Modal logics of closed domains on Minkowski plane ⋆

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ABSTRACT. *In this paper we study modal logics of closed domains on the real plane ordered by the chronological future relation. For the modal logic determined by an arbitrary closed convex domain with a smooth bound, we present a finite axiom system and prove the finite modal property.*

KEYWORDS: *chronological future modality, completeness, finite model property.*

1. Introduction

Chronological ($\prec$) and causal ($\preceq$) accessibility are two basic relations in Minkowski spacetime. The causal future of a point-event x consists of all those points y, to which a signal from x can be sent; $x \prec y$ if this signal is slower than light.

In (Goldblatt, 1980) the logic of Minkowski spacetime ordered by the causal future relation is described: $\mathbf{L}(\mathbb{R}^n, \preceq)$ is $\mathbf{S4.2}$ for $n \geq 2$. Next, in (Shehtman, 1983) modal logics of domains on $(\mathbb{R}^2, \preceq)$ were considered. In particular, it was shown that the logic of an open connected domain U with a smooth bound is $\mathbf{S4}$, and $\mathbf{S4.1}$ is the logic of the closure of U.

The problem of modal axiomatization of $(\mathbb{R}^n, \prec)$ (one of three problems put by R.Goldblatt in (Goldblatt, 1980)) remained open for more then twenty years and recently has been solved in (Shapirovsky *et al.*, 2003). In the same paper modal logics of open domains on $(\mathbb{R}^2, \prec)$ were considered.

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This paper is a continuation of (Shapirovsky *et al.*, 2003): here we axiomatize the modal logic $\mathbf{L}_3$ determined by an arbitrary closed convex domain with a smooth bound on $(\mathbb{R}^2, \prec)$. An essential part of the proof is the finite model property (FMP) of $\mathbf{L}_3$. To prove the FMP we use the Kripke–Gabbay method of selective filtration (see e.g. (Blackburn *et al.*, 2001)) in a combination with the method of maximal points (Fine, 1985). Following the lines of (Shehtman, 1983), we also consider some other kinds of closed domains.

2. Preliminaries

Let us recall some definitions and notation from (Shapirovsky *et al.*, 2003).

We consider normal monomodal propositional logics containing **K4**. For a modal logic Λ and a modal formula A, $\Lambda + A$ denotes the smallest modal logic containing $\Lambda \cup \{A\}$. The notation $\Lambda \vdash A$ means $A \in \Lambda$.

PV denotes a fixed countable set of propositional variables. Modal formulas are built using the basic connectives $\Diamond,\ \to,\ \bot$. Connectives $\Box,\ \neg,\ \vee,\ \wedge,\ \top$ are defined in the usual way, in particular $\Box A := \neg\Diamond\neg A$. In addition, $\Box^+ A$ abbreviates $\Box A \wedge A$, $\Diamond^+ A$ abbreviates $\Diamond A \vee A$.

We consider the following modal axioms:

$$\begin{array}{ll}
A4 := \Diamond\Diamond p \to \Diamond p, & AD := \Diamond\top, \\
AT := p \to \Diamond p, & A2 := \Diamond\Box p \to \Box\Diamond p, \\
A1 := \Box\Diamond p \to \Diamond\Box p, & Ad := Ad_1 = \Diamond p \to \Diamond\Diamond p, \\
AM := \Diamond^+\Box\bot, & Ad_n := \Diamond p_1 \wedge \ldots \wedge \Diamond p_n \to \Diamond(\Diamond p_1 \wedge \ldots \wedge \Diamond p_n),
\end{array}$$

and modal logics:

$$\begin{array}{lll}
\mathbf{K4} := \mathbf{K} + A4, & \mathbf{S4} := \mathbf{K4} + AT, & \\
\mathbf{S4.1} := \mathbf{S4} + A1, & \mathbf{S4.2} := \mathbf{S4} + A2, & \mathbf{S4.1.2} := \mathbf{S4.2} + A1, \\
\mathbf{L}_0 := \mathbf{K4} + Ad_2, & \mathbf{L}_1 = \mathbf{L}_0 + AD, & \\
\mathbf{L}_2 := \mathbf{L}_1 + A2, & \mathbf{L}_3 := \mathbf{L}_0 + AM. &
\end{array}$$

By a *(Kripke) frame* we mean a non-empty set with a transitive relation (W, R). A *(Kripke) model* is a Kripke frame with a valuation: $M = (W, R, \theta)$, where $\theta :\ PV \longrightarrow 2^W$, 2^W denotes the power set of W. The sign $\vDash$ denotes the truth at a point in a Kripke model and also the validity in a Kripke frame. For a class of frames $\mathcal{F}$, $\mathbf{L}(\mathcal{F})$ denotes the set of all formulas that are valid in all frames from $\mathcal{F}$. For a single frame F, $\mathbf{L}(F)$ abbreviates $\mathbf{L}(\{F\})$.

For a model $M = (W, R, \theta)$ or a frame $F = (W, R)$, the notations $x \in M$, $x \in F$ mean $x \in W$. As usual, for $x \in W,\ V \subseteq W$ we denote

$$R(x) := \{y \mid xRy\},\ R(V) := \bigcup_{x \in V} R(x),\ R|V := R \cap (V \times V).$$

We also put

$$W^x := \{x\} \cup R(x),\ F^x := (W^x, R|W^x).$$

F^x is called a *cone* (in F). Recall that for any $x \in F$, $\mathbf{L}(F^x) \supseteq \mathbf{L}(F)$. If $F = F^x$ for some $x \in F$, then x is called a *root* of F, and $\overline{x}$ is the *initial cluster* of F.

Consider frames $F = (W, R)$ and $G = (V, S)$. A surjective map $f : W \to V$ is called a *p-morphism* from F onto G (in notation, $f : F \twoheadrightarrow G$), if for any $x \in W$ we have $f(R(x)) = S(f(x))$. The notation $F \twoheadrightarrow G$ means that there exists a p-morphism from F onto G. Recall that $F \twoheadrightarrow G$ implies $\mathbf{L}(F) \subseteq \mathbf{L}(G)$ (p-morphism lemma).

A model $M_1 = (W_1, R_1, \theta_1)$ is a *(weak) submodel* of $M = (W, R, \theta)$ (notation: $M_1 \subseteq M$) if $W_1 \subseteq W$, $R_1 \subseteq R$, $\theta_1(p) = \theta(p) \cap 2^{W_1}$ for every $p \in PV$. If also $R_1 = R|W_1$, then M_1 is called a *restriction* of M to W_1 and denoted by $M|W_1$. We put $M^x := M|W^x$.

For a frame $F = (W, R)$ let $\sim_R := (R \cap R^{-1}) \cup I_W$, where I_W is the equality relation on W. Recall that a *cluster* in (W, R) is an equivalence class under the relation $\sim_R$, a *degenerate cluster* is an irreflexive singleton. The associated relations between clusters

$$C \leq_R D := D \subseteq R(C),$$
$$C <_R D := C \leq_R D \text{ and } C \neq D$$

are transitive and antisymmetric, and $<_R$ is irreflexive. The frame $F/{\sim_R} := (W/{\sim_R}, \leq_R)$ is called the *skeleton* of F.

For a point x, $\overline{x}$ denotes its cluster. A point $x \in W$ is called *maximal* if $\overline{x}$ is maximal (with respect to $\leq_R$). For clusters C_1, C_2, we say that C_2 is a *successor* of C_1 (C_1 is a *predecessor* of C_2), if $C_1 <_R C_2$ and there is no cluster C such that $C_1 <_R C$ and $C <_R C_2$. For $x, y \in F$ we say that y is a *successor* of x (x is a *predecessor* of y), if $\overline{y}$ is a successor of $\overline{x}$.

We say that x is *serial* in a frame (W, R), if $R(x) \neq \varnothing$.

For a relation R, R^+ denotes the reflexive closure of R.

It is easy to describe the first-order correspondents for Ad_n and AM:

LEMMA 1. — *For a frame F,*

– $F \vDash Ad_n$ *iff F is* n-dense*, i.e.,*

$$\forall x \forall y_1 \ldots \forall y_n \exists z (xRy_1 \,\&\, \ldots \,\&\, xRy_n \Rightarrow xRz \,\&\, zRy_1 \,\&\, \ldots \,\&\, zRy_n);$$

– $F \vDash AM$ *iff F has the M-*property*, i.e.,*

$$\forall x \exists y (xR^+y \,\&\, R(y) = \varnothing).$$

Note that $\mathbf{K4} + Ad_2 \vdash Ad_n$ for all $n > 0$ (Shapirovsky *et al.*, 2003).

Lemma 1 implies the following

LEMMA 2. — *For a finite frame F,*

– *F is 2-dense iff every its degenerate non-maximal cluster C has a unique successor D, and D is non-degenerate.*

– *F has the M-property iff all maximal clusters in F are degenerate.*

Recall that in the transitive case $A1$ corresponds to the *McKinsey property:* $\forall x \exists y (xRy \,\&\, R(y) = \{y\})$. So AM is the irreflexive analog of $A1$, i.e. if R is transitive and irreflexive, then

$$(W, R) \vDash AM \Leftrightarrow (W, R^+) \vDash A1.$$

By Sahlqvist Theorem (see e.g. (Chagrov *et al.*, 1997)) we have completeness:

PROPOSITION 3. — *The logic $\mathbf{L}_3$ is canonical*[1].

For a set V, $|V|$ denotes the cardinality of V.

3. Finite model property

Is this section we show that the logic $\mathbf{L}_3$ has the FMP, in the same way as it was done in (Shapirovsky *et al.*, 2003) for the logics $\mathbf{L}_1$, $\mathbf{L}_2$.

DEFINITION 4. — *Let M be a Kripke model, Ψ a set of formulas closed under subformulas.*

A submodel $M_1 \subseteq M$ (with a relation R_1) is called a selective filtration of M through Ψ *if for any $x \in M_1$, for any formula A*

$$\Diamond A \in \Psi \,\&\, M, x \vDash \Diamond A \Rightarrow \exists y \in R_1(x)\; M, y \vDash A.$$

The following lemma is easily proved by induction on the complexity of A.

LEMMA 5. — *If M_1 is a selective filtration of M through Ψ, then for any $x \in M_1$, for any $A \in \Psi$*

$$M, x \vDash A \Leftrightarrow M_1, x \vDash A.$$

We also need the *maximality property* of a canonical model (Fine, 1985):

LEMMA 6. — *Let $\mathfrak{M}$ be the canonical model for a modal logic Λ, and assume that a formula B is satisfied at some $x \in \mathfrak{M}$. Consider the set of all those clusters in $\mathfrak{M}^x$, in which B is satisfied:*

$$\Gamma := \{C \subseteq \mathfrak{M}^x \mid \exists y \in C\; B \in y\}.$$

Then the model $\mathfrak{M}| \bigcup \Gamma$ contains a maximal cluster.

1. The notion of *canonicity* is defined in the usual way, see e.g. (Chagrov *et al.*, 1997)

Let $\mathcal{F}_0$ be the class of all finite 2-dense cones,

$$\mathcal{F}_M := \{F \mid F \in \mathcal{F}_0,\ F \text{ has the } M-\text{property}\}.$$

Let $\mathcal{M}_0, \mathcal{M}_M$ be the classes of Kripke models based on frames in $\mathcal{F}_0,\ \mathcal{F}_M$, respectively.

LEMMA 7. — *Let $\mathfrak{M}$ be the canonical model for a logic $\Lambda \supseteq \mathbf{L}_0$, $x_0 \in \mathfrak{M}$, and let Ψ be a finite set of formulas closed under subformulas. Then there exists a selective filtration M of $\mathfrak{M}$ through Ψ such that $M \in \mathcal{M}_0$ and $x_0 \in M$.*

PROOF. — The proof is almost the same as the proof of Theorem 10 in (Shapirovsky *et al.*, 2003).

Let R be the accessability relation on $\mathfrak{M}$. For any $x \in \mathfrak{M}$ we put:

$$\Psi_x := \{\Diamond A \mid \Diamond A \in \Psi\} \cap x.$$

Consider the relation $S \subseteq R$:

$$xSy := xRy,\ \Psi_y = \Psi_x,\ \text{and } \overline{y} <_R \overline{z} \Rightarrow \Psi_z \subsetneq \Psi_x \text{ for any } z \in \mathfrak{M}.$$

If $\Psi_x \neq \varnothing$, put $D_x := \bigwedge\limits_{\Diamond A \in \Psi_x} \Diamond A$. By Lemma 6 we have

Claim 1. If $\Psi_x \neq \varnothing$, then $S(x) \neq \varnothing$.

Claim 2. If $\Psi_x \neq \varnothing$, xSy, then yRy.

Since $\Psi_y = \Psi_x$, $D_x \in y$. Since $\mathbf{L}_0 \vdash Ad_n$ for any $n \geq 1$, we have $\Diamond D_x \in y$. Thus for some $z \in R(y)$ we have $D_x \in z$, so $\Psi_z \supseteq \Psi_x$. It follows that zRy, so yRy. Q.e.d.

By Claim 1, there exists a map f such that for any $x \in \mathfrak{M}$, $\Psi_x \neq \varnothing$, we have: if xSx, then $f(x) = x$; otherwise $f(x) \in S(x)$.

Therefore, if $\Psi_x \neq \varnothing$, then $xSf(x)$, $f(f(x)) = f(x)$, and $f(x)Rf(x)$.

Now by induction we construct a selective filtration M of $\mathfrak{M}$ through Ψ. We also define an auxiliary set X_k at every Stage k.

Stage 0.

Assume $\Psi_{x_0} = \varnothing$. Let M be the submodel of $\mathfrak{M}$ based on the frame $(\{x_0\}, \varnothing)$. Trivially, M is a selective filtration of $\mathfrak{M}$ through Ψ and $M \in \mathcal{M}_0$.

If $\Psi_{x_0} \neq \varnothing$, we put

$$W_0 := \{x_0, f(x_0)\},\ \ M_0 := \mathfrak{M}|W_0,\ X_0 := \{f(x_0)\}$$

. By Claim 2, $M_0 \in \mathcal{M}_0$.

Stage n+1.

Assume that on Stage n we have a submodel M_n of $\mathfrak{M}$ based on a frame (W_n, R_n), $M_n \in \mathcal{M}_0,\ X_n \neq \varnothing$, and for every $x \in W_n$ the following holds:

(1) $x \in X_n \Rightarrow x = f(x)$;
(2) $\Psi_x \neq \varnothing \Rightarrow f(x) \in W_n$
(3) $\Diamond A \in \Psi_x \Rightarrow \exists y \in R_n(x)(A \in y \vee (y \in X_n \;\&\; \Diamond A \in y))$.

Now we construct $W_{n+1},\ X_{n+1}$, and R_{n+1}, Fig. 1.

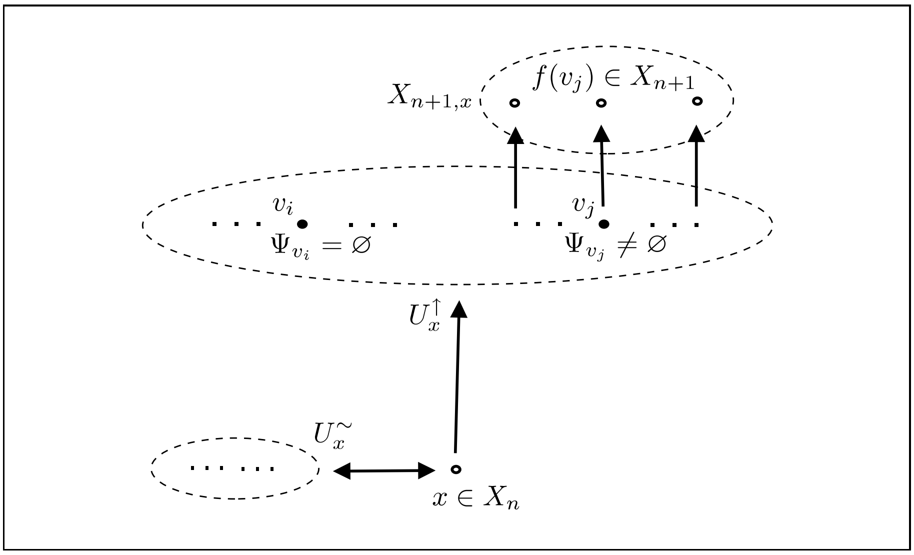

Figure 1. *$W_{n+1},\ X_{n+1}$, and R_{n+1}*

Let $x \in X_n$. We put

$$\Psi_x^{\sim} := \{\Diamond A \in \Psi_x \mid \exists t \sim_R x \;\; A \in t\}, \quad \Psi_x^{\uparrow} := \Psi_x - \Psi_x^{\sim}.$$

If $\Psi_x^{\sim} = \{\Diamond A_1, \ldots, \Diamond A_m\}$, then there exist points $u_1, \ldots, u_m$ such that $u_i \ni A_i \;\&\; u_i \sim_R x$, $i = 1 \ldots m$. We put $U_x^{\sim} := \{u_1, \ldots, u_m\}$ (if $\Psi_x^{\sim} = \varnothing$, we put $U_x^{\sim} := \varnothing$).

If $\Psi_x^{\uparrow} = \varnothing$, then we put $U_x^{\uparrow} := \varnothing$. Otherwise, for $\Psi_x^{\uparrow} = \{\Diamond B_1, \ldots, \Diamond B_l\}$ we put $U_x^{\uparrow} := \{v_1, \ldots, v_l\}$, where $v_i \ni B_i$ (and therefore $\overline{x} <_R \overline{v_i}$), $i = 1 \ldots l$.

For any $x \in X_n$, put

$$X_{n+1,x} := \{f(u) \mid u \in U_x^{\uparrow},\ \Psi_u \neq \varnothing\}.$$

We define X_{n+1} and W_{n+1} as follows.

$$X_{n+1} := \bigcup_{x \in X_n} X_{n+1,x}; \quad W_{n+1} := W_n \cup \bigcup_{x \in X_n} \left(U_x^{\sim} \cup U_x^{\uparrow} \cup X_{n+1,x}\right).$$

Now we define R_{n+1}. If $\Psi_x = \varnothing$, we put $R_{n+1}(x) := \varnothing$; otherwise we put $R_{n+1}(x) := R(f(x)) \cap W_{n+1}$. Since $W_{n+1} \supseteq W$, $R_{n+1} \supseteq R_n$.

By Claim 2, R_{n+1} is 2-dense. To show that R_{n+1} is transitive, suppose that $xR_{n+1}yR_{n+1}z$. Then $f(x)Ry$ and $f(y)Rz$. Since $yRf(y)$, we have $f(x)Rz$, and $xR_{n+1}z$. Thus $M_{n+1} \in \mathcal{M}_0$.

Due to the construction, the properties (1),(2) hold. Let us check the property (3). Assume that $x \in M_{n+1}, \Diamond A \in \Psi_x$, and consider two cases:
(i) $x \in M_n$.
If $A \in y$ for some $y \in R_n(x)$, then the statement holds because $R_{n+1} \supseteq R_n$. Otherwise $\Diamond A \in y$, xR_ny for some $y \in X_n$. Then due to the construction, $A \in u$ for some $u \in U_y^{\sim} \cup U_y^{\uparrow}$, and $xR_{n+1}u$.
(ii) $x \notin M_n$.
Then $x \in U_a^{\sim} \cup U_a^{\uparrow} \cup X_{n+1,a}$ for some $a \in X_n$. The case $x \in X_{n+1,a}$ is trivial, since $X_{n+1,a} \subseteq X_{n+1}$. If $x \in U_a^{\sim}$, then $\Diamond A \in a \in W_n$, and we proceed as in the case (i). If $x \in U_a^{\uparrow}$, then $f(x) \in X_{n+1}$, $xR_{n+1}f(x)$, and $\Diamond A \in f(x)$.

Note that if $x \in X_n,\ y \in X_{n+1}, x$ then $|\Psi_y| < |\Psi_x|$. Thus

$$\max\{|\Psi_x| \mid x \in X_{n+1}\} < \max\{|\Psi_x| \mid x \in X_n\}.$$

Therefore we obtain $X_n = \varnothing$ for some $n \leq |\Psi_{x_0}|$. Then we put $M := M_n$. Due to the property (3), M is a selective filtration of $\mathfrak{M}$ through Ψ. ■

COROLLARY 8. — $\mathbf{L}_0 = \mathbf{L}(\mathcal{F}_0)$.

THEOREM 9. — $\mathbf{L}_3 = \mathbf{L}(\mathcal{F}_M)$.

PROOF. — Let $\mathfrak{M}$ be the canonical model for a logic $\mathbf{L}_3$, and let B be an $\mathbf{L}_3$-consistent formula. Let Ψ_0 be the set of all subformulas of B, $\Psi := \Psi_0 \cup \{\Diamond^+\Box\bot\}$. Then Ψ is $\mathbf{L}_3$-consistent, and for some $x \in \mathfrak{M}$ we have $\Psi \subseteq x$. By Lemma 7, there exists a selective filtration M of $\mathfrak{M}$ through Ψ such that $M \in \mathcal{M}_0$ and $x \in M$. Since $B \in \Psi$, we have $M, x \models B$. Moreover, for any $y \in M$ we have $\mathfrak{M}, y \models AM$, so $M, y \models AM$. Since AM is a variable-free formula, the frame of M is in $\mathcal{F}_M$. ■

4. Suitable frames for $\mathbf{L}_3$

In this section we formulate some useful completeness theorems for $\mathbf{L}_3$.

DEFINITION 10. — *A* reflexive tree *is a poset (W, R) with a root, in which every subset $R^{-1}(x)$ is a chain; by a* tree *we mean a frame, whose reflexive closure is a reflexive tree. A frame F is called a* quasitree *if $F/{\sim_R}$ is a tree.*

DEFINITION 11. — *Let $F \in \mathcal{F}_M,\ \overline{x}, \overline{y} \in F/{\sim_R}$.*

A pair $(\overline{x}, \overline{y})$ is called an a-defect *in F if $\overline{x}$ is a predecessor of $\overline{y}$ and both of them are non-degenerate. $D_a(F)$ denotes the set of all a-defects in F.*

A non-degenerate cluster $\overline{x}$ is called a b-defect *in F if x does not have a non-serial successor. $D_b(F)$ denotes the set of all b-defects in F.*

We say that F is $\mathbf{L}_3$-suitable, *if F satisfies the following conditions:*

1) $F \in \mathcal{F}_M$*;*
2) F *is a quasitree and its initial cluster is non-degenerate;*
3) F *has no defects:* $D_a(F) = D_b(F) = \varnothing$.

Let $\mathcal{G}_M$ *be the class of all* $\mathbf{L}_3$*-suitable frames.*

LEMMA 12. — $\mathbf{L}(\mathcal{G}_M) = \mathbf{L}_3$.

PROOF. — Let $\mathcal{G}^*$ be the class of all quasitrees from $\mathcal{F}_M$. Let us first prove that $\mathbf{L}(\mathcal{G}^*) = \mathbf{L}_3$. By Theorem 9, $\mathbf{L}(\mathcal{F}_M) = \mathbf{L}_3$. Since $\mathcal{G}^* \subseteq \mathcal{F}_M$, it is sufficient to show that for any $F = (W, R) \in \mathcal{F}_M$ there exists $G \in \mathcal{G}^*$ such that $G \twoheadrightarrow F$. Let C_0 be the initial cluster of F,

$$W' := \{(C_0, \dots, C_k) \mid C_i \in F/{\sim_R},\ C_i \text{ is a successor of } C_{i-1}\}$$

For $\alpha, \beta \in W'$, put $\alpha R' \beta$ iff either α is a prefix of β and $\alpha \neq \beta$, or $\alpha = \beta$ and the last cluster of α is non-degenerate. We define a frame $G = (V, S)$ as follows:

$$\begin{gathered} V := \{(C_0, \dots, C_k, w) \mid (C_0, \dots, C_k) \in W', w \in C_k\}, \\ (C_0, \dots, C_k, w) S (C_0, \dots, C_l, v) \text{ iff } (C_0, \dots, C_k) R' (C_0, \dots, C_l). \end{gathered}$$

Trivially, G is a finite quasitree; moreover, using Lemma 2, one can see that $G \vDash Ad_2,\ G \vDash AM$, so $G \in \mathcal{G}^*$. Let $f(C_1, \dots, C_k, w) := w$. By a standard unravelling argument, $f : G \twoheadrightarrow F$. Therefore, $\mathbf{L}(\mathcal{G}^*) = \mathbf{L}(\mathcal{F}_M) = \mathbf{L}_3$.

Let $\mathcal{G}_0^*$ be the class of all quasitrees from $\mathcal{F}_M$ with non-degenerate initial clusters. Since $\mathcal{G}_0^* \subseteq \mathcal{G}^*$, $\mathbf{L}(\mathcal{G}_0^*) \supseteq \mathbf{L}(\mathcal{G}^*)$. On the other hand, for $G \in \mathcal{G}^*$ and a reflexive singleton E, we have $\mathbf{L}(E + G) \subseteq \mathbf{L}(G)$, $E + G \in \mathcal{G}_0^*$ (here $+$ denotes the ordinal sum of frames). Thus $\mathbf{L}(\mathcal{G}_0^*) \subseteq \mathbf{L}(\mathcal{G}^*)$, and so $\mathbf{L}(\mathcal{G}_0^*) = \mathbf{L}_3$.

Consider a frame $F = (W, R) \in \mathcal{G}_0^*$ such that $|D_a(F)| = k > 0$ and $(\overline{x}, \overline{y}) \in D_a(F)$. Let $F_1 = (W_1, R_1)$ be the frame obtained by inserting a new irreflexive point z between x and y:

$$R_1(z) := R(y),\ R_1^{-1}(z) := R^{-1}(x).$$

It is easy to see that $F_1 \twoheadrightarrow F$ (to obtain a p-morphism, we merge z with y), $F_1 \in \mathcal{G}_0^*$ and $|D_a(F_1)| = k - 1$. After removing all a-defects in this way, we obtain a sequence:

$$F_k \twoheadrightarrow \dots \twoheadrightarrow F_1 \twoheadrightarrow F,\ |D_a(F_k)| = 0,\ F_k \in \mathcal{G}_0^*.$$

Now let $F = (W, R) \in \mathcal{G}_0^*,\ |D_a(F)| = 0,\ |D_b(F)| = k > 0,\ \overline{x} \in D_b(F)$. Let $F_1 = (W_1, R_1)$ be a frame obtained by adding a new non-serial successor of x called z. Since $F \vDash AM$, for some $y \in R(x)$ $R(y) = \varnothing$. It is easy to see that $F_1 \twoheadrightarrow F$ (merging z with y), $F_1 \in \mathcal{G}_0^*,\ |D_a(F_1)| = 0$ and $|D_b(F_1)| = k - 1$. After removing all b-defects in this way, we obtain a sequence:

$$F_k \twoheadrightarrow \dots \twoheadrightarrow F_1 \twoheadrightarrow F,\ |D_a(F_k)| = |D_b(F_k)| = 0,\ F_k \in \mathcal{G}_0^*.$$

Since $|D_a(F_k)| = |D_b(F_k)| = 0$, we have $F_k \in \mathcal{G}_M$.

Thus, for any $F \in \mathcal{G}_0^*$ there exists $G \in \mathcal{G}_M$ such that $G \twoheadrightarrow F$. It follows that $\mathbf{L}(\mathcal{G}_M) \subseteq \mathbf{L}(\mathcal{G}_0^*)$. Conversely, $\mathcal{G}_M \subseteq \mathcal{G}_0^*$, so $\mathbf{L}(\mathcal{G}_M) \supseteq \mathbf{L}(\mathcal{G}_0^*)$. Thus $\mathbf{L}(\mathcal{G}_M) = \mathbf{L}_3$. ■

DEFINITION 13. — *Let $T_\mathbb{Z}$ ($T_\mathbb{Z}^\infty$) be the set of all finite (infinite) sequences of integers, T_n (T_n^∞) the set of all finite (infinite) sequences of numbers $\{1, 2, \dots, n\}$. The sequences are ordered in a standard way: $\sigma_1 \sqsubseteq \sigma_2$ iff σ_1 is an initial part of σ_2. Put $\sigma_1 \sqsubset \sigma_2$ iff $\sigma_1 \sqsubseteq \sigma_2$ and $\sigma_1 \neq \sigma_2$;*

Next we define the frame $\widetilde{\mathbf{I}}_\mathbf{2}$ (Fig. 2)

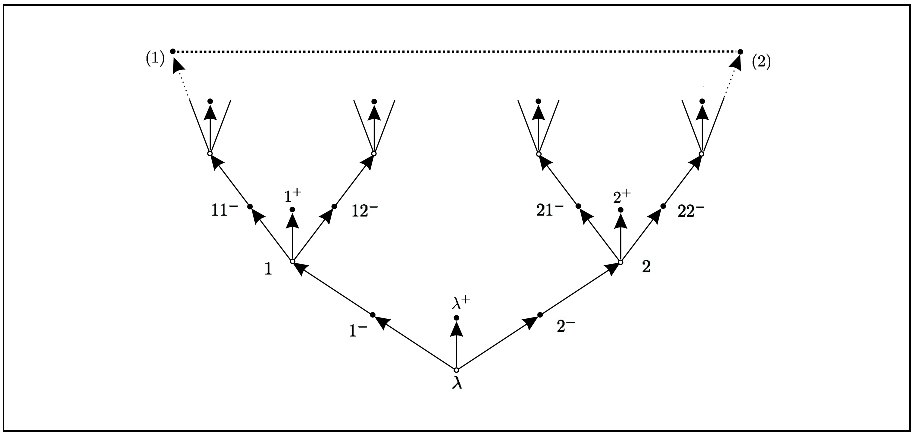

Figure 2. *The frame* $\widetilde{\mathbf{I}}_\mathbf{2}$

Let $\sigma, \sigma_1, \sigma_2 \in T_n$. $\sigma_1\sigma_2$ denotes the concatenation of sequences σ_1 and σ_2, $\sigma^1 := \sigma$, $\sigma^{i+1} := \sigma^i\sigma$; (σ) denotes the infinite sequence $\sigma\sigma\sigma\dots$; λ denotes the empty sequence.

For $\tau \in T_2$ let $\tau^- := (\tau, -)$, $\tau^+ := (\tau, +)$. Put

$$I_2 := T_2 \cup \{\tau^- \mid \tau \in T_2 - \{\lambda\}\},\ \widetilde{I_2} := I_2 \cup \{\tau^+ \mid \tau \in T_2\} \cup T_2^\infty$$

Now we define the relation $\lessdot$ on $\widetilde{I_2}$, an extension of $\sqsubseteq$. Let $\tau \in T_2$, $\tau \neq \lambda$, $\xi \in T_2^\infty$. We put

$$\lessdot(\tau^-) := \lessdot(\tau) := \bigcup_{\rho \in T_2,\ \tau \sqsubset \rho} \{\rho, \rho^-, \rho^+\} \cup \{\tau, \tau^+\},$$

$$\lessdot(\tau^+) := \lessdot(\lambda^+) := \lessdot(\xi) := \varnothing,$$

$$\lessdot(\lambda) := \widetilde{I_2}.$$

Let $\widetilde{\mathbf{I_2}} := (\widetilde{I_2}, \lessdot)$, $\mathbf{I_2} := (I_2, \lessdot|I_2)$.

In (Shapirovsky *et al.*, 2003) it was shown that $\mathbf{L}_1 = \mathbf{L}(\mathbf{I_2})$. Let us prove an analogous completeness theorem for $\mathbf{L}_3$.

DEFINITION 14. — *Consider a frame* $F \in \mathcal{F}_M$. *Let* Σ *be a* $<_R$*-chain in* $F/\sim_R$. $rh(\Sigma)$ *denotes the number of non-degenerate clusters in* Σ*;*

$$rh(F) := \max\{rh(\Sigma) \mid \Sigma \textit{ is a } <_R\textit{-chain in } F/\sim_R\}.$$

Note that for every $G \in \mathcal{G}_M$ we have $rh(G) > 0$, since its initial cluster is non-degenerate.

THEOREM 15. — $\mathbf{L}(\widetilde{\mathbf{I_2}}) = \mathbf{L}_3$.

PROOF. — Using Lemma 1, one can check that $\mathbf{L}(\widetilde{\mathbf{I_2}}) \supseteq \mathbf{L}_3$.

To prove the converse inclusion, it is sufficient to show that for any $G = (W, R) \in \mathcal{G}_M$ there exists a p-morphism $f : \widetilde{\mathbf{I_2}} \twoheadrightarrow G$.

Let $\uparrow^s(x)$ be the set of all serial successors of x, $\uparrow^{ns}(x)$ be the set of all non-serial successors of x.

Now we argue by induction on $rh(G)$.

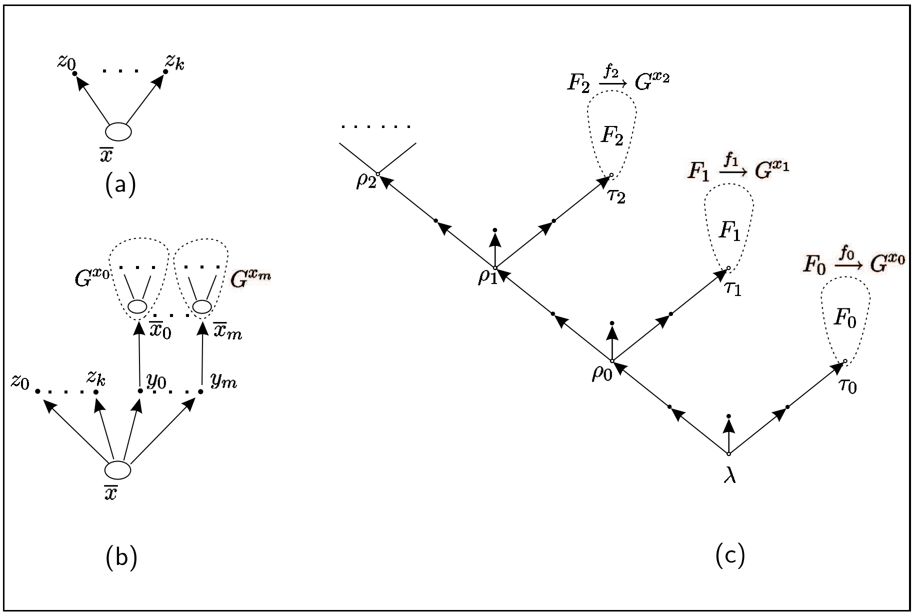

Figure 3. *Construction of the p-morphism* $f : \widetilde{\mathbf{I_2}} \twoheadrightarrow G$

Suppose $rh(G) = 1$. It follows that $W/\sim_R = \{\overline{x}, \overline{z}_0, \ldots, \overline{z}_k\}$, where $R(x) = W$, $\uparrow^{ns}(x) = \{z_0, \ldots, z_k\}$ (Fig. 3, a). Let $\overline{x} = \{a_0, \ldots, a_s\}$. For any $t \in T_2$ we put

$f(\tau) := a_i,\ f(\tau^+) := z_j$, where $i \equiv l(mod\, s+1)$, $j \equiv l(mod\, k+1)$, l is the length of τ. For all other $\rho \in \widetilde{I}_2$ we put: $f(\rho) := a_0$ if ρ is serial, and $f(\rho) := z_0$ otherwise.

Due to this construction, we have: if $\rho \in \widetilde{I}_2$ is serial, $f(\lessdot(\rho)) = W = R(f(\rho))$; if $\rho \in \widetilde{I}_2$ is non-serial, $f(\lessdot(\rho)) = \varnothing = R(f(\rho))$.

Suppose $rh(G) = n > 1$. Let $\overline{x} = \{a_0, \ldots, a_s\}$ be the initial cluster of G. Since $rh(G) > 1$, we have $\uparrow^s(x) \neq \varnothing$. Since G has no b-defects, $\uparrow^{ns}(x) \neq \varnothing$. Let $\uparrow^s(x) = \{y_0, \ldots, y_m\},\ \ \uparrow^{ns}(x) = \{z_0, \ldots, z_k\}$. G has no a-defects, thus $y_0, \ldots, y_m$ are irreflexive. By Lemma 2, every $\overline{y}_i$ has a unique non-degenerate successor $\overline{x}_i$, $i = 0 \ldots m$ (Fig. 3, b).

Consider two sequences of points from T_2 (Fig. 3, c):

$$\tau_0 = 2,\ \tau_1 = 12, \ldots, \tau_l = 1^l 2, \ldots,$$

$$\rho_0 = 1,\ \rho_1 = 11, \ldots, \rho_l = 1^l 1, \ldots$$

Let F_l be a cone in $\widetilde{\mathbf{I}}_\mathbf{2}$ with an initial point τ_l. Observe that F_l is isomorphic to $\widetilde{\mathbf{I}}_\mathbf{2}$. By the induction hypothesis there exist maps $f_0, \ldots, f_m$ such that $f_i :\ F_i \twoheadrightarrow G^{x_i}$, $i = 0 \ldots m$.

Now we construct a p-morphism $f : \widetilde{\mathbf{I}}_\mathbf{2} \twoheadrightarrow G$. For every integer $l > 0$ there exist integers i, j, r such that $0 \leq i \leq m,\ 0 \leq j \leq k,\ 0 \leq r \leq s$ and

$$i \equiv l(mod\, m+1),\ j \equiv l(mod\, k+1),\ r \equiv l(mod\, s+1).$$

We define f on F_l as follows. Let $\tau = \tau_l \rho,\ \rho \in T_2$. We put $f(\tau) := f_i(\rho),\ f(\tau^+) := f_i(\rho^+)$; if $\rho \neq \lambda$, $f(\tau^-) := f_i(\rho^-)$. For $\rho \in T_2^\infty$, $f(\tau) := f_i(\rho)$.

We also put $f(\tau_l^-) := y_i,\ f(\rho_l) := f(\rho_l^-) := a_r,\ f(\rho_l^+) := z_j,\ f(\lambda) := a_0,\ f((1)) := f(\lambda^+) := z_0$.

By a straightforward argument, $R(f(\rho))$=$f(\lessdot(\rho))$ for every $\rho \in \widetilde{\mathbf{I}}_\mathbf{2}$.

For example, consider the case $\rho = \rho_l$. $f(\rho_l) \in \overline{x}$, and thus $R(f(\rho_l)) = W$. On the other hand, $\lessdot(\rho_l) \supset \{\rho_l, \ldots, \rho_{l+s}\}$ and thus $f(\lessdot(\rho_l)) \supset \{a_0, \ldots, a_s\}$. Similarly,

$$f(\lessdot(\rho_l)) \supset \{z_0, \ldots, z_k, y_0, \ldots, y_m\},\ f(\lessdot(\rho_l)) \supset W^{x_0} \cup \ldots \cup W^{x_m},$$

and thus $f(\lessdot(\rho_l)) = W = R(f(\rho_i))$.

f is surjective, so $f : \widetilde{\mathbf{I}}_\mathbf{2} \twoheadrightarrow G$. ■

5. Closed domains on Minkowski plane

Let us recall the definition of the chronological future relation $\prec$ and the causal future relation $\preceq$ on the real plane $\mathbb{R}^2$.

DEFINITION 16. — *Let* $X, Y \in \mathbb{R}^2,\ X = (x_1, t_1),\ Y = (x_2, t_2)$. *We put*

$$\mu(X,Y) := (t_2 - t_1)^2 - (x_2 - x_1)^2,$$

$$X \prec Y \Leftrightarrow \mu(X,Y) > 0 \ \&t_2 > t_1,$$

$$X \preceq Y \Leftrightarrow \mu(X,Y) \geq 0 \ \&t_2 \geq t_1.$$

It is well-known that the logic of $(\mathbb{R}^2, \preceq)$ is $\mathbf{S4.2}$ (Goldblatt, 1980), and the logic of $(\mathbb{R}^2, \prec)$ is $\mathbf{L}_2$ (Shapirovsky *et al.*, 2003).

For $D \subseteq \mathbb{R}^2$ let $(D, \prec)$ abbreviate $(D, \prec\mid D)$, $(D, \preceq)$ abbreviate $(D, \preceq \mid D)$. $\overline{D}$ denotes the closure of D.

Let D be an open connected domain in $\mathbb{R}^2$ bounded by a closed smooth curve. Then $\mathbf{L}(D, \preceq) = \mathbf{S4},\ \mathbf{L}(\overline{D}, \preceq) = \mathbf{S4.1}$ (Shehtman, 1983); $\mathbf{L}(D, \prec) = \mathbf{L}_1$ (Shapirovsky *et al.*, 2003). Our aim is to consider the case of $(\overline{D}, \prec)$.

Let us first describe some constructions.

Let $P,\ Q \in \mathbb{R}^2,\ P \neq Q$. For every $m \in \mathbb{Z}$ we define the point $S_m(P,Q)$ on the segment $\left[P,Q\right]$ as follows (Fig. 4, a):

- $S_0(P,Q)$ is the midpoint of $\left[P,Q\right]$;
- $S_m(P,Q)$ is the midpoint of $\left[P, S_{m+1}(P,Q)\right]$ for $m < 0$;
- $S_m(P,Q)$ is the midpoint of $\left[S_{m-1}(P,Q), Q\right]$ for $m > 0$.

For $\sigma \in T_{\mathbb{Z}}$ we define the points A^σ, B^σ by induction on the length of σ:

- $A^\lambda := P,\ B^\lambda := Q$;
- $A^{\sigma m} := S_m(A^\sigma, B^\sigma)$;
- $B^{\sigma m} := S_{m+1}(A^\sigma, B^\sigma)$.

Next we define A^σ for $\sigma \in T_{\mathbb{Z}}^\infty$. Observe that for $\sigma_1, \sigma_2, \ldots \in T_{\mathbb{Z}}$, $\sigma_1 \sqsubset \sigma_2 \sqsubset \ldots$ implies $\left[A^{\sigma_1}, B^{\sigma_1}\right] \supset \left[A^{\sigma_2}, B^{\sigma_2}\right] \supset \ldots$, and the lengths of these segments converge to zero. So by Cantor's principle of nested segments there exists a unique $X \in \left[P,Q\right]$, such that $\{X\} = \bigcap\limits_{\rho \sqsubset \sigma} \left[A^\rho, B^\rho\right]$. We put $A^\sigma := B^\sigma := X$.

Let Pr_x and Pr_t be the first and the second standard projections $\mathbb{R}^2 \longrightarrow \mathbb{R}$: $Pr_x(x_1, x_2) := x_1,\ Pr_t(x_1, x_2) := x_2$.

Now assume that $\mu(P,Q) < 0$.

Consider the right triangle PQR, such that $\mu(P,R) = \mu(Q,R) = 0,\ Pr_t(R) < Pr_t(P)$, Fig. 4, b. Let g be a real-valued differentiable function whose domain contains $\left[Pr_x(P), Pr_x(Q)\right]$ (without any loss of generality, we assume that $Pr_x(P) < Pr_x(Q)$). Let γ be the graph of g, $P, Q \in \gamma$, and

$$\forall x \in \ ]Pr_x(P), Pr_x(Q)[\ \ |g'(x)| < 1.$$

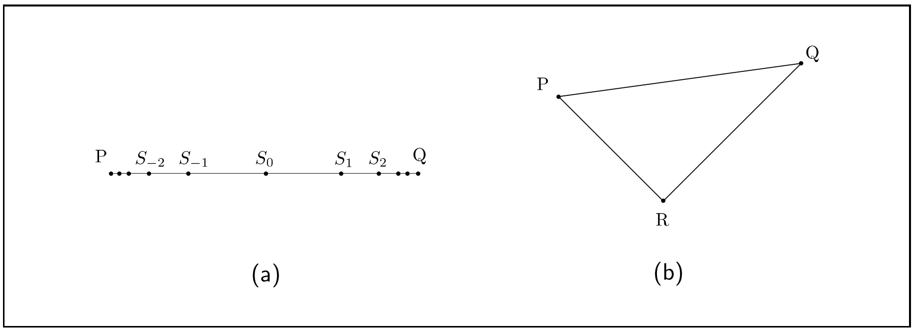

Figure 4. (a) $S_m(P,Q)$, $m \in \mathbb{Z}$ **(b)** $\mu(P,Q) < 0$, $\mu(P,R) = \mu(Q,R) = 0$

Let $\gamma(P,Q)$ be the graph of the restriction g to $\left]Pr_x(P), Pr_x(Q)\right[$, D an open domain bounded by $\left[R,P\right]$, $\left[R,Q\right]$ and $\gamma(P,Q)$. We put

$$\nabla_\gamma(P,Q) := D \cup \{R\},\ \widetilde{\nabla}_\gamma(P,Q) := \nabla_\gamma(P,Q) \cup \gamma(P,Q),$$

$$\widehat{\nabla}_\gamma(P,Q) := \nabla_\gamma(P,Q)\ \cup\ \left]P,R\right[\ \cup\ \left]R,Q\right[.$$

LEMMA 17. — $(\widetilde{\nabla}_\gamma(P,Q), \prec) \twoheadrightarrow \widetilde{\mathbf{I}}_\mathbf{2}$

PROOF. — To prove the statement of the lemma, we modify the construction proposed in (Shapirovsky *et al.*, 2003).

First we define a correspondence between γ and $T_\mathbb{Z} \cup T_\mathbb{Z}^\infty$. For $\sigma \in T_\mathbb{Z} \cup T_\mathbb{Z}^\infty$ we put

$$A_\gamma^\sigma := (Pr_x(A^\sigma), g(Pr_x(A^\sigma))),\ B_\gamma^\sigma := (Pr_x(B^\sigma), g(Pr_x(B^\sigma))).$$

Note that $A_\gamma^\lambda = P,\ B_\gamma^\lambda = Q,$ and for $\sigma \neq \lambda$ we have $A_\gamma^\sigma, B_\gamma^\sigma \in \gamma(P,Q)$.

Let

$$\nabla^\sigma := \widehat{\nabla}_\gamma(A_\gamma^\sigma, B_\gamma^\sigma),\ W^\sigma := \bigcup_{m\in\mathbb{Z}} \nabla^{\sigma m}.$$

Then for every $X \in \nabla_\gamma(P,Q)$ there exists a unique $\sigma(X) \in T_\mathbb{Z}$ such that $X \in \nabla^{\sigma(X)}$ and $X \notin W^{\sigma(X)}$ (see (Shapirovsky *et al.*, 2003) for details).

For $\sigma = m_1 m_2 \ldots \in T_\mathbb{Z} \cup T_\mathbb{Z}^\infty$ we put $[\sigma] := \tilde{m}_1\tilde{m}_2\ldots$, where $\tilde{m}_i \in \{1,2\}$, $\tilde{m}_i \equiv m_i \pmod 2$.

We define a map $f_0 : \nabla_\gamma(P,Q) \longrightarrow I_2$ as follows:

$$f_0(X) := \begin{cases} [\sigma(X)] & \text{if } X \text{ is an interior point of } \nabla^{\sigma(X)}; \\ \lambda & \text{if } X = R; \\ [\sigma(X)]^- & \text{otherwise.} \end{cases}$$

In (Shapirovsky *et al.*, 2003) it was shown that $f_0 : (\nabla_\gamma(PQ), \prec) \twoheadrightarrow \widetilde{\mathbf{I}}_\mathbf{2}$.

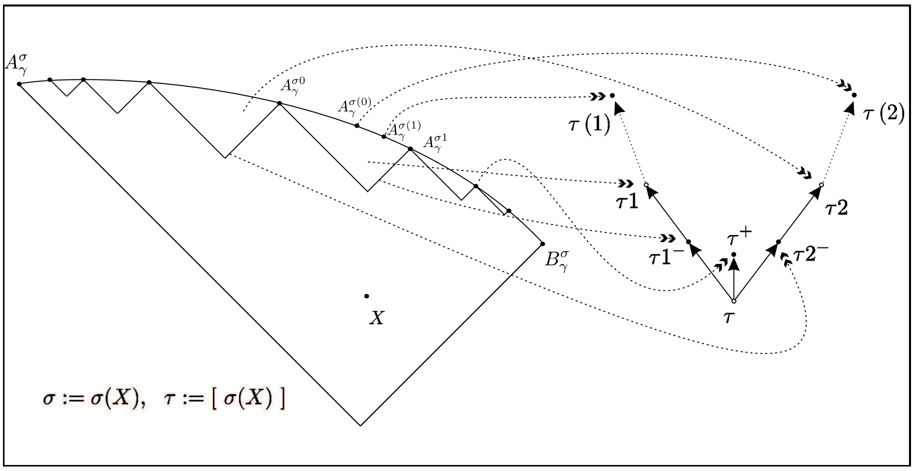

Figure 5. $f : (\widetilde{\nabla}_\gamma(P,Q), \prec) \twoheadrightarrow \widetilde{\mathbf{I}}_\mathbf{2}$

Now we extend f_0 to a p-morphism $f : (\widetilde{\nabla}_\gamma(P,Q), \prec) \twoheadrightarrow \widetilde{\mathbf{I}}_\mathbf{2}$ (Fig. 5). Note that for every $X \in \gamma(P,Q)$ there exists a unique $\sigma \in T_\mathbb{Z} \cup T^\infty_\mathbb{Z} - \{\lambda\}$, such that $X = A^\sigma_\gamma$. We put for $X \in \widetilde{\nabla}_\gamma(P,Q)$:

$$f(X) := \begin{cases} f_0(X) & X \in \nabla_\gamma(P,Q); \\ [\sigma]^+ & X = A^{\sigma m}_\gamma,\ \sigma \in T_\mathbb{Z},\ m \in \mathbb{Z}; \\ [\sigma] & X = A^\sigma_\gamma,\ \sigma \in T^\infty_\mathbb{Z}. \end{cases}$$

To check the monotonicity, it is sufficient to show that $f(X) \lessdot f(Y)$ for any $X \in \nabla_\gamma(P,Q),\ Y \in \gamma(P,Q)$. It is not difficult to see that if $Y = A^\sigma_\gamma$, then $\sigma(X) \sqsubseteq \sigma$. By definition of f, $f(X) = \sigma(X)$ or $f(X) = \sigma(X)^-$, $f(Y) = \sigma$ or $f(Y) = \sigma^+$, thus $f(X) \lessdot f(Y)$.

To check the lift property for f, suppose that $X \in \widetilde{\nabla}_\gamma(P,Q),\ \rho \in \widetilde{I_2}$ and $f(X) \lessdot \rho$. Thus $X \in \nabla_\gamma(P,Q)$. We have to show that $X \prec Y,\ f(Y) = \rho$ for some Y. Let us consider three cases.

Case a: $\rho \in \widetilde{I_2} - I_2$. The statement holds because f_0 has the lift property.

Case b: $\rho = \tau^+$ for some $\tau \in T_2$. Then $f(X) \lessdot \tau$, and there exists $Y_0 \in \nabla_\gamma(P,Q)$, such that $f(Y_0) = \tau,\ X \prec Y_0$. Thus we have for some m: $Y_0 \prec A^{\sigma(Y_0)m}_\gamma$. Let $Y := A^{\sigma(Y_0)m}_\gamma$. Then $X \prec Y$ and $f(Y) = \rho$.

Case c: $\rho \in T^\infty_2$. Since $f(X) \lessdot \rho$, we have $\rho = [\sigma(X)]t_1 t_2 \ldots,\ t_i \in \{1,2\}$. Let $\rho_n = [\sigma(X)]t_1 \ldots t_n$. Due to the lift property of f_0, there exist $Y_1, Y_2, \ldots$, such that $X \prec Y_1 \prec Y_2 \prec \ldots$ and $f(Y_n) = \rho_n$ for every n. Thus $\sigma(X) \sqsubset \sigma(Y_1) \sqsubset \sigma(Y_2) \ldots$. Consider $\sigma \in T^\infty_\mathbb{Z}$ such that $\sigma(Y_n) \sqsubset \sigma$ for every n. Then $X \prec A^\sigma_\gamma$ and $f(A^\sigma_\gamma) = \rho$.

Let us show that f is surjective. $f^{-1}(\lambda) \neq \varnothing$ (since f_0 is surjective), $\lessdot(\lambda) = \widetilde{I}_2$. Since f has the lift property, for any $\rho \in \widetilde{I}_2$ there exists $X \in \widetilde{\nabla}_\gamma(P,Q)$ such that $f(X) = \rho$.

Therefore $f : (\widetilde{\nabla}_\gamma(P,Q), \prec) \twoheadrightarrow \widetilde{\mathbf{I}}_\mathbf{2}$. ■

By Lemma 1, $\mathbf{L}(\widetilde{\nabla}_\gamma(P,Q), \prec) \supseteq \mathbf{L}_3$. By Theorem 15 and Lemma 17 we have

LEMMA 18. — $\mathbf{L}(\widetilde{\nabla}_\gamma(P,Q), \prec) = \mathbf{L}_3$.

Let $\mathbb{R}^2_- := \{(x,t) \in \mathbb{R}^2 \mid t \leq 0\}$.

THEOREM 19. — $\mathbf{L}(\mathbb{R}^2_-, \prec) = \mathbf{L}_3$.

PROOF. — By Lemma 1, $\mathbf{L}(\mathbb{R}^2_-, \prec) \supseteq \mathbf{L}_3$.

Let $F := (\mathbb{R}^2_-, \prec)$, $Z = (0,-1)$. By Lemma 18, $\mathbf{L}(F^Z) = \mathbf{L}_3$, thus $\mathbf{L}(\mathbb{R}^2_-, \prec) \subseteq \mathbf{L}_3$. ■

THEOREM 20. — *Let D be an open connected domain in $\mathbb{R}^2$ bounded by a closed smooth curve. Then $\mathbf{L}(\overline{D}, \prec) \subseteq \mathbf{L}_3$.*

PROOF. — Let γ be the boundary of D. It is not difficult to see that γ contains points $P = (x_1,t_1),\ Q = (x_2,t_2),\ x_1 < x_2$ such that $\gamma(P,Q)$ is a graph of some function $F :]x_1,x_2[\longrightarrow \mathbb{R}$, where $|F'(x)| < 1$ for $x \in]x_1,x_2[$, and $(\widetilde{\nabla}_\gamma(P,Q), \prec)$ is a cone in $\overline{D}$ (see (Shapirovsky *et al.*, 2003) for more detail). So $\mathbf{L}(\overline{D}, \prec) \subseteq \mathbf{L}(\widetilde{\nabla}_\gamma(P,Q), \prec)$, and by Lemma 18, $\mathbf{L}(\overline{D}, \prec) \subseteq \mathbf{L}_3$. ■

THEOREM 21. — *Let D be an open convex domain in $\mathbb{R}^2$ bounded by a closed smooth curve. Then $\mathbf{L}(\overline{D}, \prec) = \mathbf{L}_3$.*

PROOF. — Let γ be the boundary of D. Let us check that $\mathbf{L}(\overline{D}, \prec) \supseteq \mathbf{L}_3$.

One can see that for every serial $X \in \overline{D}$ there exists $Y \in \gamma$ such that $X \prec Y$ and $\prec(Y) \cap \overline{D} = \varnothing$. It follows that $(\overline{D}, \prec) \vDash AM$.

Let us show that $(\overline{D}, \prec)$ is 2-dense. Consider arbitrary points $X, Y_1, Y_2 \in \overline{D}$ such that $X \prec Y_1,\ X \prec Y_2$. We have to find a point $X_0 \in \overline{D}$ such that $X \prec X_0,\ X_0 \prec Y_1,\ X_0 \prec Y_2$ (Fig. 6, a). Let Z be the middle of $\left[Y_1, Y_2\right]$, and let l be the straight line connecting X and Z. Let $U := \{Y \in \mathbb{R}^2 \mid Y \prec Y_1,\ Y \prec Y_2\} \cap l$. Then U is open in the topology of $l,\ X \in U$. Thus $U \cap \left]X, Z\right[\neq \varnothing$. Let $X_0 \in U \cap \left]X, Z\right[$. It is easy to see that $X \prec X_0$. Due to the convexity we have $X_0 \in \overline{D}$, and thus $(\overline{D}, \prec) \vDash Ad_2$. ■

Note that a non-convex set may be not 2-dense (Fig. 6, b).

6. Conclusion and further results

Let us consider polygons on Minkowski plane. The logics of open and closed convex polygons on the plane ordered by the causal future relation are described in (Shehtman, 1983): if D is an open convex polygon, then $\mathbf{L}(D, \preceq)$ is either $\mathbf{S4}$ (Fig.

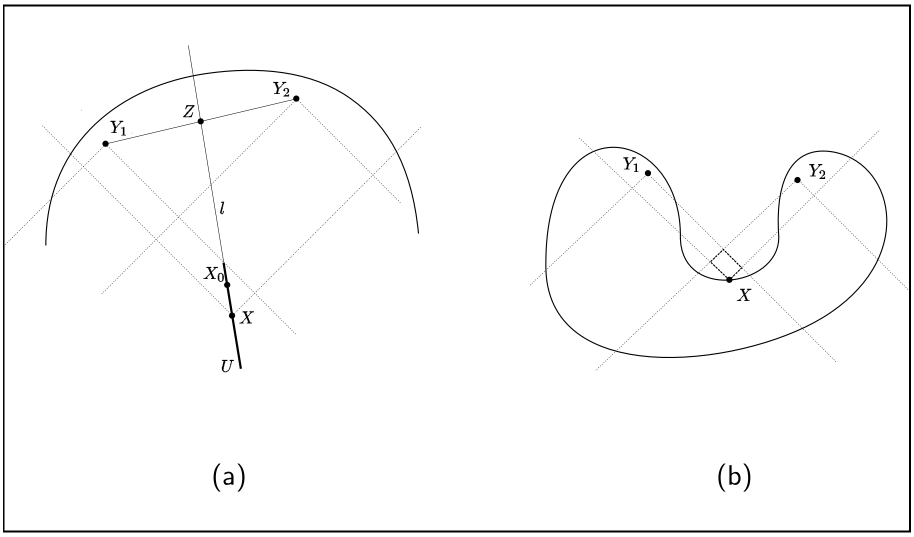

Figure 6. (a) *2-dense domain* **(b)** *not 2-dense domain*

7, a) or $\mathbf{S4.2}$ (Fig 7, b-d); $\mathbf{L}(\overline{D}, \preceq)$ is either $\mathbf{S4.1}$ or $\mathbf{S4.1.2}$. In (Shapirovsky *et al.*, 2003) it was shown that $\mathbf{L}(D, \prec) \in \{\mathbf{L}_1, \mathbf{L}_2\}$. The case of closed convex polygons on $(\mathbb{R}^2, \prec)$ is more complicated. There are four essentially different types of polygons (Fig. 7, a-d).

By Lemma 18, we have $\mathbf{L}(\overline{\mathrm{D}}_a, \prec) = \mathbf{L}_3$.

Let $\Box^\circ A := \Box(A \vee \Box\bot)$, $\Diamond^\circ A := \Diamond(A \wedge \Diamond\top)$. Consider the following "confluence-like" axioms

$$A2^+ := \Diamond\Box^+ p \to \Box\Diamond^+ p,\ A2^\circ := \Diamond^\circ\Box p \to \Box^\circ\Diamond^\circ p,$$

and their first-order correspondents

$$\begin{aligned} P2^+ &:= \forall x \forall y_1 \forall y_2 (xRy_1 \wedge xRy_2 \to \exists z(y_1R^+z \wedge y_2R^+z)), \\ P2^\circ &:= \forall x \forall y_1 \forall y_2 (xRy_1 \wedge xRy_2 \wedge R(y_1) \neq \varnothing \wedge R(y_2) \neq \varnothing \to \\ &\qquad \exists z(y_1Rz \wedge y_2Rz \wedge R(z) \neq \varnothing)). \end{aligned}$$

Let $\mathbf{L_b} := \mathbf{L}_3 + A2^+ + A2^\circ$.

Consider the polygon $\overline{\mathrm{D}}_b$. One can check that $\mathbf{L}(\overline{\mathrm{D}}_b, \prec) \supseteq \mathbf{L_b}$. Using the methods proposed in (Shapirovsky *et al.*, 2003) and in the present paper, it is possible to show that $\mathbf{L}(\overline{\mathrm{D}}_b, \prec) = \mathbf{L_b}$, the proof will be published in the sequel.

Two other cases seem to be rather difficult. For $n \geq 1$ let

$$Al_n := \bigwedge_{0 \leq i \leq n} \Box^\circ \neg p_i \to \bigvee_{0 \leq i \leq n} \Box \left(\Diamond p_i \to \bigvee_{0 \leq j \neq i \leq n} \Diamond p_j \right).$$

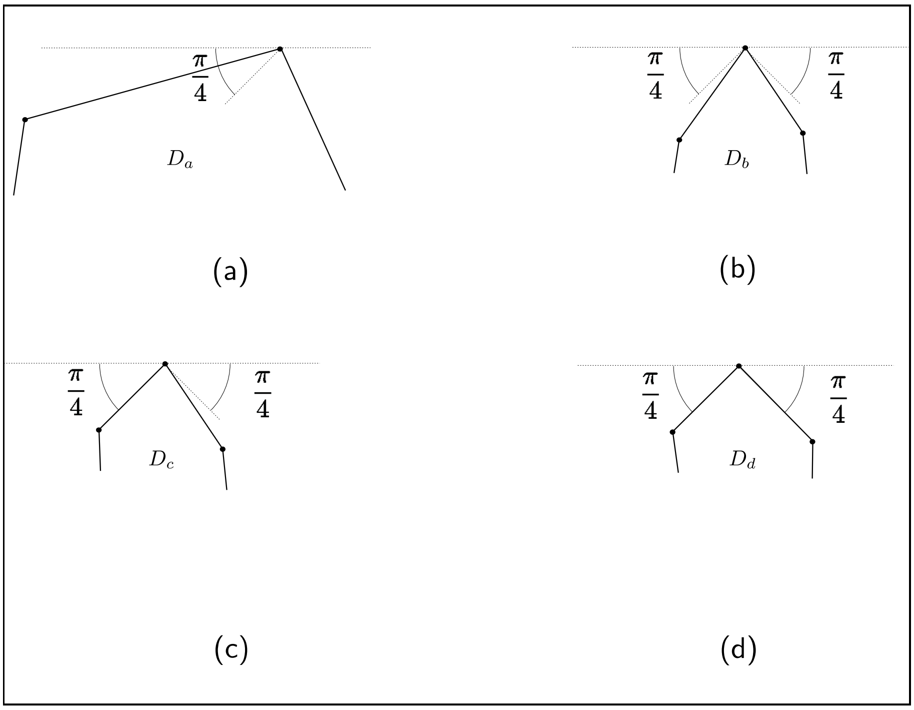

Figure 7. *Polygons on Minkowski plane*

Al_n has the following first-order correspondent:

$$\forall u \forall x_0 \ldots \forall x_n \big(\bigwedge_{0 \le i \le n} (uRx_i \wedge R(x_i) = \varnothing) \rightarrow \\ \bigvee_{0 \le i \le n} \forall z (uRz \wedge zRx_i \rightarrow \bigvee_{0 \le j \ne i \le n} zRx_j)\big).$$

Let $\mathbf{L}l_n := \mathbf{L}_3 + A2^\circ + Al_n$. By a straightforward argument, $\mathbf{L}(\overline{\mathrm{D}}_c, \prec) \supseteq \mathbf{L}l_1$, $\mathbf{L}(\overline{\mathrm{D}}_d, \prec) \supseteq \mathbf{L}l_2$. But to find axiomatization for $\mathbf{L}(\overline{\mathrm{D}}_c, \prec)$ and $\mathbf{L}(\overline{\mathrm{D}}_d, \prec)$ is an open problem.

The logics arising in the study of domains on Minkowski plane can be interpreted as fragments of interval temporal logics, (Shehtman, 1987), (Shapirovsky *et al.*, 2005).

Let I be the set of all non-strict intervals on the real line:

$$\mathrm{I} = \{(a, b) \in \mathbb{R}^2 \mid a \le b\}.$$

Consider the following relation between intervals:

$$(a_1, b_1) \rhd (a_2, b_2) := a_1 < a_2 \ \& \ b_2 < b_1.$$

For an interval $i = (a, b) \in \mathrm{I}$ we define the point $X_i \in \mathbb{R}^2_-$ as follows:

$$X_i := \left(\frac{a+b}{2}, \frac{a-b}{2}\right).$$

One can easily check that for any intervals $i, j \in \mathrm{I}$

$$i \rhd j \Leftrightarrow X_i \prec X_j$$

and the frames $(\mathbb{R}^2_-, \prec)$ and $(\mathrm{I}, \rhd)$ are isomorphic. So by Theorem 19, we obtain

COROLLARY 22. — $\mathbf{L}(\mathrm{I}, \rhd) = \mathbf{L}_3$.

Finally we make some remarks about the computational complexity of the logic $\mathbf{L}_3$. Since $\mathbf{L}_3$ has the FMP, it is decidable. The complexity of 2-dense logics was first studied in (Shapirovsky, 2005), where the proof of PSPACE-completeness for $\mathbf{L}_1$, $\mathbf{L}_2$ was given. A slight modification of this proof yields the PSPACE-completeness for $\mathbf{L}_3$.

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