Habib University
Course Code: EE366/CE366/CS380
Course Title: Introduction to Robotics
Instructor name: Dr. Basit Memon
Examination: Final Exam Exam Date: May 7, 2024
Total Marks: 100 Duration: 180 minutes

Instructions

1. You can refer to course slides, your notes, homework solutions, or any of the books included in the course syllabus. Where appropriate, you can cite the slides and don't have to redo what has already been done.
2. Use of internet or AI chat is not permitted.
3. You are welcome to utilize MATLAB in whichever way you find useful. You can submit the response to a question in the form of a MATLAB live script as well, but you cannot use numerical methods where question explicitly asks you to employ analytical methods. In the case that you're utilizing it, make sure that your answers are organized.
4. The exam will be administered under HU student code of conduct (see Chapter 3 of `https://habibuniversity.sharepoint.com/sites/Student/code-of-conduct`). You may not talk, discuss, compare, copy from, or consult with any other human on this planet (except me) on questions or your responses during the exam.
5. Make sure that you explain your process of arriving at a final response. Most of the points for any question will be given based on the followed process.
6. Kindly use a black/blue pen to write your hand-written solutions, so that they are legible.
7. The questions or their associated points are not arranged by complexity of material or the time required to complete it.

Questions

Problem 1
CLO1-C3
20 points

Figure 1 shows a concrete mixing truck that you may have seen around construction sites in Karachi. It has a rotating drum at the back that keeps on rotating the concrete inside. Assume that the driver is in an adventurous mood and starts moving this truck in circles inside a parking lot such that the truck is rotating in a perfect circle. If you were a fly sitting on the rim of the rotating drum (you'll rotate along with the drum), then discuss the direction

Figure 1: A concrete mixing truck

of the angular velocity at which you're rotating.

Solution 1 The discussion can be simplified if the situation is modeled using frames. Assume that a world frame, $\{0\}$, is fixed at the center of the circle in the parking lot. A frame, $\{1\}$, is now attached to the bed of the truck in such a way that the z-axis is pointed upwards and the y-axis is directed from the back of the truck towards the front. The final frame, $\{2\}$, is attached at the point where the fly is seated. (Add figure of frames)

The rotation of the truck in the parking lot can be modeled as the rotation of frame $\{1\}$ about $\hat{z}_0$. The rotation of the drum can be modeled as the rotation of frame $\{2\}$ about an axis $\hat{k}$ in the plane $y_1 - z_1$ making an angle α with $-\hat{y}_1$. Since we know that angular velocities between successive frames can be added, we have that:

$$^0\omega_2^0 = {}^0\omega_1^0 + {}^0R_1\,{}^1\omega_2^1$$

Ignoring the magnitudes of the angular velocities,

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -\cos\alpha \\ \sin\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \sin\theta_1\cos\alpha \\ -\cos\theta_1\cos\alpha \\ 1+\sin\alpha \end{bmatrix}$$

This vector is the direction of the angular velocity in the $\{0\}$ frame.

Problem 2
CLO4-C3
60 points

Context: Consider a 2R planar robot with link lengths $l_1 = 0.5$ and $l_2 = 0.4$ meters, illustrated in Figure 2. The robot is to catch a ball moving at a constant speed $v = 0.3$ m/sec along a line passing through the point $P_0 = (-0.8, 1.1)$ meters and making an angle $\beta = -20^\circ$ with the axis x_0. The robot begins its motion as soon as the ball enters the robot's workspace.

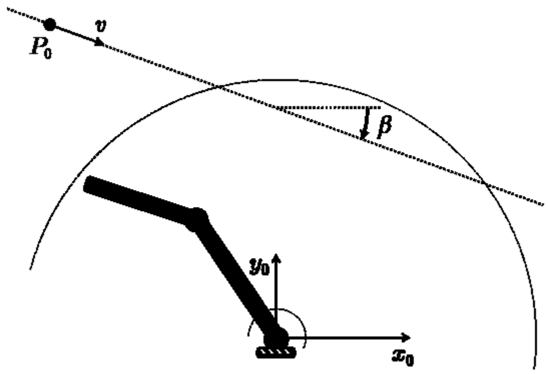

Figure 2: A 2R rugby player

Problem Statement: You're required to perform trajectory planning for both joints of this robot so that the robot moves to catch the ball. The robot begins at rest from the configuration[1] $q_0 = (\pi, 0)$ and catches the ball after $T = 2$s. At the instant the robot catches the ball, its end-effector and the ball should have the same Cartesian velocity (magnitude and direction). Provide expressions for trajectories of both the joints of the 2R robot, ensuring successful execution of the task.

Helpful fact: The equation of this line is:

$$(x - x_0)\sin\beta - (y - y_0)\cos\beta = 0.$$

Solution 2

We're to construct trajectories of the two joint angles, θ_1 and θ_2, in the interval $[0, T]$. We'll first collect all the constraints on these two variables before constructing a trajectory.

First set of constraints. We are provided the starting configuration of the robot, i.e. $\theta_1(0) = \pi$ and $\theta_2(0) = 0$.

Second set of constraints. If $(x(t), y(t))$ is the position of the robot's end-effector at time t, then we are also provided that the end position of the robot's end-effector, $(x(T), y(T))$, should be the position at which the ball will be at the 2 seconds mark of entering the robot's workspace. We can then use inverse kinematics to determine the end joint angles, i.e. $\theta_1(T)$ and $\theta_2(T)$.

[1]Angles are defined in the standard way.

Let's us first determine the point at which the ball enters the robot's workspace, which will be a circle of radius 0.9m centered at the robot's base. This point is solution to the following pair of equations:

$$x^2 + y^2 = 0.9^2 \tag{1}$$
$$(x + 0.8)\sin(-20°) - (y - 1.1)\cos(-20°) = 0 \tag{2}$$

Rearranging (2),

$$y = \frac{-0.342(x + 0.8)}{0.9397} + 1.1 = 0.81 - 0.3639x$$

Let's substitute y from (2) in (1):

$$\begin{aligned}
x^2 + (0.81 - 0.36x)^2 &= 0.81 \\
1.13x^2 - 0.58x - 0.15 &= 0 \\
\Rightarrow x &= \frac{0.58 \pm \sqrt{0.34 + 0.68}}{2.26} = \{0.7, -0.19\} \\
y &= \{0.56, 0.88\}
\end{aligned}$$

It is evident from the figure that the ball enters the workspace at $(-0.19, 0.88)$. Now, we need to determine the location of the ball after two seconds. It is provided that the ball is traveling along the direction $\begin{bmatrix} \cos(-20°) \\ \sin(-20°) \end{bmatrix}$ at a constant speed of 0.3m/s. Thus, at time $t = T$, the ball will be at:

$$\begin{bmatrix} -0.19 \\ 0.88 \end{bmatrix} + vT \begin{bmatrix} \cos(-20°) \\ \sin(-20°) \end{bmatrix} = \begin{bmatrix} 0.37 \\ 0.67 \end{bmatrix}.$$

Thus, $x(T) = 0.37$ and $y(T) = 0.67$. We can use inverse kinematics to determine the corresponding $\theta_1(T)$ and $\theta_2(T)$. We can use either the elbow-up or elbow-down solution. Let us use the elbow-up solution.

$$\begin{aligned}
\theta_1(T) &= \arctan 2(y(T), x(T)) + \arccos\left(\frac{0.5^2 + x^2(T) + y^2(T) - 0.4^2}{2(0.5)\sqrt{x^2(T) + y^2(T)}}\right) \\
&= 1.54 rad \\
\theta_2(T) &= -\pi + \arccos\left(\frac{0.5^2 + 0.4^2 - x^2(T) - y^2(T)}{2(0.5)(0.4)}\right) \\
&= -1.09 rad
\end{aligned}$$

Third set of constraints. We're also provided that the motion begins from rest, i.e. the initial jointe velocities are zero.

$$\dot{\theta}_1(0) = 0$$
$$\dot{\theta}_2(0) = 0$$

Fourth set of constraints. The final constraint is on the end-effector velocity at $t = T$. Specifically, it should be equal to the ball velocity at that time, i.e.

$$\begin{bmatrix} \dot{x}(T) \\ \dot{y}(T) \end{bmatrix} = 0.3 \begin{bmatrix} \cos(-20°) \\ \sin(-20°) \end{bmatrix}.$$

This is equivalent to corresponding constraints on the joint velocities, $\dot{\theta}_1(T)$ and $\dot{\theta}_2(T)$. These constraints can be determined through the Jacobian.

$$\begin{aligned}
\begin{bmatrix} \dot{\theta}_1(T) \\ \dot{\theta}_2(T) \end{bmatrix} &= J^{-1}\Big|_{(\theta_1,\theta_2)=(\theta_1(T),\theta_2(T))} \begin{bmatrix} \dot{x}(T) \\ \dot{y}(T) \end{bmatrix} \\
&= \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2) & -l_2 \sin(\theta_1+\theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) & l_2 \cos(\theta_1+\theta_2) \end{bmatrix}^{-1} \Bigg|_{(\theta_1,\theta_2)=(\theta_1(T),\theta_2(T))} \begin{bmatrix} \dot{x}(T) \\ \dot{y}(T) \end{bmatrix} \\
&= \begin{bmatrix} -0.47 \\ 0.21 \end{bmatrix}
\end{aligned}$$

Constructing the trajectories. We have 4 constraints on each of the joint variables. We can construct a cubic trajectory for each of the joint variables.

$$\theta_1(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

where,

$$\begin{aligned}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 4 & 8 \\ 0 & 1 & 4 & 12 \end{bmatrix}^{-1} \begin{bmatrix} \pi \\ 0 \\ 1.54 \\ -0.47 \end{bmatrix} \\
&= \begin{bmatrix} \pi \\ 0 \\ -0.97 \\ 0.28 \end{bmatrix}
\end{aligned}$$

Similarly,

$$\theta_2(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

where,

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 4 & 8 \\ 0 & 1 & 4 & 12 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ -1.09 \\ 0.21 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 0 \\ -0.92 \\ 0.32 \end{bmatrix}$$

Problem 3
CLO4-C3
20 points

Context: Once the robot of the previous problem has caught the ball, it will now move along the same line trajectory that the ball was supposed to follow. In this problem, you'll design a kinematic controller for this robot so that it follows the line trajectory from the moment that it has caught the ball. Kinematic controller means that you forget about all the dynamic effects, e.g. the impulse experienced when catching the ball, the delays associated with transitioning the system, the motor dynamics, etc.

Problem Statement: Assume that the robot is commanded by the joint velocity $\dot{q} \in \mathbb{R}^2$, i.e. the control command that is generated is the vector of joint velocities (perhaps stepper motors are installed on the robot). Design a feedback control law[2] that will ensure that the robot follows the line trajectory drawn in Figure 2 from the moment of catching onwards so that the position errors in x and y of the end-effector of the robot converge to zero. Provide explicit expression for the control law, i.e. the joint velocity commands for both joints, and argue that the error in both x and y will converge to zero.

How to proceed: In class, we have looked at control in terms of the joint variables, i.e. we're provided a desired joint angle trajectory and we minimize the error with respect to it. In this case, you're provided a desired end-effector trajectory. The broader idea of kinematic control still remains the same and will be based on Slide 11-28. You're expected to understand this slide and base your solution on it.

Solution 3

Based on the mentioned slide, our controller will be of the form:

$$\dot{q} = J^{-1}(q)\left[\dot{\mathbf{x}}_d + K_p(\mathbf{x}_d - \mathbf{x})\right],$$

where $\mathbf{x}_d = \begin{bmatrix} x_d \\ y_d \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$. It can be easily shown that the error will converge to zero

[2]It could include feedforward terms.

with this controller. Let $e_x = x_d - x$. Then,

$$\begin{aligned} \dot{e}_x &= \dot{x}_d - \dot{x} \\ &= \dot{x}_d - J(q)\dot{q} \\ &= \dot{x}_d - J(q)\, J^{-1}(q)\, [\dot{x}_d + K_p(x_d - x)] \\ &= -K_p e_x \end{aligned}$$

Let $K_p = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}$ for some $p_1 > 0$ and $p_2 > 0$. Then,

$$\begin{aligned} x_d - x &= e^{-p_1 t} \\ y_d - y &= e^{-p_2 t}. \end{aligned}$$

So, the error in both the coordinates decays to zero.

Let's now compute the required terms in the control expression. We've already determined the Jacobian, J, in the previous part. If our system can measure the end-effector position directly, then $\mathbf{x}$ will be acquired from the sensors. Instead, if the joint angles. q, are being measured, then

$$\begin{aligned} x &= l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ y &= l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2). \end{aligned}$$

Now, the only undetermined terms in the expression are $\mathbf{x}_d$ and $\dot{\mathbf{x}}_d$. We can rest the time and say that $t = 0$ when the robot has catched the ball. We know that

$$\mathbf{x}_d(0) = \begin{bmatrix} x(T) \\ y(T) \end{bmatrix} = \begin{bmatrix} 0.37 \\ 0.67 \end{bmatrix}.$$

Correspondingly,

$$\begin{aligned} \mathbf{x}_d(t) &= \begin{bmatrix} 0.37 \\ 0.67 \end{bmatrix} + vt \begin{bmatrix} \cos(-20°) \\ \sin(-20°) \end{bmatrix} \\ \dot{\mathbf{x}}_d(t) &= v \begin{bmatrix} \cos(-20°) \\ \sin(-20°) \end{bmatrix} \end{aligned}$$

If the dynamic models of the motors, gears, and the 2R mechanical structure were to be included, then model the entire robotic system and design a control law for the same task outlined in the previous problem, now when the system is commanded by motor voltages.

Problem 4
CLO4-C3

Extra Credit 20 points