Problem. *(Sylvester's problem)*
Let $a_1, a_2, \ldots, a_m$ be the points in $\mathbb{R}^d$. Given these points we need to find an spheroid, that contains all points with the minimum volume (or equivalently, with the minimum radius)

Solution:
In other words we need to find solution of the minimization task:

$$\varphi(x) = \max_{i=\overline{1,m}} (\|a_i - x\|) \to \min, \ x \in \mathbb{R}^d$$

Note that $\varphi(x)$ is point-wise maximum of m convex function it means that we can use subdifferential calculus for searching the extremum point of $\varphi(x)$.

We will use the following theorems from subdifferential calculus:

Theorem 1. *(Dubovitsky - Milutin)*
Let $f_i(x)$, $i = \overline{1,n}$ is convex functions $S \to \mathbb{R}^d$ (where S is open and conves set) and all of these functions are continuous at a point $x_0 \in S$ and $f(x) = \max_{i=\overline{1,n}}(f_i(x))$, then

$$\partial f(x_0) = ch\left(\bigcup_{j \in I} \partial f_j(x_0)\right),$$

where $I = \{j : f_j(x_0) = f(x_0)\}$ – set of active indexes, ch is convex hull.

Theorem 2. *(Fermat Lemma in subdifferential calculus)*
x_0 is point of extremum of $f(x)$ if and only if $0 \in \partial f(x_0)$, where $f(x)$ is convex function.

proposition 1.

$$\partial\left(\|a - x\|\right)(x_0) = \begin{cases} B_1(0), & \text{if } x_0 = a \\ \dfrac{a - x}{\|a - x\|}, & \text{if } x_0 \neq a \end{cases},$$

where $B_1(0) = \left\{x \in \mathbb{R}^d : \|x\| \leqslant 1\right\}$[1]

proposition 2.
Let $\hat{x}$ is a point of extremum $\varphi(x)$ (we consider that $m > 1$) , then $|I| > 1$ where I is set of active indexes (see theorem 1)

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Indeed, if $|I| = 1$ (let $I = \{j^*\}$) then only a_{j^*} belongs to the boundary of the spheroid, but according to Fermat Lemma (theorem 2) and proposition 1 $x = a_{j^*}$, it means $r = 0$ which is

[1] In general, we we have a case of dual space, but in this article the case of $\mathbb{R}^d$ with standard metric is considered, so $\left(\mathbb{R}^d\right)^* = \mathbb{R}^d$

not possible in the case of $m > 1$.

$\triangleleft$

Let's first consider the case of $\mathbb{R}^2$. A circle can be defined by two points on the diameter or by three points belonging to it. If two points belong to a circle, then it is easy to see that the theorem 1 holds, but we need to check that all other points belong to this circle. Consider three arbitrary points of the set a_1, a_2, a_3 (these points do not belong to one line). Let the circle built on them has a center c and the radius r. If this circle contains all the other points of the set then we have to check the fulfillment of the theorem 1:

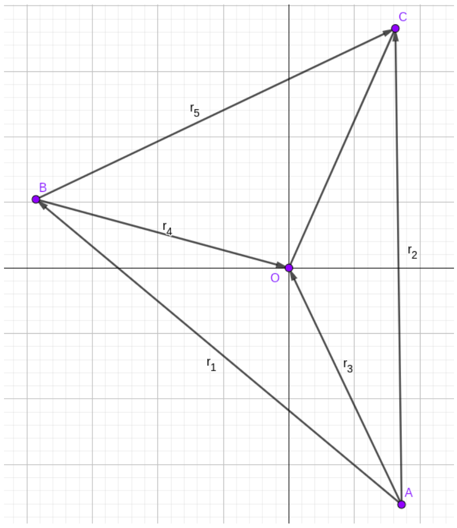
$$0 \in \triangle ABC = \operatorname{ch}\left\{\frac{a_i - c}{\|a_i - c\|},\ i = 1, 2, 3\right\}$$

The following is an algorithm for checking whether a point belongs to a triangle. Point O belongs $\triangle ABC$ if and only if $S_{ABC} = S_{ABO} + S_{ACO} + S_{BCO}$.

The square of the triangle can be calculated through the determinant of the matrix formed by the vectors of the sides of the triangle:

$$S_{ABC} = \frac{1}{2}\left|\det\begin{pmatrix} r_1 & r_2 \end{pmatrix}\right|,\ S_{ABO} = \frac{1}{2}\left|\det\begin{pmatrix} r_1 & r_3 \end{pmatrix}\right|$$

the remaining squares can be expressed similarly.



Thus, we present a complete solution algorithm in case $\mathbb{R}^2$

1. Iterate through all possible pairs of points. On each such pair, build a circle with a diameter with the ends of these points. If such a circle contains all the other points, then this is the desired circle

2. Iterate through all possible triples of points. Build a circle on each such triple (if these points do not lie on the same straight line).

   (a) Check whether the constructed circle contains all points of the set

   (b) If the previous paragraph is fulfilled, check the fulfillment of the condition of the theorem 1

The presented algorithm is generalized in an understandable way to the case $\mathbb{R}^d,\ d > 2$