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Finite model property of modal logics of finite depth

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In this paper we consider *propositional normal modal logics* [2]. A modal logic has the finite model property (fmp, for short) if it is complete with respect to a class of finite frames. By Harrop's theorem, finitely axiomatizable logics with the fmp are decidable. Despite the fact that the fmp of modal logics has been systematically studied for about fifty years, the picture is still incomplete even for the basic language (that is, when we have a single modal operator). Many modal logics are known to have the fmp (cf. [2], [1]). There are logics without the fmp, but in the unimodal case such examples are usually quite artificial. There are also natural examples of unimodal logics for which the fmp is unknown. The most well-known example is perhaps the logic K_3^2 axiomatized by the simple formula $\Box\Box p \to \Box\Box\Box p$. In general, the finite model property and even the decidability of the logics K_n^m axiomatized by the formulae $\Box^m p \to \Box^n p$ is an old open problem (cf. [2], problem 11.2 and [8], problem 6), and the answer is unknown for all $m, n > 1$ with $m \neq n$.

The modal operator corresponding to the transitive reflexive closure of a binary relation is expressible in the logics K_n^m for $n > m$. We call such logics *pretransitive*. Formally, L is pretransitive if there exists a formula $\chi(p)$ such that for any model M with $M \vDash \mathrm{L}$, and for any point w in this model we have $M, w \vDash \chi(p) \iff \forall u\,(wR^*u \Rightarrow M, u \vDash p)$, where R^* is the transitive reflexive closure of the relation R of the model M.

A logic is pretransitive if and only if for some $k \geqslant 0$ it includes the k-*transitivity* formula $\Box^{\leqslant k} p \to \Box^{k+1} p$, where $\Box^{\leqslant k}\varphi = \bigwedge_{i=0}^{k} \Box^i \varphi$ (cf. [3]). In particular, for $n > m$ the logic K_n^m is $(n-1)$-transitive. Let $\mathrm{K}_{\leqslant m}$ be the logic axiomatized by the formula $\Box^{\leqslant m} p \to \Box^{m+1} p$. Thus, $\mathrm{K}_{\leqslant m}$ is the minimal m-transitive logic. For $m > 1$, the fmp of $\mathrm{K}_{\leqslant m}$ is also unknown.

For a Kripke complete logic L, the fmp means that if a formula is satisfiable in an L-frame, then it is satisfiable in a finite L-frame. Sahlqvist's theorem implies that all the logics K_n^m and $\mathrm{K}_{\leqslant m}$ are Kripke complete. The frames of these logics have a simple first-order characterization; for example, K_3^2-frames are characterized by the first-order condition $R \circ R \circ R \subseteq R \circ R$, where $\circ$ stands for the composition of relations. In general, K_n^m-frames are characterized by the condition $R^n \subseteq R^m$ and $\mathrm{K}_{\leqslant m}$-frames by the condition $R^{m+1} \subseteq \bigcup_{i \leqslant m} R^i$. Hence, there are no known ways to transform an infinite frame into a finite one preserving the satisfiability of a given formula while maintaining the properties mentioned. We were able to construct such transformations in the case when the depth of the frames (that is, the maximal cardinality of chains in the partial order induced by the relation R) is finite.

Lemma 1. *Let* L *be one of the logics* K_n^m, $\mathrm{K}_{\leqslant m}$ *with* $n > m \geqslant 1$. *If a formula is satisfiable in an* L*-frame of finite depth, then it is also satisfiable in a finite* L*-frame of the same depth.*

There are modal formulae restricting the depth of preorders [2]. Let us define analogues of such formulae for the pretransitive case. Put $\Box^*\varphi = \Box^{\leqslant k}\varphi$ for the smallest k such that $\mathrm{K}_{\leqslant k} \subseteq \mathrm{L}$. In particular, $\Box^*\varphi = \Box^{\leqslant n-1}\varphi$ for the logic K_n^m with $n > m$. We also put

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$\Diamond^*\varphi = \neg\Box^*\neg\varphi$. We define B_h by induction: $B_1 = p_1 \to \Box^*\Diamond^* p_1$, and $B_{h+1} = p_{h+1} \to \Box^*(\Diamond^* p_{h+1} \vee B_h)$.

Proposition 2. *For a pretransitive logic* L *and an* L*-frame* F, $F \vDash B_h$ *if and only if the depth of* F *is not greater than or equal to* h.

The extension of L with B_h is denoted by $\mathrm{L}.B_h$.

Theorem 3. *Let* L *be a pretransitive logic. Then*: 1) $\mathrm{L}.B_1 \supseteq \mathrm{L}.B_2 \supseteq \mathrm{L}.B_3 \supseteq \cdots \supseteq \mathrm{L}$; 2) *if* L *is consistent, then each* $\mathrm{L}.B_h$ *with* $h \geqslant 1$ *is consistent*; 3) *if* L *is canonical, then each* $\mathrm{L}.B_h$ *with* $h \geqslant 1$ *is canonical.*

Corollary 4. *For all* $n > m \geqslant 1$ *and* $h \geqslant 1$, *the logics* $\mathrm{K}_n^m.B_h$ *and* $\mathrm{K}_{\leqslant m}.B_h$ *are canonical and hence Kripke complete.*

In general, pretransitive logics of finite depth are much more complex than their analogues above the logic of preorders S4. Thus, the logics $\mathrm{S4}.B_h$ are locally tabular, and the logic $\mathrm{S4}.B_1 = \mathrm{S5}$ is pretabular, whereas even the 'simplest' logic $\mathrm{K}_{\leqslant 2}.B_1$ is neither pretabular nor locally tabular [7], [5]. The same is true for all the logics $\mathrm{K}_n^m.B_h$ and $\mathrm{K}_{\leqslant m}.B_h$ $(n > m \geqslant 2)$, since they are included in $\mathrm{K}_{\leqslant 2}.B_1$. Nevertheless, these logics have the finite model property. Even in the case of depth 1 the proof is non-trivial (especially for $\mathrm{K}_n^m.B_1$ with $n > m+1$); the fmp for $\mathrm{K}_{\leqslant m}.B_1$ was proved in [4], and for the pretransitive logics $\mathrm{K}_n^m.B_1$ it was proved in [6]. The following theorem generalizes these results to the case of arbitrary finite depth.

Theorem 5. *For all* $n > m \geqslant 1$ *and* $h \geqslant 1$ *the logics* $\mathrm{K}_n^m.B_h$ *and* $\mathrm{K}_{\leqslant m}.B_h$ *have the finite model property.*

An important corollary of this result is the following criterion.

Corollary 6. *Let* L *be one of the logics* K_n^m *and* $\mathrm{K}_{\leqslant m}$ *with* $n > m \geqslant 1$. *Then*

$$\mathrm{L} \textit{ has the finite model property} \quad \Longleftrightarrow \quad \mathrm{L} = \bigcap_{h \geqslant 1} \mathrm{L}.B_h.$$

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