Sufficient conditions for local finiteness of a polymodal logic

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This talk is about normal propositional modal logics, the equational theories of modal algebras.

Part I. Preliminaries.

Part II. Two sufficient conditions for local finiteness.

Language

Fix a set $\mathcal{A}$ for the *alphabet of modal operators*.
$\mathcal{A}$-formulas: a countable set $\textsc{Var}$ (propositional variables), Boolean connectives, unary connectives $\Diamond \in \mathcal{A}$.

Normal modal logics: Definition 1

A set of modal $\mathcal{A}$-formulas L is a *normal modal logic*, if for all $\Diamond \in \mathcal{A}$, L contains

classical tautologies

$$\Diamond\bot \leftrightarrow \bot, \quad \Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q$$

and is closed under MP, Sub, and *Mon*:
if $(\varphi \to \psi) \in L$, then $(\Diamond\varphi \to \Diamond\psi) \in L$.

Normal modal logics: Definition 2

A *modal algebra* is a Boolean algebra B endowed with unary operations that distributes over finite disjunctions.

A set of modal formulas L is a *normal modal logic*, if L is the logic of a modal algebra B: $L = \{\varphi \mid \varphi = 1 \text{ holds in B}\}$

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Kripke semantics

A *(Kripke) frame* F: $(W, (R_\Diamond)_{\Diamond\in\mathcal{A}})$, where $R_\Diamond$ are binary relations on a set W.
A *model* M on F is a pair (F, θ) where $\theta : \textsc{Var} \to \mathcal{P}(W)$.

$$M, x \vDash p \text{ iff } x \in \theta(p), \qquad M, x \vDash \Diamond\varphi \text{ iff } M, y \vDash \varphi \text{ for some } y \text{ with } xR_\Diamond y.$$

$\mathrm{Log}(\mathrm{F}) = \{\varphi \mid \mathrm{F} \vDash \varphi\}$, where $\mathrm{F} \vDash \varphi$ means that $M, x \vDash \varphi$ for all M on F and all x in M.

The *algebra $\mathrm{Alg}(\mathrm{F})$ of an $\mathcal{A}$-frame* $(W, (R_\Diamond)_{\Diamond\in\mathcal{A}})$ is the powerset algebra of W with the unary operations $f_\Diamond$: for $Y \subseteq W$, $f_\Diamond(Y)$ is $R_\Diamond^{-1}[Y] = \{x \mid \exists y \in Y \; xR_\Diamond y\}$.

$$\mathrm{F} \vDash \varphi \quad \text{iff} \quad \varphi = 1 \text{ holds in } \mathrm{Alg}(\mathrm{F})$$

A logic L is *Kripke complete*, if L is the logic of a class $\mathcal{C}$ of Kripke frames: $L = \bigcap\{\mathrm{Log}(\mathsf{F}) \mid \mathsf{F} \in \mathcal{C}\}$.

A logic L has the *finite model property*, if L is the logic of a class $\mathcal{C}$ of finite frames (algebras, models).

Fact

If L has the fmp and is finitely axiomatizable, then it is decidable.

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A logic L is *locally finite* (aka *locally tabular*), if for all $k < \omega$ there are only finitely many formulas in k variables (up to $\leftrightarrow_L$).

TFAE:

L is locally finite.	Every finitely generated free (aka Lindenbaum-Tarski) algebra of L is finite.	The variety of L-algebras is *locally finite*, i.e., all finitely generated L-algebra are finite.

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normal modal logics $\supsetneq$
  Kripke complete logics $\supsetneq$
    logics with the finite model property $\supsetneq$
      logics whose all extensions have the fmp $\supsetneq$
        locally finite logics

Local finiteness for modal logics and their relatives (sound but incomplete list):

Segerberg, K., "An Essay in Classical Modal Logic," 1971.

Kuznetsov, A. *Some properties of the structure of varieties of pseudo-Boolean algebras*, 1971.

Maksimova, L. *Modal logics of finite slices*, 1975.

Komori, Y. *The finite model property of the intermediate propositional logics on finite slices*, 1975.

Byrd, M. *On the addition of weakened L-reduction axioms to the Brouwer system*, 1978.

Makinson, D. *Non-equivalent formulae in one variable in a strong omnitemporal modal logic*, 1981.

Mardaev, S. *The number of prelocally tabular superintuitionistic propositional logics*, 1984.

Citkin, A. *Finite axiomatizability of locally tabular superintuitionistic logics*, 1986.

...

Bezhanishvili, G. and Grigolia, R. *Locally tabular extensions of MIPC*, 1998.

Bezhanishvili, G., *Locally finite varieties*, 2001.

Bezhanishvili, N. *Varieties of two-dimensional cylindric algebras. Part I: Diagonal-free case*, 2002.

Shehtman, V. *Canonical filtrations and local tabularity*, 2014.

Shapirovsky, I. and Shehtman, V. *Local tabularity without transitivity*, 2016

Shapirovsky, I. *Glivenko's theorem, finite height, and local tabularity*, 2021.

Dzik W., Kost S., Wojtylak P. *Finitary unification in locally tabular modal logics characterized*, 2022

...

The *height* of a frame $(W,(R_\Diamond)_{\Diamond\in\mathcal{A}})$ is the height of the preorder (W,R), were R is the transitive reflexive closure of $\bigcup_{\mathcal{A}} R_\Diamond$.

Unimodal transitive case

$\Diamond\Diamond p \to \Diamond p$ expresses the transitivity of a binary relation.

Formulas of *finite height*:

$$B_0 = \bot, \quad B_{i+1} = p_{i+1} \to \Box(\Diamond p_{i+1} \vee B_i)$$

($\Box$ abbreviates $\neg\Diamond\neg$)

A transitive logic is locally finite iff it is of finite height:

[Segerberg 1971; Maksimova 1975]
Let L be a unimodal logic containing $\Diamond\Diamond p \to \Diamond p$. Then

L is LF iff L has a formula of finite height.

Unimodal non-transitive case and poly-modal case are much less clear...

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Unimodal nontransitive case

Sometimes an analog of finite height criterion holds in non-transitive case:

[Shehtman & Sh, 2016] Let L be a unimodal logic containing $\Diamond \ldots \Diamond p \to \Diamond p \vee p$. Then

L is LF iff L has a formula of finite height.

(Formulas of finite height will be slight modifications of B_i's.)

But in general, in non-transitive case, finite height is not sufficient:

[Byrd 1978; Makinson, 1981] There a reflexive symmetric structure $\mathrm{F} = (W, R)$ with $R \circ R = W \times W$ (so it is of height 1) s.t. its logic is not LF. Moreover, its one-variable fragment is infinite.

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Polymodal case: (non)examples

The logic of monadic Boolean algebras (aka $\mathrm{S5}$) is one of the simplest examples of locally finite modal logics. However, the logic $\mathrm{S5} * \mathrm{S5}$ with two monadic operators is not locally finite; moreover, its one-variable fragment is infinite.

[Shehtman & Sh, 2016] If the 1-variable fragment of an $\mathcal{A}$-logic L is finite, then:
1. For some finite m, L has an axiom expressing the following on frames:
If there is a path from x to y, then there is a path of length $\leq m$ from x to y.
2. L has an axiom of finite height.

$\mathrm{S5} * \mathrm{S5}$ is the logic of frames $(W, \sim_1, \sim_2)$ with two equivalence relations. (1) does not hold for $\mathrm{S5} * \mathrm{S5}$: we need axioms that make all $\Diamond \in \mathcal{A}$ dependant.

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[N. Bezhanishvili, 2002] Every extension of

$$\mathrm{S5}*\mathrm{S5} + \Diamond_1\Diamond_2 p \leftrightarrow \Diamond_2\Diamond_1 p$$

is locally finite.

Part II. Two sufficient conditions for local finiteness of a polymodal logic.

Many important normal modal logics can by characterized as logics of *sums* of relational structures.

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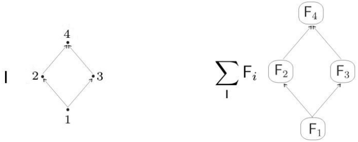
Given a family $(\mathsf{F}_i \mid i \text{ in } \mathsf{I})$ of $\mathcal{A}$-frames indexed by elements of another $\mathcal{A}$-frame I, the *sum of the frames* F_i*'s over* I is obtained from the disjoint union of F_i's by connecting elements of i-th and j-th distinct components according to the relations in I.

Unimodal case (for simplicity only):
frame of indices $\mathsf{I} = (I, S)$;
frames-summands $\mathsf{F}_i = (W_i, R_i)$, i in I.



For classes $\mathcal{I}$, $\mathcal{F}$ of frames, $\sum_{\mathcal{I}} \mathcal{F}$ is the class of all sums $\sum_{i \in \mathsf{I}} \mathsf{F}_i$ such that $\mathsf{I} \in \mathcal{I}$ and $\mathsf{F}_i \in \mathcal{F}$ for every i in I.

Many important normal modal logics can by characterized as logics of *sums* of relational structures.

Idea: In many cases, the modal logic of a class of sums inherits "good" properties of the logics of summands/indices.

This is not a new approach:
In classical model theory, *"composition theorems"* reduce the theory (FO, MSO) of a compound structure to theories of its components [Mostowski, 1952], [Feferman–Vaught 1959], [Shelah 1975], [Gurevich 1979], ...

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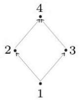
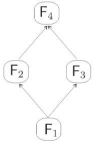


$\sum_{\mathsf{I}} \mathsf{F}_i$



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Transfer results for sums in modal logic:

Axiomatization [Kracht 1993; Beklemishev 2007; Balbiani 2009; Balbiani & Mikulás 2013; Balbiani & Sh, 2014; Balbiani & Fernández-Duque 2016]

Finite model property and decidability [Babenyshev & Rybakov 2010; Sh 2018]

Computational complexity [Sh 2008; Sh 2020]

Local finiteness [this talk]

Let the set $\mathcal{A}$ of modal operators be finite.

Main result

Let $\mathcal{F}$ and $\mathcal{I}$ be classes of $\mathcal{A}$-frames. If the logics $\mathrm{Log}(\mathcal{F})$ and $\mathrm{Log}(\mathcal{I})$ are locally finite, then the logic $\mathrm{Log}(\sum_{\mathcal{I}} \mathcal{F})$ is.

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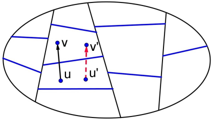
Let $\mathsf{F} = (W, (R_\Diamond)_{\Diamond \in \mathcal{A}})$. A partition $\mathcal{S}$ of W is *tuned* if for all $U, V \in \mathcal{S}$, $\Diamond \in \mathcal{A}$

$$\exists u \in U \exists v \in V \; uR_\Diamond v \;\Rightarrow\; \forall u \in U \exists v \in V \; uR_\Diamond v.$$



F is said to be *tunable* if every finite partition $\mathcal{S}$ of F admits a finite tuned refinement.

Observation The algebra of F is locally finite iff F is tunable.

Remark. An analog of this observation can be stated for any powerset modal algebra.

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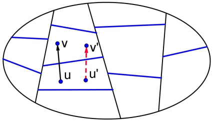

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Key step in proving main result: given a finite partition of a sum, construct its finite tuned refinement via tuned refinements of the index and summands; control the size.

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Auxiliary step

For a frame $F = (W, (R_\Diamond)_{\Diamond \in \mathcal{A}})$, let F^{r} be the frame $(W, (R^{\mathrm{r}}_\Diamond)_{\Diamond \in \mathcal{A}})$, where R^{r}_a is the reflexive closure of $R_\Diamond$. For a class $\mathcal{F}$ of frames, $\mathcal{F}^{\mathrm{r}} = \{F^{\mathrm{r}} \mid F \in \mathcal{F}\}$.

Expectable theorem.

Let $\mathcal{F}$ be a class of frames. The $\mathrm{Log}(\mathcal{F})$ is locally finite iff $\mathrm{Log}(\mathcal{F}^{\mathrm{r}})$ is locally finite.

“Only if” is trivial. “If” is based on the following lemma (with unexpectedly convoluted proof)

Lemma. Let F be an irreflexive $\mathcal{A}$-frame. Assume that the logic of the frame F is locally finite. Then for every $k < \omega$, every k-generated subalgebra of $\mathrm{Alg}(F)$ is contained in a $(k + 3|\mathcal{A}|)$-generated subalgebra of $\mathrm{Alg}(F^{\mathrm{r}})$.

Question. Should we expect that Expectable theorem holds for locally finite algebras (not varieties)?

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Main result

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A version of the main result for algebras:

Let $(\mathsf{F}_i)_{i\in I}$ be a family of $\mathcal{A}$-frames, $\mathsf{I} = (I, (S_\Diamond)_{\Diamond\in\mathcal{A}})$ be an $\mathcal{A}$-frame. If all $S_\Diamond$ are *irreflexive*, algebras $\mathrm{Alg}(\bigsqcup_I \mathsf{F}_i)$ and $\mathrm{Alg}(\mathsf{I})$ are locally finite, then $\mathrm{Alg}(\sum_{\mathsf{I}} \mathsf{F}_i)$ is locally finite.

Question Can irreflexivity condition be omitted?

Auxiliary step

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$$\Diamond_b\Diamond_a p \to \Diamond_a p, \Diamond_a\Diamond_b p \to \Diamond_a p, \Diamond_a p \to \Box_b\Diamond_a p$$

with $\Diamond_a$ in $\mathcal{A}$ and $\Diamond_b$ in $\mathcal{B}$.

Let L_1 be a logic in the language of $\mathcal{A}$ given by a set of axioms Ψ_1, and L_2 be a logic in the language of $\mathcal{B}$ given by a set of axioms Ψ_2. We define $L_1 \oplus L_2$ as the logic in the language of $\mathcal{A} \cup \mathcal{B}$ given by the axioms

$$\Psi_1 \cup \Psi_2 \cup \Phi(\mathcal{A},\mathcal{B})$$

Theorem

Let L_1 and L_2 be locally finite. Then:

1. If $L_1 \oplus L_2$ is Kripke complete, then it is locally finite.
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$\mathsf{I} = (I,(S_a)_{a\in\mathcal{A}})$ is an $\mathcal{A}$-frame, $(\mathsf{F}_i)_{i\in I}$ are $\mathcal{B}$-frames, $\mathsf{F}_i = (W_i,(R_{i,b})_{b\in\mathcal{B}})$.

The *lexicographic sum* $\sum^{\text{lex}}_{\mathsf{I}}\mathsf{F}_i$ is the $(\mathcal{A}\cup\mathcal{B})$-frame $(\bigsqcup_{i\in I} W_i, (S^{\text{lex}}_a)_{a\in\mathcal{A}}, (R_b)_{b\in\mathcal{B}})$:

$$(i,w)S^{\text{lex}}_a(j,u) \quad \text{iff} \quad iSj,$$
$$(i,w)R_b(j,u) \quad \text{iff} \quad i=j \;\&\; wR_{i,b}u.$$

For logics L_1, L_2, $\sum^{\text{lex}}_{L_1} L_2$ is the logic of lexicographic sums of their frames.

Theorem. If L_1 and L_2 are locally finite, then the logic $\sum^{\text{lex}}_{L_1} L_2$ is locally finite.

This is an easy corollary of the Main result.

[Balbiani and others] In many cases, $\sum^{\text{lex}}_{L_1} L_2 = L_1 \oplus L_2$.

Observation. For all logics, $\sum^{\text{lex}}_{L_1} L_2 \subseteq L_1 \oplus L_2$ provided that $L_1 \oplus L_2$ is Kripke complete.

What could be other operations on frames that preserve good properties of their logics? For instance:

Does the direct product of finitely many frames preserve local finiteness?

For example, let $(\omega, \preceq)^n$ be the n-th direct power of $(\omega, \leq)$.

[2019] For all finite n, $\mathrm{Alg}(\omega^n, \preceq)$ is locally finite.

$n = 1$: it is easy to tune partitions in $(\omega, \leq)$; $n > 1$: not that easy.

Let $\mathsf{F} = (W, (R_\Diamond)_{\Diamond \in \mathcal{A}})$. A partition $\mathcal{S}$ of W is *tuned* if for all $U, V \in \mathcal{S}$, $\Diamond \in \mathcal{A}$

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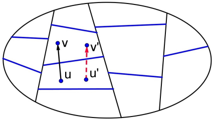

Consider tunable frames F_1 and F_2. Is the direct product $\mathsf{F}_1 \times \mathsf{F}_2$ tunable?
For quasi-orders? Partial orders? For well-founded orders?
At least, for well-orders is must be true...

What could be other operations on frames that preserve good properties of their logics? For instance:

Does the direct product of finitely many frames preserve local finiteness?

For example, let $(\omega, \preceq)^n$ be the n-th direct power of $(\omega, \leq)$.

[2019] For all finite n, $\mathrm{Alg}(\omega^n, \preceq)$ is locally finite.

$n = 1$: it is easy to tune partitions in $(\omega, \leq)$; $n > 1$: not that easy.

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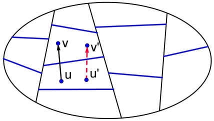

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Thank you!