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# Application of Enhanced Extended Observer in Station-Keeping of a Quadrotor with Unmeasurable Pitch and Roll Angles ⋆

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**Abstract:** In this paper we provide an example of application of the design of a control law that makes use of an enhanced version of the so-called extended observer. The enhancement in question reposes on the extension of a classical Lemma due to Dayawansa to the case of nonlinear systems whose normal form includes time-varying (and measurable) gains. The presented example concerns a station-keeping problem of a quadrotor with unmeasurable pitch and roll angles.



*Keywords:* Dayawansa's lemma, Robust observers, Nonlinear systems, High-gain observers, Quadrotors

## 1. INTRODUCTION

*Extended observers* are acknowledged to be a powerful tool in the design of robust output-feedback control laws in the presence of dynamic input uncertainties. The advantage of adding an extra state variable to the standard high-gain observer so as to accommodate for input uncertainties has been highlighted in various recent papers, such as those of Praly and Jiang (1998), Freidovich and Khalil (2008), Han (2009),Guo and Zhao (2013) and Chen et al. (2016). In particular, Freidovich and Khalil (2008) have demonstrated the possibility of asymptotically recovering the performances achievable by means of a state-feedback linearizing control. The design of extended observers for multivariable systems has been studied by Isidori (2017) and Khalil (2017). Recently, Isidori et al. (2019) have shown how the domain of applicability of the method in question can be enhanced so as to include the case of nonlinear systems whose normal form is characterized by time-dependent gains in the string of integrators between input and output. The enhancement in question is made possible by an extension of a classical Lemma due to Dayawansa, whose relevance in the design of high-gain observers has been earlier highlighted in Gauthier and Kupka (2001).

Sensorless control is a relatively new, actively developing scientific direction in the control systems community. The main reason is that physical sensors are no longer reliable due to their sensitivity to environmental hazards. Therefore, the design of approaches which reduce the number of measurements needed to be fed into the controller (especially including the regulated signals) is of high interest to researchers. Motivated by this challenge, in the current paper we show how extended observers can be profitably used in the problem of control design for station-keeping in a quadrotor with unmeasurable pitch and roll angles and possibly uncertain parameters.

Station-keeping is a basic flight mode, which represents automatically handled maneuvers of keeping the quadrotor in the given position and orientation. This is relevant to a wide range of applications. For example, a quadcopter hovering above a boat can be used to provide a live feed from the onboard camera. This video signal can be then used to observe the area around the boat and to assist in visual navigation. Another example being increasingly trending is water quality monitoring by means of quadrotors hovering in the given points on the lake or river when sampling.

As it happens in various problems of attitude control of spacecrafts, the model that we consider in this paper is characterized by a normal form in which time-variable (but measurable) gains appear in the string of integrators between inputs and outputs. Such time-variable gains, being measurable, can be duplicated in the observer, but they eventually affect the dynamics of the observation error. It is for this reason that the choice of the observer parameters suggested in the original contributions on extended observers (see e.g. Freidovich and Khalil (2008)) might no longer guarantee asymptotic decay of the observation error. Motivated by this problem, in the paper Isidori et al. (2019) the authors have presented a different, a bit

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more sophisticated, methodology for the selection of design parameters so as to guarantee asymptotic decay of the observation error. The purpose of the present paper is to describe an example of application of this newly-developed methodology, in the design of controls for station-keeping in a quadrotor.

## 2. PRELIMINARIES

### 2.1 Model Transformations

Consider the quadrotor model[1] given in Altuğ et al. (2005) as

$$
\begin{aligned}
\ddot{x} &= \frac{1}{m}\left(\sum_{i=1}^{4} F_i\right)[c_\phi s_\theta c_\psi + s_\phi s_\psi],\\
\ddot{y} &= \frac{1}{m}\left(\sum_{i=1}^{4} F_i\right)[s_\phi s_\theta c_\psi - c_\phi s_\psi],\\
\ddot{z} &= \frac{1}{m}\left(\sum_{i=1}^{4} F_i\right)c_\theta c_\psi - g,\\
\ddot{\theta} &= \frac{\ell}{J_\theta}[-F_1 - F_2 + F_3 + F_4],\\
\ddot{\psi} &= \frac{\ell}{J_\psi}[-F_1 + F_2 + F_3 - F_4],\\
\ddot{\phi} &= \frac{C}{J_\phi}[F_1 - F_2 + F_3 - F_4],
\end{aligned} \tag{1}
$$

in which $x$, $y$, $z$ are the Cartesian coordinates of the center of mass, $\theta$, $\psi$, $\phi$ are the Euler angles that characterize the attitude, $F_i$, $i = \{1, 2, 3, 4\}$ are the lift forces of the rotors, $m$ is the mass, $g = 9.81\,\mathrm{m/s^2}$ is the gravitational acceleration, $J_\theta$, $J_\psi$, $J_\phi$ are the moments of inertia, $\ell$ is the distance between the center of gravity and rotors, $C$ is the scaling factor between the force and torque, $s_{(\cdot)} \equiv \sin(\cdot)$, $c_{(\cdot)} \equiv \cos(\cdot)$. The problem is to design the controls $F_i$ so as to asymptotically steer $x$, $y$, $z$ and $\theta$, $\psi$, $\phi$ to zero.

Define new control inputs

$$
\begin{aligned}
u_0' &= \frac{1}{m}\left(\sum_{i=1}^{4} F_i\right) := (g + u_0),\\
u_1 &= \frac{\ell}{J_\theta}[-F_1 - F_2 + F_3 + F_4],\\
u_2 &= \frac{\ell}{J_\psi}[-F_1 + F_2 + F_3 - F_4],\\
u_3 &= \frac{C}{J_\phi}[F_1 - F_2 + F_3 - F_4]
\end{aligned}
$$

and rewrite (1) as

$$
\begin{aligned}
\ddot{x} &= (g + u_0)[c_\phi s_\theta c_\psi + s_\phi s_\psi],\\
\ddot{y} &= (g + u_0)[s_\phi s_\theta c_\psi - c_\phi s_\psi],\\
\ddot{z} &= (g + u_0)c_\theta c_\psi - g,\\
\ddot{\theta} &= u_1,\\
\ddot{\psi} &= u_2,\\
\ddot{\phi} &= u_3.
\end{aligned} \tag{2}
$$

Once the model is decoupled, the yaw angle $\phi$ can be independently stabilized to 0 by appropriate design of $u_3$. Let us then assume that this has been done (e.g. by a regular PD controller), in which case the model (2) with $\phi = \dot{\phi} = \ddot{\phi} = 0$ reduces to

$$
\begin{aligned}
\ddot{x} &= (g + u_0)s_\theta c_\psi,\\
\ddot{y} &= -(g + u_0)s_\psi,\\
\ddot{z} &= (g + u_0)c_\theta c_\psi - g,\\
\ddot{\theta} &= u_1,\\
\ddot{\psi} &= u_2.
\end{aligned} \tag{3}
$$

Let us then proceed with the regulation of the Cartesian coordinates $(x, y, z)$ through the pitch and roll angles $(\theta, \psi)$, which are assumed to be unmeasurable.

### 2.2 Hovering Control

Set $u_0$ to stabilize the coordinate $z$ as

$$
u_0 = \mathrm{sat}_L(-r_0 z - r_1 \dot{z}) \tag{4}
$$

with $L = \alpha g$ and $\alpha < 1$.

It can be shown that - if the parameters $r_0, r_1$ are appropriately chosen – the resulting vertical dynamics, namely

$$
\begin{aligned}
\dot{z}_1 &= z_2\\
\dot{z}_2 &= (g + \mathrm{sat}_{\alpha g}(-r_0 z_1 - r_1 z_2))c_\theta c_\psi - g
\end{aligned}
$$

in which $z_1 = z$, viewed as a system with input $(\theta, \psi)$ and state $(z_1, z_2)$, is *input-to-state stable*, subject to the restriction $|c_\theta c_\psi| \geq 1/(1 + \alpha)$.

### 2.3 Nominal Feedback Linearization

Define, for $i = 1, 2$, new variables $\xi_i = (\xi_{i1}, \xi_{i2}, \xi_{i3}, \xi_{i4})$ as

$$
\begin{aligned}
\xi_{11} &= x,\\
\xi_{12} &= \dot{x},\\
\xi_{13} &= g s_\theta c_\psi,\\
\xi_{14} &= g c_\theta c_\psi \dot{\theta} - g s_\theta s_\psi \dot{\psi},
\end{aligned}
$$

and

$$
\begin{aligned}
\xi_{21} &= y,\\
\xi_{22} &= \dot{y},\\
\xi_{23} &= -g s_\psi,\\
\xi_{24} &= -g c_\psi \dot{\psi}.
\end{aligned}
$$

If $|\theta| < \pi/2$ and $|\psi| < \pi/2$ this change of variables is well-defined. In fact, the map

$$
\begin{pmatrix}\theta\\ \psi\end{pmatrix} \mapsto \begin{pmatrix}\xi_{13}\\ \xi_{23}\end{pmatrix} = \begin{pmatrix}g s_\theta c_\psi\\ -g s_\psi\end{pmatrix}
$$

is invertible and so is the map

$$
\begin{pmatrix}\dot{\theta}\\ \dot{\psi}\end{pmatrix} \mapsto \begin{pmatrix}\xi_{14}\\ \xi_{24}\end{pmatrix} = \begin{pmatrix}g c_\theta c_\psi & -g s_\theta s_\psi\\ 0 & -g c_\psi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\psi}\end{pmatrix}.
$$

In the new coordinates, the systems reads as

$$
\begin{aligned}
\dot{\xi}_{11} &= \xi_{12},\\
\dot{\xi}_{12} &= \beta(t)\xi_{13},\\
\dot{\xi}_{13} &= \xi_{14},\\
\dot{\xi}_{14} &= q_1(\theta, \dot{\theta}, \psi, \dot{\psi}) + \begin{pmatrix}g c_\theta c_\psi & -g s_\theta s_\psi\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{\xi}_{21} &= \xi_{22},\\
\dot{\xi}_{22} &= \beta(t)\xi_{23},\\
\dot{\xi}_{23} &= \xi_{24},\\
\dot{\xi}_{24} &= q_2(\psi, \dot{\psi}) + \begin{pmatrix}0 & -g c_\psi\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix},
\end{aligned}
$$

[1] Here we assume the drag coefficients are negligibly small, which is reasonable at low speeds.

in which $q_1(\theta, \dot\theta, \psi, \dot\psi)$ and $q_2(\psi, \dot\psi))$ are appropriately-defined functions (whose specific expression is not need in what follows and hence not relevant) and the function

$$\beta(t) = \left(1 + \frac{u_0(t)}{g}\right) \tag{5}$$

satisfies

$$0 < \beta_{min} \le \beta(t) \le \beta_{max} \tag{6}$$

due to the choice $\alpha < 1$ in the saturated PD controller (4).

Clearly, the dynamics of $(\xi_1, \xi_2)$ can be feedback-linearized. The nominal "feedback-linearizing" control law $(u_1, u_2)$ is the control that forces

$$\begin{aligned}\dot\xi_{14} &= -k_1\xi_{11} - k_2\xi_{12} - k_3\xi_{13} - k_4\xi_{14} := K\xi_1,\\ \dot\xi_{24} &= -k_1\xi_{21} - k_2\xi_{22} - k_3\xi_{23} - k_4\xi_{24} := K\xi_2\end{aligned}$$

and hence is defined as

$$\begin{pmatrix}u_1\\u_2\end{pmatrix} = B^{-1}(\theta,\psi)\begin{pmatrix}-q_1(\theta,\dot\theta,\psi,\dot\psi) + K\xi_1\\ -q_2(\psi,\dot\psi) + K\xi_2\end{pmatrix}, \tag{7}$$

in which $B(\theta, \psi)$ is the matrix

$$B(\theta,\psi) = \begin{pmatrix}b_{11}(\theta,\psi) & b_{12}(\theta,\psi)\\ 0 & b_{22}(\theta,\psi)\end{pmatrix} = \begin{pmatrix}gc_\theta c_\psi & -gs_\theta s_\psi\\ 0 & -gc_\psi\end{pmatrix}.$$

Applying the control law (7) one obtains two identical systems of the form

$$\begin{aligned}\dot\xi_{i1} &= \xi_{i2},\\ \dot\xi_{i2} &= \beta(t)\xi_{i3},\\ \dot\xi_{i3} &= \xi_{14},\\ \dot\xi_{i4} &= -k_1\xi_{i1} - k_2\xi_{i2} - k_3\xi_{i3} - k_4\xi_{i4} := K\xi_i.\end{aligned}$$

Since $\beta(t)$ is bounded from below (by a positive number) and from above (see (6)), it is possible to find a choice of $K$, a matrix $P = P^T > 0$ and a number $\lambda$, that only depend on the bounds $\beta_{min}, \beta_{max}$, such that the derivative of $V(\xi_i) = \xi_i^T P\xi_i$ along trajectories is bounded as

$$\dot V(\xi_i) \le -\lambda|\xi_i|^2$$

and hence $\xi_i(t)$ exponentially decays to 0. Proof of the existence of such $K$ can be obtained by a simple adaptation of the arguments presented in Gauthier and Kupka (2001) to prove the "standard version" of Dayawansa's Lemma.

Clearly, if the controls $u_0, u_1, u_2$ are chosen in this way (and the constraints $|c_\theta c_\psi| \ge 1/(1+\alpha)$, $|\theta| < \pi/2$, $|\psi| < \pi/2$ are respected), the state of the controlled system asymptotically (and locally exponentially) converges to 0 as $t \to \infty$.

## 3. ENHANCED EXTENDED OBSERVER DESIGN

Since parameters and states that define the nominal control law (7) are not available, we choose instead – following the design paradigm proposed in Freidovich and Khalil (2008) and extended in Isidori et al. (2019) – the controls as

$$u = \begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}\mathrm{sat}_N(\hat u_1)\\ \mathrm{sat}_N(\hat u_2)\end{pmatrix}, \tag{8}$$

with

$$\begin{pmatrix}\hat u_1\\ \hat u_2\end{pmatrix} = \bar B^{-1}\begin{pmatrix}K\hat\xi_1 - \sigma_1\\ K\hat\xi_2 - \sigma_2\end{pmatrix}, \tag{9}$$

where $N$ is a design parameter, $\bar B$ is a nonsingular matrix chosen in such a way that

$$\|[B(\theta,\psi) - \bar B]\bar B^{-1}\|_1 \le \delta < 1, \tag{10}$$

and in which $\hat\xi_i = (\hat\xi_{i1}, \hat\xi_{i2}, \hat\xi_{i3}, \hat\xi_{i3})$ and $\sigma_i$, for $i = 1, 2$, are the states of the pair of "enhanced" extended observers defined by the equations

$$\begin{aligned}\dot{\hat\xi}_{11} &= \hat\xi_{12} + \kappa a_4(x - \hat\xi_{11}),\\ \dot{\hat\xi}_{12} &= \beta(t)\hat\xi_{13} + \kappa^2 a_3(x - \hat\xi_{11}),\\ \dot{\hat\xi}_{13} &= \hat\xi_{14} + \kappa^3 a_2(x - \hat\xi_{11}),\\ \dot{\hat\xi}_{14} &= \sigma_1 + \bar b_1 u + \kappa^4 a_1(x - \hat\xi_{11}),\\ \dot\sigma_1 &= \kappa^5 a_0(x - \hat\xi_{11})\end{aligned} \tag{11}$$

and

$$\begin{aligned}\dot{\hat\xi}_{21} &= \hat\xi_{22} + \kappa a_4(y - \hat\xi_{21}),\\ \dot{\hat\xi}_{22} &= \beta(t)\hat\xi_{23} + \kappa^2 a_3(y - \hat\xi_{21}),\\ \dot{\hat\xi}_{23} &= \hat\xi_{24} + \kappa^3 a_2(y - \hat\xi_{21}),\\ \dot{\hat\xi}_{24} &= \sigma_2 + \bar b_2 u + \kappa^4 a_1(y - \hat\xi_{21}),\\ \dot\sigma_2 &= \kappa^5 a_0(y - \hat\xi_{21}),\end{aligned} \tag{12}$$

in which by $\bar b_1, \bar b_2$ we have denoted the two rows of the matrix $\bar B$. [2] In this respect, we stress that the time-varying coefficient $\beta(t)$, defined as in (5), *is available for measurement*.

In order to analyze the resulting closed-loop system, it is convenient to introduce "estimation errors" $\varepsilon_i = (\varepsilon_{i1}, \varepsilon_{i2}, \ldots, \varepsilon_{i5})$, for $i = 1, 2$, defined as

$$\begin{aligned}\varepsilon_{i1} &= \kappa^4(\xi_{i1} - \hat\xi_{i1}),\\ \varepsilon_{i2} &= \kappa^3(\xi_{i2} - \hat\xi_{i2}),\\ \varepsilon_{i3} &= \kappa^2(\xi_{i3} - \hat\xi_{i3}),\\ \varepsilon_{i4} &= \kappa(\xi_{i4} - \hat\xi_{i4}),\\ \varepsilon_{i5} &= q_i(\theta,\dot\theta,\psi,\dot\psi) + [b_i(\theta,\psi) - \bar b_i]\bar u(\xi_1,\xi_2,\sigma_1,\sigma_2) - \sigma_i,\end{aligned}$$

in which $b_i(\theta, \psi)$ denotes the $i$-th row of $B(\theta, \psi)$ and $\bar u(\xi_1, \xi_2, \sigma_1, \sigma_2)$ is the function obtained replacing $\hat\xi_1, \hat\xi_2$ by $\xi_1, \xi_2$ in the expressions (8)–(9). It can be shown that map $(\hat\xi_1, \hat\xi_2, \sigma_1, \sigma_2) \mapsto (\varepsilon_1, \varepsilon_2)$ is globally invertible (see, e.g., Lemma 10.1 of Isidori (2017)).

Appropriate calculations show that $\varepsilon = (\varepsilon_1, \varepsilon_2)$ is characterized by the dynamics

$$\dot\varepsilon = \kappa[A(t) + B\Delta_0(\xi,\varepsilon)C]\varepsilon + D\Delta_1(\xi,\varepsilon) + B\Delta_2(\xi,\varepsilon,t) \tag{13}$$

in which

$$A(t) = \begin{pmatrix}A_1(t) & 0\\ 0 & A_2(t)\end{pmatrix}$$

$$A_1(t) = A_2(t) = \begin{pmatrix}-a_4 & 1 & 0 & 0 & 0\\ -a_3 & 0 & \beta(t) & 0 & 0\\ -a_2 & 0 & 0 & 1 & 0\\ -a_1 & 0 & 0 & 0 & 1\\ -a_0 & 0 & 0 & 0 & 0\end{pmatrix},$$

$$B = \begin{pmatrix}B_1 & 0\\ 0 & B_2\end{pmatrix},\quad C = \begin{pmatrix}C_1 & 0\\ 0 & C_2\end{pmatrix},\quad D = \begin{pmatrix}D_1 & 0\\ 0 & D_2\end{pmatrix}$$

[2] Loosely speaking, the matrix $\bar B$ can be regarded as a fixed "guess" of $B(\theta, \psi)$. If the system were single-input single-output, with $B(\theta, \psi)$ written as $b(x)$, the condition in question would be written as $|b(x) - \bar b||\bar b|^{-1} \le \delta < 1$, which indeed holds, for some $\bar b$, if $|b(x)|$ is bounded from above and from below. Condition (10) is a multivariable version of such condition.

$$B_1 = B_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, D_1 = D_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, C_1^T = C_2^T = \begin{pmatrix} a_0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and $\Delta_0(\xi,\varepsilon), \Delta_1(\xi,\varepsilon), \Delta_2(\xi,\varepsilon,t)$ are appropriately-defined functions of $\xi = (\xi_1,\xi_2)$, $\varepsilon = (\varepsilon_1,\varepsilon_2)$ and $t$. It can be shown (see, e.g., Lemma 10.2 of Isidori (2017)) that there exist numbers $\delta_0 < 1$ and $\delta_1$ such that

$$\begin{aligned} \|\Delta_0(\xi,\varepsilon)\|_1 &\le \delta_0 < 1 && \text{for all } (\xi,\varepsilon) \text{ and all } \kappa \\ \|\Delta_1(\xi,\varepsilon)\| &\le \delta_1\|\varepsilon\| && \text{for all } (\xi,\varepsilon) \text{ and all } \kappa\,. \end{aligned} \tag{14}$$

Moreover, for each $R > 0$ there is a number $M_R$ such that

$$\|\xi\| \le R \quad \Rightarrow \quad \|\Delta_2(\xi,\varepsilon,t)\| \le M_R \quad \text{for all } \varepsilon, t \text{ and all } \kappa. \tag{15}$$

For the analysis of the asymptotic properties of the closed-loop system, it is important to determine appropriate "practical stability" properties of the dynamics of $\varepsilon$. It is in this context that the enhanced version of Dayawansa's Lemma plays a fundamental role.

The extension of the Lemma of Dayawansa (see Isidori et al. (2019)) considers a system with input $u$ and output $y$, modeled as

$$\begin{aligned} \dot{x} &= A(t)x + Bu \\ y &= C(t)x \end{aligned} \tag{16}$$

in which $A(t)$ is the matrix

$$A(t) = \begin{pmatrix} -a_{n-1}g_1(t) & g_2(t) & 0 & \cdots & 0 & 0 \\ -a_{n-2}g_1(t) & 0 & g_3(t) & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ -a_1 g_1(t) & 0 & 0 & \cdots & 0 & g_n(t) \\ -a_0 g_1(t) & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{17}$$

and $B, C$ are defined as

$$\begin{aligned} B^{\mathrm{T}} &= (0\ 0\ \cdots\ 0\ 1) \\ C(t) &= (a_0 g_1(t)\ 0\ \cdots\ 0\ 0) \end{aligned} \tag{18}$$

and can be stated as follows.

*Lemma 1.* Consider a system of the form (16) in which $A(t)$ and $B, C(t)$ have the form (17) and, respectively, (18). Suppose the $g_i(t)$'s are continuous functions of time satisfying, for some pair $(g_{\min}, g_{\max})$, the bounds

$$0 < g_{\min} \le g_i(t) \le g_{\max} \qquad \forall t \ge 0, \tag{19}$$

for all $1 \le i \le n$. Then, for every choice of a number $\gamma > 1$, there exist a set of real numbers $a_0, a_1, \ldots, a_{n-1}$, a number $\lambda > 0$ and a continuous positive definite function $V(x)$, bounded from below as $\underline{a}\|x\| \le V(x)$ for some $\underline{a} > 0$ and satisfying $|V(x_1) - V(x_2)| \le L\|x_1 - x_2\|$ for some $L > 0$, such that, along trajectories of (16), [3]

$$D^+_{(16)}V \le -\lambda V(x) + \gamma|u| - |y|\,. \tag{21}$$

This Lemma essentially says that a strict *dissipation inequality* of the form (21) can be forced *for any* number $\gamma$ arbitrarily close to 1, but larger than 1, if the parameters $a_0, a_1, \ldots, a_{n-1}$ are properly chosen.

[3] With $x(t)$ the state trajectory of a system modeled by

$$\dot{x} = f(x,u) \tag{20}$$

$D^+_{(20)}V(x(t))$ denotes the Dini derivative of $V(x(t))$, which is defined as

$$D^+_{(20)}V(x(t)) = \limsup_{h\to 0^+} \frac{1}{h}[V(x(h+t)) - V(x(t))].$$

Dini derivative is used because the function $V(x)$ that we consider is only Lipschitz continuous and not continuously differentiable.

With this in mind, observe that system (13), if $\kappa = 1$ and both the "perturbation terms" $\Delta_1(\xi,\varepsilon)$ and $\Delta_2(\xi,\varepsilon,t)$ are zero, reduces to the system

$$\dot{\varepsilon} = [A(t) + B\Delta_0(\xi,\varepsilon)C]\varepsilon \tag{22}$$

that can be interpreted as the interconnection of

$$\begin{aligned} \dot{\varepsilon}_1 &= A_1(t)\varepsilon_1 + B_1 u_1 \\ \dot{\varepsilon}_2 &= A_2(t)\varepsilon_2 + B_2 u_2 \\ y_1 &= C_1\varepsilon_1 \\ y_2 &= C_2\varepsilon_2 \end{aligned}$$

and

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \Delta_0(\xi,\varepsilon)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Observe now that the matrices $A_i(t), B_i, C_i$ are special cases of the matrices (17) and (18) and that the coefficient $\beta(t)$ satisfies a bound of the form (19). Thus, according to Lemma 1, for any $\gamma > 1$, there exists a set of real numbers $a_0, a_1, \ldots, a_4$, a number $\lambda > 0$ and a positive definite and proper continuous function $V(\varepsilon_i)$ satisfying an inequality of the form

$$D^+_{(23)}V \le -\lambda V(\varepsilon_i) + \gamma|u_i| - |y_i|$$

along the trajectories of

$$\begin{aligned} \dot{\varepsilon}_i &= A_i(t)\varepsilon_i + B_i u_i \\ y_i &= C_i\varepsilon_i. \end{aligned} \tag{23}$$

Since, by assumption, the number $\delta_0$ in (14) is less than 1, one can pick $\gamma$ satisfying

$$1 < \gamma < \frac{1}{\delta_0}\,.$$

Thus, the function $U(\varepsilon) = V(\varepsilon_1) + V(\varepsilon_2)$, along the trajectories of (22), satisfies (set $u = \mathrm{col}(u_1,u_2)$ and $y = \mathrm{col}(y_1,y_2)$ and use the bound (14))

$$\begin{aligned} D^+_{(22)}U &\le -\lambda U(\varepsilon) + \gamma|u|_1 - |y|_1 \\ &\le -\lambda U(\varepsilon) + \gamma|\Delta_0(\xi,\varepsilon)|_1|y|_1 - |y|_1 \\ &\le -\lambda U(\varepsilon)) + \gamma\delta_0|y|_1 - |y|_1 \\ &\le -\lambda U(\varepsilon)\,. \end{aligned}$$

Taking now the derivative of the function $U(\varepsilon)$ along the trajectories of the full system (13), and bearing in mind bounds (14)–(15) for $\Delta_1(\xi,\varepsilon), \Delta_2(\xi,\varepsilon,t)$, one obtains that, so long as $\|\xi(t)\| \le R$, an inequality holds of the form

$$D^+_{(13)}U \le -\kappa\lambda U(\varepsilon) + \ell_1\delta_1\|\varepsilon\| + \ell_2 M_R$$

in which $\ell_1, \ell_2$ are appropriate constants. From this, one can deduce the desired "practical stability" properties of the dynamics of $\varepsilon$. In fact, from this inequality it is easily deduced that, so long as $\|\xi(t)\| \le R$, given any arbitrarily small time $T_0 > 0$ and any arbitrarily small number $\epsilon > 0$, there is a value $\kappa^*$ such that, if $\kappa > \kappa^*$, then $\|\varepsilon(t)\| \le \epsilon$ for all $t \ge T_0$. This being the case, following a rather standard paradigm (see e.g. Teel and Praly (1995)), this result can be used to conclude that *semiglobal* stability of the equilibrium $(\xi,\varepsilon) = (0,0)$ is achieved. In fact, assuming that the initial conditions are taken in a fixed (but otherwise arbitrary) compact set, it can be shown that - for $t \ge T_0$ – the dynamics of $\xi_1, \xi_2$ under the controls (8)–(9) differ from the nominally feedback-linearized dynamics by a quantity of order $\epsilon$. This establishes the property of (semiglobal) practical stability of the overall system. For small $\xi$ and $\varepsilon$, the overall system can be seen as pure feedback interconnection of two exponentially stable systems, with a loop-gain that can be made arbitrarily small by increasing $\kappa$ and this also establishes the desired property of asymptotic stability.

## 4. SIMULATION

Let us now simulate a station-keeping system of a quadrotor modeled by (3) starting from different initial conditions to illustrate the enhanced extended observer technique.

Apply the control law (4) with use

$$r_0 = 5, \quad r_1 = 10, \quad \alpha = 0.9.$$

to stabilize the coordinate $z$ and the control law (8), (9) based on the enhanced extended observers (11), (12) with

$$K = (-100\ -100\ -100\ -10)\,,$$
$$a_0 = 0.01,\ a_1 = 1,\ a_2 = 1000,\ a_3 = 1000,\ a_4 = 10,$$
$$\bar{B} = \begin{pmatrix} g & 0 \\ 0 & -g \end{pmatrix}, \quad N = 100, \quad \kappa = 5$$

to stabilize the coordinates $x$ and $y$.

Run the system multiple times with different initial conditions provided in table 1.

Table 1. Initial conditions

| Run | $x(0)$ | $y(0)$ | $z(0)$ | $\theta(0)$ | $\psi(0)$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | $\pi/6$ | $\pi/6$ |
| 2 | 1 | 2 | 3 | $\pi/6$ | $\pi/4$ |
| 3 | $-1$ | $-1$ | $-1$ | $-\pi/6$ | $-\pi/6$ |
| 4 | $-1$ | $-2$ | $-3$ | $-\pi/6$ | $-\pi/4$ |

The simulation results are shown in Fig. 1–4. As expected, all the Cartesian coordinates converge to 0, while all the constraints $|c_\theta c_\psi| \geq 1/(1+\alpha)$, $|\theta| < \pi/2$, $|\psi| < \pi/2$ are respected.

## 5. CONCLUSION

In this paper we have provided an applied example of application the enhanced extended observer technique described in Isidori et al. (2019). Its practical value has been shown by the simulation of station-keeping of a quadrotor with uncertain parameters and unmeasurable pitch and roll angles.

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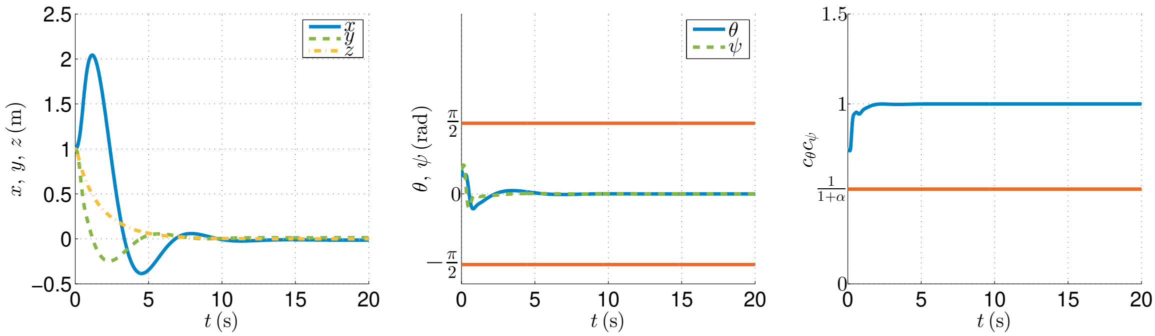


Fig. 1. Simulation results for $x(0) = 1$, $y(0) = 1$, $z(0) = 1$, $\theta(0) = \pi/6$, $\psi(0) = \pi/6$

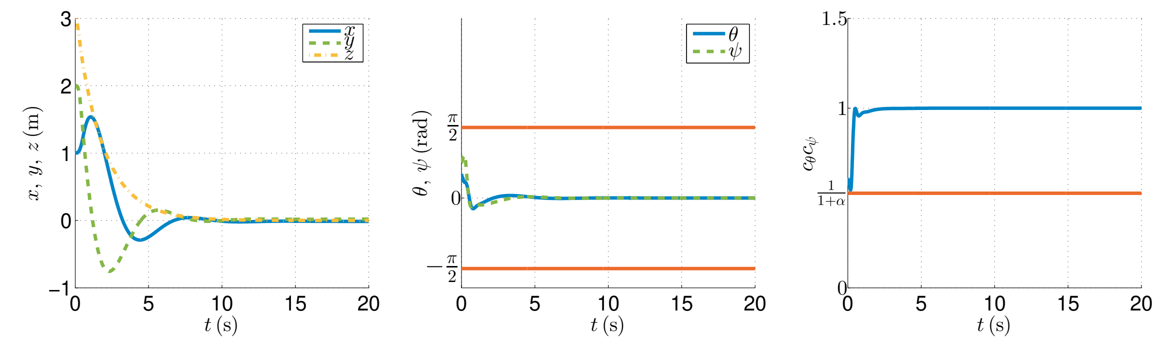


Fig. 2. Simulation results for $x(0) = 1$, $y(0) = 2$, $z(0) = 3$, $\theta(0) = \pi/6$, $\psi(0) = \pi/4$

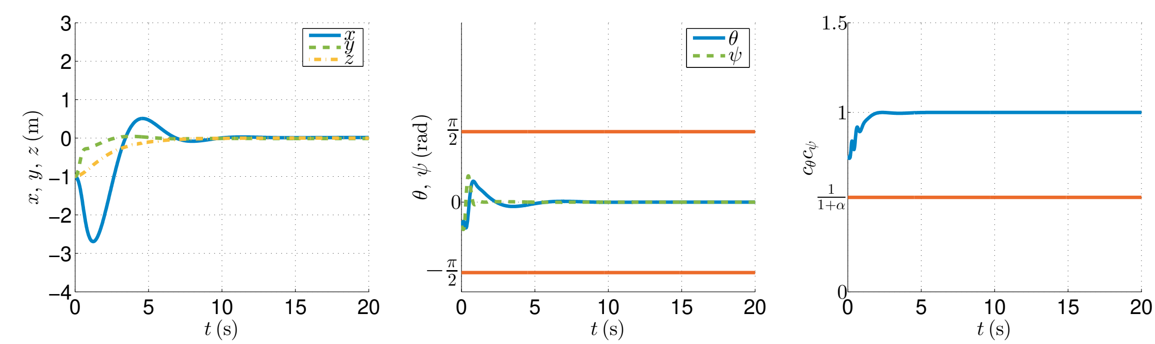


Fig. 3. Simulation results for $x(0) = -1$, $y(0) = -1$, $z(0) = -1$, $\theta(0) = -\pi/6$, $\psi(0) = -\pi/6$

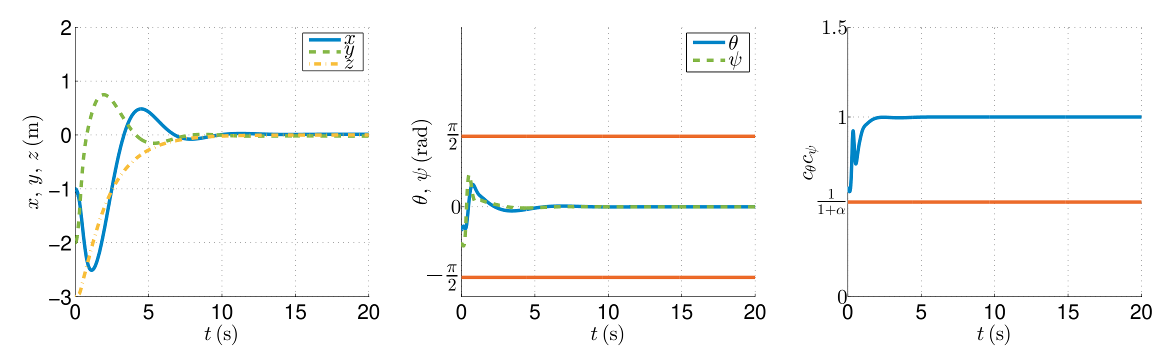


Fig. 4. Simulation results for $x(0) = -1$, $y(0) = -2$, $z(0) = -3$, $\theta(0) = -\pi/6$, $\psi(0) = -\pi/4$