Numerical Methods for PDE mandatory 1

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1.2.3

We want to show that $u(t,x,y) = e^{i(k_x x + k_y y - \omega t)}$ is an exact solution to the wave equation.
We have the following equations

$$\frac{\partial^2 u}{\partial t^2} = -\omega^2 e^{i(k_x x + k_y y - \omega t)}$$
$$\frac{\partial^2 u}{\partial x^2} = -k_x^2 e^{i(k_x x + k_y y - \omega t)}$$
$$\frac{\partial^2 u}{\partial y^2} = -k_y^2 e^{i(k_x x + k_y y - \omega t)}$$

When we put everything together we obtain

$$-\omega^2 e^{i(k_x x + k_y y - \omega t)} = c^2(-k_x^2 e^{i(k_x x + k_y y - \omega t)} - k_y^2 e^{i(k_x x + k_y y - \omega t)})$$

We have that $-\omega^2 = c^2(-k_x^2 - k_y^2)$ which leads to $\omega = c\sqrt{k_x^2 + k_y^2}$ or $c = \frac{\omega}{\sqrt{k_x^2 + k_y^2}}$. This means that this choice of $u(t,x,y)$ is an exact solution of the wave equation if and only if $c = \frac{\omega}{\sqrt{k_x^2 + k_y^2}}$.

1.2.4

We want to show that if $C = \frac{1}{\sqrt{2}}$ then we have that $\tilde{\omega} = \omega$. We do this by looking at the equation given on page 159 of the book Finite difference computing with PDE's by Hans Petter Langtangen and Svein Linge, which is

$$\frac{4}{\Delta t^2}\sin^2(\frac{\tilde{\omega}\Delta t}{2}) = c^2(\frac{4}{h^2}\sin^2(\frac{kh}{2} + \frac{kh}{2}))$$

Using the fact that $C^2 = \frac{c^2 \Delta t^2}{h^2}$ and by dividing by 4 on both sides we obtain

$$\sin^2(\frac{\tilde{\omega}\Delta t}{2}) = 2C^2 \sin^2(\frac{kh}{2})$$

Furthermore, we know that $C = \frac{1}{\sqrt{2}}$ which means $C^2 = \frac{1}{2}$, which leads us to the following equations

$$\sin^2(\frac{\tilde{\omega}\Delta t}{2}) = \sin^2(\frac{kh}{2})$$
$$\sin\left(\frac{\tilde{\omega}\Delta t}{2}\right) = \sin\left(\frac{kh}{2}\right)$$
$$\frac{\tilde{\omega}\Delta t}{2} = \frac{kh}{2}$$

After some rewriting the last equation becomes

$$\tilde{\omega} = \frac{kh}{\Delta t}$$

From question 1.2.3 we have that $\omega = ck\sqrt{2}$ (when taking $k_x = k_y = k$) and from the definition of C we have

$$\begin{aligned}\frac{c\Delta t}{h} &= \frac{1}{\sqrt{2}}\\ c\Delta t &= \frac{h}{\sqrt{2}}\\ c &= \frac{h}{\Delta t\sqrt{2}}\end{aligned}$$

Using this outcome for c we obtain that $\omega = \frac{kh}{\Delta t}$, which is the same as $\tilde{\omega}$. Hence we have shown that if $C = \frac{1}{\sqrt{2}}$ then $\tilde{\omega} = \omega$.