

Linear methods of classification

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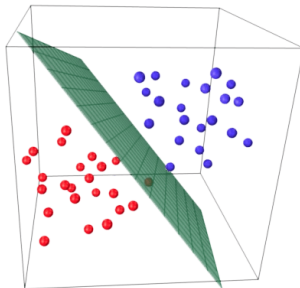


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Multiclass general classifier

- Define discriminant functions $g_c(x)$ for classes $c = \overline{1, C}$.
- General classifier:

$$\hat{y}(x) = \arg \max_c g_c(x)$$

- Boundary between classes i and j :

$$\{x : g_i(x) = g_j(x)\}$$

- Margin (quality of classification):

$$M(x, y) = g_y(x) - \max_{c \neq y} g_c(x)$$

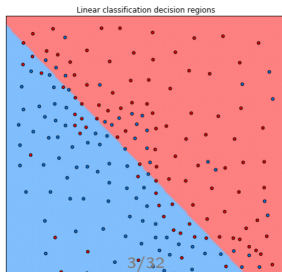
Linear classifier

- Include constant feature: $x = [1, x^2, x^3, \dots, x^D]$ when omit bias w_0 .
- Linear classifier:

$$g_c(x) = w_c^T x, \quad c = \overline{1, C}.$$

- Boundary between classes i and j is linear:

$$\left\{ x : w_i^T x = w_j^T x \right\}$$



Binary general classifier

- $y \in \{+1, -1\}$. Discriminant functions: $g_{+1}(x), g_{-1}(x)$.
- Define $g(x) = g_{+1}(x) - g_{-1}(x)$ - preference for class +1.
- Binary classifier:

$$\hat{y}(x) = \arg \max_{c \in \{+1, -1\}} g_c(x) = \text{sign}(g_{+1}(x) - g_{-1}(x)) = \text{sign}(g(x))$$

- Margin:

$$M(x, y) = g_y(x) - g_{-y}(x) = y(g_{+1}(x) - g_{-1}(x)) = yg(x)$$

Binary linear classifier

- Discriminant functions: $w_{+1}^T x$, $w_{-1}^T x$.
- Define $w = w_{+1} - w_{-1}$.
- Binary linear classifier:

$$\hat{y}(x) = \arg \max_{c \in \{+1, -1\}} w_c^T x = \text{sign} \left(w_{+1}^T x - w_{-1}^T x \right) = \text{sign} \left(w^T x \right)$$

- Margin:

$$M(x, y) = w_y^T x - w_{-y}^T x = y \left(w_{+1}^T x - w_{-1}^T x \right) = y w^T x = w^T x y$$

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Orthogonal vector to hyperplane

$\hat{y}(x) = \text{sign}(w^T x + w_0)$ binary linear classifier (with explicit bias term)

Theorem 1

Vector w is orthogonal to hyperplane $w^T x + w_0 = 0$

Proof. Consider arbitrary $x_A, x_B \in \{x : w^T x + w_0 = 0\}$:

$$w^T x_A + w_0 = 0 \quad (1)$$

$$w^T x_B + w_0 = 0 \quad (2)$$

By subtracting (2) from (1), obtain $w^T(x_A - x_B) = 0$, so w is orthogonal to hyperplane. □

Distance from point to hyperplane

Theorem 2

Distance from point x to hyperplane $w^T x + w_0 = 0$ is equal to $\frac{w^T x + w_0}{\|w_0\|}$.

Proof. Project x on the hyperplane, let the projection be p and complement $h = x - p$, orthogonal to hyperplane. Then

$$x = p + h$$

Since p lies on the hyperplane,

$$w^T p + w_0 = 0$$

Since h is orthogonal to hyperplane and according to theorem 1

$$h = r \frac{w}{\|w\|}, \quad r \in \mathbb{R} \text{ - distance to hyperplane.}$$

Distance from point to hyperplane

$$x = p + r \frac{w}{\|w\|}$$

After multiplication by w and addition of w_0 :

$$w^T x + w_0 = w^T p + w_0 + r \frac{w^T w}{\|w\|} = r \|w\|$$

because $w^T p + w_0 = 0$ and $\|w\| = \sqrt{w^T w}$. So we get, that

$$r = \frac{w^T x + w_0}{\|w\|}$$

Comments:

- From one side of hyperplane $r > 0 \Leftrightarrow w^T x + w_0 > 0$
- From the other side $r < 0 \Leftrightarrow w^T x + w_0 < 0$.
- Distance from hyperplane to origin 0 is $\frac{w_0}{\|w\|}$. So w_0 accounts for hyperplane offset.

Binary linear classifier geometric interpretation

Binary linear classifier:

$$\hat{y}(x) = \text{sign} \left(w^T x + w_0 \right)$$

divides feature space by hyperplane $w^T x + w_0 = 0$.

- Confidence of decision is proportional to distance to hyperplane $\frac{|w^T x + w_0|}{\|w\|}$.
- $w^T x + w_0$ is the confidence that class is positive.

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Estimation of w

- Let's omit w_0 and include constant feature.
- Select w to minimize # of misclassifications:

$$\sum_{n=1}^N \mathbb{I} \left[w^T x_n y_n < 0 \right] \rightarrow \min_w$$

- Piecewise constant criterion, gradient=0 almost everywhere.
 - can't use optimization methods to find minimum!
- Use decreasing \mathcal{L} with non-trivial gradients:

$$\sum_{n=1}^N \mathcal{L} \left(w^T x_n y_n \right) \rightarrow \min_w$$

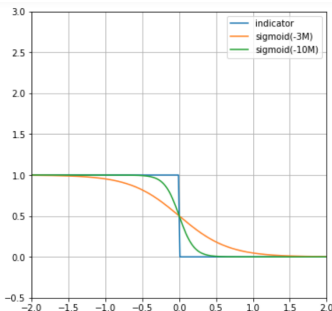
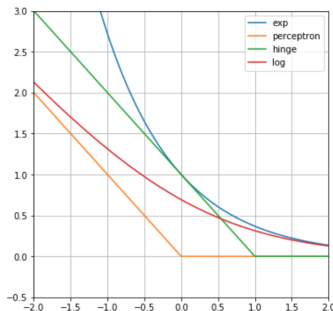
- For general binary classifier can use:

$$\sum_{n=1}^N \mathcal{L} (y_n g_w(x_n)) \rightarrow \min_w, \quad w\text{-parameters of } g(x)$$

Common loss functions

$$\mathcal{L}_{exp}(M) = e^{-M} \quad \mathcal{L}_{perceptron}(M) = [-M]_+$$

$$\mathcal{L}_{hinge}(M) = [1 - M]_+ \quad \mathcal{L}_{log}(M) = \ln(1 + e^{-M})$$



Check properties: robustness to outliers, convexity, penalization for small positive margin.

SGD examples

$$\mathcal{L}(M) = e^{-M}$$

$$w \leftarrow w + \varepsilon e^{-\langle w, x_i \rangle y_i} x_i y_i$$

$$\mathcal{L}(M) = [-M]_+$$

$$w \leftarrow w + \begin{cases} 0, & \langle w, x_i \rangle y_i \geq 0 \\ \varepsilon x_i y_i & \langle w, x_i \rangle y_i < 0 \end{cases}$$

Prove that margin for sampled object (x_i, y_i) can only increase.

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Regularization

- Insert additional requirement for regularizer $R(\beta)$ to be small:

$$\sum_{n=1}^N \mathcal{L}(M(x_n, y_n | w)) + \lambda R(w) \rightarrow \min_{\beta}$$

- $\lambda > 0$ - hyperparameter.
- $R(w)$ penalizes complexity of models.

$$R(\beta) = \|w\|_1 \quad L_1 \text{ regularization}$$

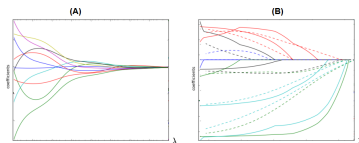
$$R(\beta) = \|w\|_2^2 \quad L_2 \text{ regularization}$$

$$R(\beta) = \alpha \|w\|_1 + (1 - \alpha) \|w\|_2^2 \quad \text{combination with } \alpha \in (0, 1)$$

- Not only *accuracy* matters for the solution but also *model simplicity*!
- λ controls complexity of the model: $\uparrow \lambda \Leftrightarrow \text{complexity} \downarrow$.

Comments

- Dependency of β from λ for L_2 (A) and L_1 (B) regularization:



- L_2 can be used for automatic feature selection.
- λ is usually found using cross-validation on exponential grid, e.g. $[10^{-6}, 10^{-5}, \dots, 10^5, 10^6]$.
- It's always recommended to use regularization because
 - it gives smooth control over model complexity.
 - reduces ambiguity for multiple solutions case.

Combination

- Combined regularizer:

$$R(w) = \alpha \|w\|_1 + (1 - \alpha) \|w\|_2^2$$

$\alpha \in (0, 1)$ - hyperparameter, controlling impact of each part.

- If two features x^i and x^j are equal:
 - L_1 may take only one of them
 - L_2 will take both with equal weight
 - but it doesn't remove useless features
 - Combination removes useless features but gives equal weight for useful equal features
 - good, because feature equality may be due to chance on this particular training set

Different account for different features

- Traditional approach regularizes all features uniformly:

$$\sum_{n=1}^N \mathcal{L}(M(x_n, y_n | w)) + \lambda R(w) \rightarrow \min_w$$

- Suppose we have K groups of features with indices:

$$I_1, I_2, \dots, I_K$$

- We may control the impact of each feature group by minimizing:

$$\sum_{n=1}^N \mathcal{L}(M(x_n, y_n | w)) + \lambda_1 R(\{w_i | i \in I_1\}) + \dots + \lambda_K R(\{w_i | i \in I_K\})$$

- $\lambda_1, \lambda_2, \dots, \lambda_K$ can be set using cross-validation
- In practice use common regularizer but with different feature scaling.

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Binary classification

- Linear classifier:

$$\text{score}(y = 1|x) = w^T x$$

- +relationship between score and class probability is assumed:

$$p(y = 1|x) = \sigma(w^T x)$$

where $\sigma(z) = \frac{1}{1+e^{-z}}$ - sigmoid function

Binary classification: estimation

Using the property $1 - \sigma(z) = \sigma(-z)$ obtain that

$$p(y = +1|x) = \sigma(w^T x) \implies p(y = -1|x) = \sigma(-w^T x)$$

So for $y \in \{+1, -1\}$

$$p(y|x) = \sigma(y\langle w, x \rangle)$$

Therefore ML estimation can be written as:

$$\prod_{i=1}^N \sigma(\langle w, x_i \rangle y_i) \rightarrow \max_w$$

Loss function for 2-class logistic regression

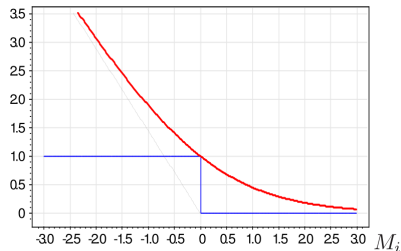
For binary classification $p(y|x) = \sigma(\langle w, x \rangle y)$ $w = [\beta'_0, \beta]$,
 $x = [1, x_1, x_2, \dots, x_D]$.

Estimation with ML:

$$\prod_{i=1}^N \sigma(\langle w, x_i \rangle y_i) \rightarrow \max_w$$

which is equivalent to

$$\sum_{i=1}^N \ln(1 + e^{-\langle w, x_i \rangle y_i}) \rightarrow \min_w$$



It follows that logistic regression is linear discriminant estimated with loss function $\mathcal{L}(M) = \ln(1 + e^{-M})$.

Multiple classes

Multiple class classification:

$$\begin{cases} \text{score}(y = 1|x) = w_1^T x \\ \text{score}(y = 2|x) = w_2^T x \\ \dots \\ \text{score}(y = C|x) = w_C^T x \end{cases}$$

+relationship between score and class probability is assumed:

$$p(\omega_c|x) = \text{softmax}(w_c^T x | x_1^T x, \dots, x_C^T x) = \frac{\exp(w_c^T x)}{\sum_i \exp(w_i^T x)}$$

Multiple classes

Weights ambiguity:

w_c , $c = 1, 2, \dots, C$ defined up to shift v :

$$\frac{\exp((w_c - v)^T x)}{\sum_i \exp((w_i - v)^T x)} = \frac{\exp(-v^T x) \exp(w_c^T x)}{\sum_i \exp(-v^T x) \exp(w_i^T x)} = \frac{\exp(w_c^T x)}{\sum_i \exp(w_i^T x)}$$

To remove ambiguity usually $v = w_C$ is subtracted.

Estimation with ML:

$$\begin{cases} \prod_{n=1}^N \text{softmax}(w_{y_n}^T x_n | x_1^T x, \dots, x_C^T x) \rightarrow \max_{w_1, \dots, w_{C-1}} \\ w_C = 0 \end{cases}$$

SoftMax

- SoftMax converts class ratings to class probabilities:

$$\text{Softmax}_{\tau}(z_1, \dots, z_K) =$$
$$= \left[\frac{e^{z_1/\tau}}{e^{z_1/\tau} + \dots + e^{z_K/\tau}}, \frac{e^{z_2/\tau}}{e^{z_1/\tau} + \dots + e^{z_K/\tau}}, \dots, \frac{e^{z_K/\tau}}{e^{z_1/\tau} + \dots + e^{z_K/\tau}} \right]$$

- τ - "temperature" parameter, controlling probability contrast¹.

¹how?

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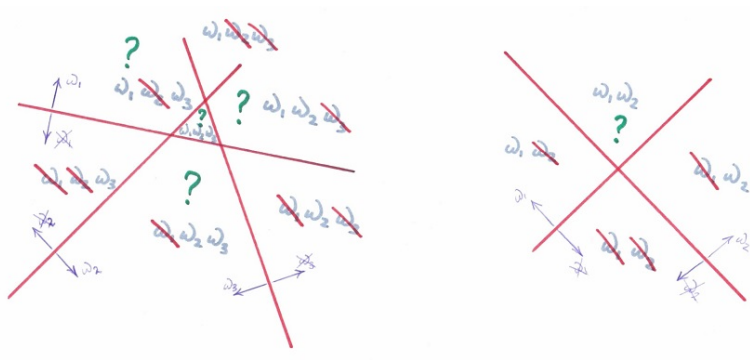
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Multiclass classification with binary classifiers

- Task - make C -class classification using many binary classifiers.
- Approaches:
 - **one-versus-all**
 - for each $c = 1, 2, \dots, C$ train binary classifier on all objects and output $\mathbb{I}[y_n = c]$,
 - assign class, getting the highest score in resulting C classifiers.
 - **one-versus-one**
 - for each $i, j \in [1, 2, \dots, C], i \neq j$ learn on objects with $y_n \in \{i, j\}$ with output y_n
 - assign class, getting the highest score in resulting $C(C - 1)/2$ classifiers.
 - **error correcting codes**

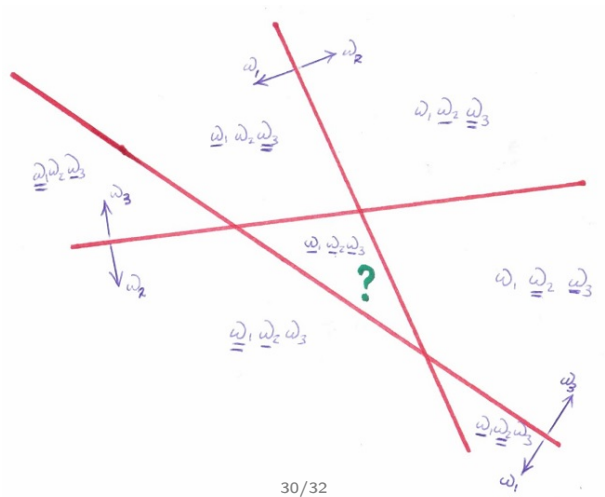
One versus all - ambiguity

Classification among three classes: $\omega_1, \omega_2, \omega_3$



One versus one - ambiguity

Classification among three classes $\omega_1, \omega_2, \omega_3$ depending only on halfspace may be ambiguous:



Error correcting codes

- Used in classification
- Each class i is coded as a binary codeword $W_i \in \mathbb{R}^B$:

$$\text{class } i \rightarrow W_i$$

- Minimum sufficient amount of bits to code C classes is $\lceil \log_2 C \rceil$
- Given x , B binary classifiers predict each bit of the class codeword.
- Class is predicted as

$$\hat{c}(x) = \arg \min_c \sum_{b=1}^B |W_{cb} - \hat{p}_b(x)|$$

- where W_{cb} is the b -th bit of codeword, corresponding to class c .
- More bits are used to make classification more robust to errors of individual binary classifiers.
- Codewords are selected to have maximum mutual Hamming distance or randomly.

Summary

- Linear classifier - classifier with linear discriminant functions.
- Binary linear classifier: $\hat{y}(x) = \text{sign}(w^T x + w_0)$.
- Perceptron, logistic, SVM - linear classifiers estimated with different loss functions.
- Weights are selected to minimize total loss on margins.
- Regularization gives smooth control over model complexity.
- L_1 regularization automatically selects features.