Robust Kernel Hypothesis Testing under Data Corruption

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*equal contributions

General Robust Testing Framework

- Space of distribution: $\mathscr{P}$ partitioned into disjoint $\mathscr{P}_0$ and $\mathscr{P}_1$

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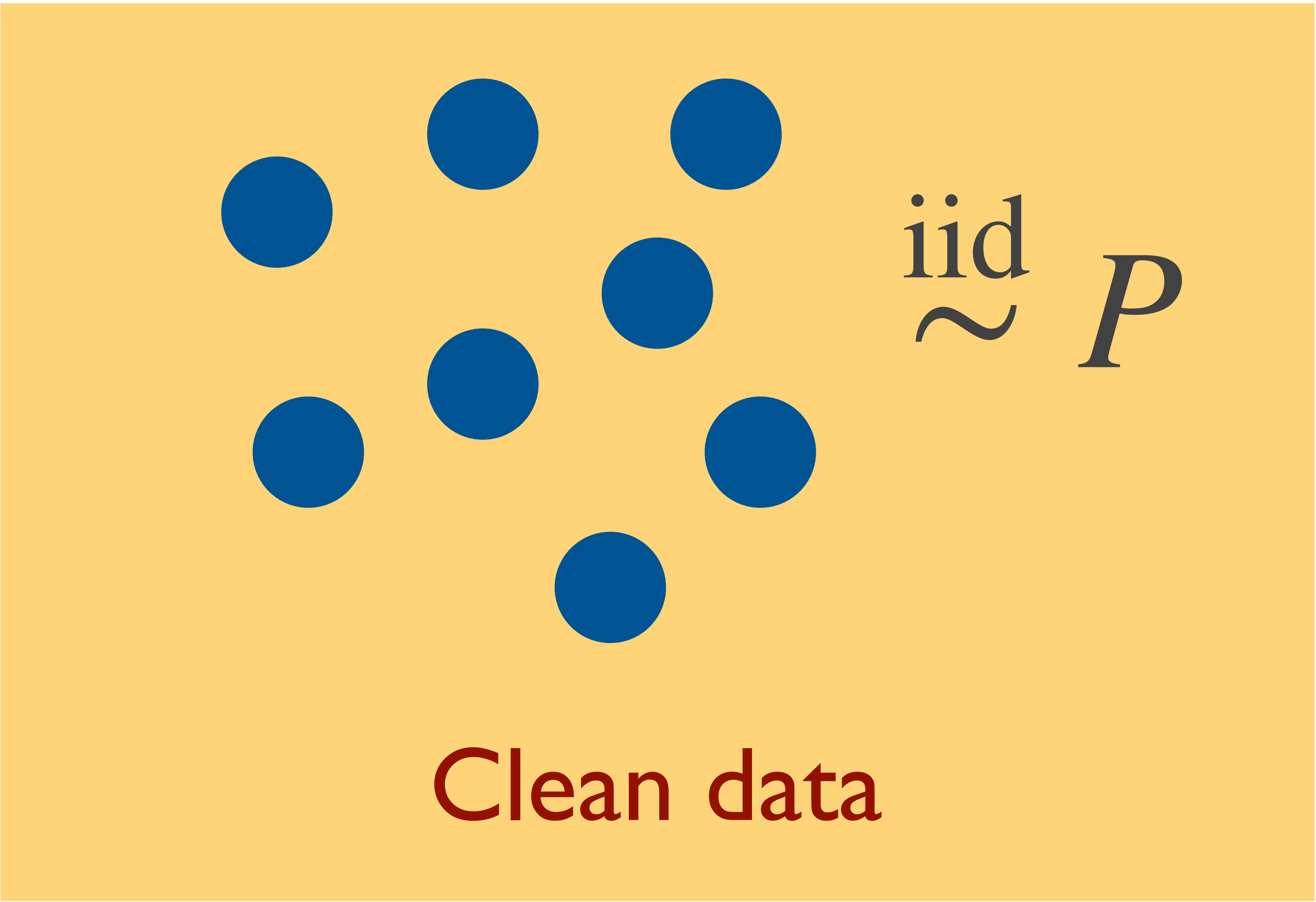
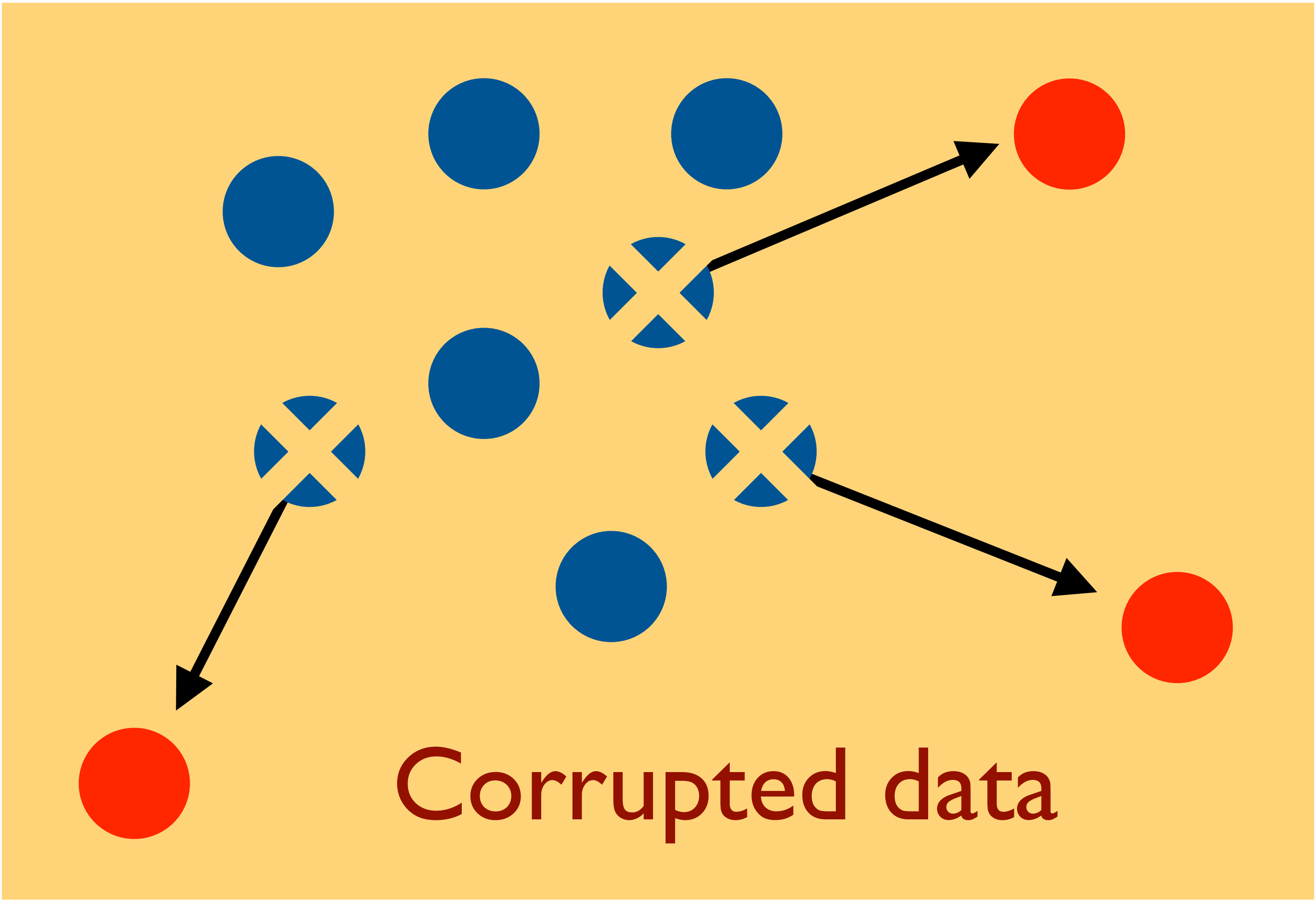
Standard framework



vs

Robust framework



Up to r samples may have been corrupted arbitrarily

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- **(Specific) Goal:** given $\tilde{X}_1, \ldots, \tilde{X}_N$ related to some fixed $P \in \mathscr{P}$, determine whether

$\mathscr{H}_0: \tilde{X}_1, \ldots, \tilde{X}_N$ are exchangeable or $\mathscr{H}_1: \tilde{X}_1, \ldots, \tilde{X}_N$ are not exchangeable

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- **Challenge:** we don't observe $\tilde{X}_1, \ldots, \tilde{X}_N$ but observe $X_1, \ldots, X_N$ where
  - Up to r samples might have been corrupted arbitrarily
  - $N - r$ samples are from P

♣ Robustness parameter r is specified by the user depending on the application

DC Procedure for Robust Testing

- Global sensitivity: maximum possible change in T when one data point is arbitrarily changed

$$\Delta_T \coloneqq \sup_{\boldsymbol{\pi} \in \boldsymbol{\Pi}_n} \sup_{\mathcal{X}_n, \mathcal{Y}_n \,:\, d_{\mathrm{ham}}(\mathcal{X}_n, \mathcal{Y}_n) \leq 1} \big| T(\mathcal{X}_n^{\boldsymbol{\pi}}) - T(\mathcal{Y}_n^{\boldsymbol{\pi}}) \big|$$

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$T_0 > q + 2r\Delta_T$

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Adjustment factor for data corruption

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Adjustment factor for data corruption

We coin this as the "DC test":
A permutation test under data corruption

DC Procedure for Robust Testing

We prove two fundamental results for the DC test

- Level: the DC test is well-calibrated non-asymptotically

$$\mathbb{P}_{P_0}\big(\text{DC rejects } \mathscr{H}_0 \mid r \text{ corrupted data}\big) \ \leq \ \alpha \quad \begin{cases} \text{for any distribution } P_0 \in \mathscr{P}_0 \\ \text{for any value } \alpha \in (0,1) \\ \text{for any } N \geq 1 \end{cases}$$

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- **Consistency:** the DC test is consistent in the sense that

$$\lim_{N\to\infty} \mathbb{P}_{P_1}\big(\text{DC rejects } \mathscr{H}_0 \mid r \text{ corrupted data}\big) = 1 \text{ for any fixed distribution } P_1 \in \mathscr{P}_1$$

whenever $\lim_{N\to\infty} \mathbb{P}_{P_1}\big(T(\mathbb{X}_n) > T(\mathbb{X}_n^{\pi}) + 4r\Delta_T\big) = 1$

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Robust Two-Sample Testing

- Two-sample problem: Given mutually independent
  - i.i.d. samples $\tilde{X}_1, \ldots, \tilde{X}_m$ from a distribution P
  - i.i.d. samples $\tilde{Y}_1, \ldots, \tilde{Y}_n$ from a distribution Q

test whether $\mathscr{H}_0: P = Q$ or $\mathscr{H}_1: P \neq Q$

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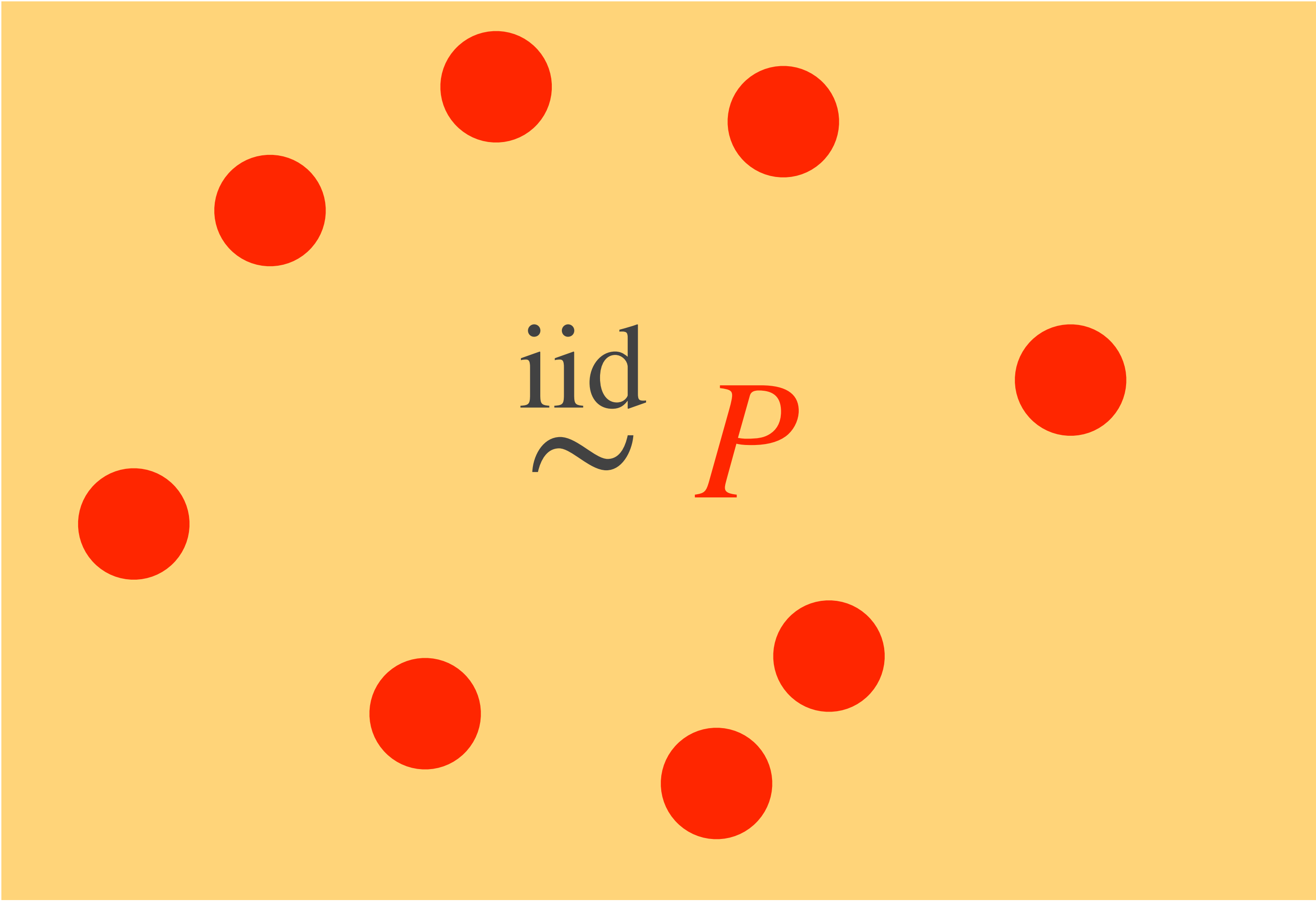
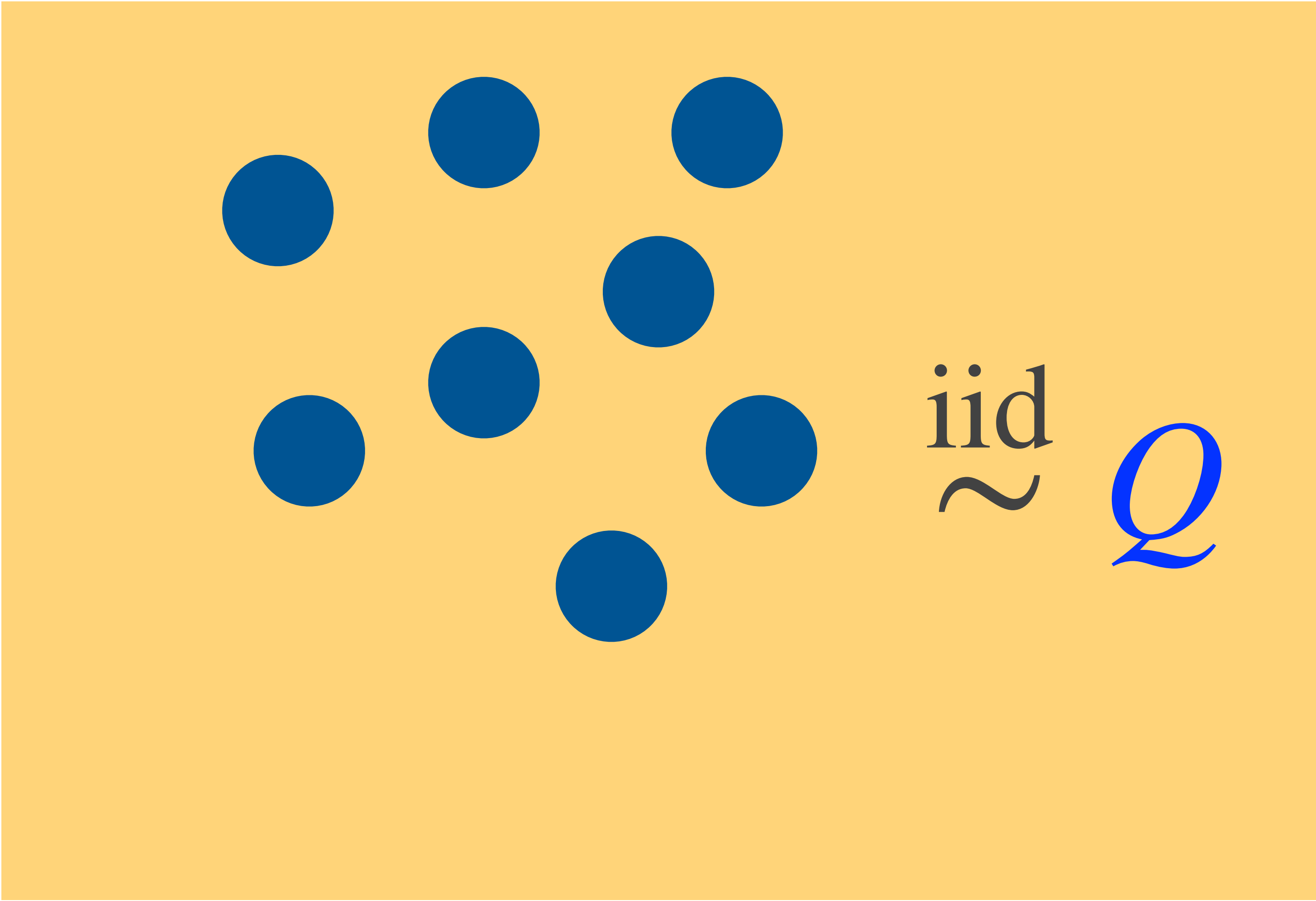


i.i.d. Samples from P



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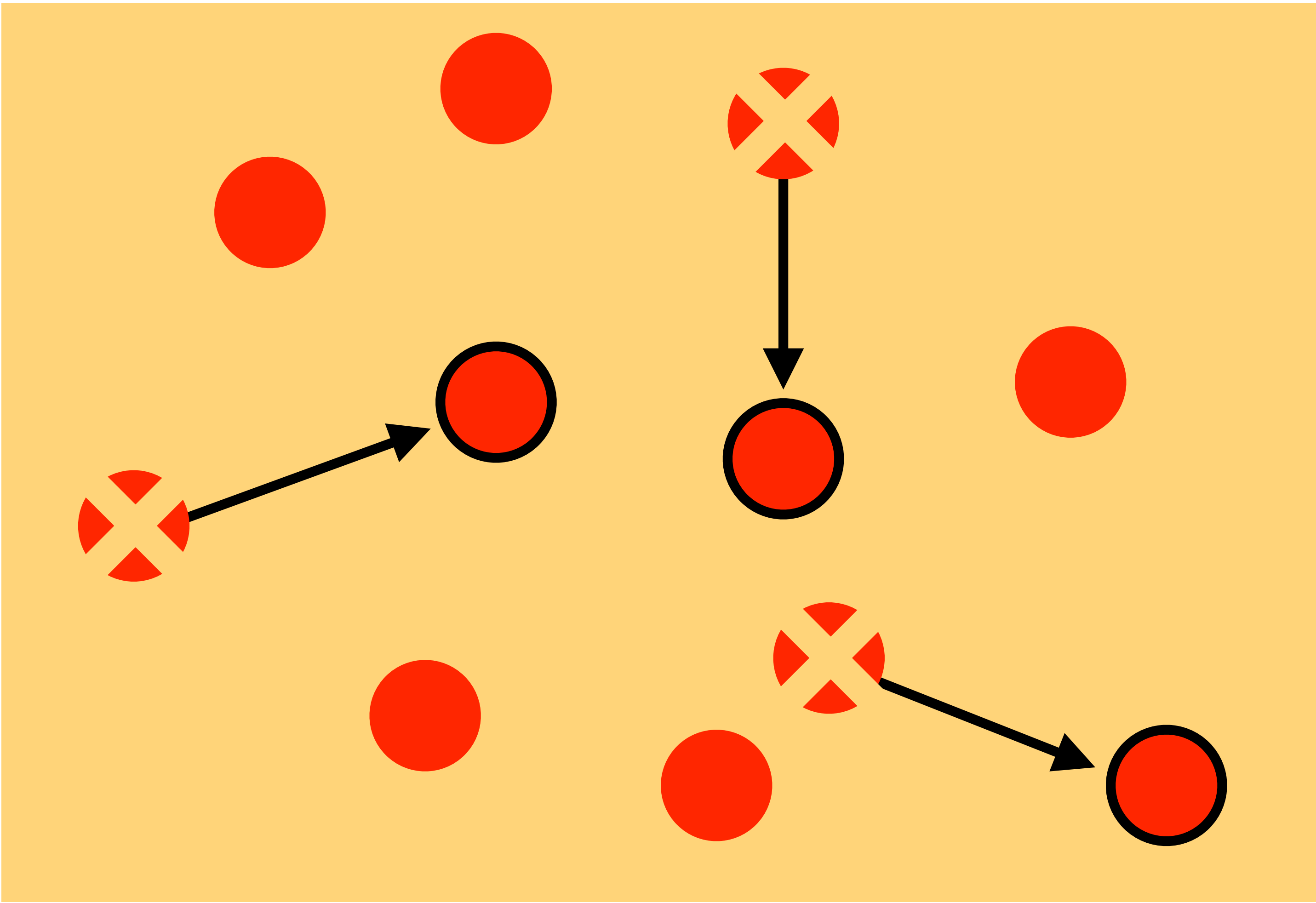
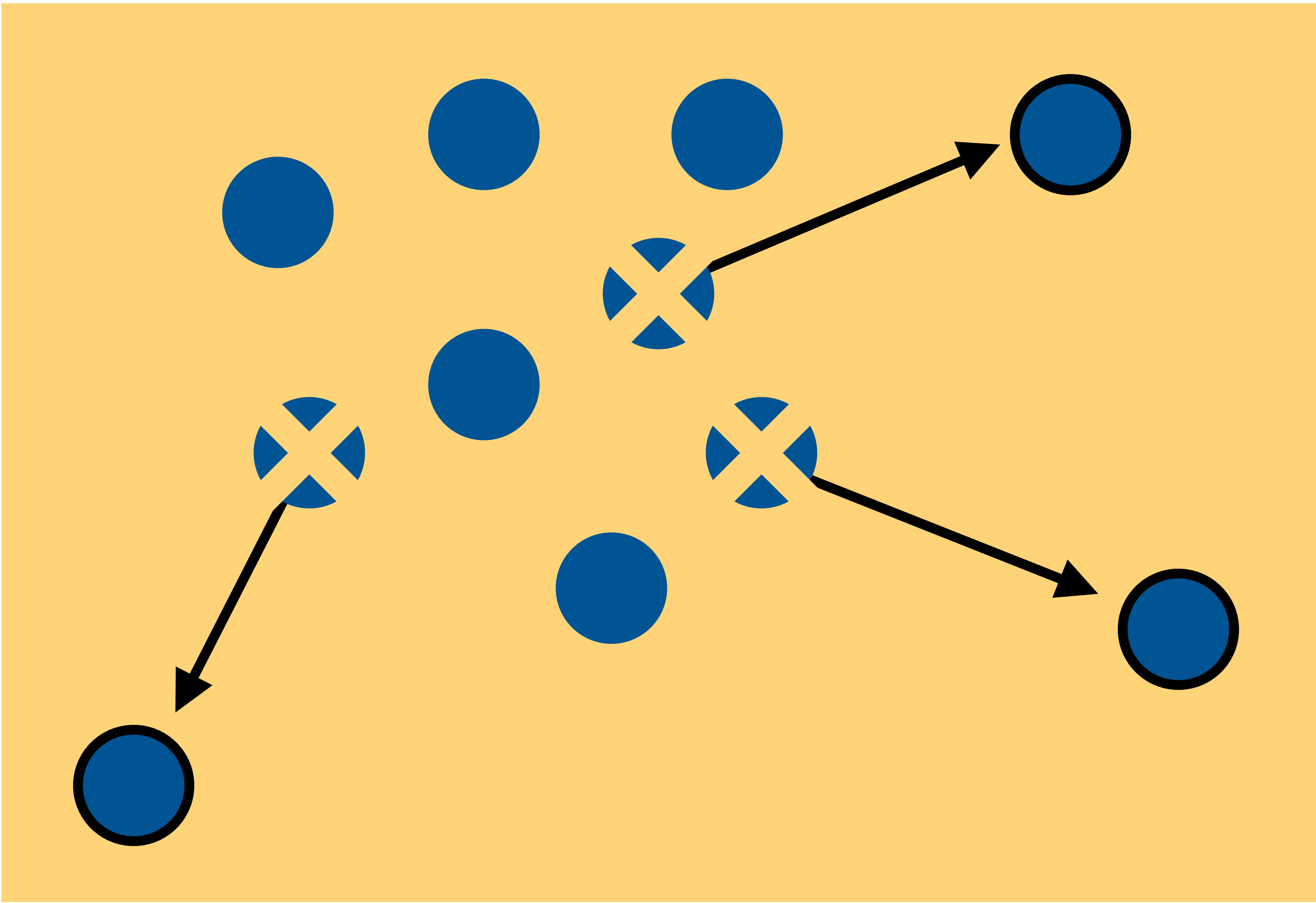


Corrupted Samples from P



Corrupted Samples from Q

dcMMD Procedure

- Maximum Mean Discrepancy:

$$\mathrm{MMD} = \sqrt{\mathbb{E}_{P,P}[k(X, X')] - 2\mathbb{E}_{P,Q}[k(X, Y)] + \mathbb{E}_{Q,Q}[k(Y, Y')]}$$

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- Statistic (plug-in estimator):

$$\widehat{\mathrm{MMD}} = \sqrt{\frac{1}{m^2} \sum_{1 \le i, i' \le m} k(X_i, X_{i'}) - \frac{2}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} k(X_i, Y_j) + \frac{1}{n^2} \sum_{1 \le j, j' \le n} k(Y_j, Y_{j'})}$$

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- dcMMD test: Apply DC procedure with $\widehat{\mathrm{MMD}}$ and $\Delta_{\widehat{\mathrm{MMD}}}$

dcMMD Guarantees

- **Level:** for any distribution P and any sample size

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dcMMD achieves high test power

$$\mathbb{P}_{P,Q}\big(\text{dcMMD rejects } \mathscr{H}_0 \mid r \text{ corrupted data}\big) \ \geq \ 1 - \beta$$

This rate is minimax optimal with respect to m, n, r, α, β.

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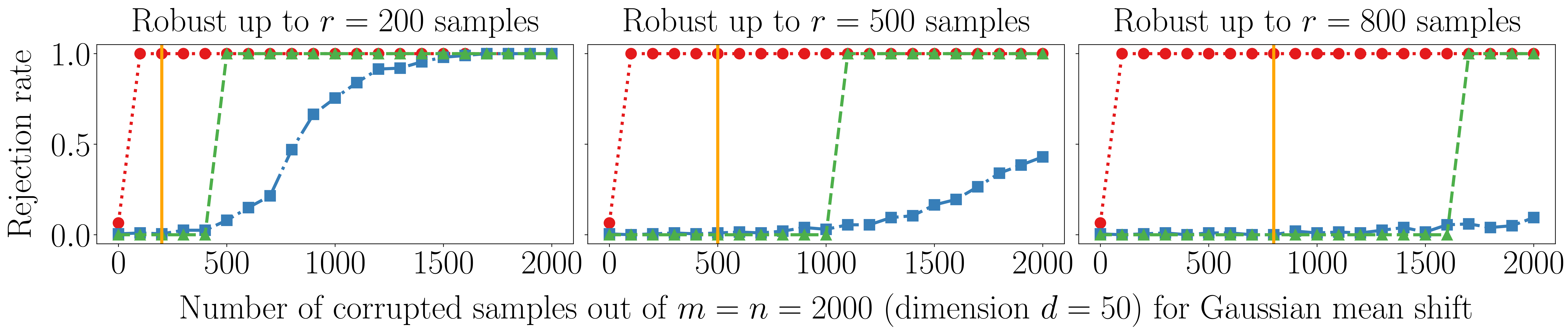
Experiments

dcMMD Experiments: Gaussian Mean Shift



- Generate two samples

  $\tilde{X}_1, \ldots, \tilde{X}_m \overset{\text{iid}}{\sim} N_d(0, 0.1)$

  $\tilde{Y}_1, \ldots, \tilde{Y}_n \overset{\text{iid}}{\sim} N_d(0, 0.1)$

- Corrupt one sample using

  $Z_1, \ldots, Z_k \overset{\text{iid}}{\sim} N_d(1000, 0.1)$

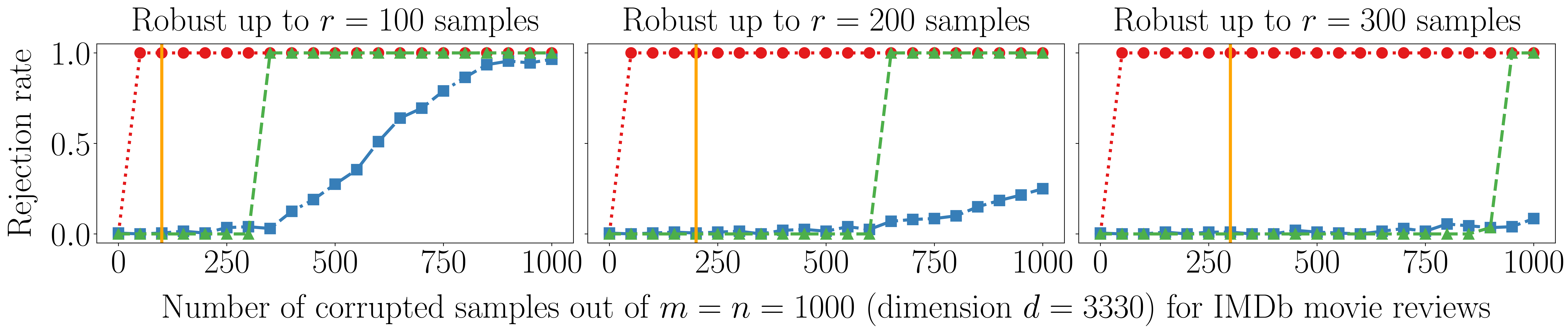
dcMMD: Our proposal

dpMMD: Procedure via differential privacy (Kim & Schrab)

MMD: Standard non-robust MMD

r : Robustness parameter

dcMMD Experiments: IMDb movie reviews



- Generate two samples

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- Corrupt one sample using

  $Z_1, \ldots, Z_k \overset{\text{iid}}{\sim}$ Geometric(3330)

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Summary

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- DC procedure: a general approach for constructing robust tests under data corruption
- Non-asymptotic validity and consistency under r data corruption
- Construct dcMMD and dcHSIC for two-sample and independence robust testing
- Prove that dcMMD/dcHSIC are minimax rate optimal
- Provide public implementations and illustrate the practicality

Any Question?

Paper:

Code: