Locally Minimax Optimal and Dimension-Agnostic Discrete Argmin Inference

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Joint work with

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Problem Setup: Discrete Argmin Inference

- Suppose $X_1, \ldots, X_{2n} \overset{\text{i.i.d.}}{\sim} P$ in $\mathbb{R}^d$ with mean $\mu = (\mu_1, \ldots, \mu_d)^\top$ and let

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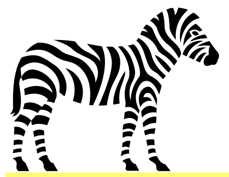
Each $r \in \Theta$ is included in $\widehat{\Theta}$ with high probability:

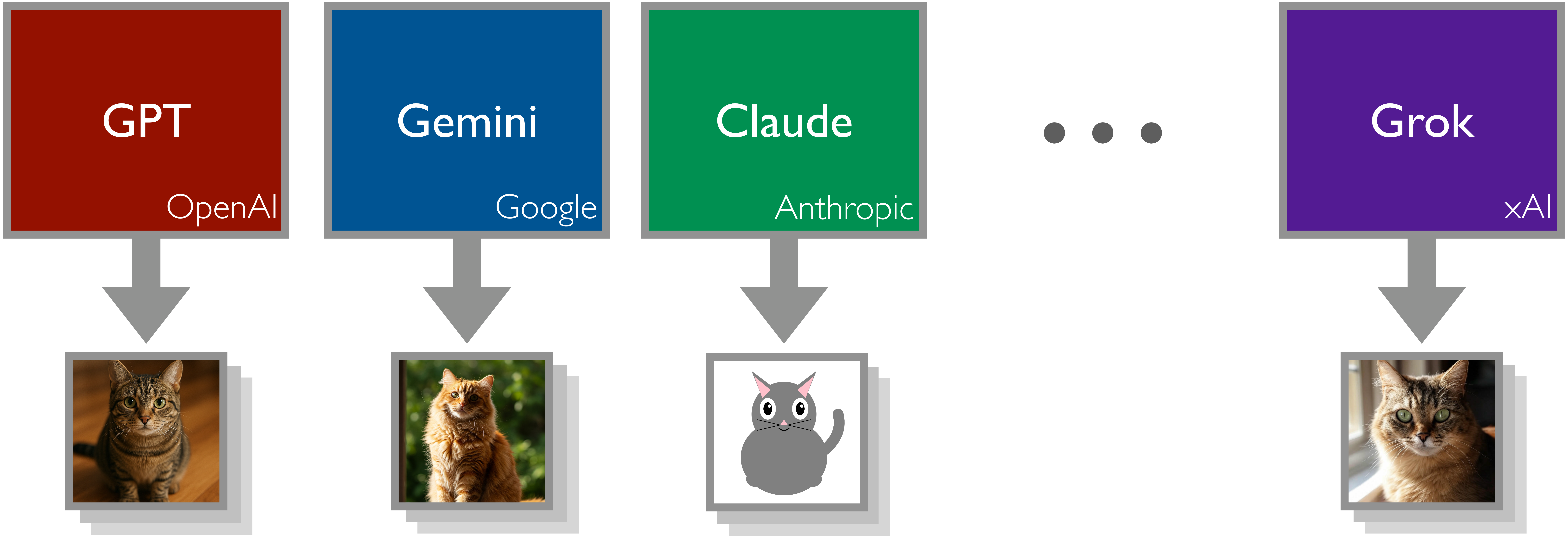
$$\inf_{P \in \mathscr{P}} \inf_{r \in \Theta(P)} P\left(r \in \widehat{\Theta}\right) \geq 1 - \alpha$$



$\mathscr{P}$: class of distributions
α: target error level

Modern Application

Prompt: "Generate cat images"



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- But **the population risk is unknown** → **the empirical risk**
- We must **account for statistical uncertainty** to determine which models are plausibly **optimal with statistical confidence**

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→ Sample Splitting + Studentization (Kim and Ramdas, 2024)

Dimension-Agnostic Argmin Inference

Formal (Primal) Goal.

- We seek a **dimension-agnostic confidence set** $\widehat{\Theta}$ for Θ that such that

$$\liminf_{n\to\infty} \inf_{P\in\mathscr{P}} \inf_{r\in\Theta(P)} P\left(r \in \widehat{\Theta}\right) \geq 1-\alpha, \quad \text{regardless of the sequence } (d_n)_{n=1}^{\infty}$$

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- We also seek a **confidence set** $\widehat{\Theta}$ for Θ such that its **expected cardinality** is small and **ideally optimal**

Formal (Dual) Goal.

- Given some fixed $r \in [d]$, consider the **null** and **alternative** hypotheses:

$$\boxed{H_0 : r \in \Theta \quad \text{versus} \quad H_1 : r \notin \Theta}$$

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- We seek a **dimension-agnostic test** ψ_r such that

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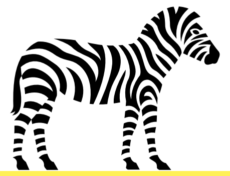

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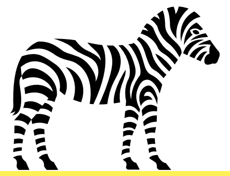
- We can construct a **DA confidence set** using **DA tests**



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DA tests yield a DA confidence set via duality

- Suppose that each ψ_k satisfies the **dimension-agnostic (DA)** property:

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- We run d such DA tests $\psi_1, \ldots, \psi_d$ and let

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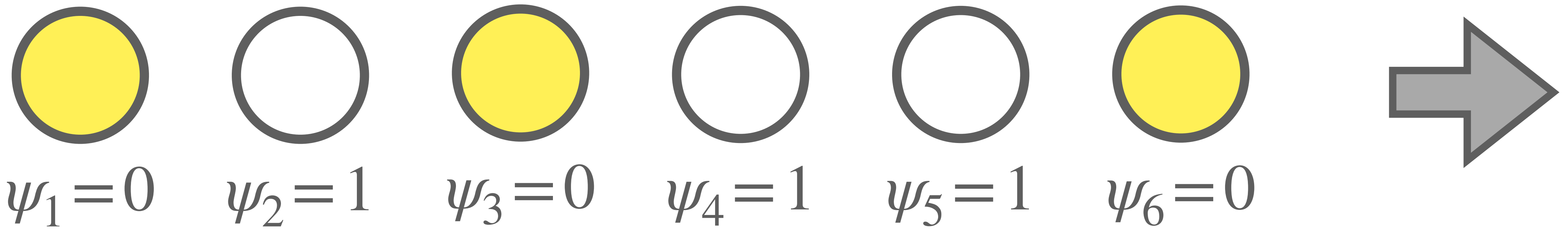

$$\widehat{\Theta} = \{1,3,6\}$$

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which results in the DA confidence set

$$\liminf_{n\to\infty} \inf_{P\in\mathscr{P}} \inf_{r\in\Theta(P)} P\left(r \in \widehat{\Theta}\right) \ge 1-\alpha, \quad \text{regardless of the sequence } (d_n)_{n=1}^{\infty}$$

Procedures

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$$\frac{\overline{X}_r - \overline{X}_s}{\hat{\sigma}_{r,s}} > z_{1-\alpha}$$

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- Hard to calibrate $c_{1-\alpha}$ in **high-dimensions**

Dimension-Agnostic Argmin Test

💡**Idea:** use $\mathscr{D}_1$ to estimate s and use $\mathscr{D}_2$ for inference

Step I (Sample Splitting)

- Split the data into two

$$\mathscr{D}_1 \quad \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{n-1} \\ X_n \end{bmatrix} \qquad \mathscr{D}_2 \quad \begin{bmatrix} X_{n+1} \\ X_{n+2} \\ \vdots \\ X_{2n-1} \\ X_{2n} \end{bmatrix}$$

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- Based on $\mathscr{D}_1$, compute

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2. Noise-adjusted version

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- Given $\hat{s} \in [d] \backslash \{r\}$, compute a one-sided t-statistic

$$T = \frac{\overline{X}_r^{(1)} - \overline{X}_{\hat{s}}^{(1)}}{\hat{\sigma}_{r,\hat{s}}^{(1)}}$$

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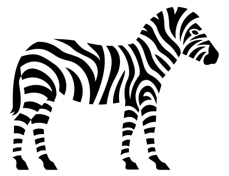
Step IV (Decision)

- **Reject** the null if

$$T > z_{1-\alpha}$$

- **Accept** the null o.w.



$\Phi(z_{1-\alpha}) = 1 - \alpha$

Theoretical Properties

1. Asymptotic Validity
2. Power Analysis

Asymptotic Validity

$X = (X^{(1)}, \ldots, X^{(d)})$

Assumption (Truncated 2nd Moment Condition)

Let $W_k := (X^{(r)} - \mu_r) - (X^{(k)} - \mu_k)$ and

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- **Arbitrary dependence** among the components except $X^{(r)}$

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Theorem Kim and Ramdas (2025)

Let $\mathscr{P}_{0,r}$ be the class of null distributions with $H_0 : r \in \Theta$ that satisfy the **truncated 2nd moment condition.** Then the **DA test** is asymptotically **valid as**

$$\limsup_{n \to \infty} \sup_{P \in \mathscr{P}_{0,r}} P\big(\psi_r = 1\big) \leq \alpha, \quad \text{regardless of the sequence } (d_n)_{n=1}^{\infty}$$

Theoretical Properties

1. Asymptotic Validity
2. Power Analysis

Confusion Set

- The problem **difficulty** depends on the cardinality of a **confusion set** $\mathbb{C}_r$

$$\mathbb{C}_r := \left\{ k \in [d] \setminus \{r\} : \frac{\mu_r - \mu_\star}{2} \leq \mu_k - \mu_\star \leq C_n \sqrt{\frac{\log(d)}{n}} \right\}$$

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When μ_k is far from $\mu_\star$, it is unlikely that $\hat{s} = k$

$\therefore$ **Well-separated set**

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- **Intuition for $\mathscr{B}^c$**

$$\frac{\mu_r - \mu_\star}{2} > \mu_k - \mu_\star \iff \mu_r - \mu_k > \frac{\mu_r - \mu_\star}{2}$$

$\therefore$ Comparable signal

Power Analysis

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- Define a class of **alternative** distributions

$$\mathscr{P}_{1,r}(\varepsilon;\tau) := \left\{P \in \mathscr{P} : \mu_r - \mu_\star \geq \varepsilon \text{ and } |\mathbb{C}_r| = \tau\right\}$$

Power Analysis

- Let $\mathscr{P}$ be a collection of **sub-Gaussian** distributions with σ^2

- Define a class of **alternative** distributions

$$\mathscr{P}_{1,r}(\varepsilon; \tau) := \left\{ P \in \mathscr{P} : \mu_r - \mu_\star \geq \varepsilon \text{ and } |\mathbb{C}_r| = \tau \right\}$$

- The **critical radius** $\varepsilon^\star$ is defined as

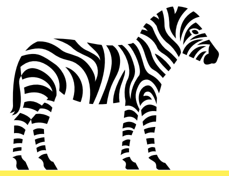
$$\varepsilon^\star = \varepsilon^\star(\tau) = \sqrt{\frac{1 \vee \log(\tau)}{n}}$$

Power Analysis

Theorem Kim and Ramdas (2025)

For any τ, suppose that $\varepsilon \gg \varepsilon^\star$. Then the asymptotic uniform power of the **DA test** is equal to one:

$$\lim_{n\to\infty} \inf_{P\in\mathscr{P}_{1,r}(\varepsilon;\tau)} P\big(\psi_r = 1\big) = 1$$



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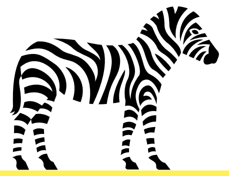

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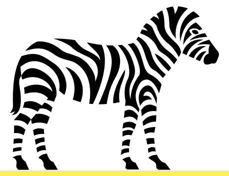
Question. Can we further **improve** the **separation rate**?



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Local Minimax Optimality

Theorem Kim and Ramdas (2025)

Let Ψ_α be the set of all asymptotic level-α tests over $\mathscr{P}_{0,r}$

$$\Psi_\alpha := \left\{ \psi : \limsup_{n\to\infty} \sup_{P\in\mathscr{P}_{0,r}} P(\psi = 1) \leq \alpha \right\}$$

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If $\varepsilon \ll \varepsilon^\star$, then the **asymptotic type II error** is at least β:

$$\liminf_{n\to\infty} \inf_{\psi\in\Psi_\alpha} \sup_{P\in\mathscr{P}_{1,r}(\varepsilon;\tau)} P(\psi = 0) \ge \beta$$

$$\varepsilon^\star = \sqrt{\frac{1 \vee \log(\tau)}{n}}$$

Summary

- We have introduced a DA method for **high-dimensional argmin inference** problem based on **sample splitting** and **studentization**
- The proposed method achieves the **locally minimax separation rate** and adapts to the intrinsic difficulty of the problem characterized by the **confusion set**
- We have demonstrated its **strong empirical performance** under various settings

Thank you!

Coverage Guarantees

1. Pointwise coverage

Each $r \in \Theta$ is included in $\widehat{\Theta}$ with high probability:

$$\inf_{P \in \mathscr{P}} \inf_{r \in \Theta(P)} P\left(r \in \widehat{\Theta}\right) \geq 1 - \alpha$$

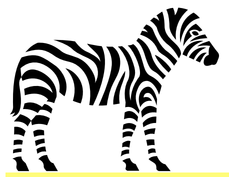
- **Less** demanding → **smaller** expected set size
- **Higher** power but **weaker** protection

2. Uniform coverage

The entire set Θ is contained in $\widehat{\Theta}$ with high probability:

$$\inf_{P \in \mathscr{P}} P\left(\Theta \subseteq \widehat{\Theta}\right) \geq 1 - \alpha$$

- **More** demanding → **larger** expected set size
- **Stronger** protection but **more** conservative



$\mathscr{P}$: class of distributions
α: target error level

Related Work

- **Classical work:** Bechhofer (1954), Gupta (1956, 1965), Futschik and Pflug (1995) etc

  → rely on parametric models, independence between coordinates, absence of ties

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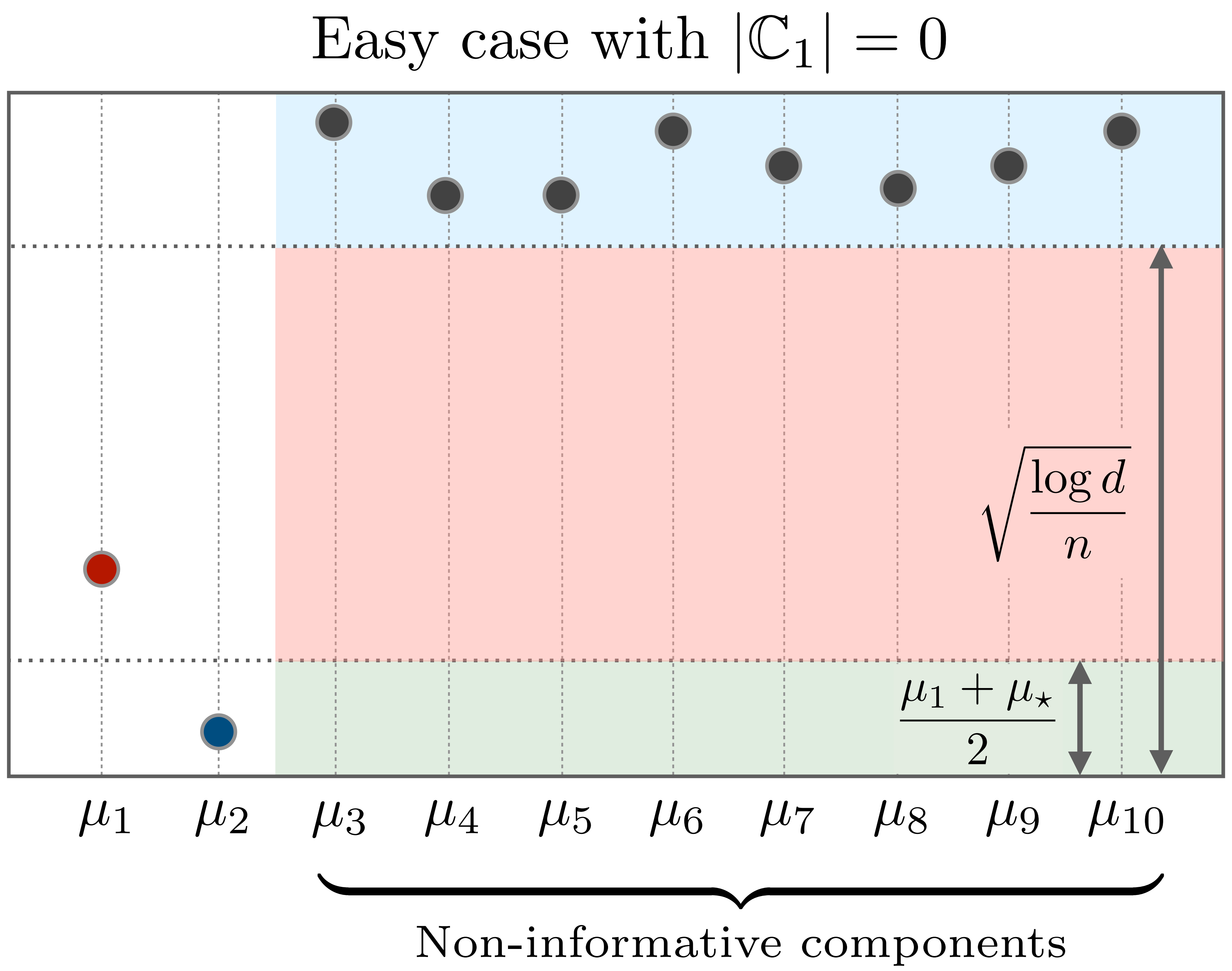
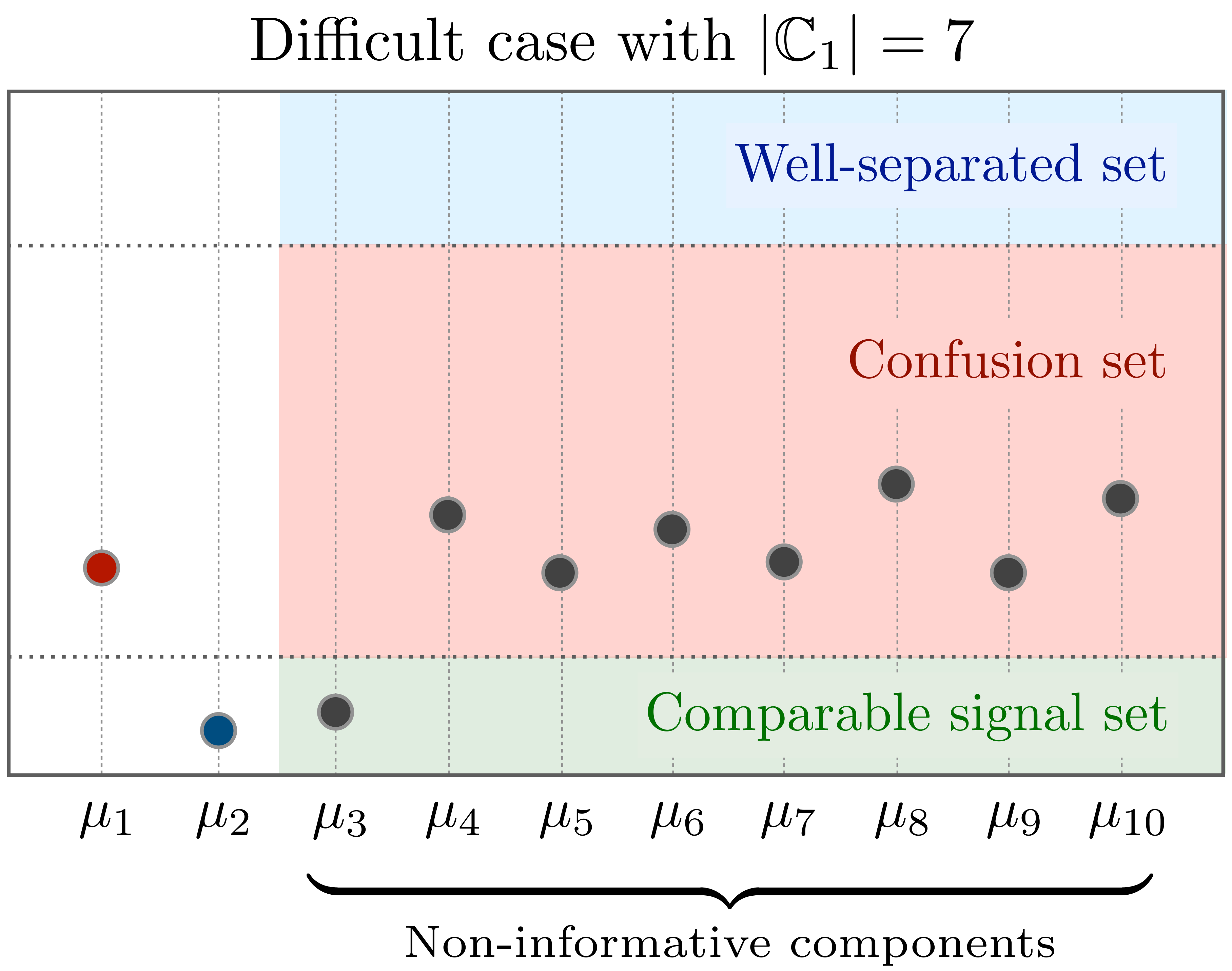
- **Zhang et al. (2024):** introduce a cross-validation + privacy approach for argmin inference

  → require careful tuning and fall short of minimax optimality

Confusion Set

- Set $\mu_r = \mu_1$ and $\mu_\star = \mu_2$





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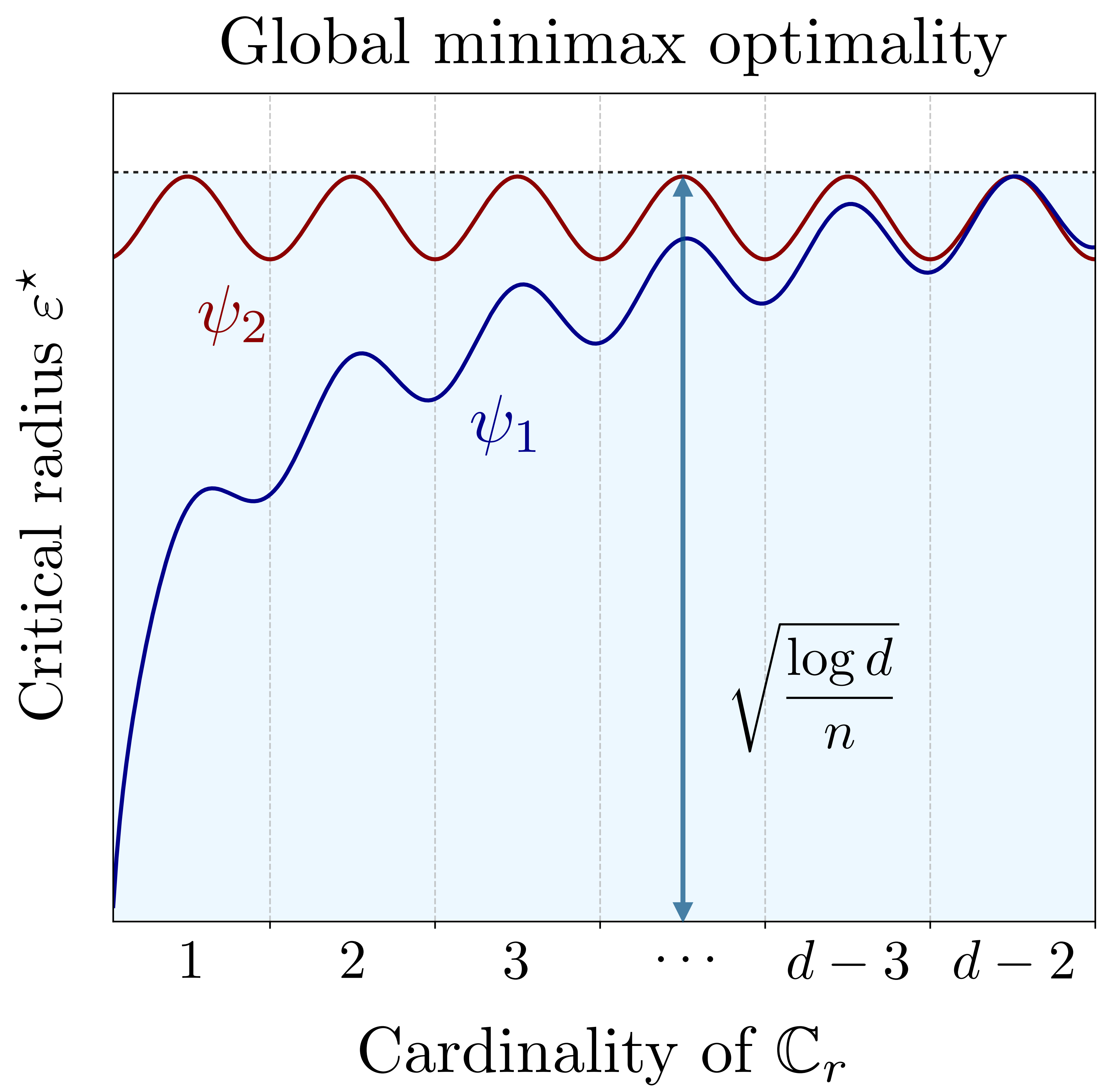
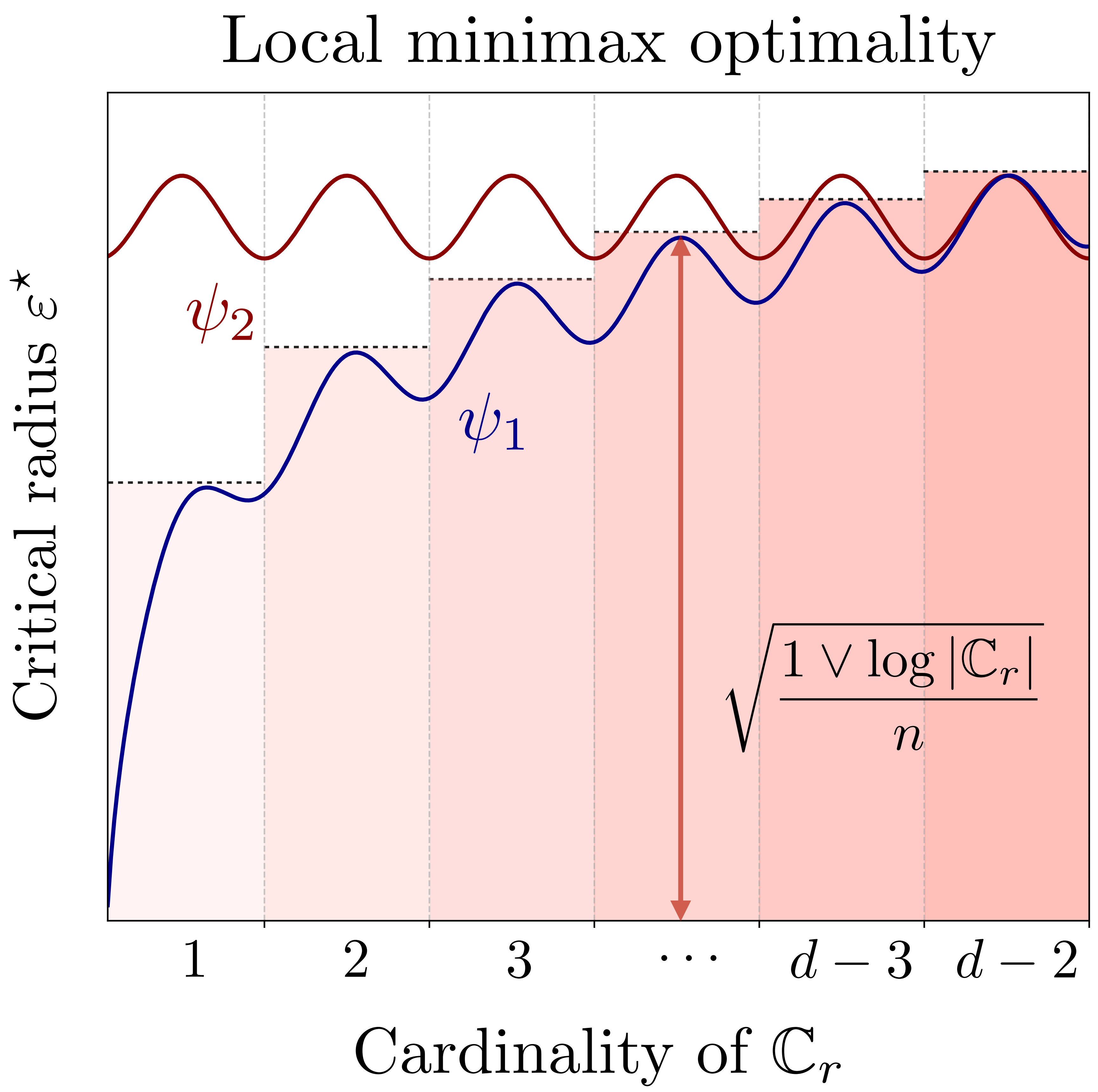
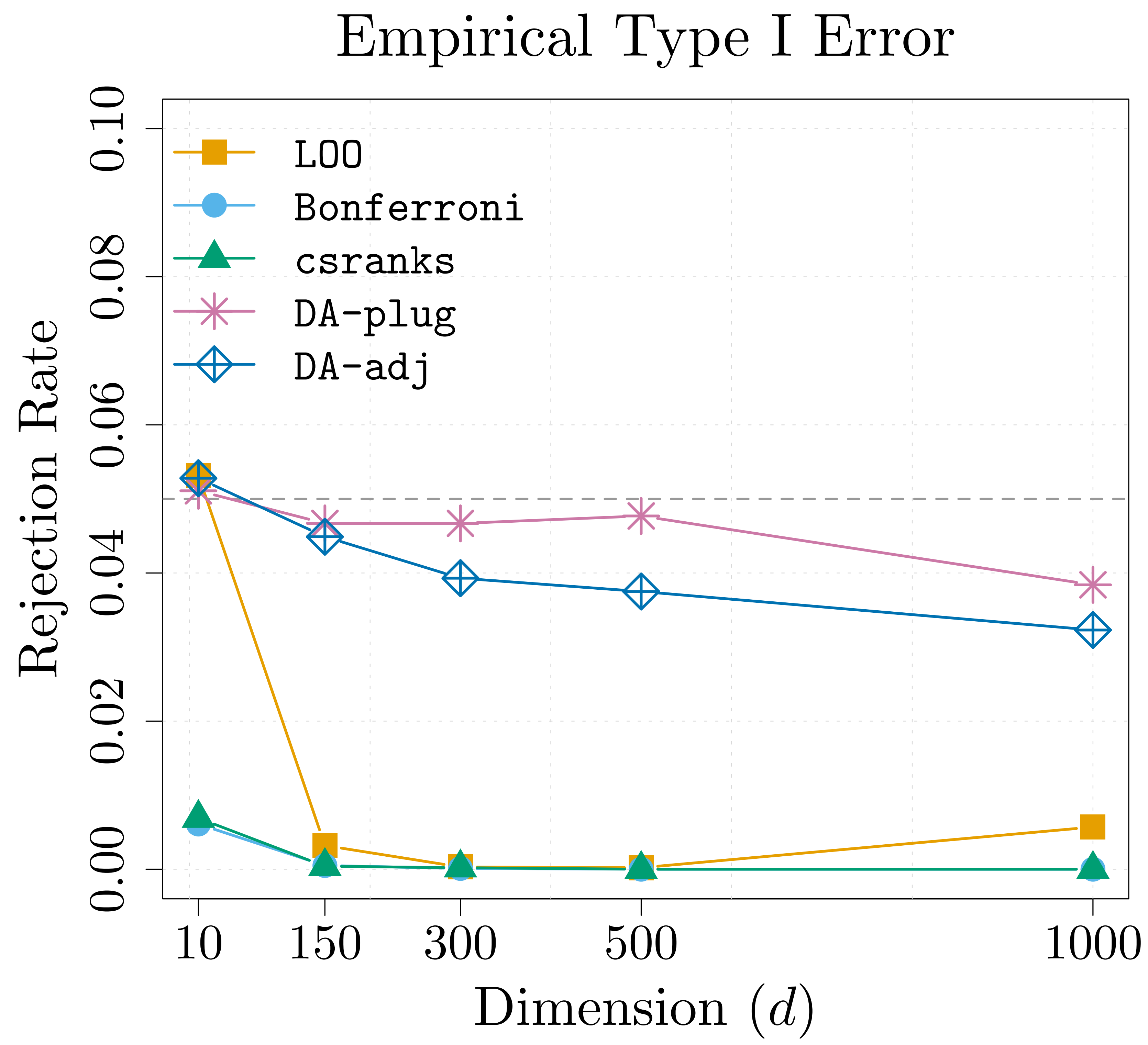
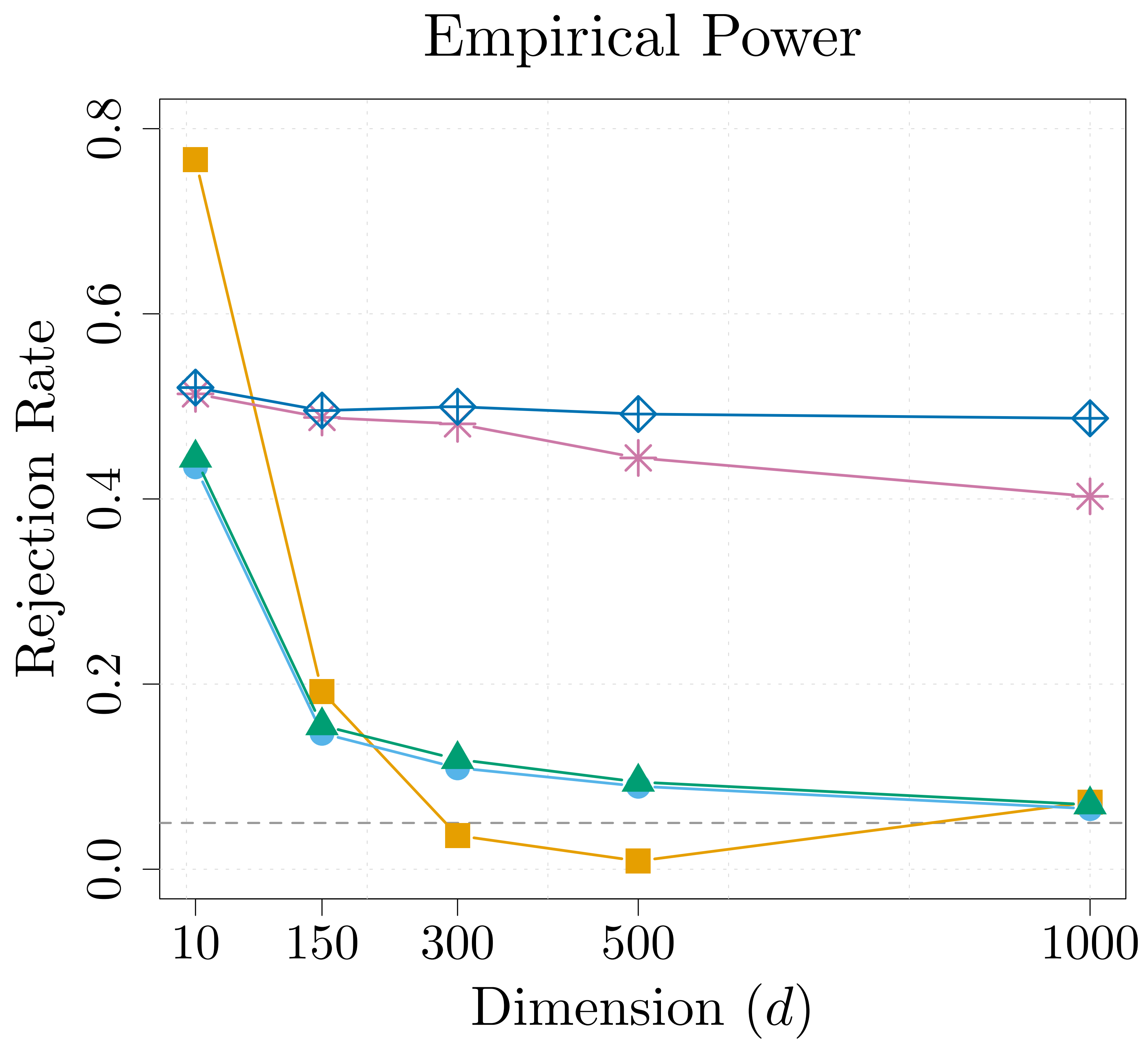




Empirical Results

Power and Validity in High-Dimensional Settings





- **LOO:** Zhang et al. (2024)
- **Bonferroni:** multiple correction
- **csranks:** Mogstad et al. (2024)
- **DA-plug:** plug-in $\hat{s}$
- **DA-adj:** noise-adjusted $\hat{s}$

- Null Setting

$\mu = (0,0,1,\ldots,1)^\top$

$\Sigma_{11} = \Sigma_{22} = 1 \ \& \ \Sigma_{33} = \ldots = \Sigma_{dd} = 20$

- Alternative Setting

$\mu = (0.15,0,1,\ldots,1)^\top$

$\Sigma_{11} = \Sigma_{22} = 1 \ \& \ \Sigma_{33} = \ldots = \Sigma_{dd} = 20$

Power under Various Settings

Method	$\boldsymbol{\mu}^{(a)}$ + unequal variance			$\boldsymbol{\mu}^{(b)}$ + unequal variance			$\boldsymbol{\mu}^{(c)}$ + unequal variance		
	$\rho = 0$	$\rho = 0.4$	$\rho = 0.8$	$\rho = 0$	$\rho = 0.4$	$\rho = 0.8$	$\rho = 0$	$\rho = 0.4$	$\rho = 0.8$
`LOO`	0.084	0.115	0.380	0.000	0.001	0.181	**0.258**	**0.351**	**0.703**
`Bonferroni`	0.171	0.130	0.055	0.166	0.103	0.030	0.017	0.006	0.003
`csranks`	**0.184**	**0.381**	**0.962**	0.162	0.363	0.961	0.019	0.041	0.223
`MCS`	0.004	0.002	0.004	0.000	0.000	0.000	0.140	0.156	0.166
`DA-plug`	0.049	0.052	0.042	0.062	0.067	0.059	0.098	0.128	0.202
`DA-plug`$^{\times 10}$	0.050	0.052	0.050	0.080	0.080	0.073	0.125	0.145	0.240
`DA-adj`	0.122	0.259	0.841	0.217	0.384	0.916	0.135	0.188	0.462
`DA-adj`$^{\times 10}$	0.160	0.343	0.946	**0.294**	**0.517**	**0.982**	0.164	0.251	0.605

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- DA-plug: plug-in $\hat{s}$
- DA-adj: noise-adjusted $\hat{s}$

- DA-plug$^{\times 10}$: 10 splits + average
- DA-adj$^{\times 10}$: 10 splits + average

Guo and Shah (2025)

1. **Multiple splits** improve the power
2. **Noise-adjusted** version performs better

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DA-adj$^{\times 10}$ achieves **the best or second-best** performance
(with only a **small** margin)

Mean Structures

$\mu^{(a)} = (0.1, 0, 0.1, \ldots, 0.1)^\top \qquad \mu^{(b)} = (0.1, 0.019, \ldots, 0.99, 1)^\top$

Power under Various Settings

Method	$\mu^{(a)}$ + unequal variance			$\mu^{(b)}$ + unequal variance			$\mu^{(c)}$ + unequal variance		
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LOO performs **best** and **DA-adj**$^{\times 10}$ performs **second best**

But **LOO** **does not** control the type I error rate

Mean Structure

$\boldsymbol{\mu}^{(c)} = (0.05, 0, 0, 0, 10, \ldots, 10)^{\top}$

Power under Various Settings

Method	$\mu^{(a)}$ + unequal variance			$\mu^{(b)}$ + unequal variance			$\mu^{(c,0)}$ + unequal variance		
	$\rho = 0$	$\rho = 0.4$	$\rho = 0.8$	$\rho = 0$	$\rho = 0.4$	$\rho = 0.8$	$\rho = 0$	$\rho = 0.4$	$\rho = 0.8$
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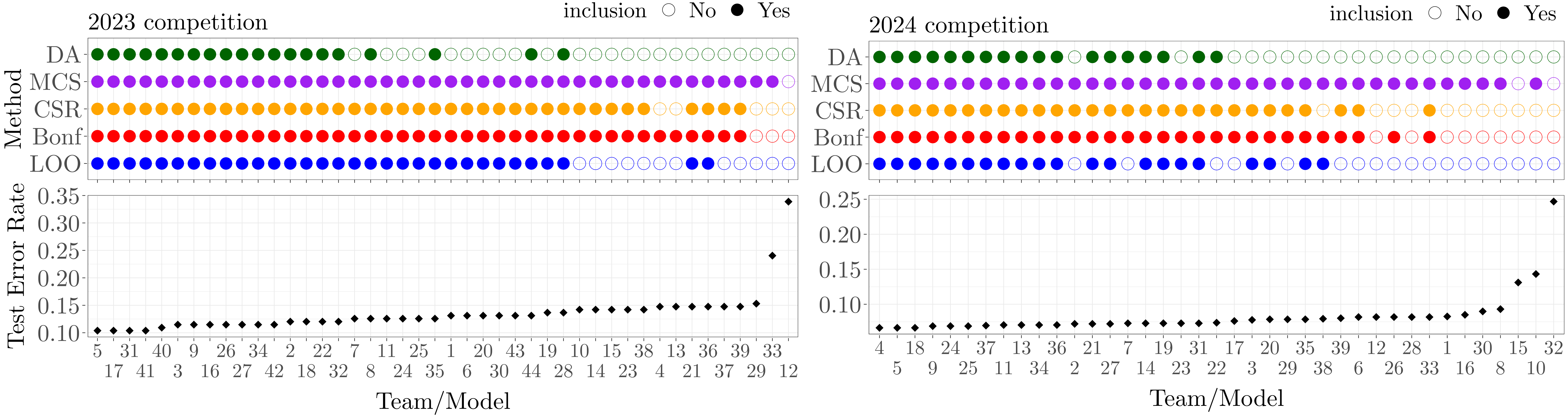
Mean Structure

$\boldsymbol{\mu}^{(c)} = (0.05, 0, 0, 0, 10, \ldots, 10)^\top$

Real-World Data

- We revisit **CMU 36-462 prediction competition** analyzed by Zhang et al. (2024)
  - Spring 2023: $n = 183, p = 44$
  - Spring 2024: $n = 1246, p = 39$
- $X_{i,k} \in \{0,1\}$: correct/incorrect prediction by team k at test point i
- μ_k: the population prediction risk of team k (*unknown*)
- Methods to compare
  - Bonferroni correction (**Bonf**)
  - Hansen et al. (2011) (**MCS**)
  - Mogstad et al. (2024) (**CSR**)
  - Zhang et al. (2024) (**LOO**)

Real-World Data



- Colored points indicate the indices included in each confidence set
- Our method (DA-adj$^{\times 10}$) produces confidence sets with smallest cardinality

Future directions

- General **rank-k** inference
- Confidence sets for $\mu_{(k)}$
- **Computationally efficient** multiple splitting approach