee \log(\tau)}{n}}$$

# Empirical Results

# Power and Validity in High-Dimensional Settings




- LOO: Zhang et al. (2024)
- Bonferroni: multiple correction
- csranks: Mogstad et al. (2024)
- DA-plug: plug-in $\widehat{s}$
- DA-adj: noise-adjusted $\widehat{s}$

- Null Setting

$\mu = (0, 0, 1, \ldots, 1)^\top$

$\Sigma_{11} = \Sigma_{22} = 1$ & $\Sigma_{33} = \ldots = \Sigma_{dd} = 20$

- Alternative Setting

$\mu = (0.15, 0, 1, \ldots, 1)^\top$

$\Sigma_{11} = \Sigma_{22} = 1$ & $\Sigma_{33} = \ldots = \Sigma_{dd} = 20$

# Power under Various Settings

| Method | $\boldsymbol{\mu}^{(a)}$ + unequal variance | | | $\boldsymbol{\mu}^{(b)}$ + unequal variance | | | $\boldsymbol{\mu}^{(c)}$ + unequal variance | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\rho = 0$ | $\rho = 0.4$ | $\rho = 0.8$ | $\rho = 0$ | $\rho = 0.4$ | $\rho = 0.8$ | $\rho = 0$ | $\rho = 0.4$ | $\rho = 0.8$ |
| LOO | 0.084 | 0.115 | 0.380 | 0.000 | 0.001 | 0.181 | **0.258** | **0.351** | **0.703** |
| Bonferroni | 0.171 | 0.130 | 0.055 | 0.166 | 0.103 | 0.030 | 0.017 | 0.006 | 0.003 |
| csranks | **0.184** | **0.381** | **0.962** | 0.162 | 0.363 | 0.961 | 0.019 | 0.041 | 0.223 |
| MCS | 0.004 | 0.002 | 0.004 | 0.000 | 0.000 | 0.000 | 0.140 | 0.156 | 0.166 |
| DA-plug | 0.049 | 0.052 | 0.042 | 0.062 | 0.067 | 0.059 | 0.098 | 0.128 | 0.202 |
| DA-plug$^{\times 10}$ | 0.050 | 0.052 | 0.050 | 0.080 | 0.080 | 0.073 | 0.125 | 0.145 | 0.240 |
| DA-adj | 0.122 | 0.259 | 0.841 | 0.217 | 0.384 | 0.916 | 0.135 | 0.188 | 0.462 |
| DA-adj$^{\times 10}$ | 0.160 | 0.343 | 0.946 | **0.294** | **0.517** | **0.982** | 0.164 | 0.251 | 0.605 |

# Power under Various Settings

| Method | $\mu^{(a)}$ + unequal variance | | | $\mu^{(b)}$ + unequal variance | | | $\mu^{(c)}$ + unequal variance | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\rho = 0$ | $\rho = 0.4$ | $\rho = 0.8$ | $\rho = 0$ | $\rho = 0.4$ | $\rho = 0.8$ | $\rho = 0$ | $\rho = 0.4$ | $\rho = 0.8$ |
| LOO | 0.084 | 0.115 | 0.380 | 0.000 | 0.001 | 0.181 | **0.258** | **0.351** | **0.703** |
| Bonferroni | 0.171 | 0.130 | 0.055 | 0.166 | 0.103 | 0.030 | 0.017 | 0.006 | 0.003 |
| csranks | **0.184** | **0.381** | **0.962** | 0.162 | 0.363 | 0.961 | 0.019 | 0.041 | 0.223 |
| MCS | 0.004 | 0.002 | 0.004 | 0.000 | 0.000 | 0.000 | 0.140 | 0.156 | 0.166 |
| DA-plug | 0.049 | 0.052 | 0.042 | 0.062 | 0.067 | 0.059 | 0.098 | 0.128 | 0.202 |
| DA-plug$^{\times 10}$ | 0.050 | 0.052 | 0.050 | 0.080 | 0.080 | 0.073 | 0.125 | 0.145 | 0.240 |
| DA-adj | 0.122 | 0.259 | 0.841 | 0.217 | 0.384 | 0.916 | 0.135 | 0.188 | 0.462 |
| DA-adj$^{\times 10}$ | 0.160 | 0.343 | 0.946 | **0.294** | **0.517** | **0.982** | 0.164 | 0.251 | 0.605 |

- DA-plug: plug-in $\hat{s}$
- DA-adj: noise-adjusted $\hat{s}$

- DA-plug$^{\times 10}$: 10 splits + average
- DA-adj$^{\times 10}$: 10 splits + average

Guo and Shah (2025)

1. **Multiple splits** improve the power
2. **Noise-adjusted** version performs better

# Power under Various Settings

| Method | $\mu^{(a)}$ + unequal variance $\rho=0$ | $\rho=0.4$ | $\rho=0.8$ | $\mu^{(b)}$ + unequal variance $\rho=0$ | $\rho=0.4$ | $\rho=0.8$ | $\mu^{(c)}$ + unequal variance $\rho=0$ | $\rho=0.4$ | $\rho=0.8$ |
|---|---|---|---|---|---|---|---|---|---|
| LOO | 0.084 | 0.115 | 0.380 | 0.000 | 0.001 | 0.181 | **0.258** | **0.351** | **0.703** |
| Bonferroni | 0.171 | 0.130 | 0.055 | 0.166 | 0.103 | 0.030 | 0.017 | 0.006 | 0.003 |
| csranks | **0.184** | **0.381** | **0.962** | 0.162 | 0.363 | 0.961 | 0.019 | 0.041 | 0.223 |
| MCS | 0.004 | 0.002 | 0.004 | 0.000 | 0.000 | 0.000 | 0.140 | 0.156 | 0.166 |
| DA-plug | 0.049 | 0.052 | 0.042 | 0.062 | 0.067 | 0.059 | 0.098 | 0.128 | 0.202 |
| DA-plug$^{\times 10}$ | 0.050 | 0.052 | 0.050 | 0.080 | 0.080 | 0.073 | 0.125 | 0.145 | 0.240 |
| DA-adj | 0.122 | 0.259 | 0.841 | 0.217 | 0.384 | 0.916 | 0.135 | 0.188 | 0.462 |
| DA-adj$^{\times 10}$ | 0.160 | 0.343 | 0.946 | **0.294** | **0.517** | **0.982** | 0.164 | 0.251 | 0.605 |

DA-adj$^{\times 10}$ achieves **the best or second-best** performance
(with only a **small** margin)

**Mean Structures**

$\mu^{(a)} = (0.1, 0, 0.1, \ldots, 0.1)^\top$ $\quad$ $\mu^{(b)} = (0.1, 0.019, \ldots, 0.99, 1)^\top$

# Power under Various Settings

| Method | $\mu^{(a)}$ + unequal variance | | | $\mu^{(b)}$ + unequal variance | | | $\mu^{(c)}$ + unequal variance | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\rho = 0$ | $\rho = 0.4$ | $\rho = 0.8$ | $\rho = 0$ | $\rho = 0.4$ | $\rho = 0.8$ | $\rho = 0$ | $\rho = 0.4$ | $\rho = 0.8$ |
| LOO | 0.084 | 0.115 | 0.380 | 0.000 | 0.001 | 0.181 | **0.258** | **0.351** | **0.703** |
| Bonferroni | 0.171 | 0.130 | 0.055 | 0.166 | 0.103 | 0.030 | 0.017 | 0.006 | 0.003 |
| csranks | **0.184** | **0.381** | **0.962** | 0.162 | 0.363 | 0.961 | 0.019 | 0.041 | 0.223 |
| MCS | 0.004 | 0.002 | 0.004 | 0.000 | 0.000 | 0.000 | 0.140 | 0.156 | 0.166 |
| DA-plug | 0.049 | 0.052 | 0.042 | 0.062 | 0.067 | 0.059 | 0.098 | 0.128 | 0.202 |
| DA-plug$^{\times 10}$ | 0.050 | 0.052 | 0.050 | 0.080 | 0.080 | 0.073 | 0.125 | 0.145 | 0.240 |
| DA-adj | 0.122 | 0.259 | 0.841 | 0.217 | 0.384 | 0.916 | 0.135 | 0.188 | 0.462 |
| DA-adj$^{\times 10}$ | 0.160 | 0.343 | 0.946 | **0.294** | **0.517** | **0.982** | 0.164 | 0.251 | 0.605 |

LOO performs **best** and DA-adj$^{\times 10}$ performs **second best**

But LOO **does not** control the type I error rate

Mean Structure

$\mu^{(c)} = (0.05, 0, 0, 0, 10, \ldots, 10)^\top$

# Power under Various Settings

| Method | $\mu^{(a)}$ + unequal variance | | | $\mu^{(b)}$ + unequal variance | | | $\mu^{(c,0)}$ + unequal variance | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\rho = 0$ | $\rho = 0.4$ | $\rho = 0.8$ | $\rho = 0$ | $\rho = 0.4$ | $\rho = 0.8$ | $\rho = 0$ | $\rho = 0.4$ | $\rho = 0.8$ |
| LOO | 0.084 | 0.115 | 0.380 | 0.000 | 0.001 | 0.181 | 0.070 | 0.064 | 0.065 |
| Bonferroni | 0.171 | 0.130 | 0.055 | 0.166 | 0.103 | 0.030 | 0.002 | 0.001 | 0.001 |
| csranks | **0.184** | **0.381** | **0.962** | 0.162 | 0.363 | 0.961 | 0.002 | 0.001 | 0.002 |
| MCS | 0.004 | 0.002 | 0.004 | 0.000 | 0.000 | 0.000 | 0.042 | 0.042 | 0.034 |
| DA-plug | 0.049 | 0.052 | 0.042 | 0.062 | 0.067 | 0.059 | 0.048 | 0.052 | 0.048 |
| DA-plug$^{\times 10}$ | 0.050 | 0.052 | 0.050 | 0.080 | 0.080 | 0.073 | 0.053 | 0.046 | 0.047 |
| DA-adj | 0.122 | 0.259 | 0.841 | 0.217 | 0.384 | 0.916 | 0.054 | 0.052 | 0.050 |
| DA-adj$^{\times 10}$ | 0.160 | 0.343 | 0.946 | **0.294** | **0.517** | **0.982** | 0.053 | 0.049 | 0.050 |

LOO performs **best** and DA-adj$^{\times 10}$ performs **second best**

But LOO **does not** control the type I error rate

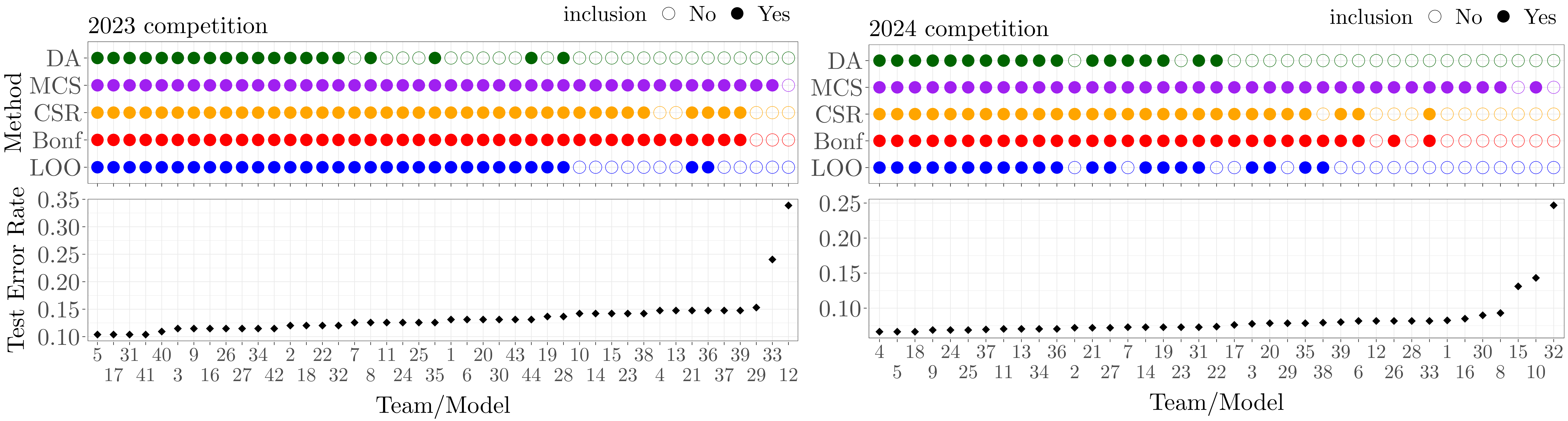
Mean Structure

$\boldsymbol{\mu}^{(c)} = (0.05, 0, 0, 0, 10, \ldots, 10)^\top$

# Real-World Data

- We revisit **CMU 36-462 prediction competition** analyzed by Zhang et al. (2024)
  - Spring 2023: $n = 183, p = 44$
  - Spring 2024: $n = 1246, p = 39$
- $X_{i,k} \in \{0,1\}$: correct/incorrect prediction by team $k$ at test point $i$
- $\mu_k$: the population prediction risk of team $k$ (*unknown*)
- Methods to compare
  - Bonferroni correction (**Bonf**)
  - Hansen et al. (2011) (**MCS**)
  - Mogstad et al. (2024) (**CSR**)
  - Zhang et al. (2024) (**LOO**)

# Real-World Data




- Colored points indicate the indices included in each confidence set
- Our method (DA-adj$^{\times 10}$) produces confidence sets with smallest cardinality

# Summary

- We have introduced a DA method for **high-dimensional argmin inference** problem based on **sample splitting** and **studentization**
- The proposed method achieves the **locally minimax separation rate** and adapts to the intrinsic difficulty of the problem characterized by the **confusion set**
- We have demonstrated its **strong empirical performance** under various settings

Thank you!

# Coverage Guarantees

## 1. Pointwise coverage

**Each** $r \in \Theta$ is included in $\widehat{\Theta}$ with high probability:

$$\inf_{P \in \mathscr{P}} \inf_{r \in \Theta(P)} P\left(r \in \widehat{\Theta}\right) \geq 1 - \alpha$$

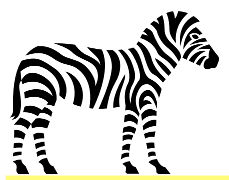
- **Less** demanding → **smaller** expected set size
- **Higher** power but **weaker** protection

## 2. Uniform coverage

**The entire set** $\Theta$ is contained in $\widehat{\Theta}$ with high probability:

$$\inf_{P \in \mathscr{P}} P\left(\Theta \subseteq \widehat{\Theta}\right) \geq 1 - \alpha$$

- **More** demanding → **larger** expected set size
- **Stronger** protection but **more** conservative



$\mathscr{P}$: class of distributions
$\alpha$: target error level

# Future directions

- General **rank-k** inference
- Confidence sets for $\mu_{(k)}$
- **Computationally efficient** multiple splitting approach