Local Permutation Tests for Conditional Independence

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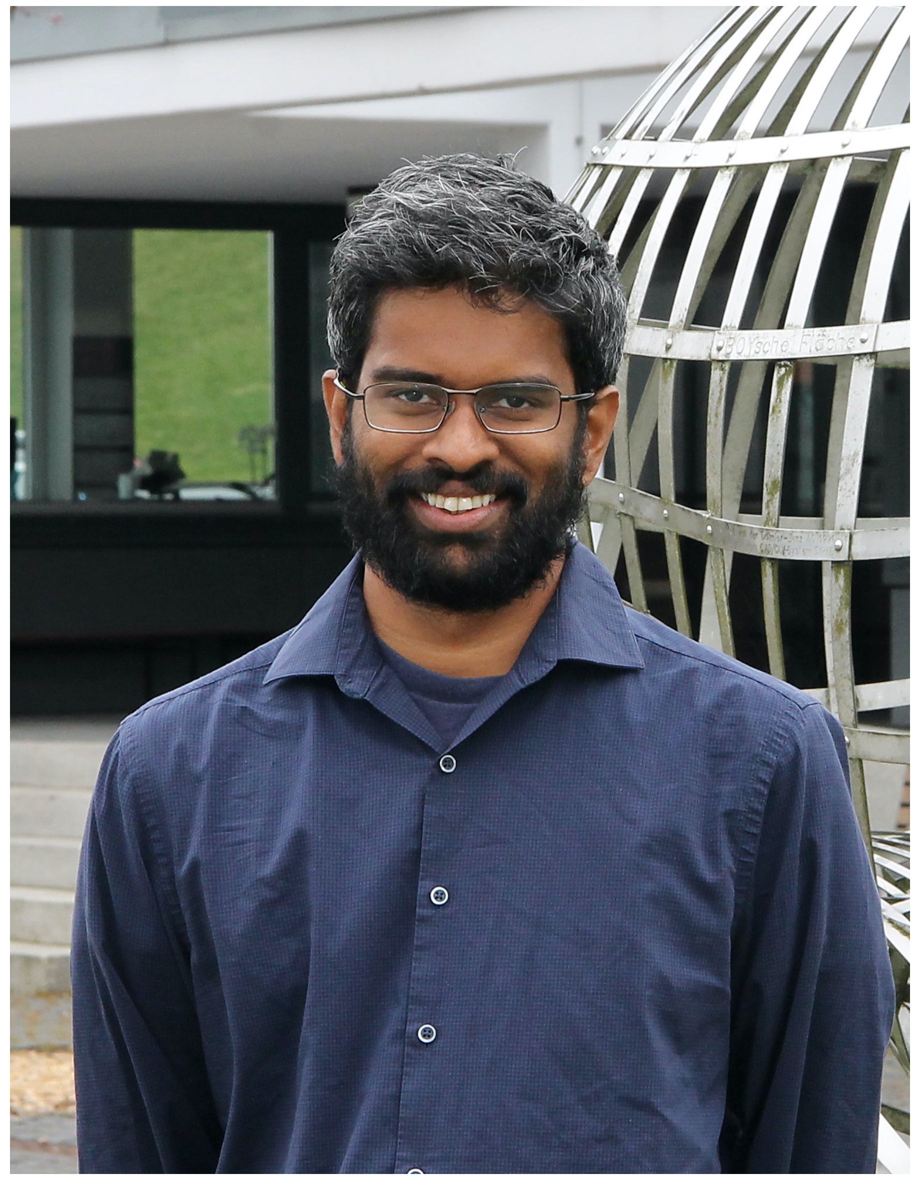
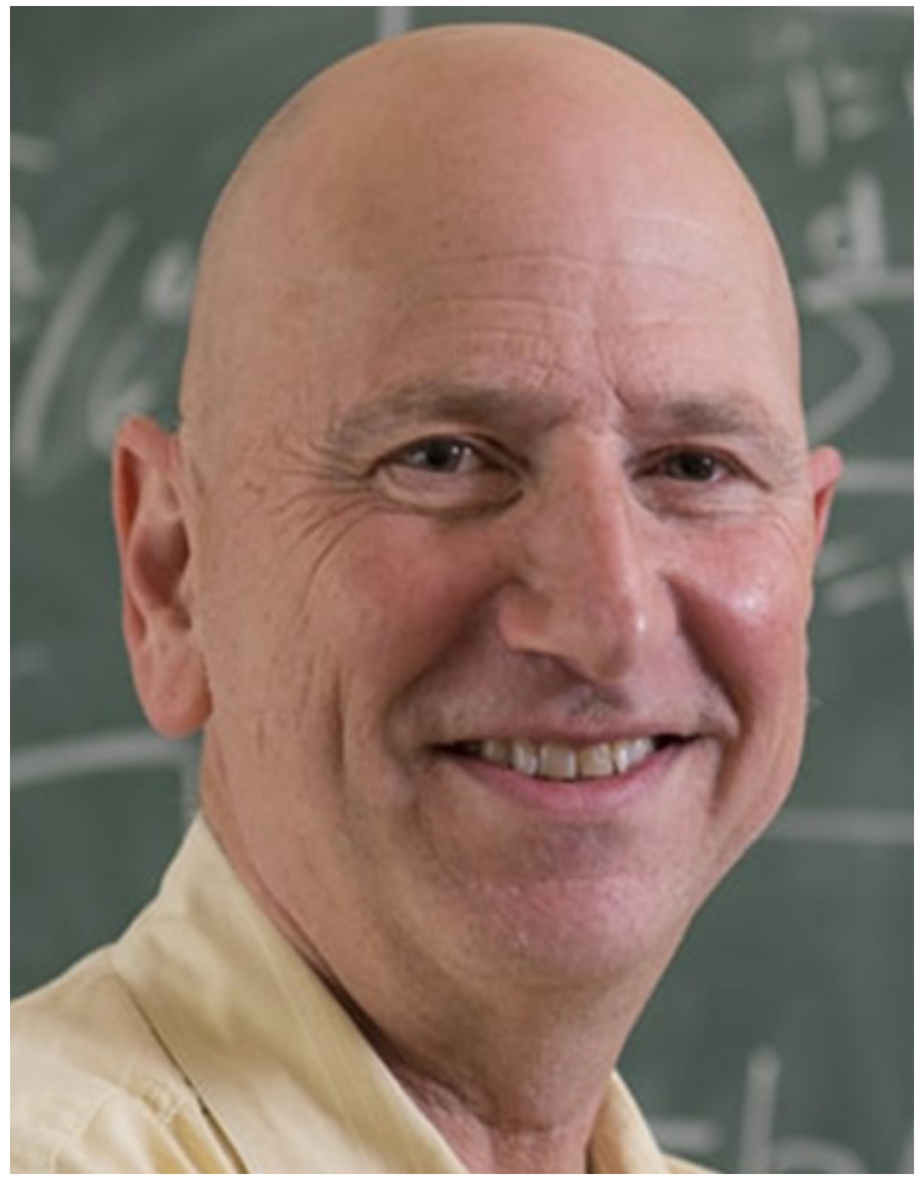
Department of Statistics & Data Science
Yonsei University

Joint work with

Matey Neykov
(Northwestern)



Sivaraman Balakrishnan
(CMU)



Larry Wasserman
(CMU)

Conditional Independence (CI) Testing

- Suppose that we observe $\{(X_i, Y_i, Z_i)\}_{i=1}^{n} \overset{i.i.d.}{\sim} P_{X,Y,Z}$
- We would like to **determine** whether

$$H_0 : X \perp\!\!\!\perp Y \,|\, Z \quad \text{versus} \quad H_1 : X \not\!\perp\!\!\!\perp Y \,|\, Z$$

$$(P_{X,Y,Z} = P_{X|Z} P_{Y|Z} P_Z) \qquad (P_{X,Y,Z} \neq P_{X|Z} P_{Y|Z} P_Z)$$

Applications: Variable Importance

Is the predictor X important to predict the response variable Y?

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1. Parametric methods

$$Y = \beta X + \gamma Z + \varepsilon$$

- e.g., F-test

$$H_0 : \beta = 0 \text{ vs. } H_1 : \beta \neq 0$$

- Not meaningful when applied beyond linear models

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- Feature importance from e.g., random forests and XGboost
- Model dependent & does not present uncertainty

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2. Blackbox methods

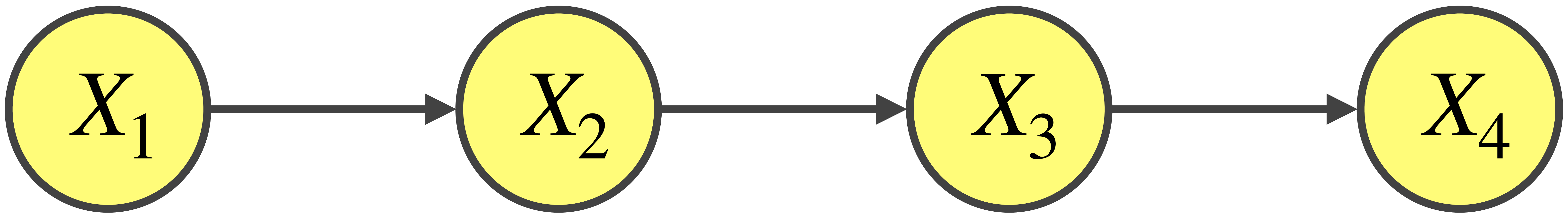
- Feature importance from e.g., random forests and XGboost
- Model dependent & does not present uncertainty

We can formulate it as a **hypothesis testing problem**!

$$H_0 : Y \perp\!\!\!\perp X \mid Z \quad \text{versus} \quad H_1 : Y \not\perp\!\!\!\perp X \mid Z$$

Applications: Bayesian network

Bayes net compactly encodes joint distributions via DAG



$$P(X_1, X_2, X_3, X_4) = P(X_1) \cdot P(X_2 \mid X_1) \cdot P(X_3 \mid X_2, X_1) \cdot P(X_4 \mid X_1, X_2, X_3)$$

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$$X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow X_4$$

$$P(X_1, X_2, X_3, X_4) = P(X_1) \cdot P(X_2 \mid X_1) \cdot P(X_3 \mid X_2, X_1) \cdot P(X_4 \mid X_1, X_2, X_3)$$

Simplify the structure via CI testing

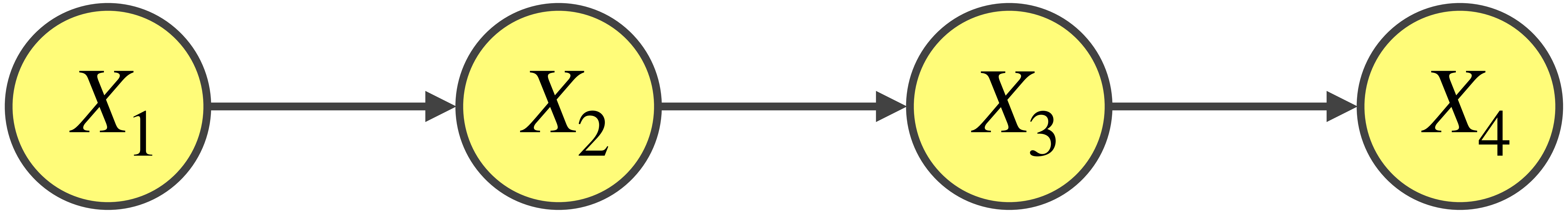
e.g., accept the following nulls

$$H_0 : X_3 \perp\!\!\!\perp X_1 \mid X_2$$

$$H_0 : X_4 \perp\!\!\!\perp (X_1, X_2) \mid X_3$$

Applications: Bayesian network

Bayes net **compactly** encodes joint distributions via DAG



$$P(X_1, X_2, X_3, X_4) = P(X_1) \cdot P(X_2 \,|\, X_1) \cdot \boxed{P(X_3 \,|\, X_2, X_1)} \cdot \boxed{P(X_4 \,|\, X_1, X_2, X_3)}$$

Replace $P(X_3 \,|\, X_2)$

Replace $P(X_4 \,|\, X_3)$

Simplify the structure via CI testing

e.g., accept the following nulls

$$H_0 : X_3 \perp\!\!\!\perp X_1 \,|\, X_2$$

$$H_0 : X_4 \perp\!\!\!\perp (X_1, X_2) \,|\, X_3$$

CI assumptions are everywhere

- Causal Inference
- Factor Analysis
- Hidden Markov Models
- Sufficiency
- Item Response Models
- D-separation
- Bayesian Inference

$\vdots$

CI assumptions are everywhere

- Causal Inference
- Factor Analysis
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- D-separation
- Bayesian Inference

⋮

J. R. Statist. Soc. B (1979),
41, *No.* 1, *pp.* 1–31

Conditional Independence in Statistical Theory

By A. P. Dawid†
University College London

[Read before the Royal Statistical Society at a meeting organized by the Research Section on Wednesday, October 18th, 1978, the Chairman Professor J. F. C. Kingman in the Chair]

Summary

Some simple heuristic properties of conditional independence are shown to form a conceptual framework for much of the theory of statistical inference. This framework is illustrated by an examination of the rôle of conditional independence in several diverse areas of the field of statistics. Topics covered include sufficiency and ancillarity, parameter identification, causal inference, prediction sufficiency, data selection mechanisms, invariant statistical models and a subjectivist approach to model-building.

Keywords: INDEPENDENCE; CONDITIONAL INDEPENDENCE; MARKOV CHAINS; SUFFICIENCY; ANCILLARITY; IDENTIFICATION; SIMPSON'S PARADOX; INVARIANCE; BAYESIAN INFERENCE; PREDICTION; ADEQUACY; TRANSITIVITY; TOTAL SUFFICIENCY; DATA SELECTION; OPTIONAL STOPPING

Prior methods

Asymptotic method

Su and White (2008)
Zhang et al. (2012)
Huang (2010)
Wang et al. (2015)
Strobl et al. (2019)
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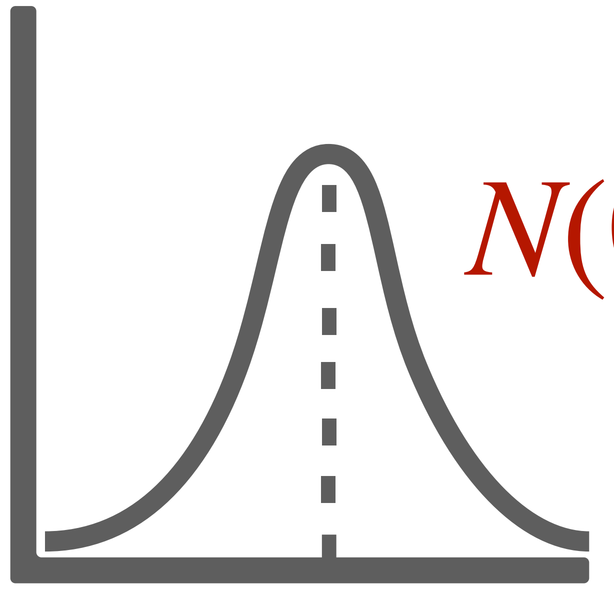
⋮

- Pivotal Statistic $\xrightarrow{d}$  $N(0,1)$

- Determine the **critical value** based on the **limiting distribution**

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Model-X framework

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⋮

- Assume $F_{X|Z}$ is known
- Generate $\tilde{X}$ from $F_{X|Z}$
- Construct an exchangeable sequence under the null

$$T, T_1, \ldots, T_B$$

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Local permutation

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We aim to enhance our understanding of

local permutation tests:

"when does it work and when does it fail?"

Local Permutation Tests

- **Partition** the domain of Z into $B_1, \ldots, B_M$ bins

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- Partition the domain of Z into $B_1, \ldots, B_M$ bins
- Transform $\{(X_i, Y_i, Z_i)\}_{i=1}^{n} \mapsto \{(X_i, Y_i, \widetilde{Z}_i)\}_{i=1}^{n}$ where $\widetilde{Z}_i = k$ if $Z_i \in B_k$

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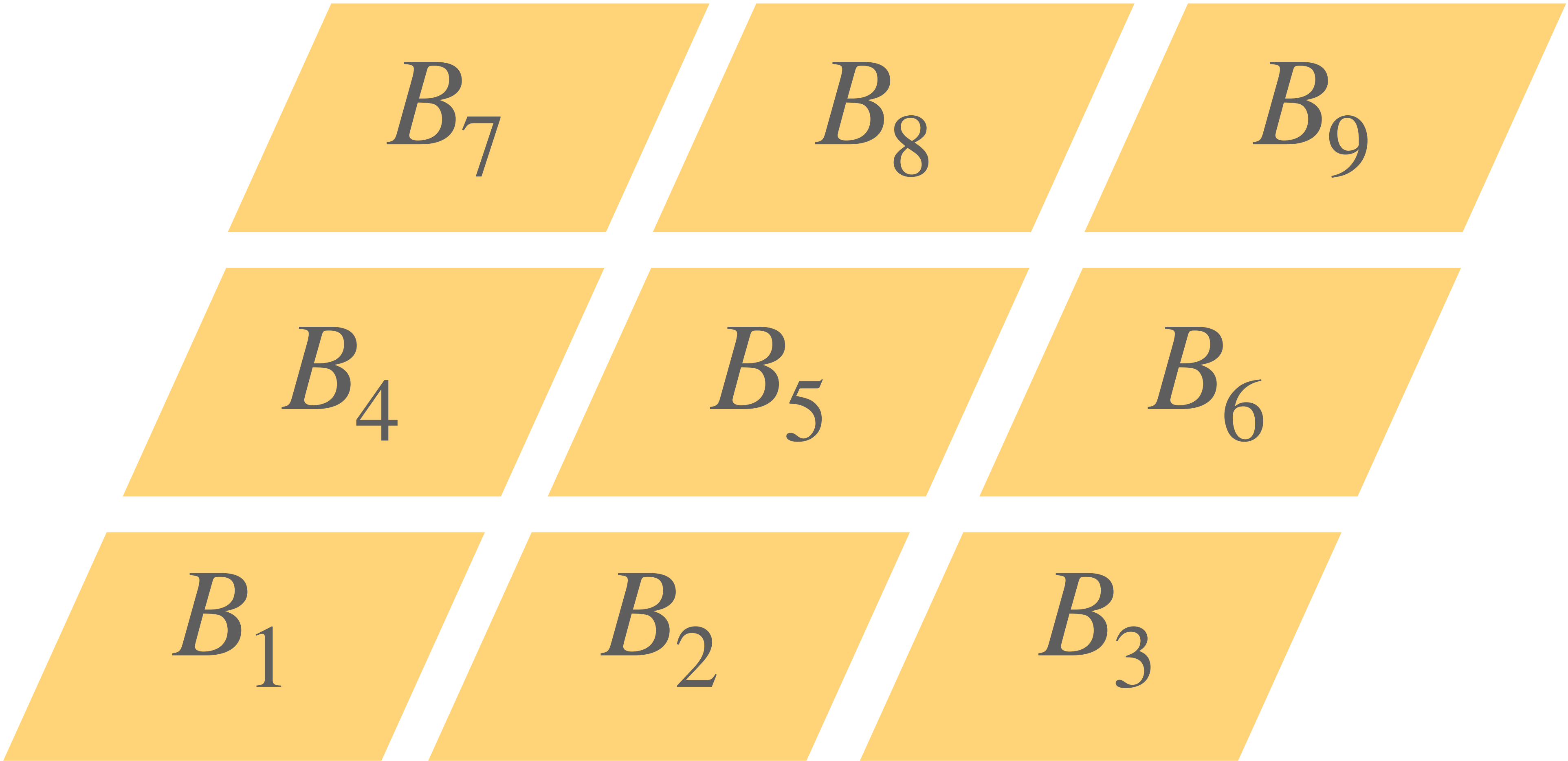

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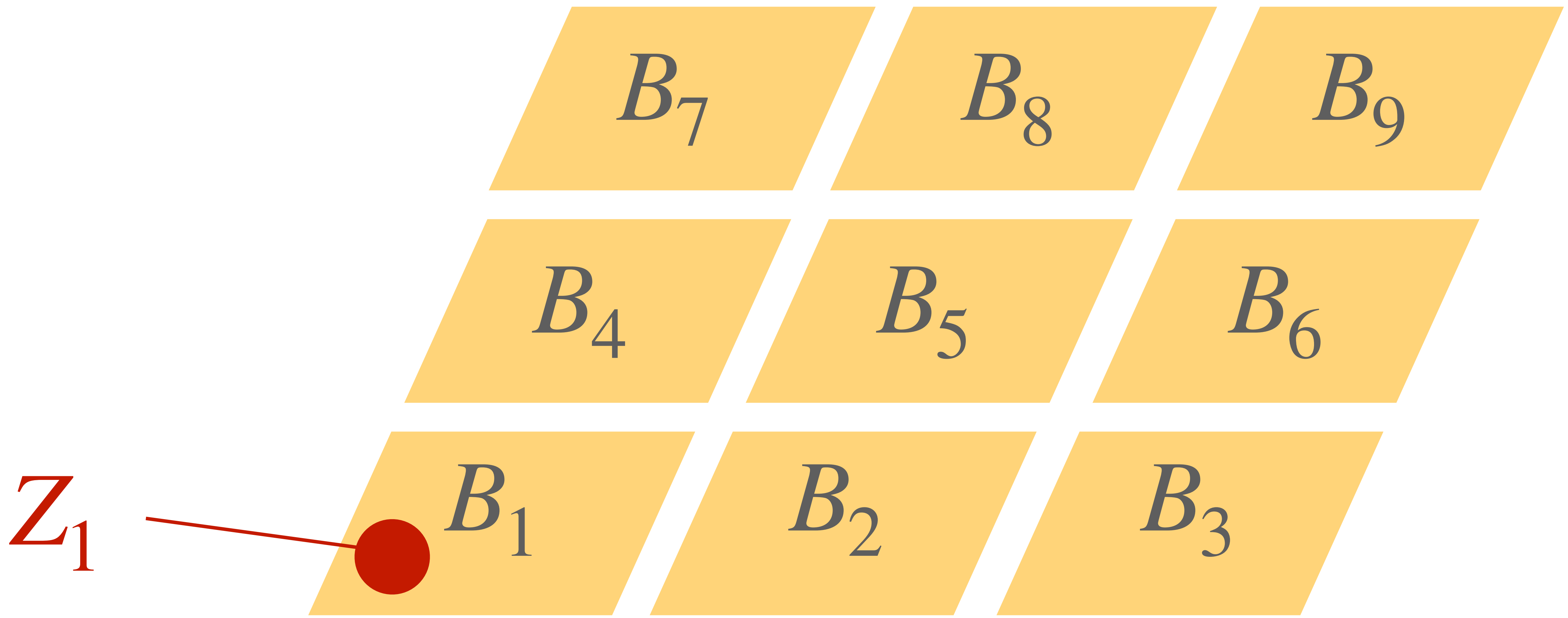

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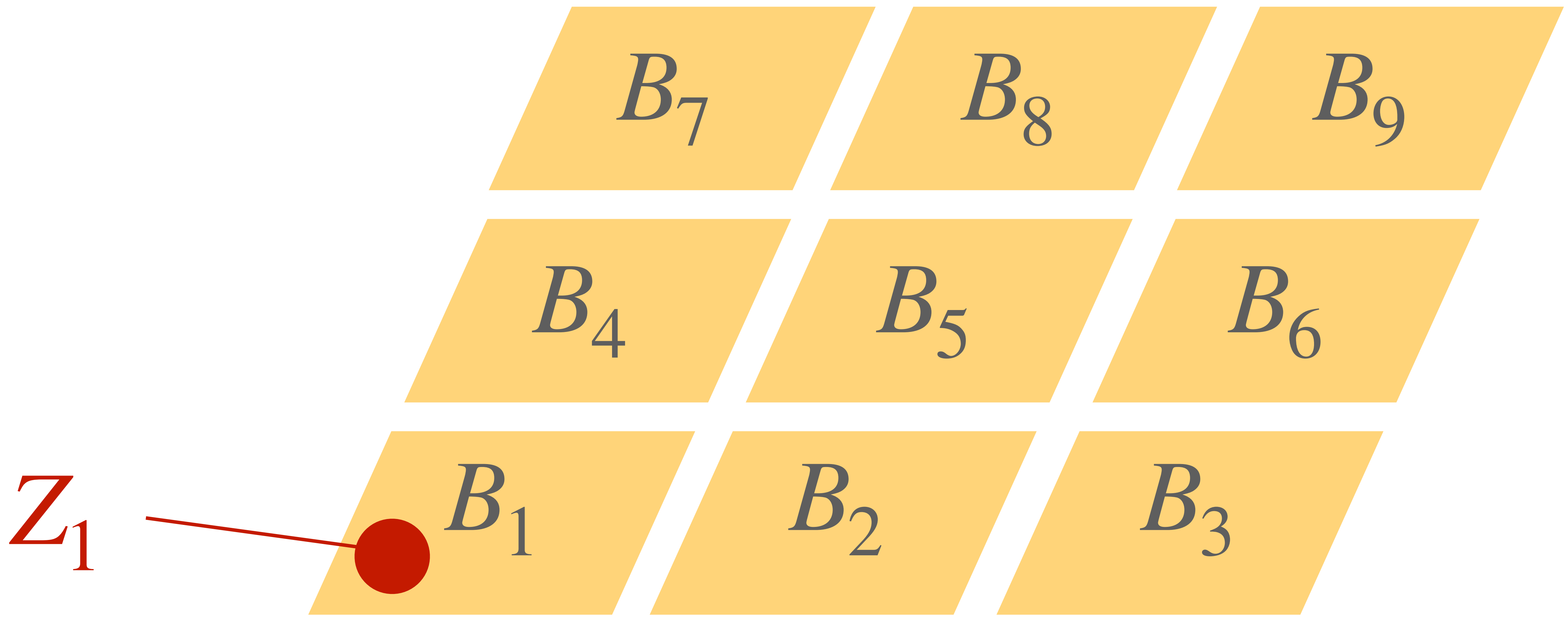

$\widetilde{Z}_1 = 1$

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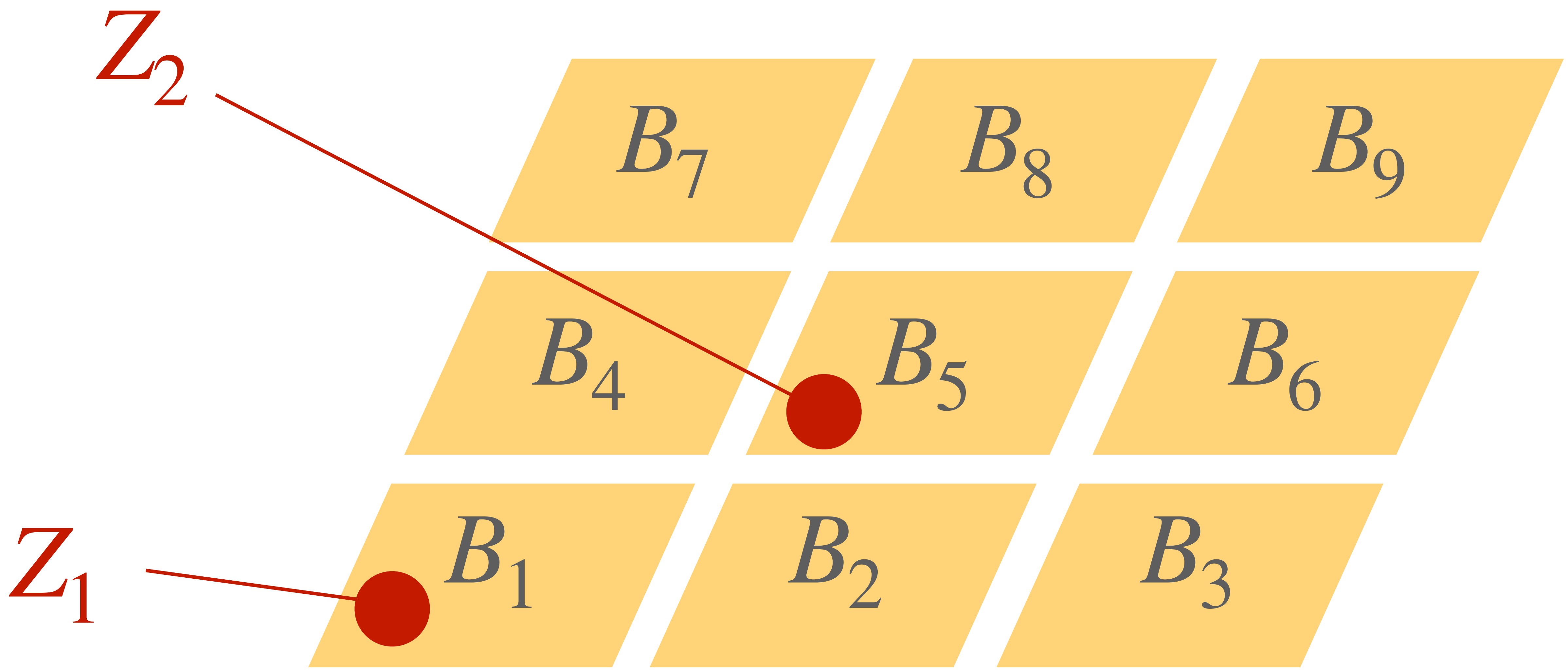

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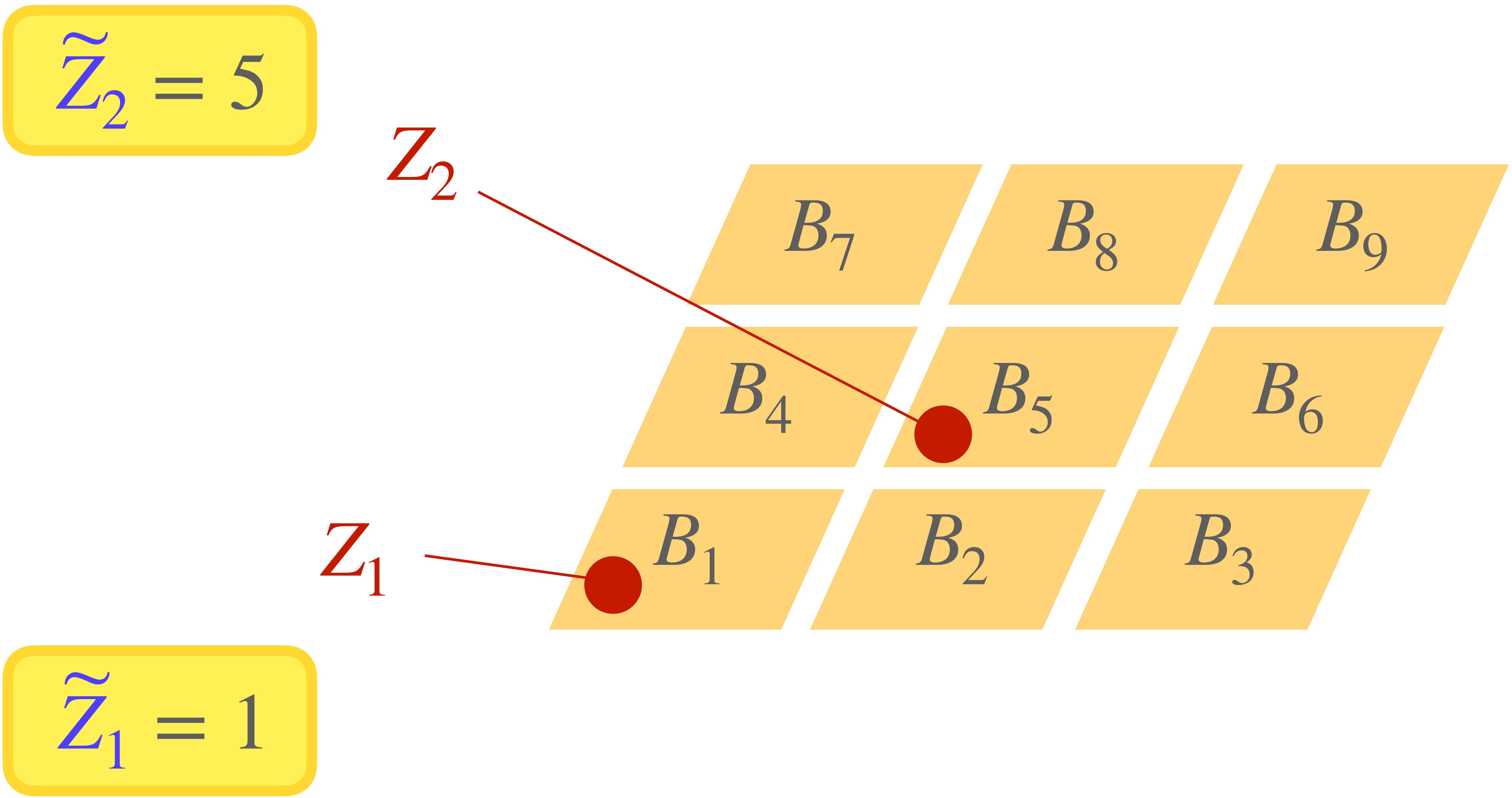

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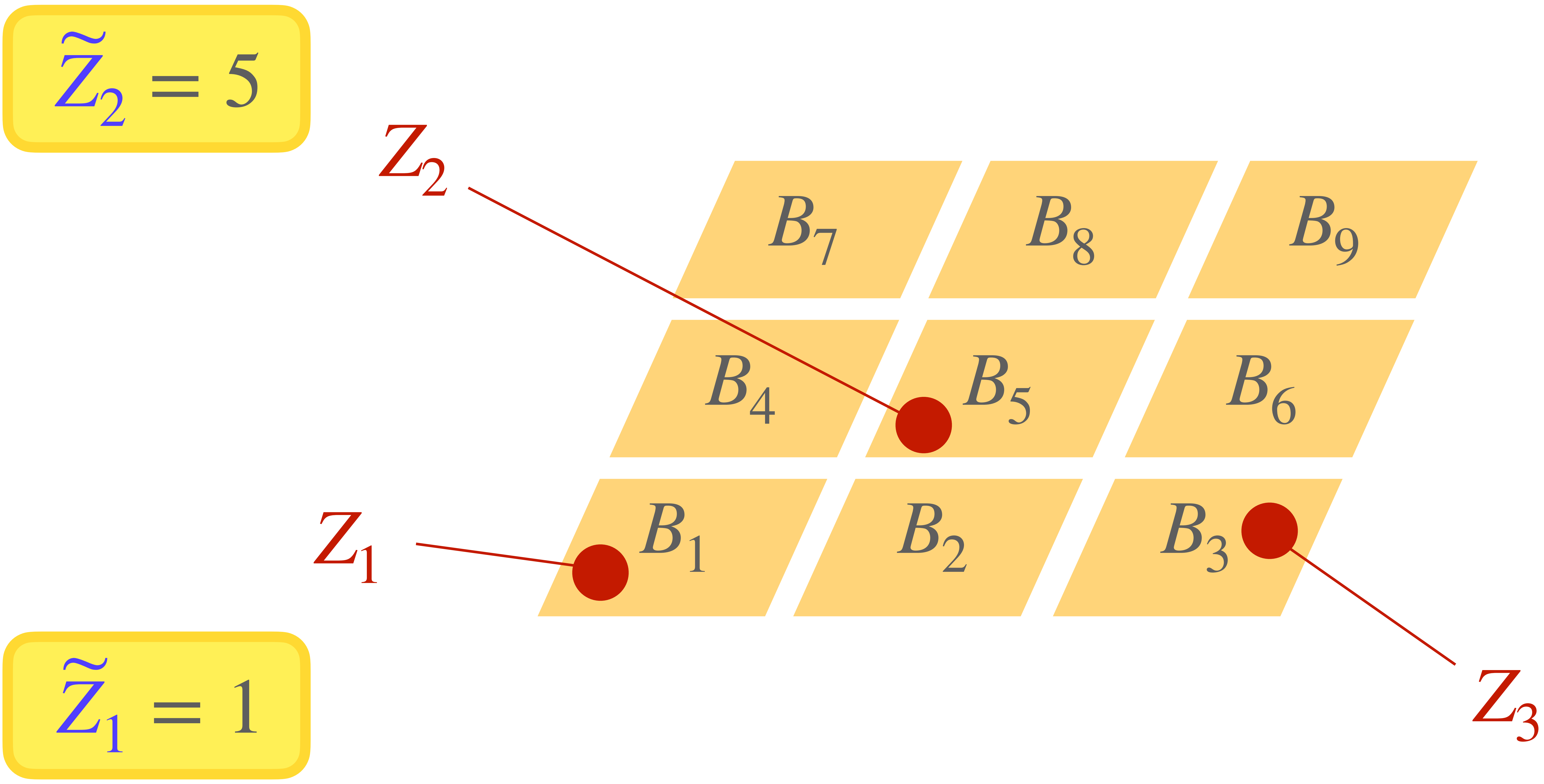

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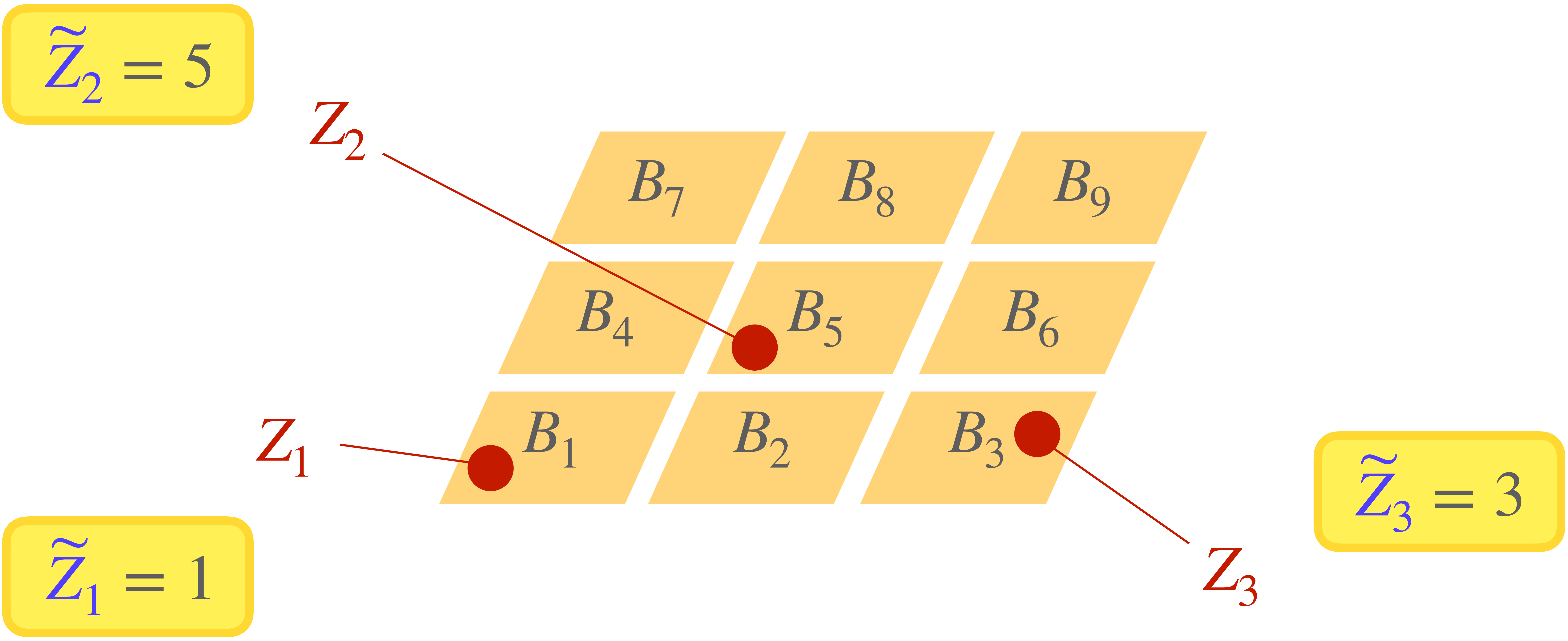

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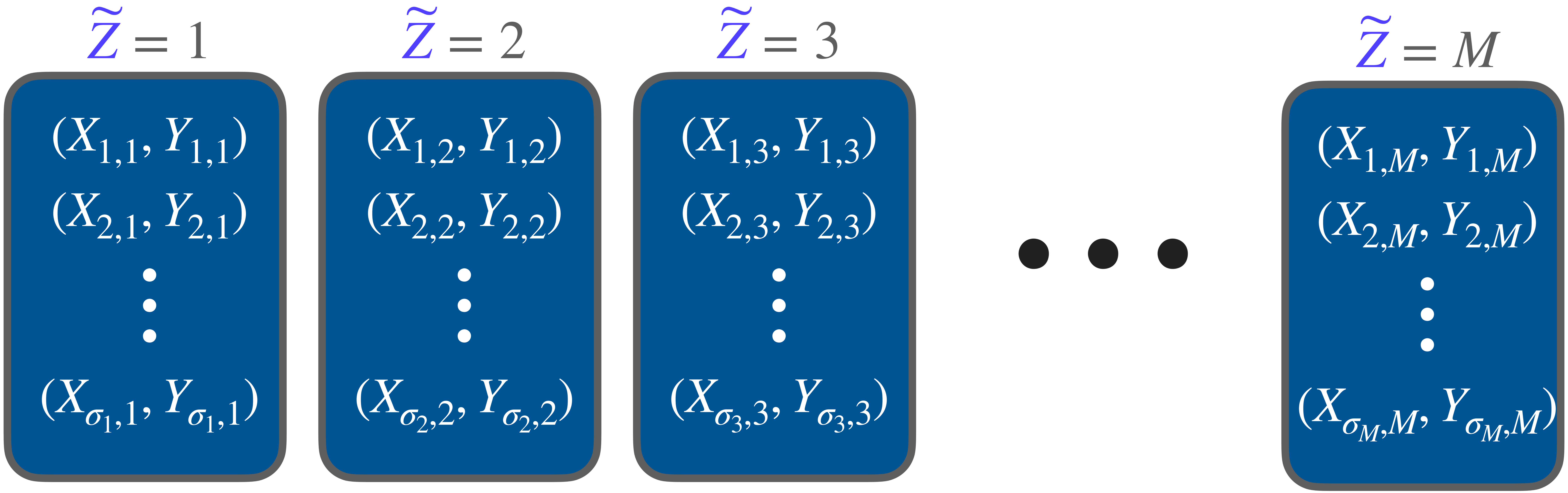

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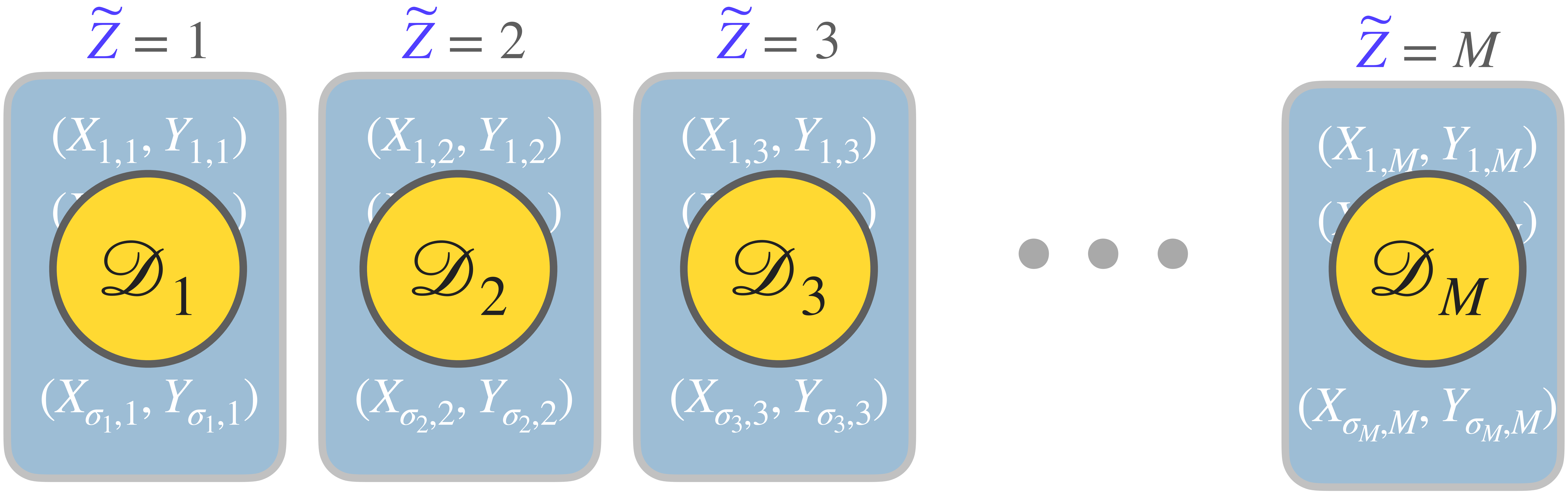
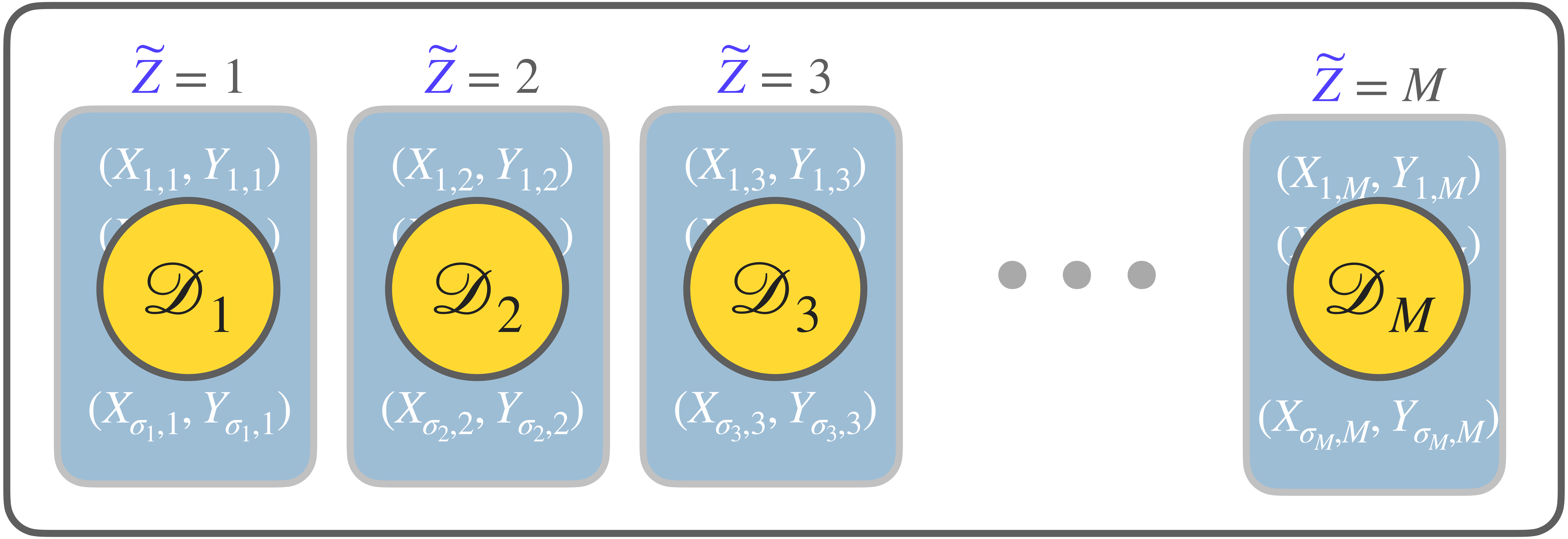


Local Permutation Tests

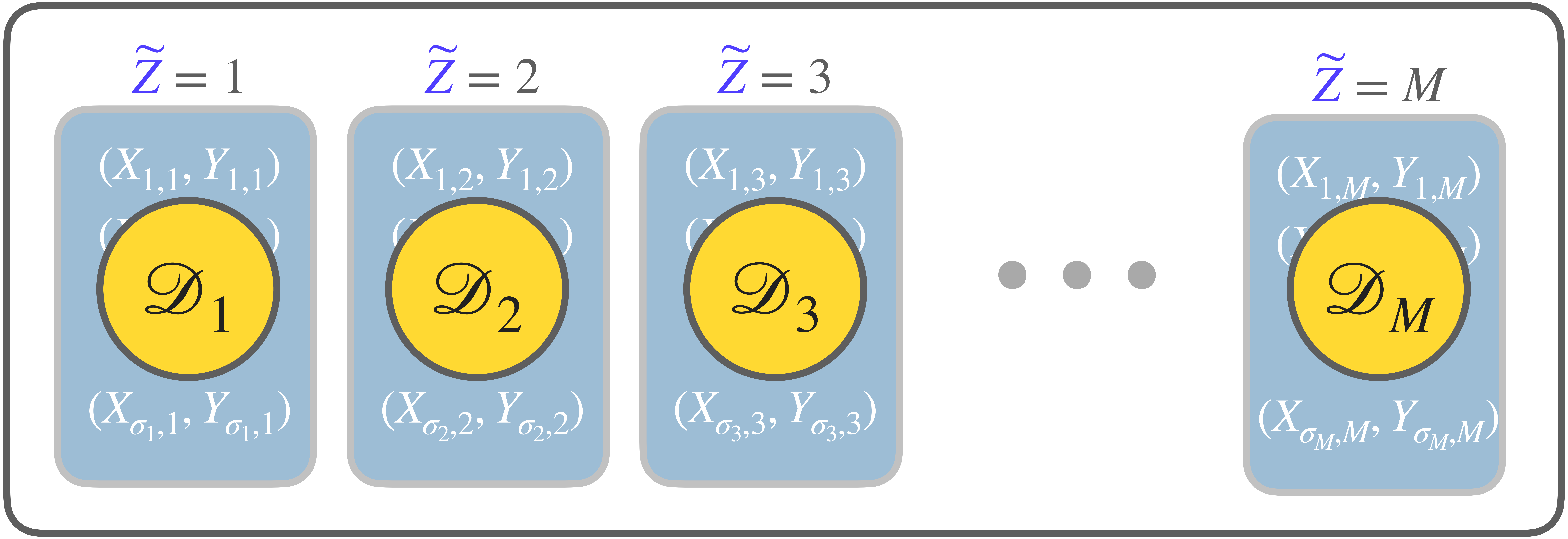
- **Compute** a binning-based test statistic $T := T(\mathscr{D}_1, \mathscr{D}_2, \ldots, \mathscr{D}_M)$



Local Permutation Tests

- **Compute** a binning-based test statistic $T := T(\mathscr{D}_1, \mathscr{D}_2, \ldots, \mathscr{D}_M)$

$$\text{e.g., } T = \sum_{i=1}^{M} \chi^2(\mathscr{D}_i) \text{ or } T = \max_{1 \le i \le M} \chi^2(\mathscr{D}_i)$$



Local Permutation Tests

$\widetilde{Z} = 1$

$(X_{1,1}, Y_{1,1})$
$(X_{2,1}, Y_{2,1})$
$\vdots$
$(X_{\sigma_1,1}, Y_{\sigma_1,1})$

$\widetilde{Z} = 2$

$(X_{1,2}, Y_{1,2})$
$(X_{2,2}, Y_{2,2})$
$\vdots$
$(X_{\sigma_2,2}, Y_{\sigma_2,2})$

$\widetilde{Z} = 3$

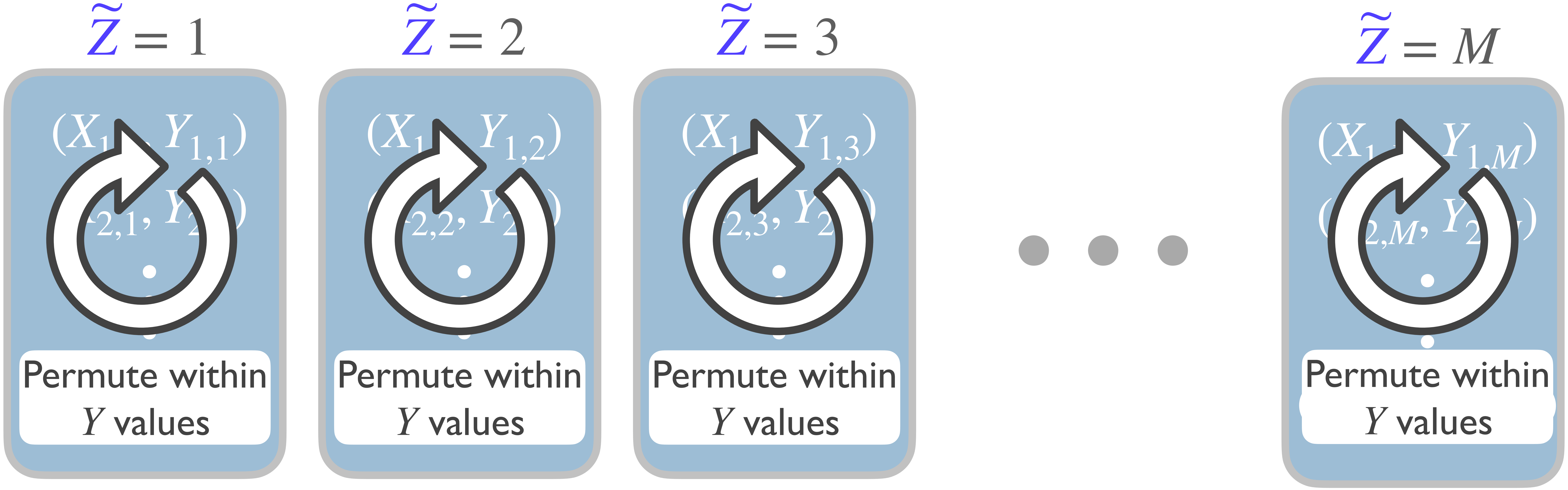
$(X_{1,3}, Y_{1,3})$
$(X_{2,3}, Y_{2,3})$
$\vdots$
$(X_{\sigma_3,3}, Y_{\sigma_3,3})$

$\cdots$

$\widetilde{Z} = M$

$(X_{1,M}, Y_{1,M})$
$(X_{2,M}, Y_{2,M})$
$\vdots$
$(X_{\sigma_M,M}, Y_{\sigma_M,M})$

Local Permutation Tests



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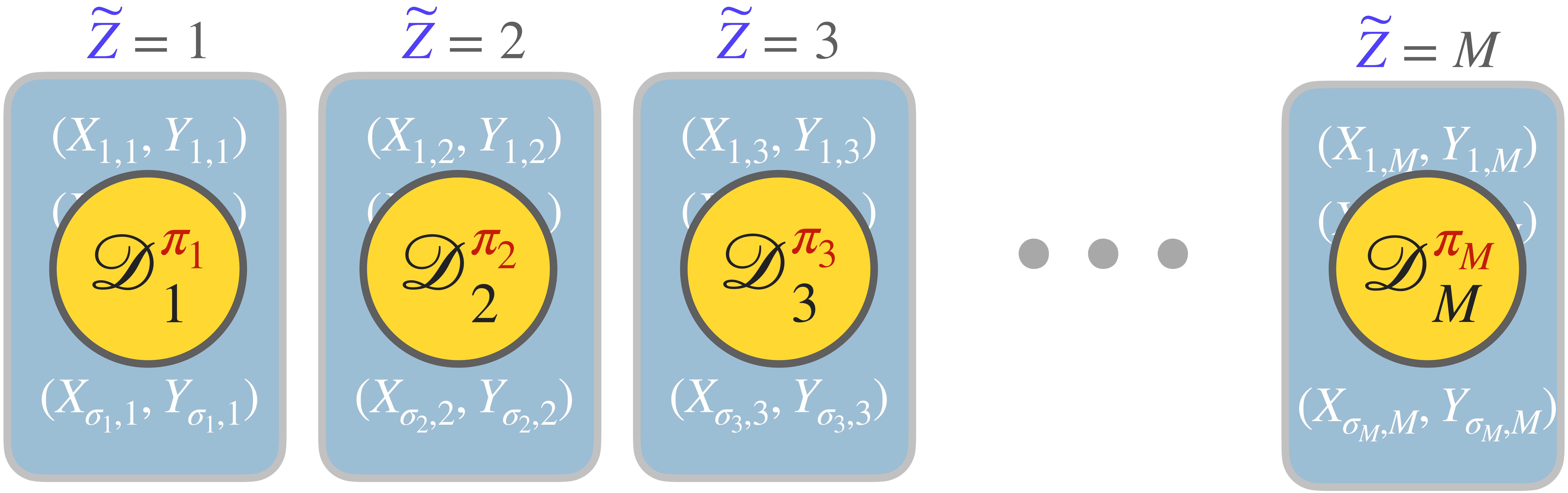
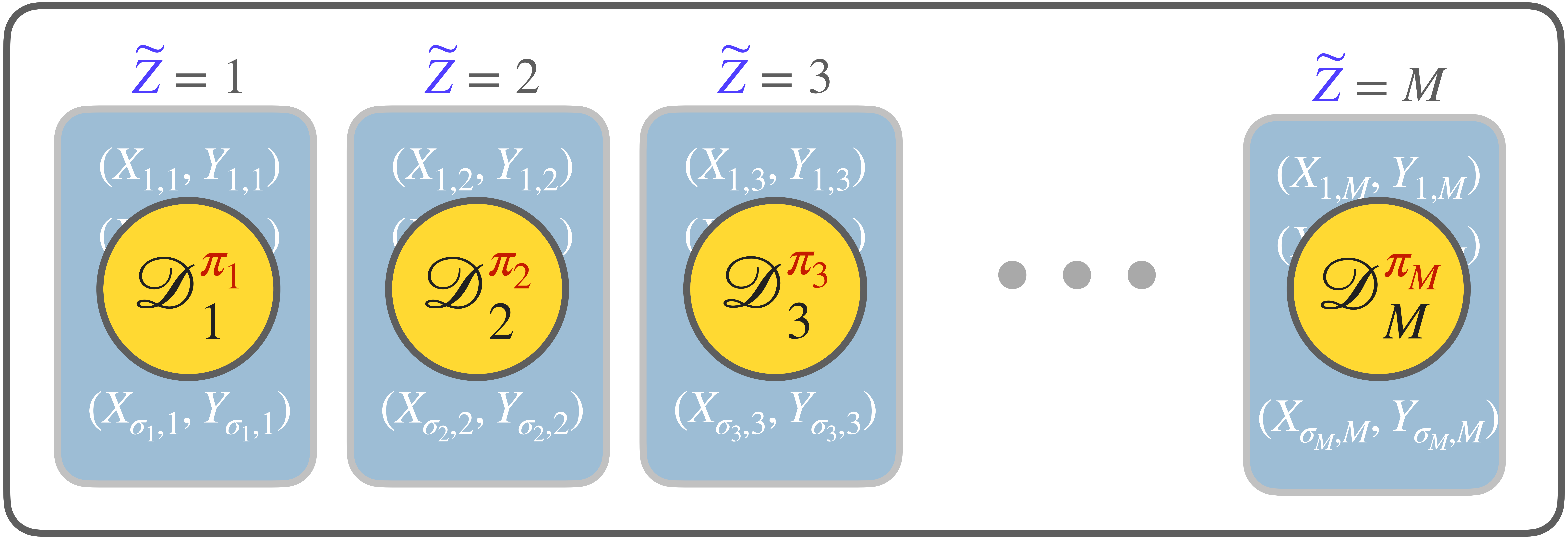


Local Permutation Tests

- **Compute** a binning-based test statistic $T^{\pi} := T(\mathscr{D}_1^{\pi_1}, \mathscr{D}_2^{\pi_2}, \ldots, \mathscr{D}_M^{\pi_M})$

e.g., $T^{\pi} = \sum_{i=1}^{M} \chi^2(\mathscr{D}_i^{\pi_1})$ or $T^{\pi} = \max_{1 \le i \le M} \chi^2(\mathscr{D}_i^{\pi_1})$



Algorithm: Local Permutation Tests

- **Compute** a binning-based test statistic $T := T(\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_M)$

For $i = 1, \ldots, K$

Permute Y values within bins

$T_i^\pi := T(\mathcal{D}_1^{\pi_1}, \mathcal{D}_2^{\pi_2}, \ldots, \mathcal{D}_M^{\pi_M})$

End

- **Reject** the null if $T > q_{1-\alpha}$

The $1 - \alpha$ **quantile of** $\{T, T_1^\pi, \ldots, T_K^\pi\}$

Algorithm: Local Permutation Tests

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> *For* $i = 1, \ldots, K$
>
> Permute Y values within bins
>
> $$T_i^\pi := T(\mathcal{D}_1^{\pi_1}, \mathcal{D}_2^{\pi_2}, \ldots, \mathcal{D}_M^{\pi_M})$$
>
> *End*

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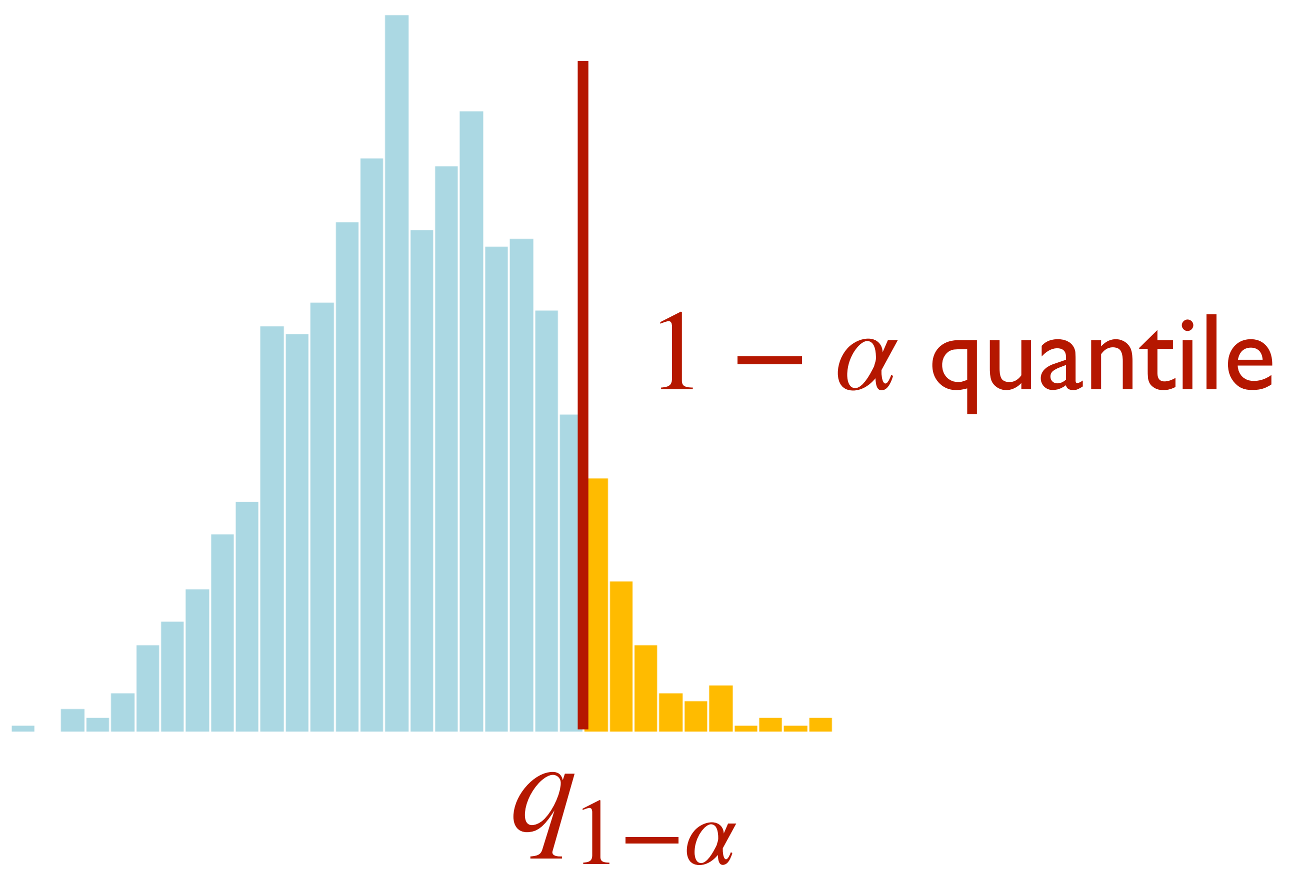

The $1 - \alpha$ quantile of $\{T, T_1^\pi, \ldots, T_K^\pi\}$

Local Permutation Tests

Remark

- Unlike the two-sample and independence testing problem, there is **no guarantee** that the **local permutation test** is **valid** under the null

- In fact, **any valid CI test** has power ***at most*** its size α against **any alternative**! *(Shah & Peters 2020)*

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Remark

- Unlike the two-sample and independence testing problem, there is **no guarantee** that the **local permutation test** is **valid** under the null

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Hardness Result for CI Testing

Theorem [Shah & Peters 2020]

- Let $\mathscr{P}_{\text{conti}}$ be the **class of continuous distributions** on $\mathbb{R}^d$, $\mathscr{P}_{0,\text{conti}}$ be the subset of $\mathscr{P}_{\text{conti}}$ such that $X \perp\!\!\!\perp Y \,|\, Z$, and $\mathscr{P}_{1,\text{conti}} = \mathscr{P}_{\text{conti}} \backslash \mathscr{P}_{0,\text{conti}}$.

Hardness Result for CI Testing

Distributions under H_0

Theorem [Shah & Peters 2020]

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Distributions under H_1

Hardness Result for CI Testing

Theorem [Shah & Peters 2020]

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- Suppose that a test ϕ satisfies

$$\sup_{P \in \mathscr{P}_{0,\mathrm{conti}}} \mathbb{E}_P[\phi] \leq \alpha \,.$$

Hardness Result for CI Testing

Theorem [Shah & Peters 2020]

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- Suppose that a test ϕ satisfies

$$\sup_{P \in \mathscr{P}_{0,\text{conti}}} \mathbb{E}_P[\phi] \leq \alpha\,.$$

Then for **any** $P \in \mathscr{P}_{1,\text{conti}}$, the power of ϕ is **bounded above** by

$$\mathbb{E}_P[\phi] \leq \alpha\,.$$

Intuition for the Hardness Result

- When Z is continuous, we never observe the same value of Z

→ *The effective sample size is one!*

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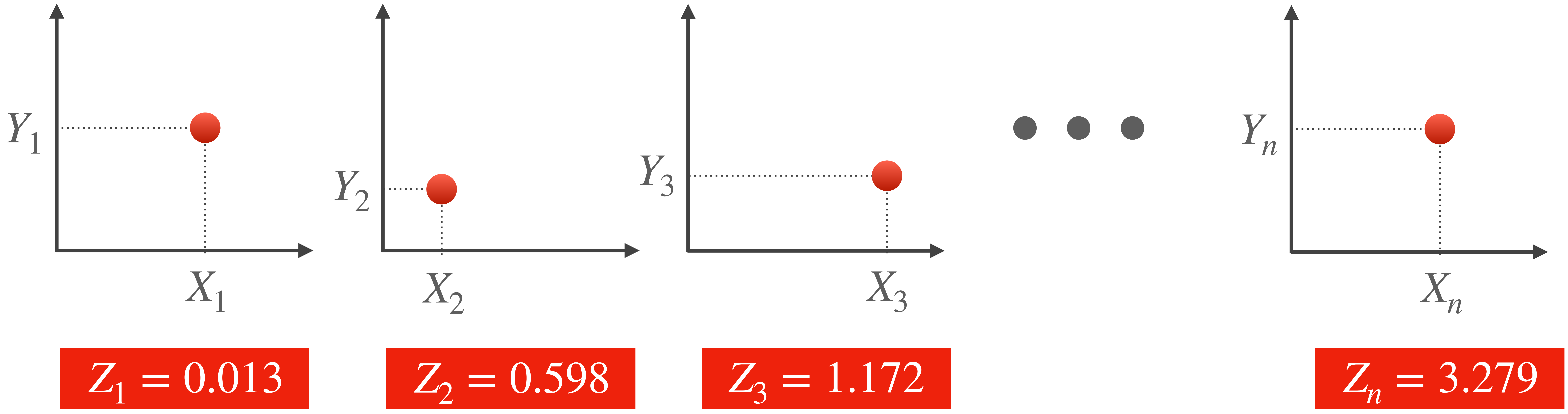

Question.

What about scenarios where Z follows a **discrete** or a **mixture** distribution? Is CI testing still **hard**?

New Hardness Result for CI Testing

Random sample from P_Z

Theorem [KNBW 2022]

- For an arbitrary $J \geq n(n-1)$, let $\rho_{J,P} = \mathbb{P}_P(Z_1, \ldots, Z_J \text{ are distinct})$

No-collision probability

New Hardness Result for CI Testing

Theorem [KNBW 2022]

- For an arbitrary $J \geq n(n-1)$, let $\rho_{J,P} = \mathbb{P}_P(Z_1, \ldots, Z_J$ are distinct)
- Suppose that a test ϕ satisfies

$$\sup_{P \in \mathscr{P}_{0,\mathrm{disc}}} \mathbb{E}_P[\phi] \leq \alpha$$

Class of null distributions including discrete P_Z

New Hardness Result for CI Testing

Theorem [KNBW 2022]

- For an arbitrary $J \geq n(n-1)$, let $\rho_{J,P} = \mathbb{P}_P(Z_1, \ldots, Z_J$ are distinct)
- Suppose that a test ϕ satisfies

$$\sup_{P \in \mathscr{P}_{0,\mathrm{disc}}} \mathbb{E}_P[\phi] \leq \alpha$$

Then for **any** $P \in \mathscr{P}_1$, the power of ϕ is **bounded above** by

$$\mathbb{E}_P[\phi] \leq \alpha \times \rho_{J,P} + (1 - \rho_{J,P}) + \frac{n(n-1)}{J}$$

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When $\rho_{J,P}$ is close to one, then the **power** becomes close to the size

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New Hardness Result for CI Testing

Examples

- **Continuous** distributions ($\rho_{J,P} = 1$ for any $J \geq 2$)

Any **valid** CI test has the power **upper bounded** by

$$\mathbb{E}_P[\phi] \leq \alpha$$

New Hardness Result for CI Testing

Examples

- Continuous distributions ($\rho_{J,P} = 1$ for any $J \geq 2$)

  Any valid CI test has the power upper bounded by

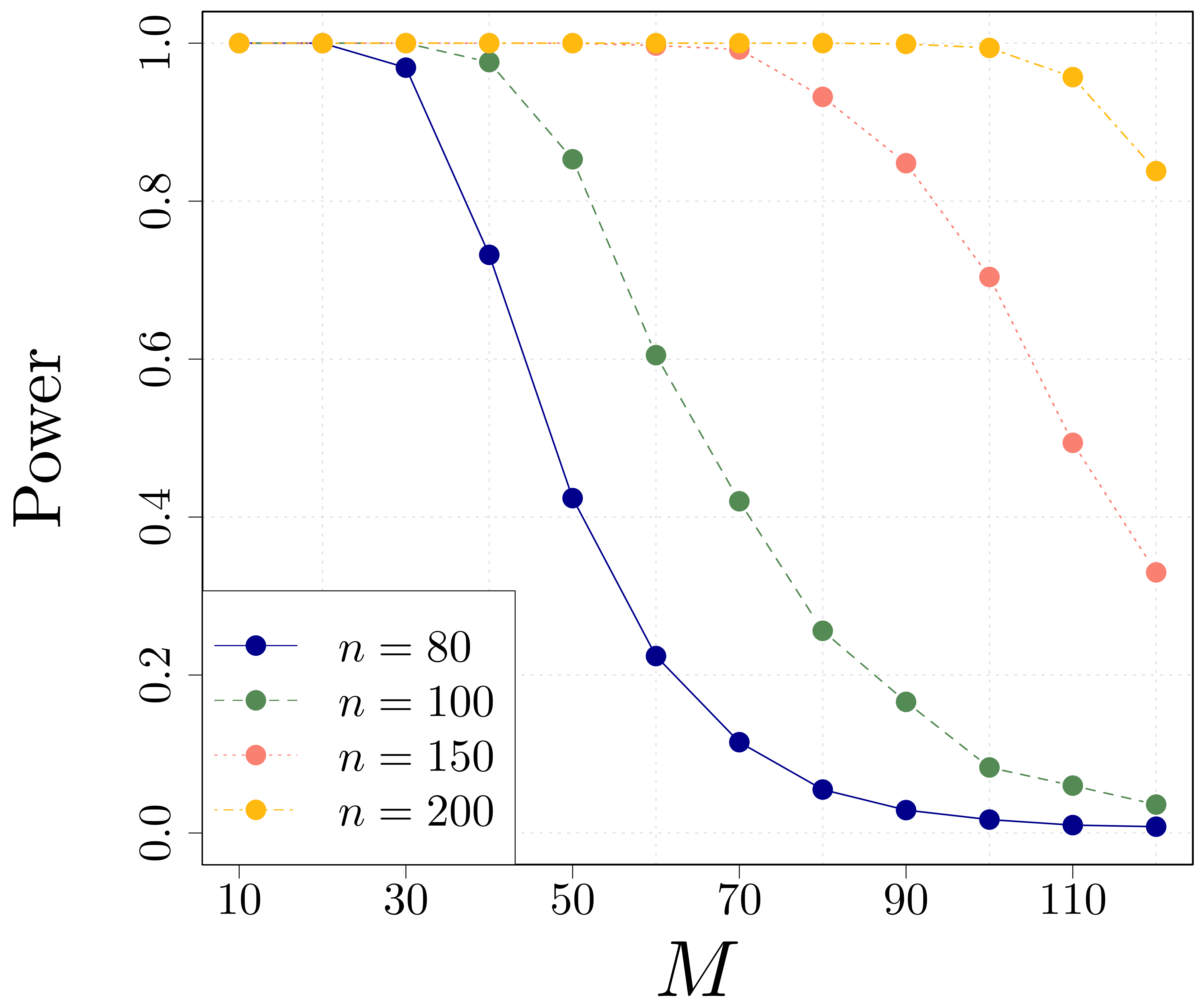
$$\mathbb{E}_P[\phi] \leq \alpha$$

- Discrete uniform distribution on $\{1,2,\ldots,M\}$ with $n^4/M \to 0$

  Any valid CI test has the power upper bounded by

$$\mathbb{E}_P[\phi] \leq \alpha + o(1)$$

CI Testing is still hard when "no-collision" probability is high

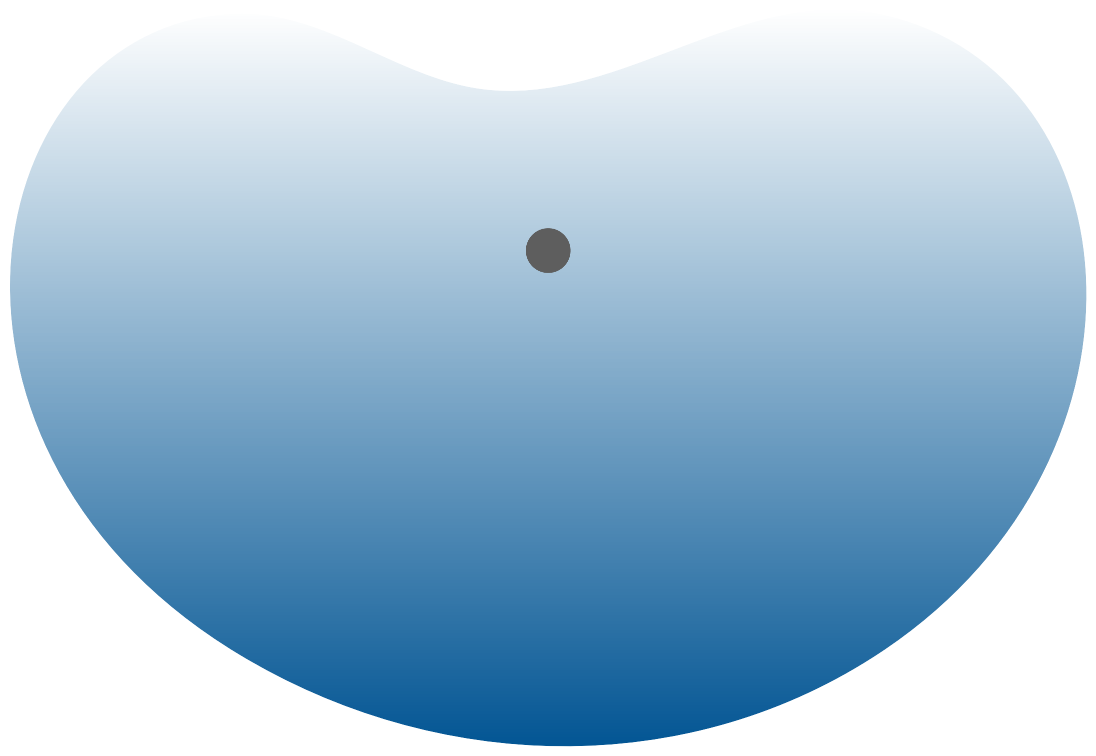


We need **assumptions** to make CI testing **feasible**
especially in continuous settings

Validity of Local Permutation Tests

Joint distribution of the original data $\{(X_i, Y_i, Z_i)\}_{i=1}^n$

$P^n_{X,Y,Z}$



Validity of Local Permutation Tests

Joint distribution of the original data
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Discretization of Z

Joint distribution of the binned data
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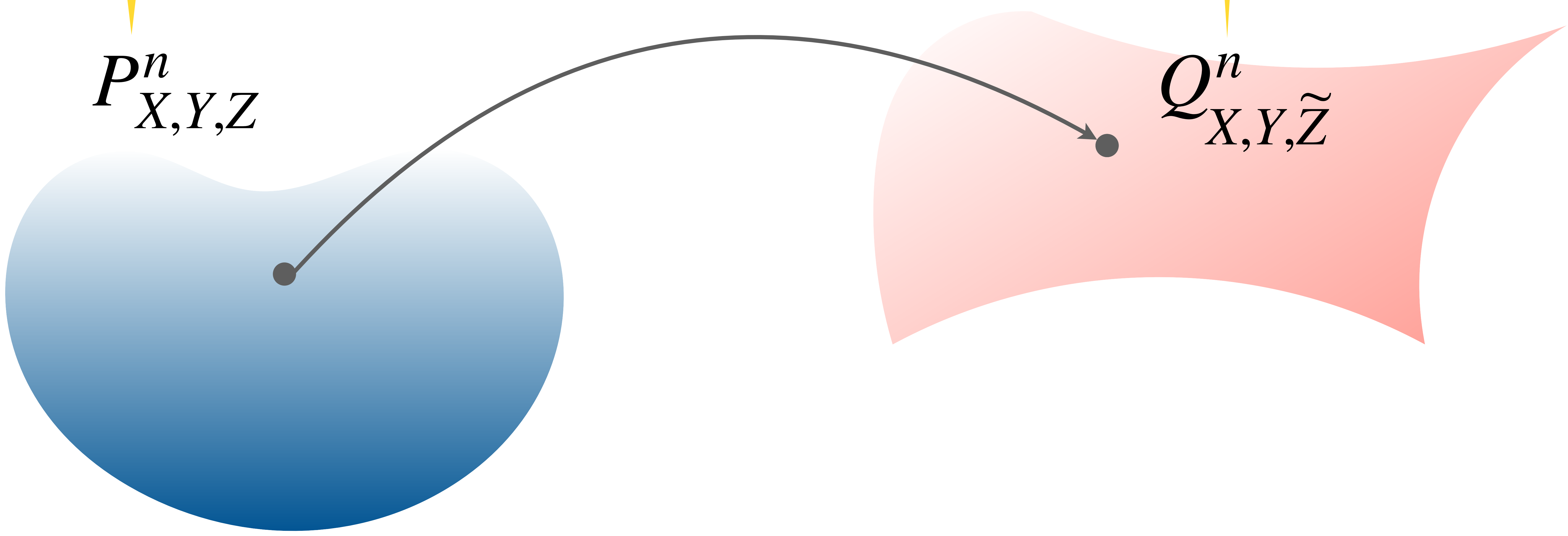

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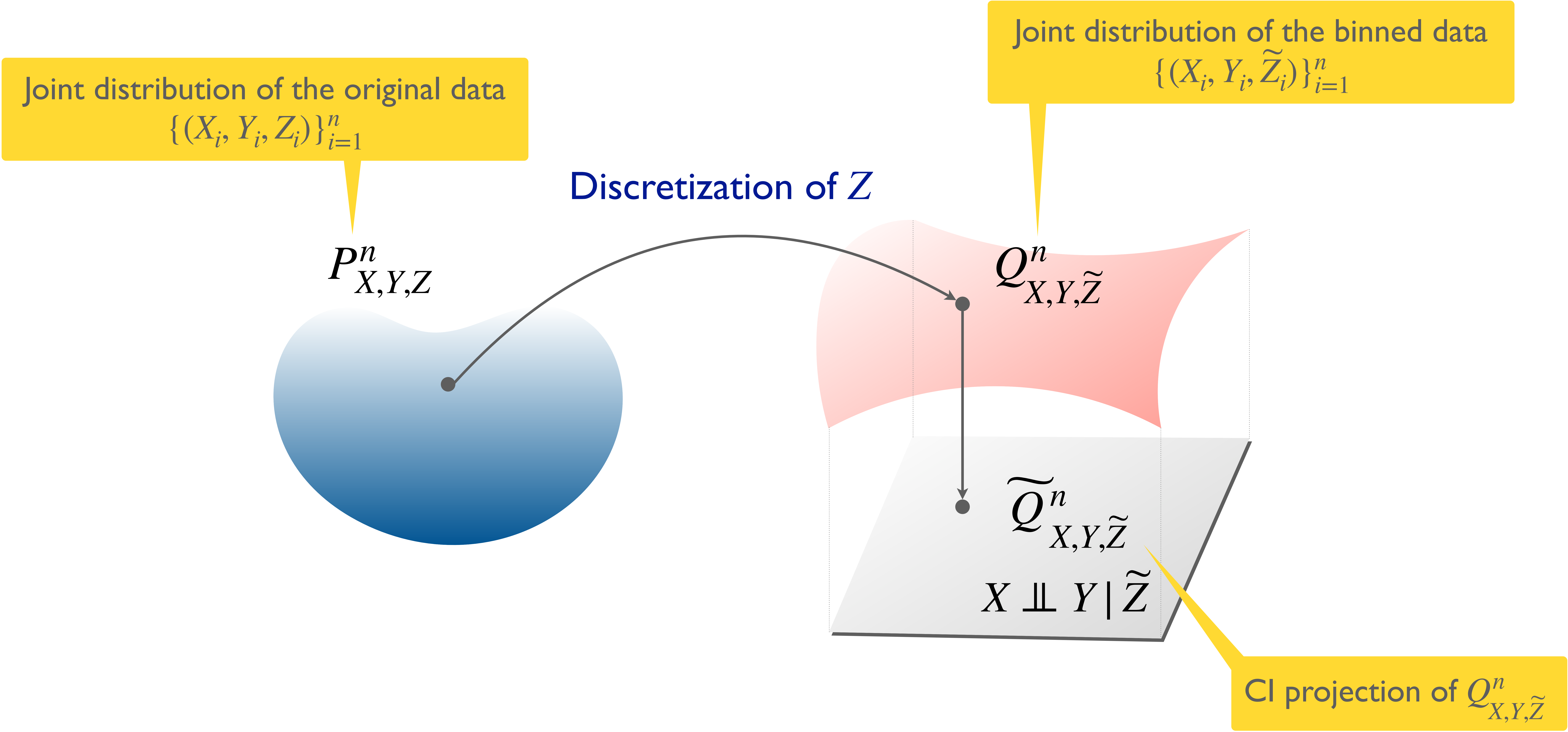

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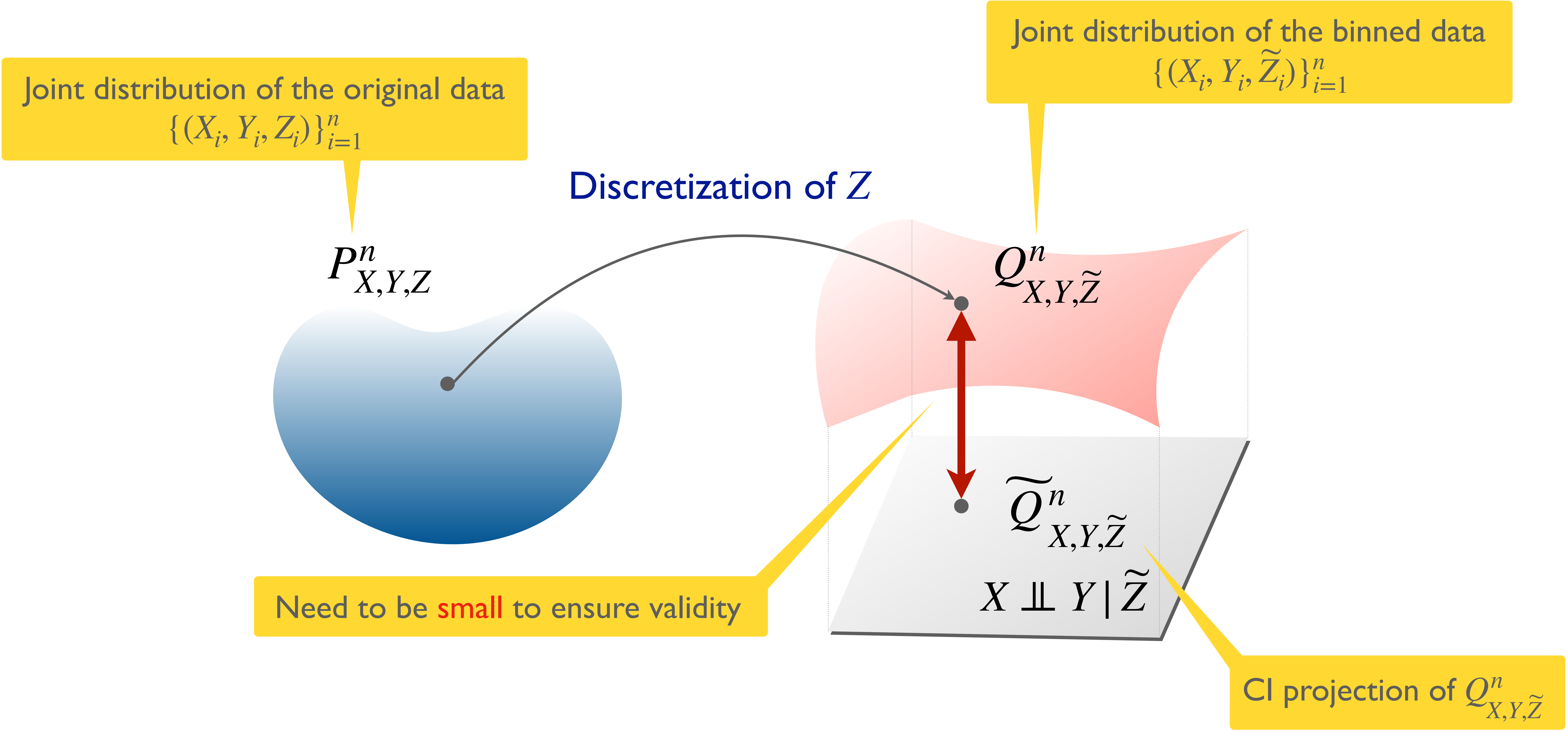

Validity of Local Permutation Tests

Lemma [KNBW 2022]

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$$\mathbb{P}_{P^n_{X,Y,Z}}(T > q_{1-\alpha}) \leq \alpha + d_{\mathrm{TV}}\left(Q^n_{X,Y,\widetilde{Z}}, \widetilde{Q}^n_{X,Y,\widetilde{Z}}\right)$$

*Total variation distance

$$d_{\mathrm{TV}}(P, Q) = \sup_{A \in \mathcal{F}} |P(A) - Q(A)| = \frac{1}{2}\|P - Q\|_1$$

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CI projection of $Q^n_{X,Y,\widetilde{Z}}$

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Discretized Distribution

Validity of Local Permutation Tests

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The bound is **universal** but **abstract**

Next goal: make this bound more **explicit** under smoothness conditions

Validity of Local Permutation Tests

Definition [Generalized Hellinger distance]

Given $\gamma \geq 1$, the **generalized Hellinger distance** with parameter γ between P and Q is defined as

$$d_\gamma(P, Q) = \left(\frac{1}{2} \int | p^{1/\gamma} - q^{1/\gamma} |^\gamma \, d\mu \right)^{1/\gamma}$$

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For example

$$d_\gamma(P, Q) = \mathrm{TV}(P, Q) \quad \text{when } \gamma = 1$$

$$d_\gamma(P, Q) = \mathrm{Hellinger}(P, Q) \quad \text{when } \gamma = 2$$

Validity of Local Permutation Tests

Definition [γ-Hellinger Lipschitzness]

Let $\mathscr{P}_{0,\gamma}(L) \subset \mathscr{P}_0$ be the collection of null distributions such that for all $z, z' \in \mathscr{Z}$

$$d_\gamma(P_{X\,|\,Z=z}, P_{X\,|\,Z=z'}) \leq L\delta(z, z') \quad \text{and} \quad d_\gamma(P_{Y\,|\,Z=z}, P_{Y\,|\,Z=z'}) \leq L\delta(z, z')$$

where $\delta(z, z')$ is a distance between z and z' (e.g., Euclidean distance)

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In a nutshell

Both $P_{X \mid Z=z}$ and $P_{Y \mid Z=z}$ are **smooth** functions with respect to z

Validity of Local Permutation Tests

Theorem [KNBW 2022]

- Let h be the maximum diameter of bins $B_1, \ldots, B_M$

Validity of Local Permutation Tests

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- Let h be the maximum diameter of bins $B_1, \ldots, B_M$
- For any $\alpha \in (0,1)$, the **Type I error** of the local permutation test over $\mathscr{P}_{0,\gamma}(L)$ is bounded above by

$$\sup_{P_{X,Y,Z} \in \mathscr{P}_{0,\gamma}(L)} \mathbb{P}_{P^n_{X,Y,Z}}(T > q_{1-\alpha}) \leq \begin{cases} \alpha + O\big(\sqrt{n} h^{\gamma}\big), & \text{if } \gamma \in [1,2] \\ \alpha + O\big(\sqrt{n} h^{2}\big), & \text{if } \gamma > 2 \end{cases}$$

Validity of Local Permutation Tests

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- If $\sqrt{n}h^{\gamma} \to \infty$, then we can construct T such that **the type I error blows up**

Validity of Local Permutation Tests

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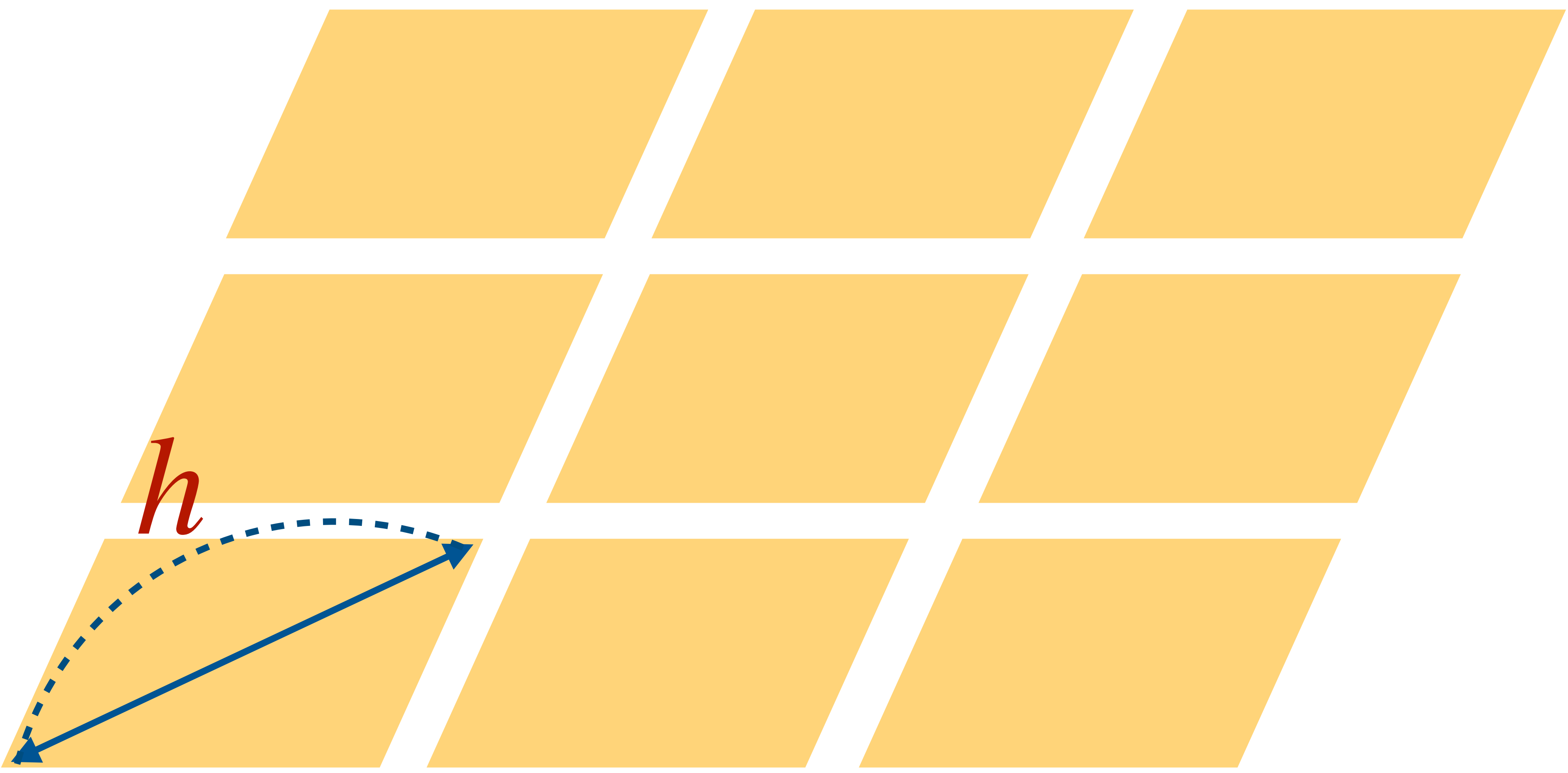
- If $\sqrt{n} h^{\gamma} \to 0$, then the local permutation test is **asymptotically valid**
- If $\sqrt{n} h^{\gamma} \to \infty$, then we can construct T such that **the type I error blows up**
- We need to take h to be small to ensure **type I error control**

Trade-off between Type I error and Type II error

Small h

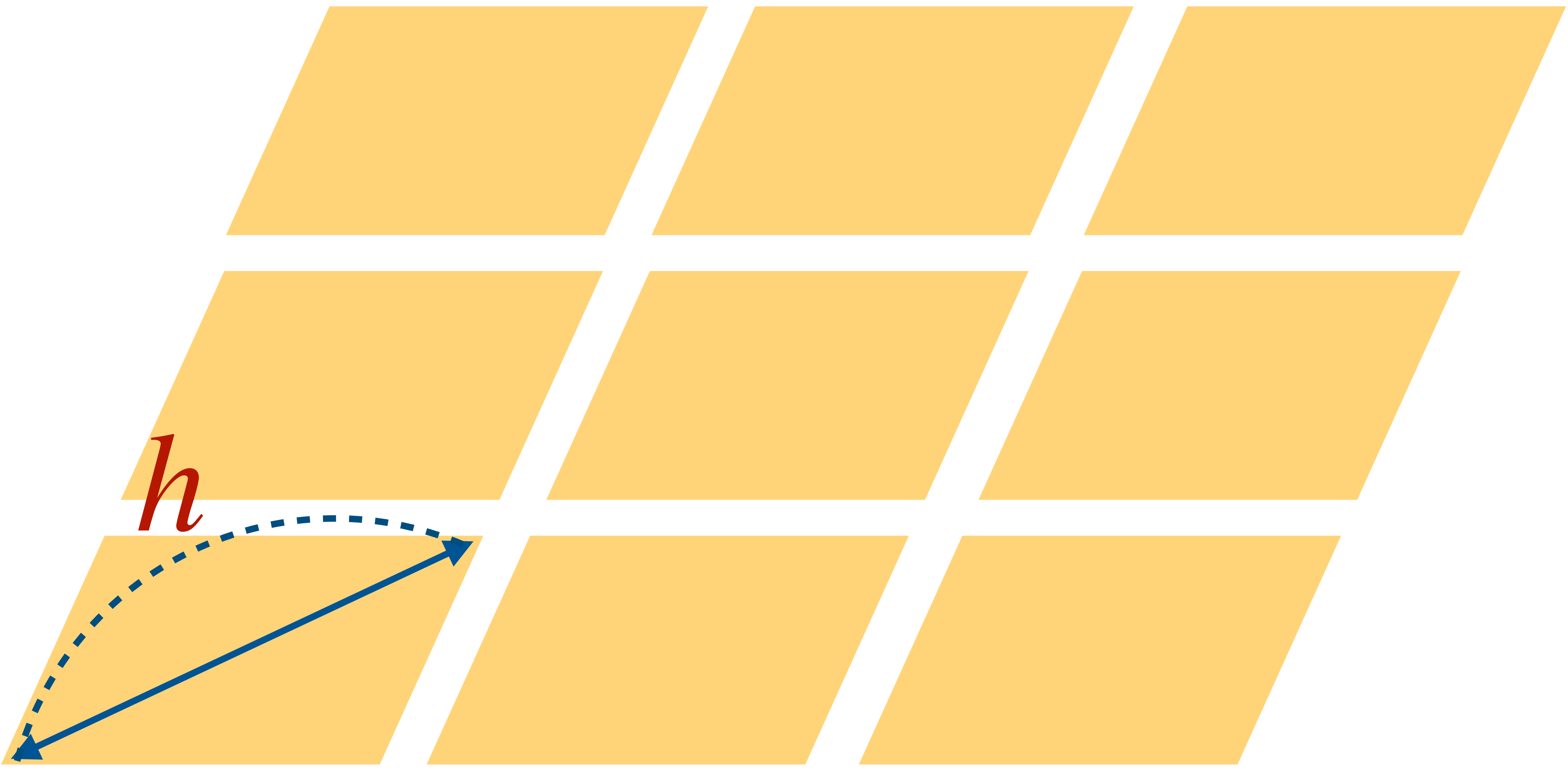


Large h

Trade-off between Type I error and Type II error

Type I error ↓ Type II error ↑

Small h



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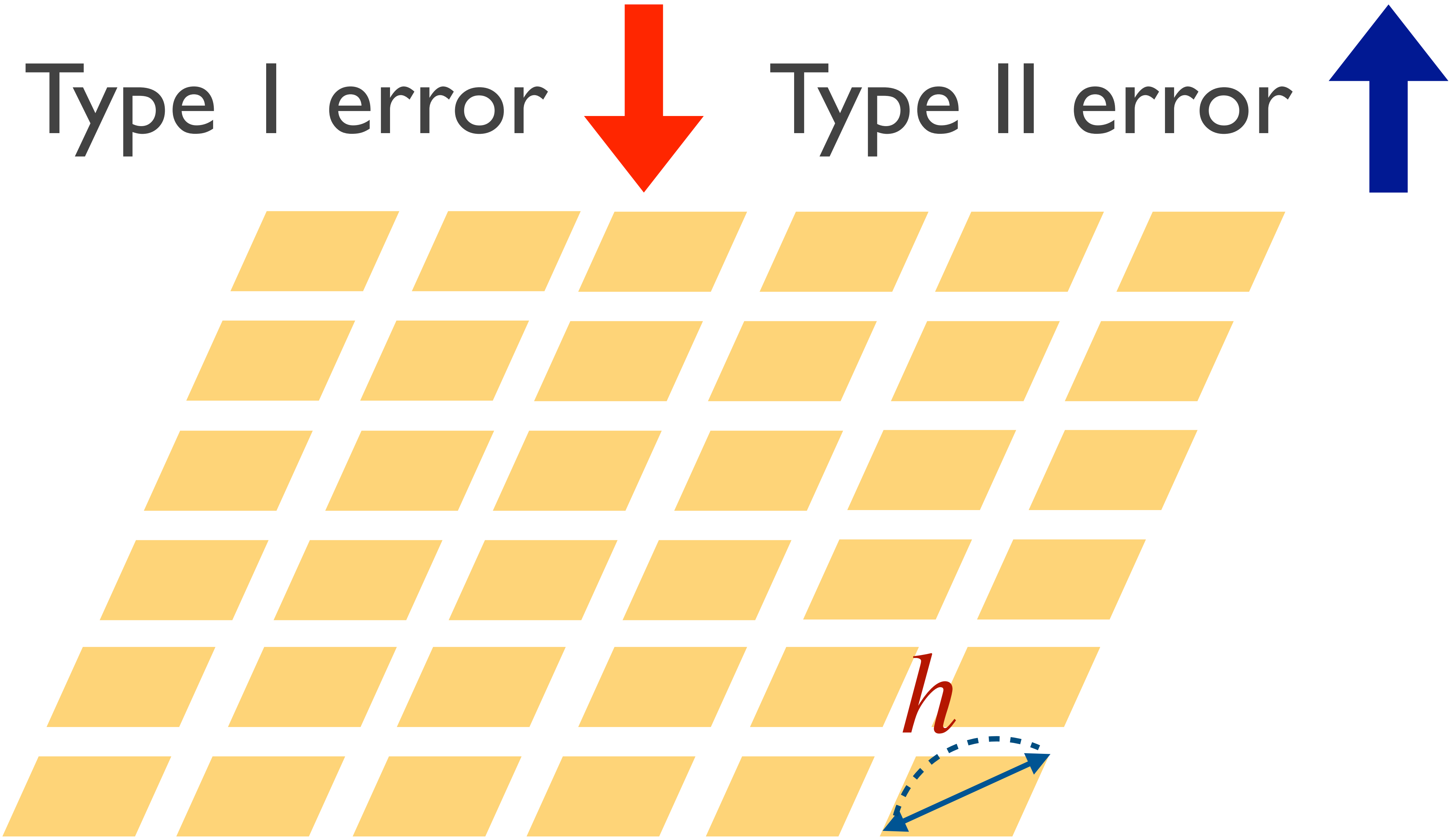

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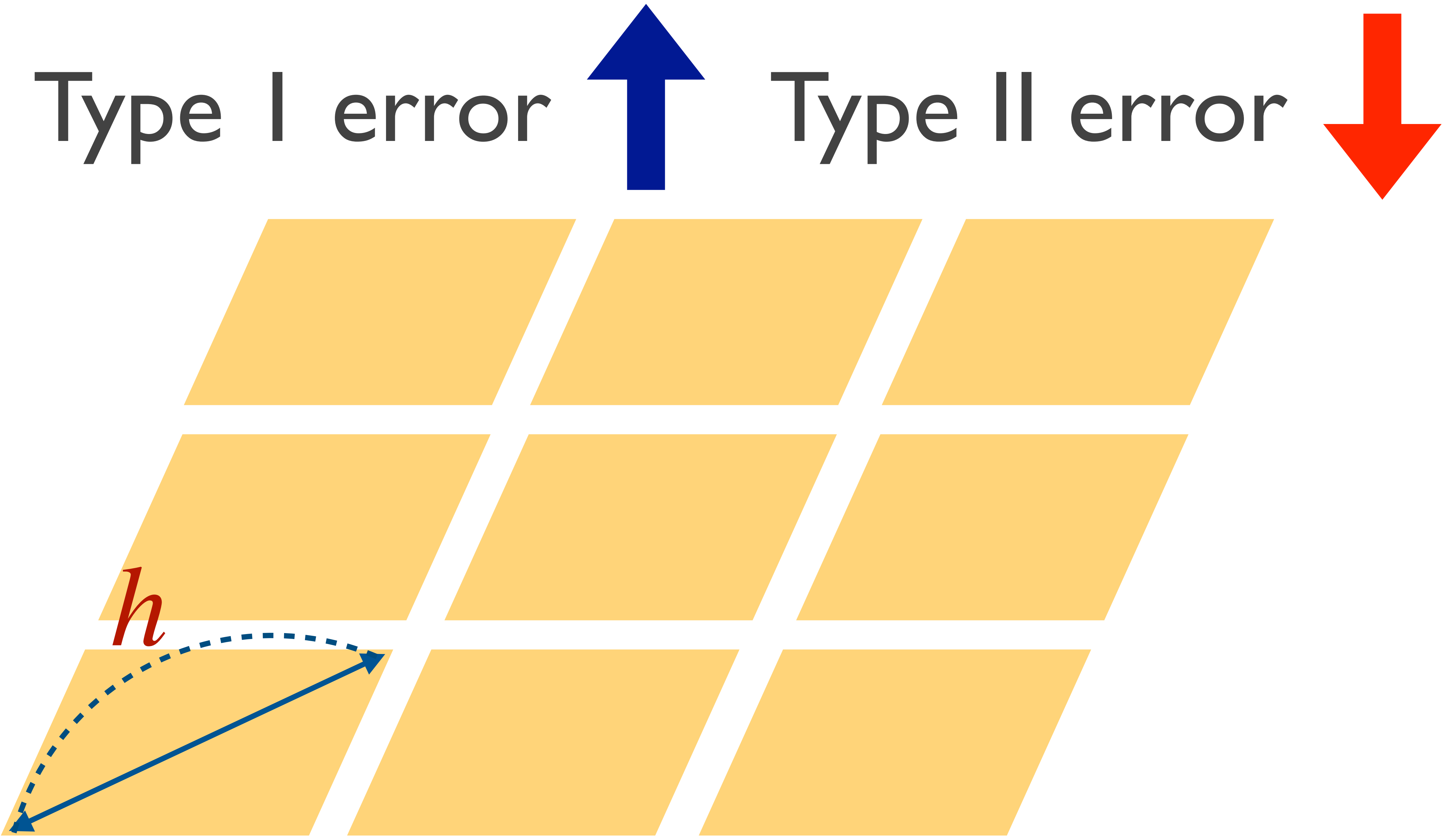

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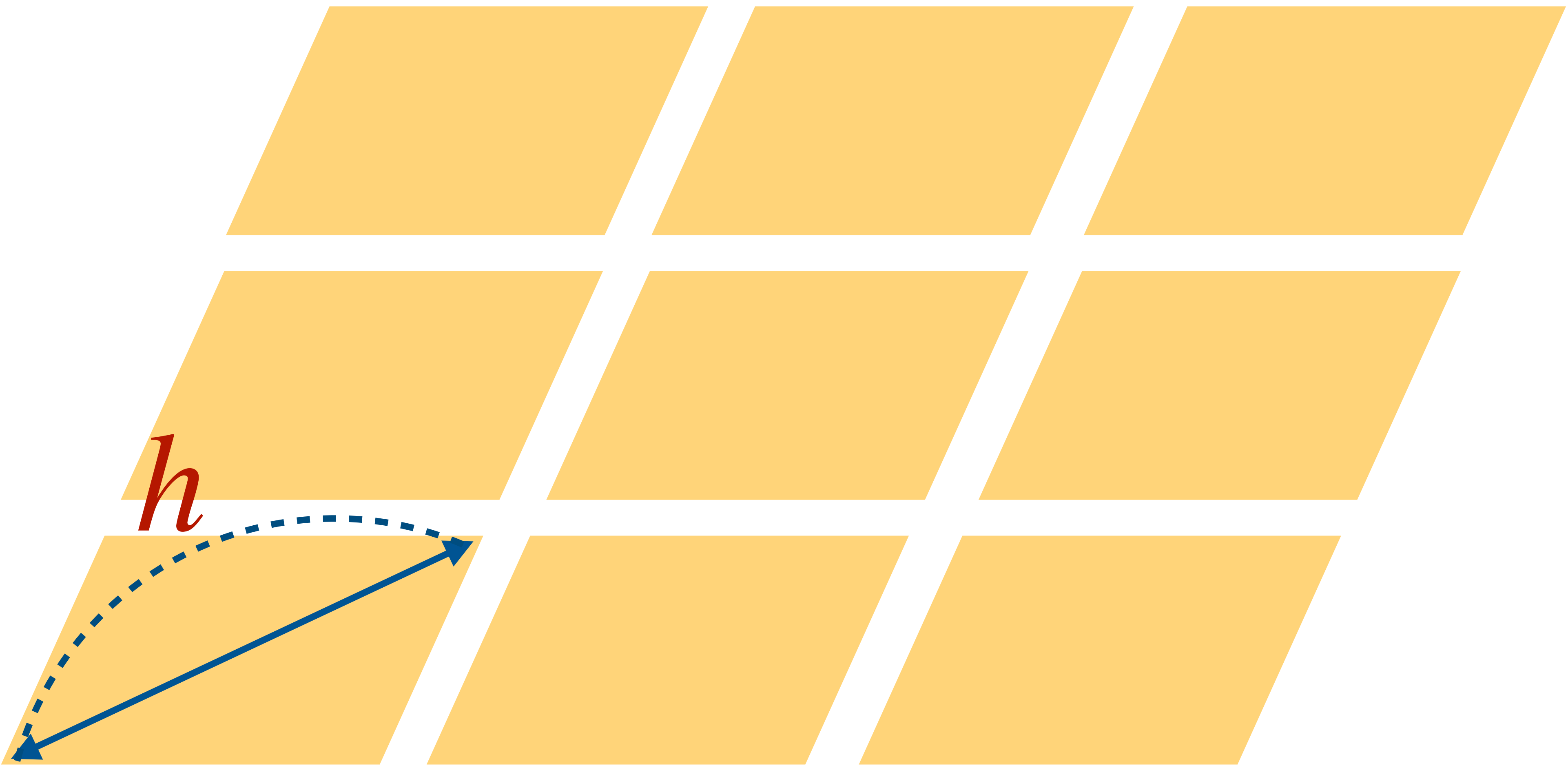


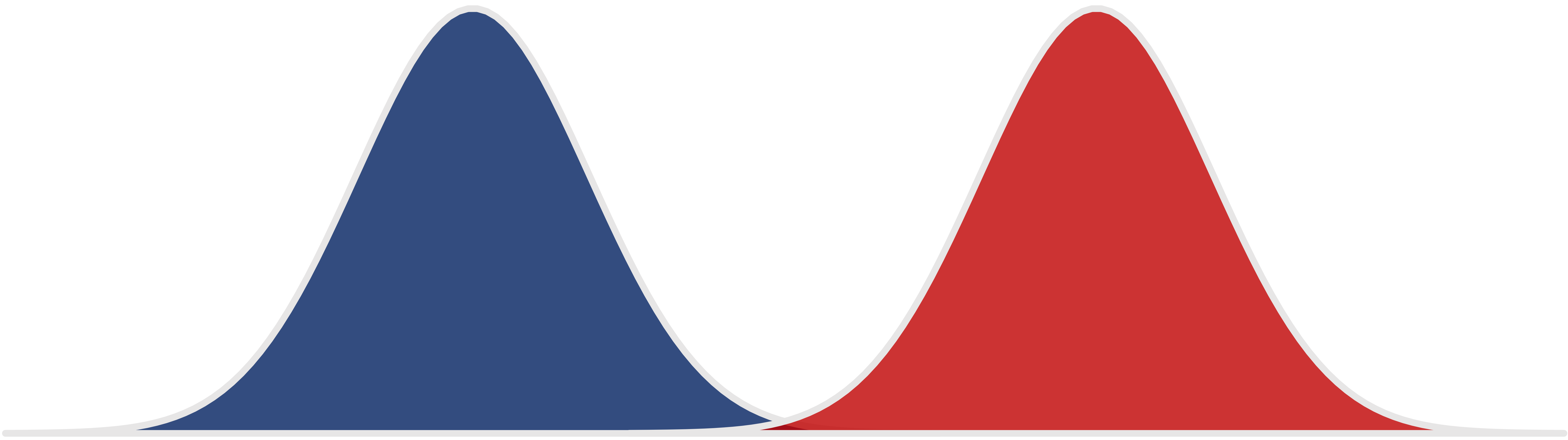
Large h

We need to balance this trade-off

Local permutation tests can achieve a certain **minimax optimality** by choosing h carefully

Detour: Minimax Testing Framework

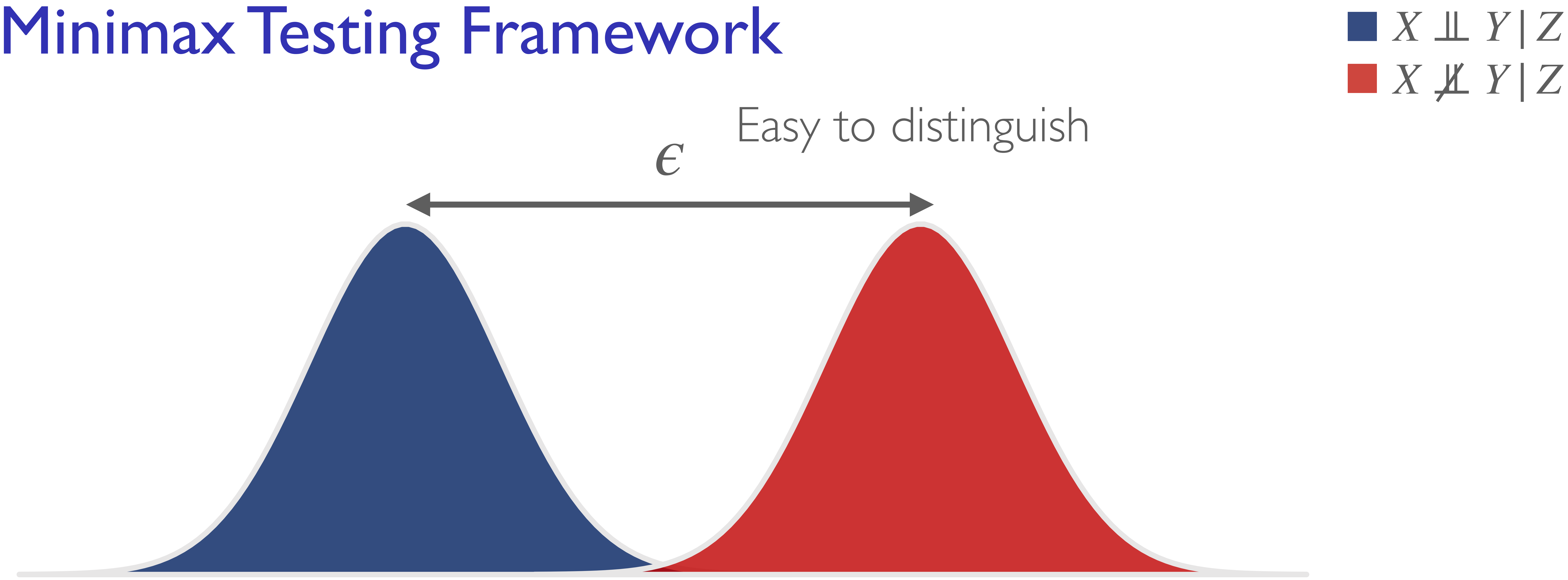
■ $X \perp\!\!\!\perp Y \mid Z$
■ $X \not\!\perp\!\!\!\perp Y \mid Z$



$$\sup_{P \in \mathscr{P}_0} \mathbb{E}_P[\phi] \quad + \quad \sup_{P \in \mathscr{P}_1(\epsilon)} \mathbb{E}_P[1 - \phi]$$

Worst-case type I error — Worst-case type II error

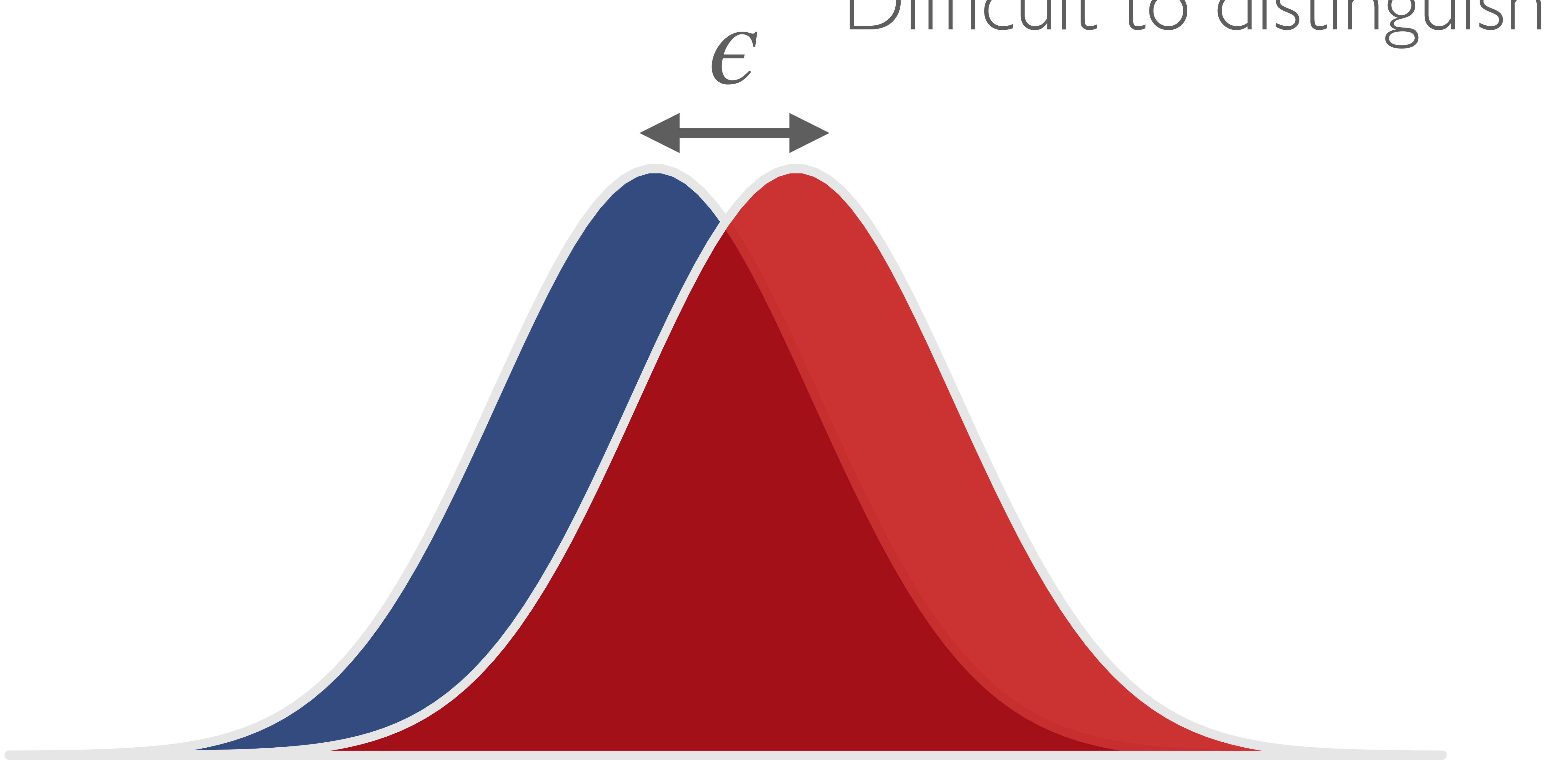
Detour: Minimax Testing Framework



$$\sup_{P \in \mathscr{P}_0} \mathbb{E}_P[\phi] + \sup_{P \in \mathscr{P}_1(\epsilon)} \mathbb{E}_P[1-\phi] \approx 0$$

Worst-case type I error   Worst-case type II error

Detour: Minimax Testing Framework



$$\underbrace{\sup_{P \in \mathscr{P}_0} \mathbb{E}_P[\phi]}_{\text{Worst-case type I error}} + \underbrace{\sup_{P \in \mathscr{P}_1(\epsilon)} \mathbb{E}_P[1 - \phi]}_{\text{Worst-case type II error}} \approx 1$$

Detour: Minimax Testing Framework

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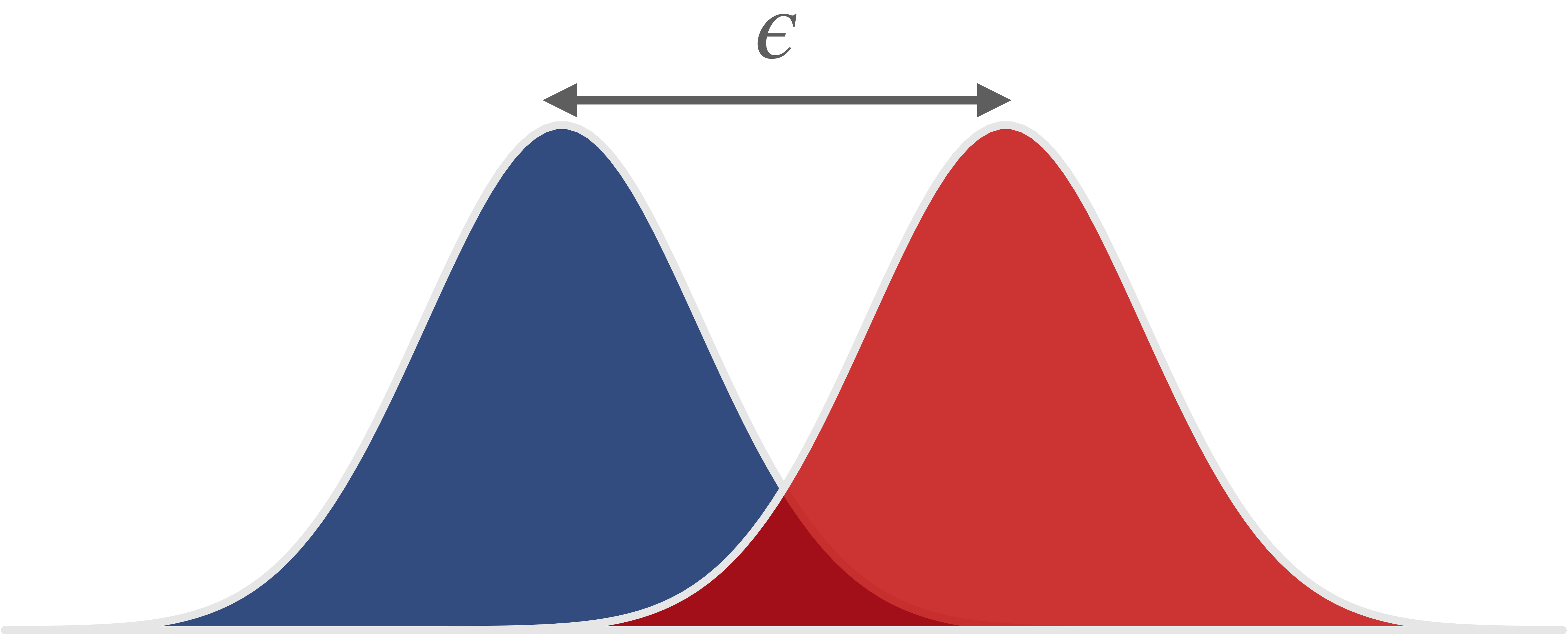

$$\sup_{P \in \mathscr{P}_0} \mathbb{E}_P[\phi] + \sup_{P \in \mathscr{P}_1(\epsilon)} \mathbb{E}_P[1 - \phi] \approx 0.1$$

Worst-case type I error   Worst-case type II error

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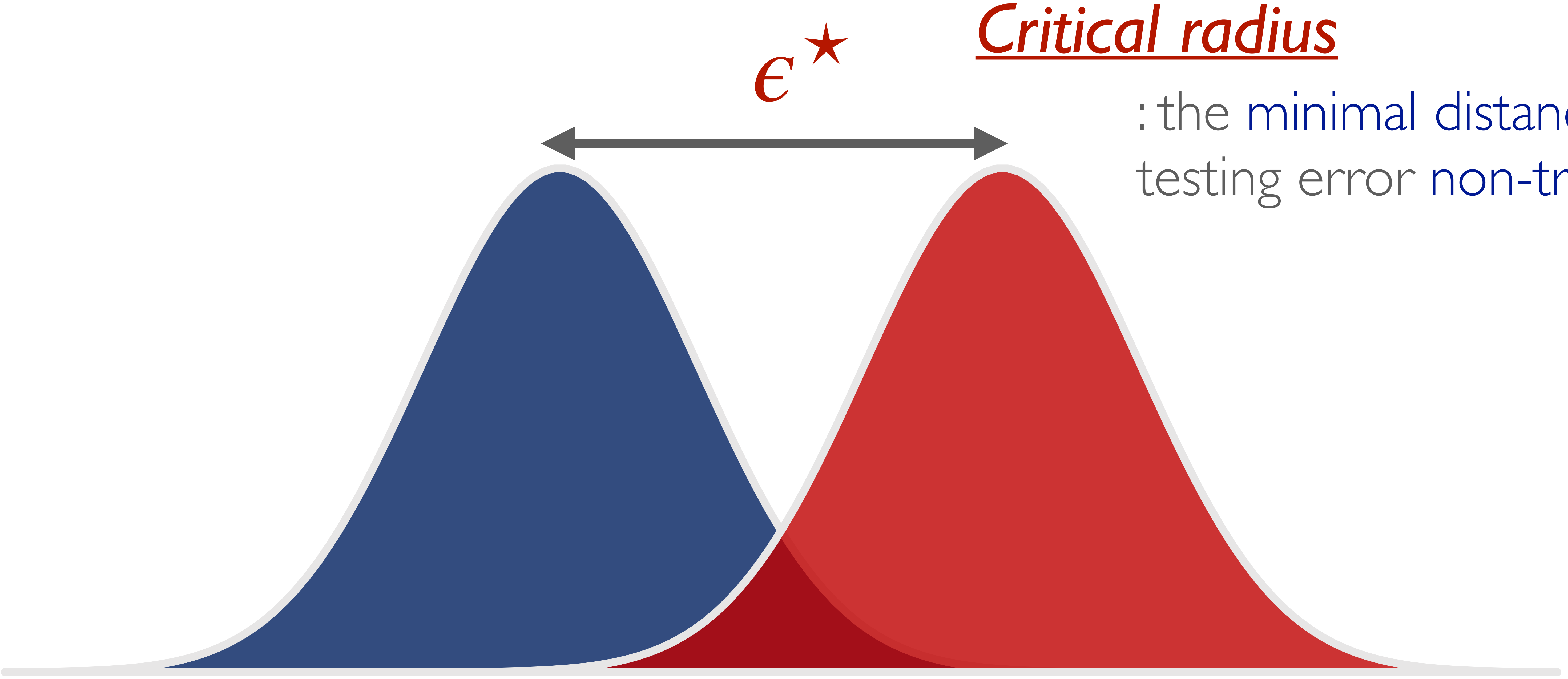


Critical radius

: the minimal distance that makes the testing error non-trivial for an optimal test

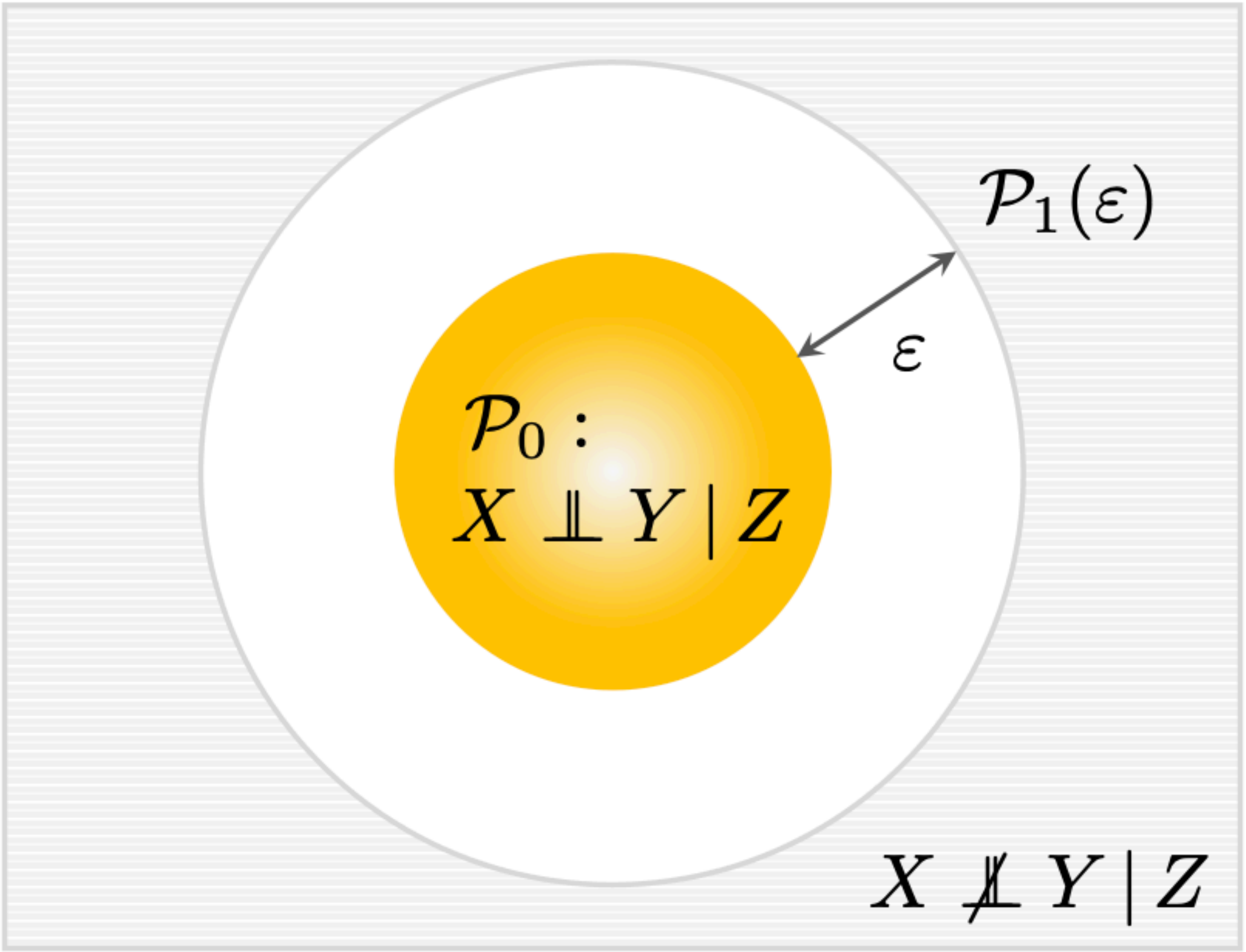
$$\inf_{\phi \in \Phi} \left\{ \sup_{P \in \mathcal{P}_0} \mathbb{E}_P[\phi] + \sup_{P \in \mathcal{P}_1(\epsilon)} \mathbb{E}_P[1 - \phi] \right\} \approx 0.1$$

Worst-case type I error | Worst-case type II error

Detour: Minimax Testing Framework

- $\mathrm{Risk}(\epsilon) = \inf_{\phi \in \Phi} \left\{ \underbrace{\sup_{P \in \mathscr{P}_0} \mathbb{E}_P[\phi]}_{\text{Worst-case type I error}} + \underbrace{\sup_{P \in \mathscr{P}_1(\epsilon)} \mathbb{E}_P[1 - \phi]}_{\text{Worst-case type II error}} \right\}$

A set of all possible tests



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Worst-case type I error

Worst-case type II error

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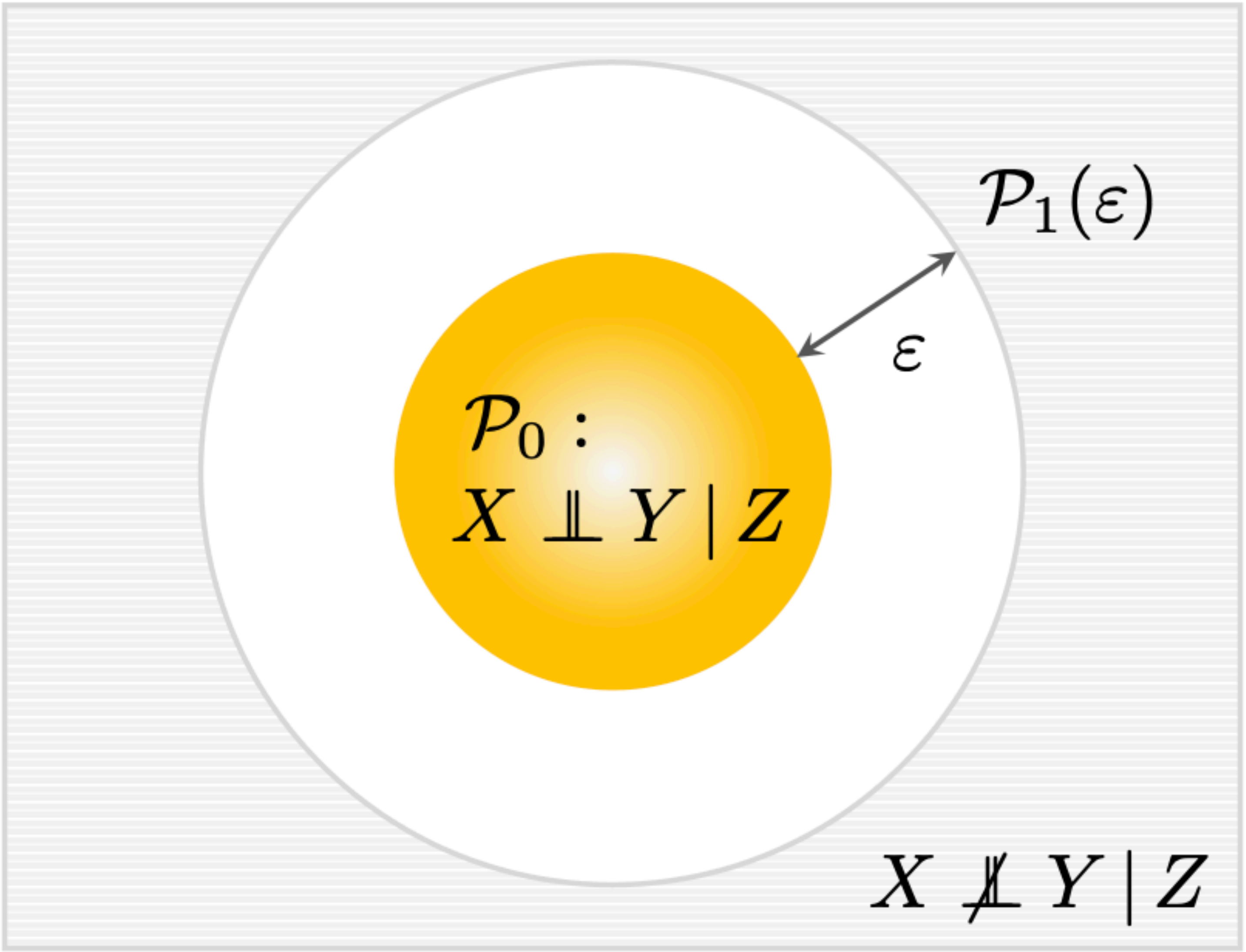
- $\epsilon^\star = \inf\left\{\epsilon : \mathrm{Risk}(\epsilon) \leq 0.1\right\}$



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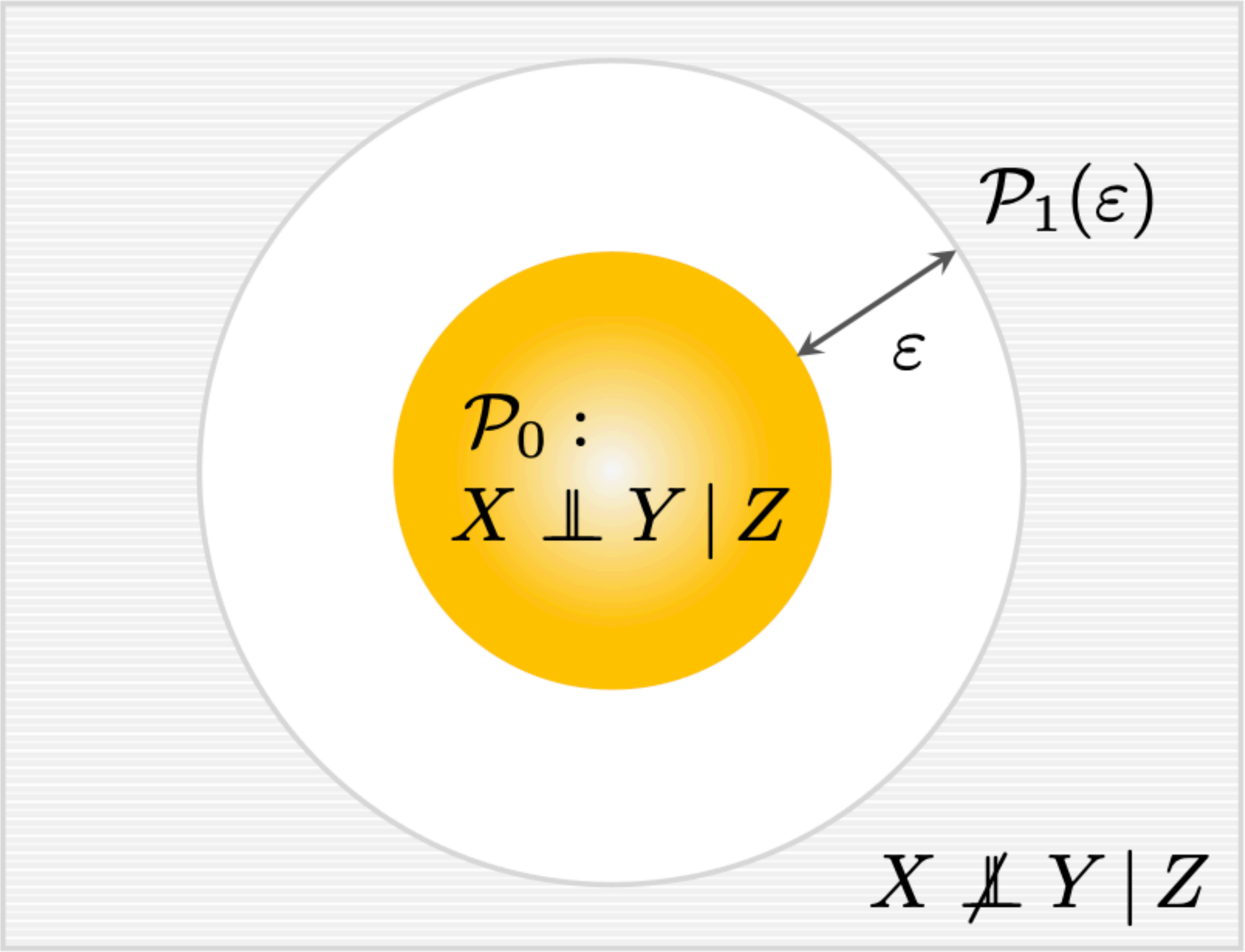
Worst-case type I error   Worst-case type II error

A set of all possible tests

- $\epsilon^\star = \inf\{\epsilon : \mathrm{Risk}(\epsilon) \le 0.1\}$

- We say that ϕ is **minimax optimal** if

$$\sup_{P \in \mathscr{P}_0} \mathbb{E}_P[\phi] + \sup_{P \in \mathscr{P}_1(\epsilon^\star)} \mathbb{E}_P[1-\phi] \le 0.1$$



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Worst-case type I error   Worst-case type II error

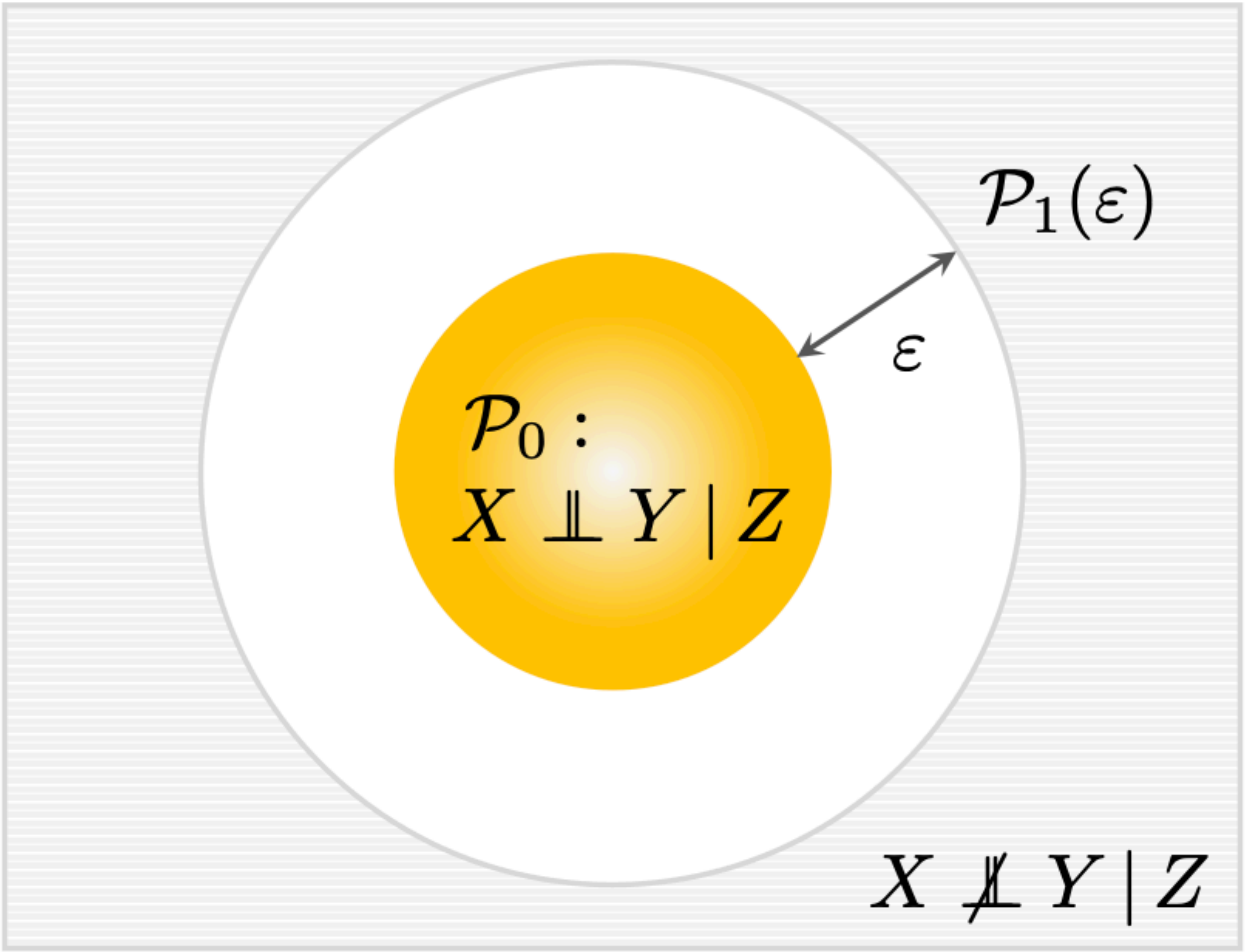
A set of all possible tests

- $\epsilon^\star = \inf\{\epsilon : \text{Risk}(\epsilon) \leq 0.1\}$

- We say that ϕ is minimax "rate" optimal if

$$\sup_{P \in \mathscr{P}_0} \mathbb{E}_P[\phi] + \sup_{P \in \mathscr{P}_1(\epsilon)} \mathbb{E}_P[1 - \phi] \leq 0.1$$

whenever $\epsilon \asymp \epsilon^\star$



Optimality results of Neykov et al. (2021)

Let $(X, Y, Z) \in [0,1]^3$

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Class of null distributions

$\mathscr{P}_{0,\mathrm{NBL}}$: TV or χ^2 smooth (Z) + $X \perp\!\!\!\perp Y \mid Z$

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Class of alternative distributions

$\mathscr{P}_{1,\mathrm{NBL}}(\epsilon)$: TV smooth ($Z$) + s-Hölder smooth (X, Y) + $\inf_{P \in \mathscr{P}_0} d_{TV}(P, Q) \geq \epsilon$

Optimality results of Neykov et al. (2021)

Let $(X, Y, Z) \in [0,1]^3$

$H_0 : P \in \mathscr{P}_{0,\mathrm{NBL}}$ versus

$H_1 : P \in \mathscr{P}_{1,\mathrm{NBL}}(\epsilon)$

Class of null distributions

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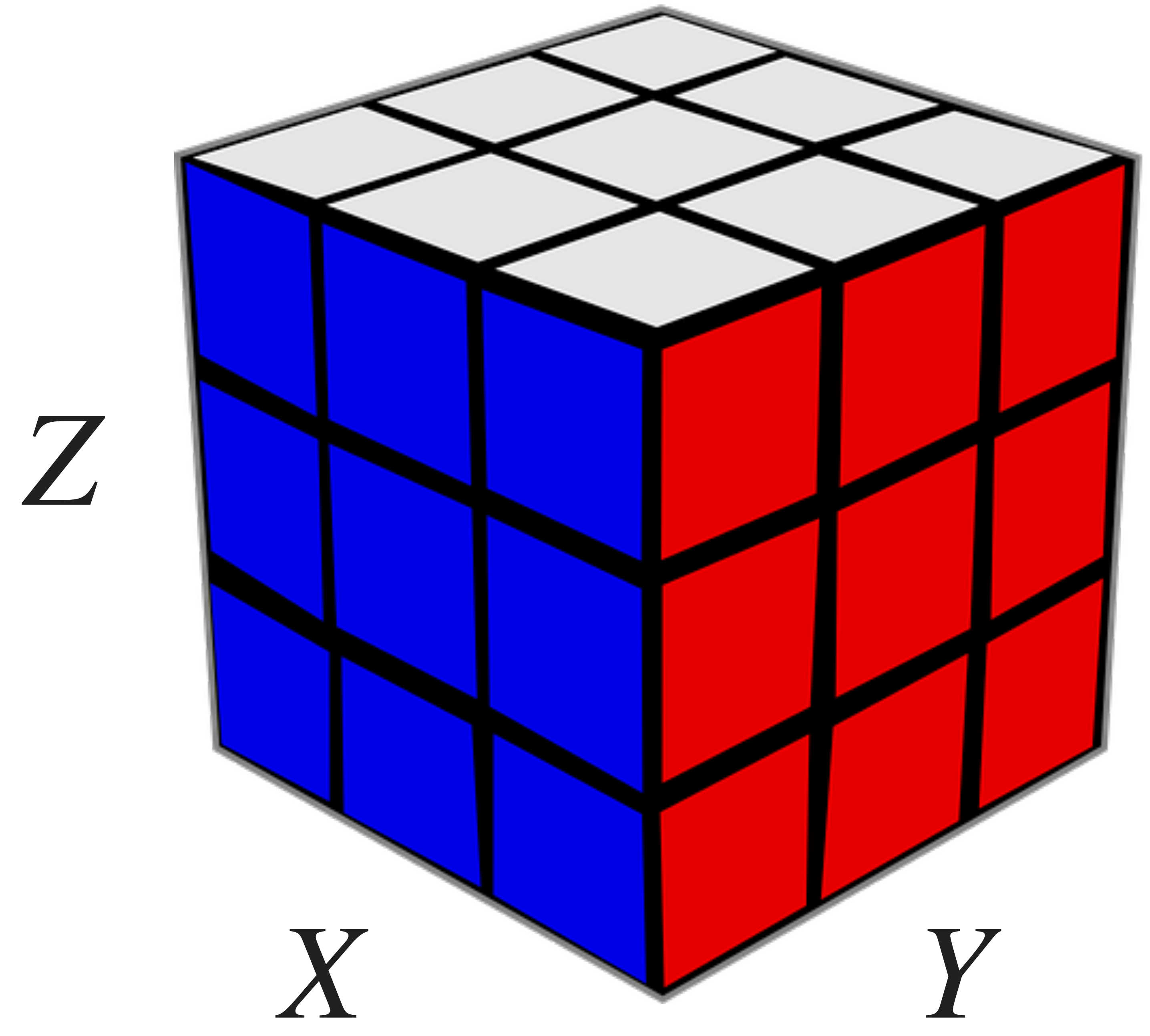
Neykov et al. (2021) prove that the **minimax testing rate** for this problem is

$$\epsilon_n^\star \asymp n^{-\frac{2s}{5s+2}}$$

Optimality results of Neykov et al. (2021)

Procedure

- Discretize (X, Y, Z) into $d_X \times d_Y \times d_Z$ bins



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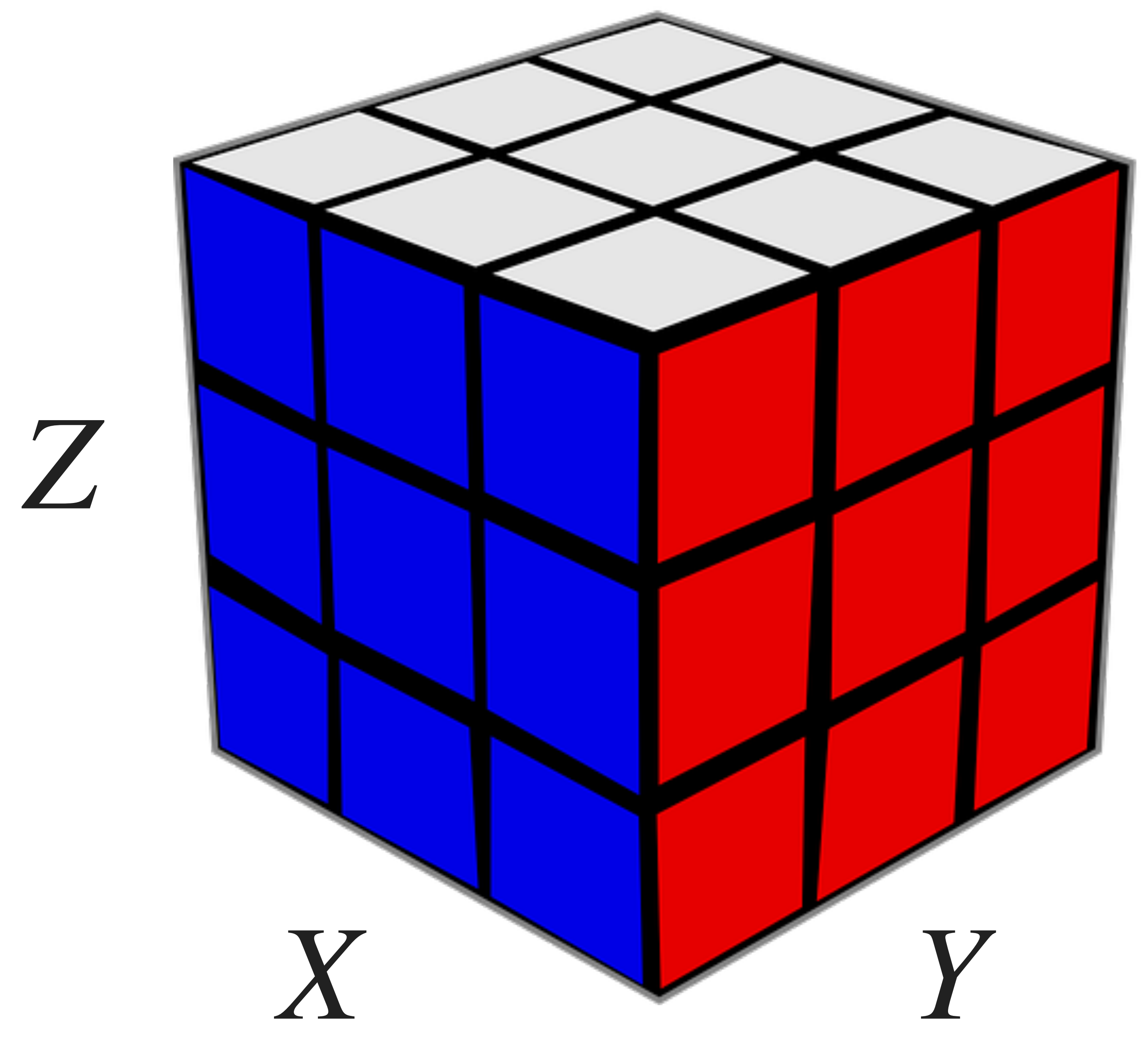

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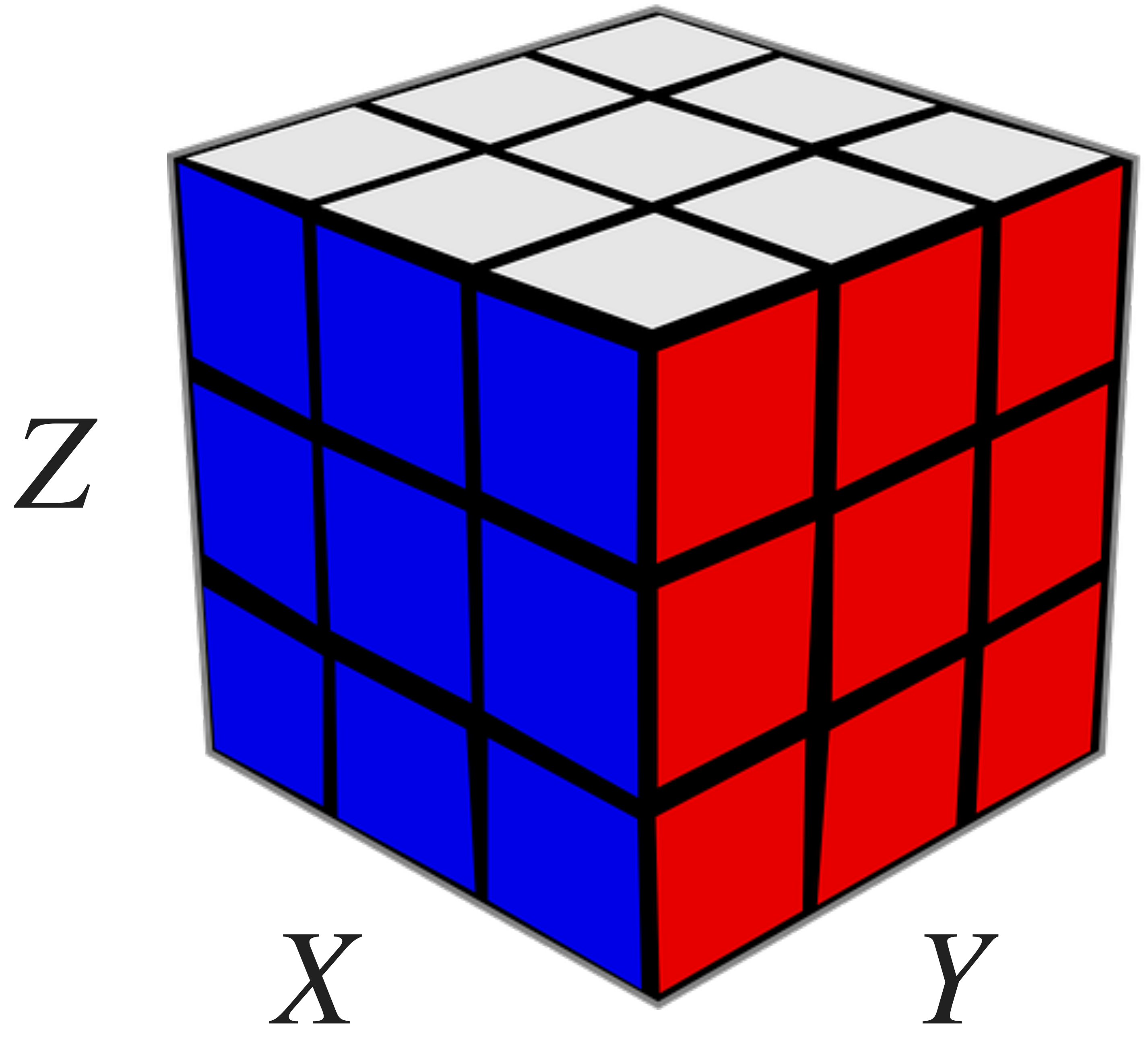
- Compute the test statistic T proposed by Canonne et al. (2018) for **discrete CI testing**



$$T = \sum_{i=1}^{d_Z} w_i U_i(\mathcal{D}_i)$$

Weighted sum of U-statistics

Optimality results of Neykov et al. (2021)

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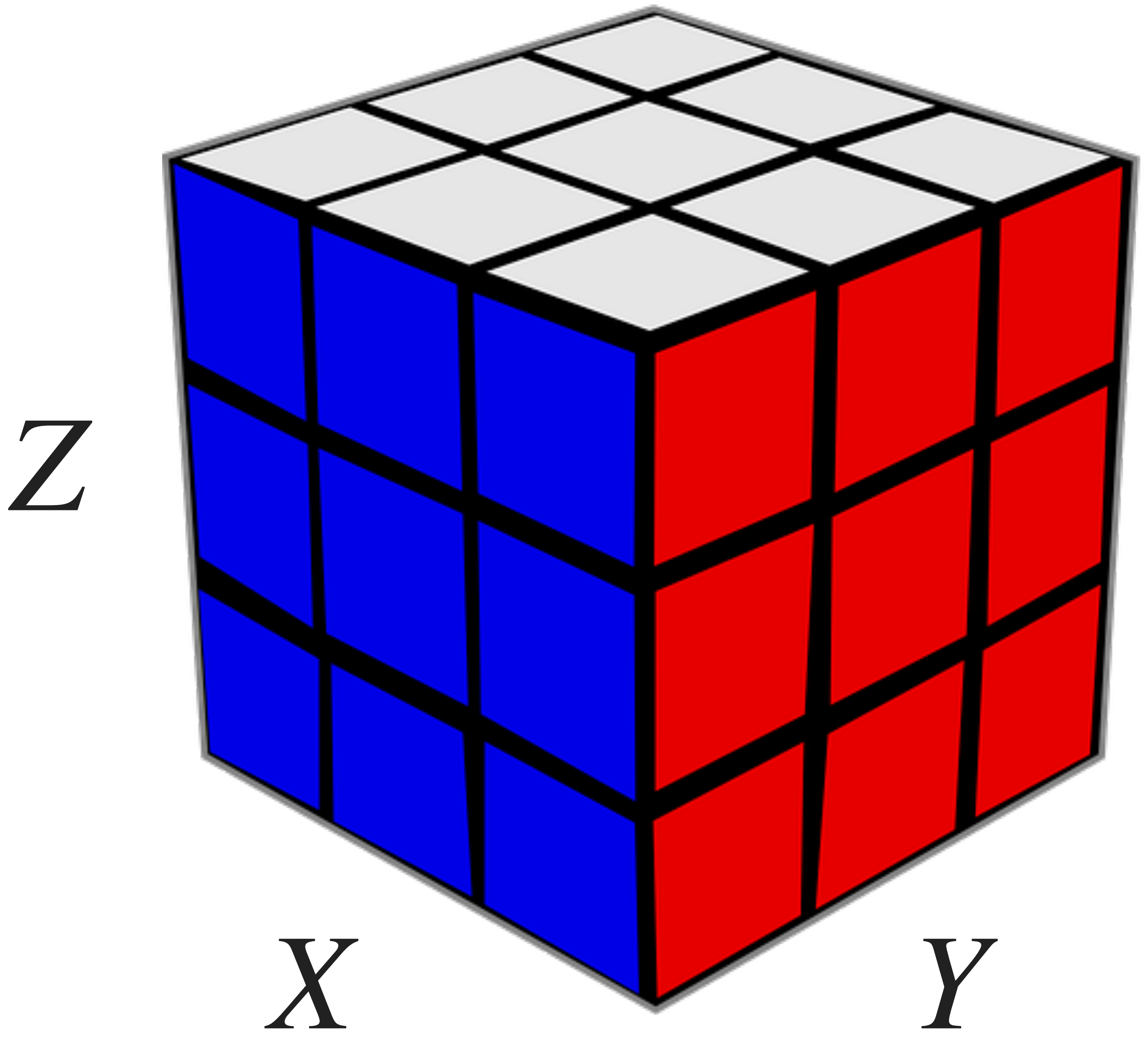

Optimality results of Neykov et al. (2021)

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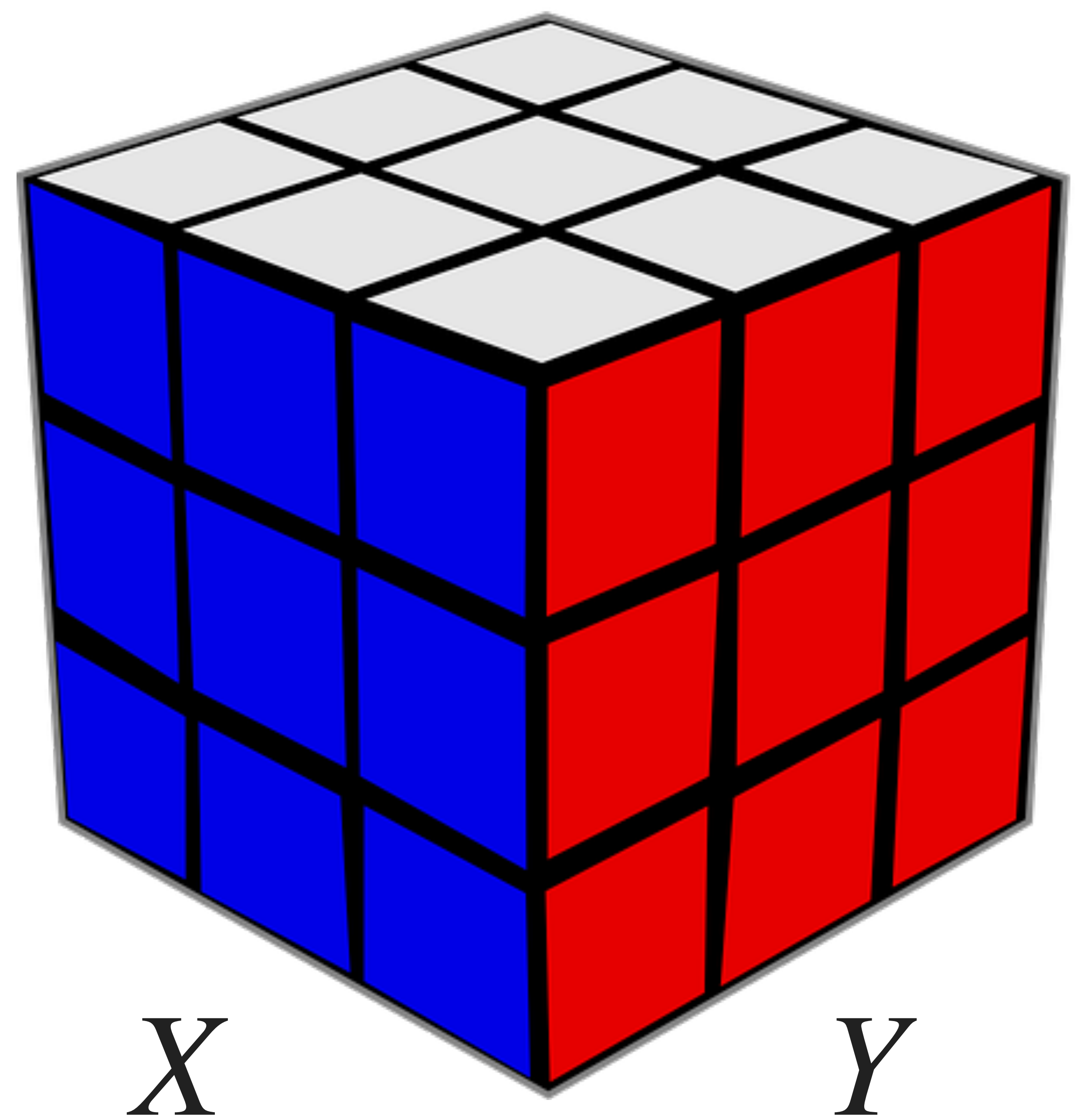

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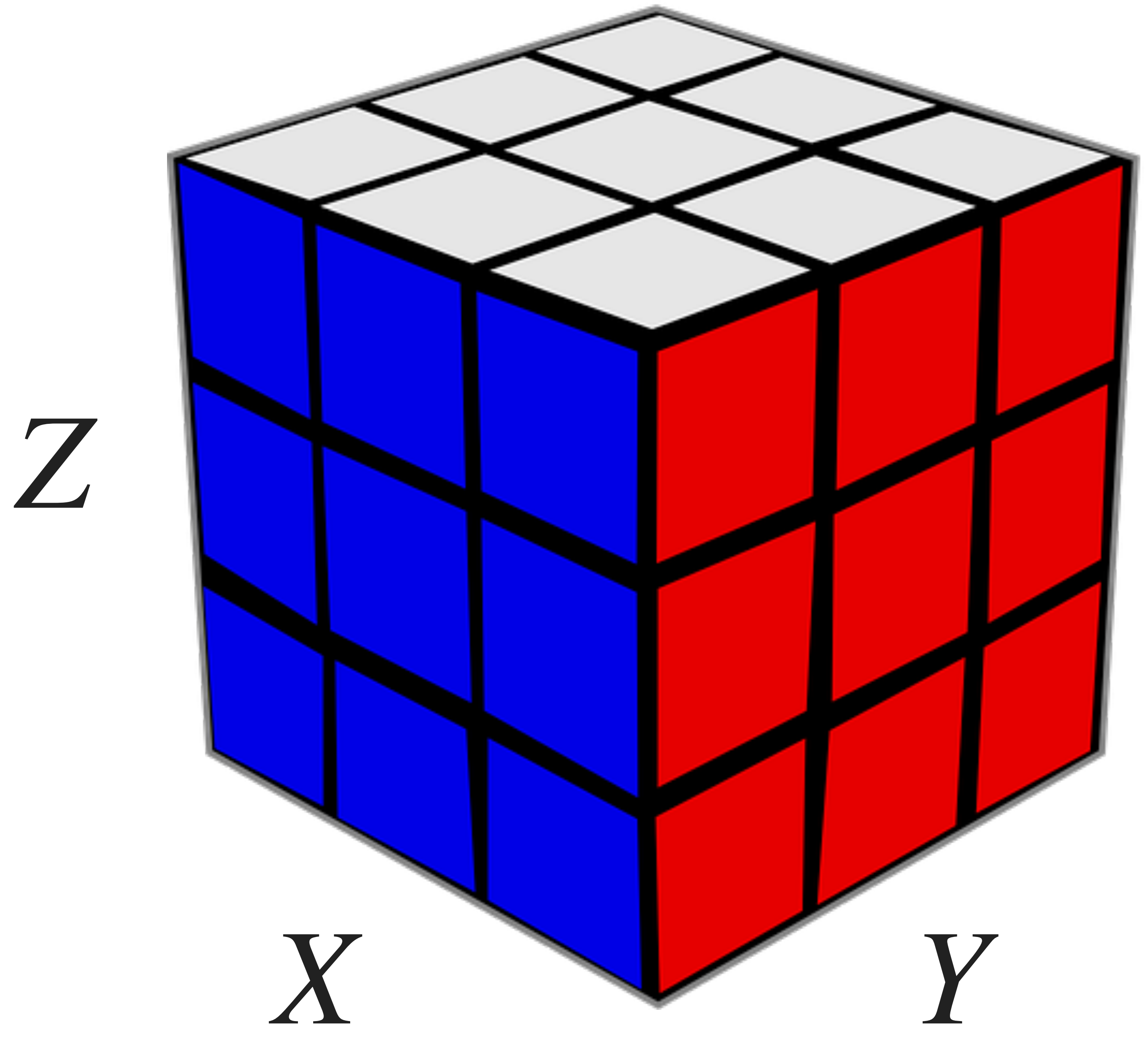
- It achieves the **minimax optimal rate**

Optimality results of Neykov et al. (2021)

Procedure

- Discretize (X, Y, Z) into $d_X \times d_Y \times d_Z$ bins
- Set $d_X = d_Y = \lceil n^{\frac{2}{5s+2}} \rceil$ and $d_Z = \lceil n^{\frac{2s}{5s+2}} \rceil$
- Compute the test statistic T proposed by Canonne et al. (2018) for **discrete CI testing**



- **Reject** the null when $T > Cd_Z^{1/2}$ for a sufficiently large C

- It achieves the **minimax optimal rate**
- **However**, the unspecified number C makes this procedure **impractical**

In our work [KNBW 2022]

- We show that the same test statistic calibrated by the **local permutation method** achieves the **same minimax rate** $\epsilon_n^\star \asymp n^{-\frac{2s}{5s+2}}$

- **However**, the optimality can be obtained over a slightly **smaller class of null distributions** (**Hellinger smoothness** instead of **TV smoothness**)

$\Longrightarrow$ **Double-binning method**

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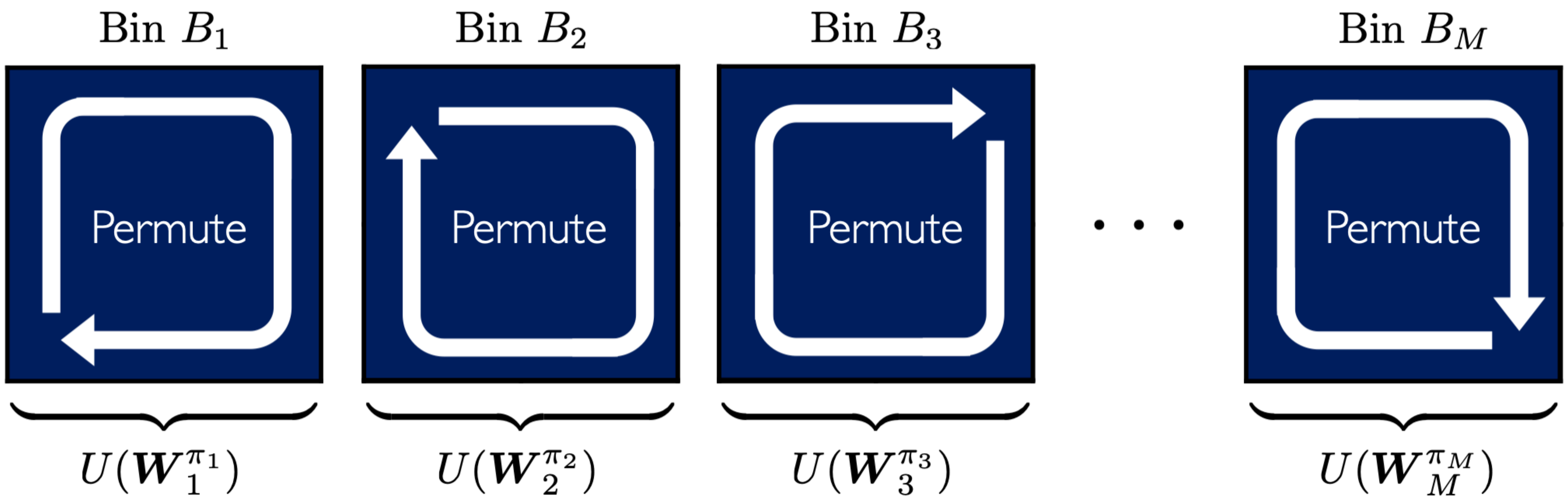
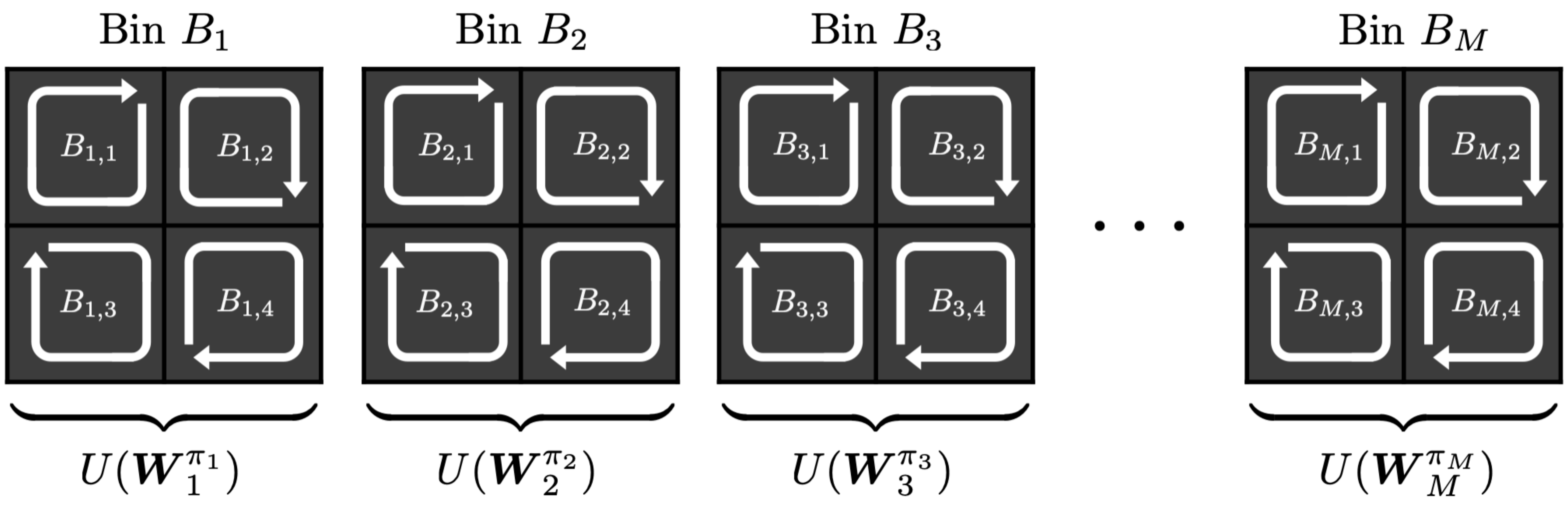
Double-binning method



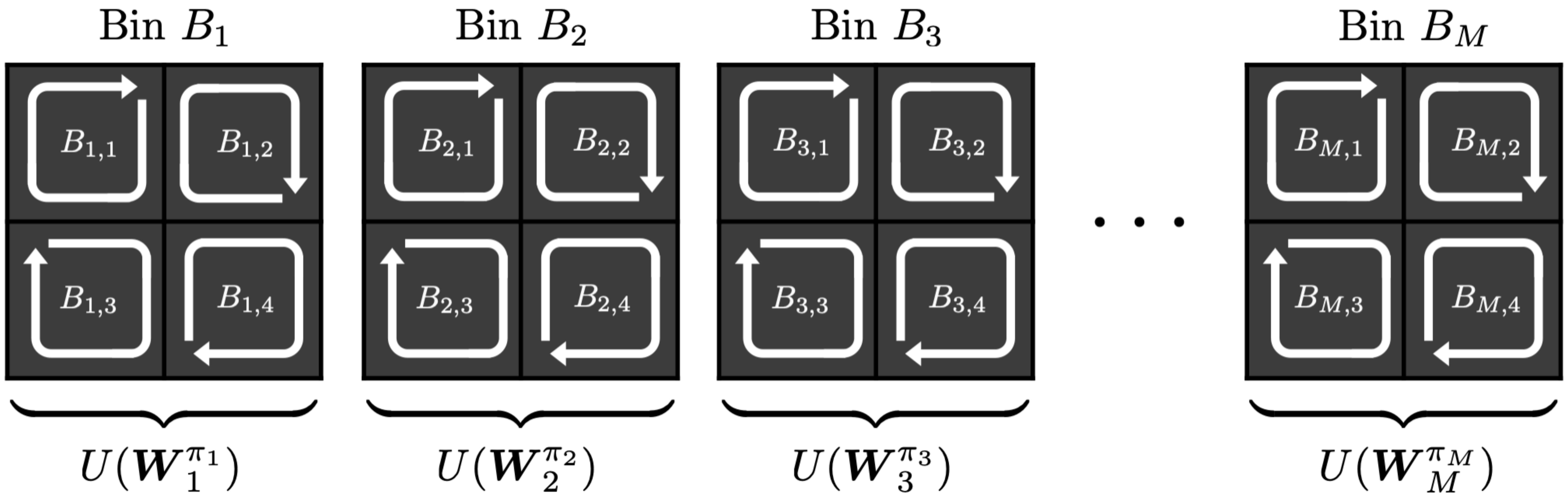
- Previously, we **permuted** a sample and **computed** a test statistic within **the same bin**

Double-binning method



- In **double-binning**, we consider bins of **two** distinct resolutions
  - **Permute** the observations within the **finer bins**
  - **Compute** test statistics over the **coarser bins**

Double-binning method



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Decrease the type I error

Increase the power

Summary

- We presented a **theoretical foundation** of local permutation tests
- We revisited the **hardness of CI testing** and showed that the hardness is determined by "**no-collision**" probability
- We illustrated that the local permutation test with a careful binning scheme can lead to **minimax optimality**

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Future Work

- **Adaptive** binning scheme for CI testing
- **Optimality** results for multivariate data
- **Hardness** results for other problems

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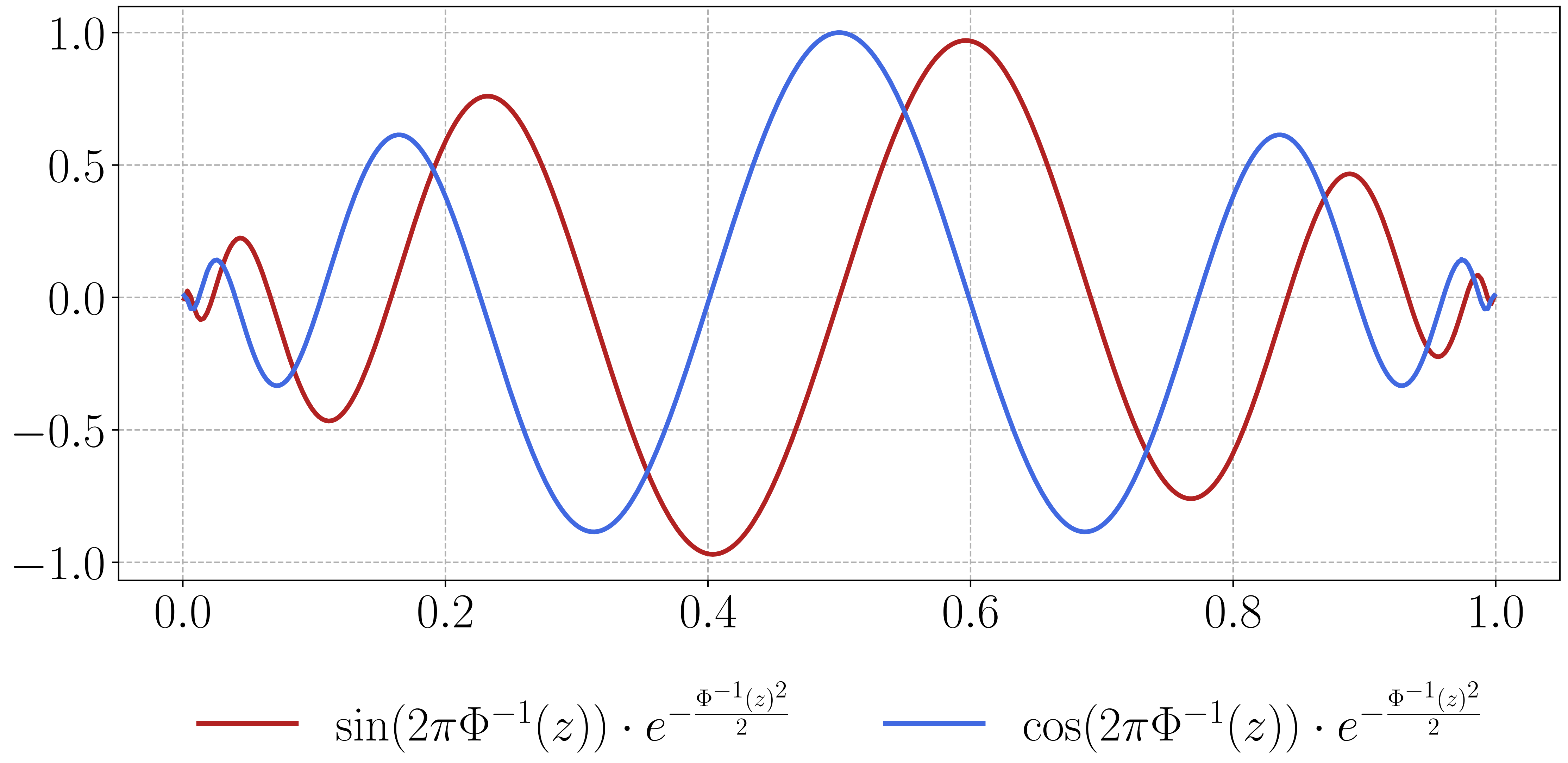
Thank you!

Simulation Results

Simulation settings

- Under the null
  - $Z \sim \text{Uniform}[0,1]$
  - $X = \sin\{2\pi\Phi^{-1}(Z)\}e^{-\Phi^{-1}(Z)^2/2} + 0.3\varepsilon_1$
  - $Y = \cos\{2\pi\Phi^{-1}(Z)\}e^{-\Phi^{-1}(Z)^2/2} + 0.3\varepsilon_1$
  - $\varepsilon_1, \varepsilon_2 \overset{i.i.d.}{\sim} N(0,1)$



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Under the alternative

- $Z \sim \text{Uniform}[0,1]$
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$U_i(W_i)$: a test statistic for independence between X and Y

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- Test statistic: $T = \sum_{i=1}^{M} U_i(W_i)$

- Sample size: $n = 1000$

$U_i(W_i)$: a test statistic for independence between X and Y

Type I error vs. Number of bins

Significance level
$\alpha = 0.05$

The number of bins

Test Statistics

	5	10	15	20	25	30	35
HSIC	1.000	0.443	0.105	0.055	0.057	0.048	0.040
Dcov	1.000	0.668	0.159	0.074	0.050	0.043	0.036
Cov	1.000	0.755	0.185	0.078	0.070	0.049	0.051
XiCov	0.986	0.103	0.058	0.043	0.065	0.058	0.053
Tau	1.000	0.679	0.158	0.073	0.066	0.054	0.048
TauStar	1.000	0.623	0.144	0.065	0.061	0.051	0.049
MIT	0.991	0.121	0.066	0.057	0.062	0.056	0.049

As # **of bins** **increases** (i.e., maximum diameter decreases), the **type I error** **decreases** consistently

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HSIC	1.000	0.996	0.870	0.653	0.501	0.410	0.369
Dcov	1.000	1.000	0.982	0.900	0.782	0.696	0.612
Cov	1.000	1.000	0.998	0.955	0.878	0.801	0.734
XiCov	1.000	0.558	0.270	0.211	0.196	0.160	0.154
Tau	1.000	1.000	0.989	0.908	0.805	0.714	0.659
TauStar	1.000	1.000	0.974	0.856	0.722	0.604	0.537
MIT	1.000	0.627	0.356	0.294	0.226	0.192	0.196

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Double-binning approach: Type I error

Significance level
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# of coarser bin: $\lceil n^{0.3} \rceil$
# of finer bin: $\lceil n^{0.3} \rceil \times \lceil n^{0.3} \rceil$

Double-binning approach: Power

Significance level
$\alpha = 0.05$

The number of bins

Test Statistics

	5	10	15	20	25	30	35	Double-Binning
HSIC	X	X	X	0.653	0.501	0.410	0.369	0.928
Dcov	X	X	X	0.900	0.782	0.696	0.612	0.987
Cov	X	X	X	0.955	0.878	0.801	0.734	0.988
XiCov	X	X	X	0.211	0.196	0.160	0.154	0.411
Tau	X	X	X	0.908	0.805	0.714	0.659	0.987
TauStar	X	X	X	0.856	0.722	0.604	0.537	0.976
MIT	X	X	X	0.294	0.226	0.192	0.196	0.480

(X) They do not control the type I error;
Hence ignored in power comparisons

\# of coarser bin: $\lceil n^{0.3} \rceil$
\# of finer bin: $\lceil n^{0.3} \rceil \times \lceil n^{0.3} \rceil$