Задание 1.

$$N_{p\neq 0} = \langle \text{в.з.} | \sum_p' a_p^+ a_p | \text{в.з.} \rangle =$$

$$= \langle \text{без в.з.} | \sum_p' U a_p^+ U^+ U a_p U^+ | \text{без. в.з.} \rangle$$

$$= \langle \text{без. в.з.} | \sum_p' (u_p a_p^+ + v_p a_{-p}) \cdot (u_p a_p + v_p a_{-p}^+)$$

$$| \text{без. в.з.} \rangle = \sum_p' v_p^2$$

$$v_p^2 = \frac{1}{2} \frac{\varepsilon_p + 2\gamma_p - \tilde{\varepsilon}_p}{\tilde{\varepsilon}_p}$$

$$\tilde{\varepsilon}_p = \sqrt{\varepsilon_p(\varepsilon_p + 4\gamma_p)}$$

$$\gamma_p = \frac{N_0}{2V} \cdot U(p) = \frac{N_0}{2V} V_0 a^3 \pi^{3/2} \exp\left(-\left(\frac{pa}{2}\right)^2\right)$$

$$N_{p\neq 0} = \frac{V}{(2\pi)^3} \int d^3p \; \frac{1}{2} \frac{\varepsilon_p + 2\gamma_p - \tilde{\varepsilon}_p}{\tilde{\varepsilon}_p} =$$

$$= \frac{V}{(2\pi)^2} \int_0^\infty p^2 dp \frac{\varepsilon_p + 2\gamma_p - \tilde{\varepsilon}_p}{\tilde{\varepsilon}_p}$$

Рассмотрим подынтегральное выражение.

При малых p:

$$\gamma_p \approx \frac{N_0}{2V} V_0 a^3 \pi^{3/2}$$

$$\varepsilon_p = \frac{p^2}{2m}$$

$$\tilde{\varepsilon}_p = \sqrt{\varepsilon_p(\varepsilon_p + 4\gamma_p)} \approx \sqrt{\varepsilon_p} \cdot \sqrt{\gamma_0} =$$

$$= p\sqrt{\frac{\gamma_0}{2m}}$$

$$p^2 \frac{\varepsilon_p + 2\gamma_p - \tilde{\varepsilon}_p}{\tilde{\varepsilon}_p} \approx p^2 \frac{2\gamma_0}{p\sqrt{\frac{\gamma_0}{2m}}} =$$

$$= 2p\sqrt{2m\gamma_0}$$

Как видим, сходимость в нуле обеспечена.

При больших p:

$$\gamma_p \to 0$$

$$\tilde{\varepsilon}_p = \sqrt{\varepsilon_p(\varepsilon_p + 4\gamma_p)} = \varepsilon_p\left(1 + 4\frac{\gamma_p}{\varepsilon_p}\right)^{\frac{1}{2}}$$

$$\approx \varepsilon_p\left(1 + 2\frac{\gamma_p}{\varepsilon_p} - 2\left(\frac{\gamma_p}{\varepsilon_p}\right)^2\right) =$$

$$= \varepsilon_p + 2\gamma_p - 2\frac{\gamma_p^2}{\varepsilon_p}$$

$$p^2 \frac{\varepsilon_p + 2\gamma_p - \tilde{\varepsilon}_p}{\tilde{\varepsilon}_p} \approx p^2 \frac{2\gamma_p^2/\varepsilon_p}{\varepsilon_p} =$$

$$= 2p^2 \frac{\gamma_p^2}{\varepsilon_p^2} = 8m^2 \frac{\gamma_p^2}{p^2}$$

Как видим, и на бесконечности сходимость обеспечена.

Основной вклад дают малые значения p, поэтому будем считать, что $\gamma_p \approx \gamma_0$

$$N_{p \neq 0} = \int_0^\infty p^2 dp \, \frac{\varepsilon_p + 2\gamma_p - \tilde{\varepsilon}_p}{\tilde{\varepsilon}_p} =$$

$$= \int_0^\infty p^2 dp \, \frac{\varepsilon_p + 2\gamma_0 - \sqrt{\varepsilon_p(\varepsilon_p + 4\gamma_0)}}{\sqrt{\varepsilon_p(\varepsilon_p + 4\gamma_0)}}$$

$$p^2 dp = 2m\varepsilon \, d\sqrt{2m\varepsilon} = (2m)^{3/2} \frac{\sqrt{\varepsilon}\, d\varepsilon}{2}$$

Получим:

$$N_{p\neq 0} = (2m)^{3/2} \int_0^\infty \frac{\varepsilon^{\frac{1}{2}} d\varepsilon}{2} \; \frac{\varepsilon + 2\gamma_0 - \sqrt{\varepsilon(\varepsilon+4\gamma_0)}}{\sqrt{\varepsilon(\varepsilon+4\gamma_0)}}$$

$$= \frac{(2m)^{3/2}}{2} \int_0^\infty \varepsilon^{\frac{1}{2}} d\varepsilon \left(\frac{\varepsilon + 4\gamma_0}{\sqrt{\varepsilon(\varepsilon+4\gamma_0)}} - \frac{2\gamma_0}{\sqrt{\varepsilon(\varepsilon+4\gamma_0)}} - 1 \right)$$

$$= \frac{(2m)^{3/2}}{2} \int_0^\infty d\varepsilon \left(\sqrt{\varepsilon + 4\gamma_0} - \frac{2\gamma_0}{\sqrt{\varepsilon+4\gamma_0}} - \varepsilon^{1/2} \right)$$

$$= \frac{(2m)^{3/2}}{2} \left(\frac{2}{3} (\varepsilon + 4\gamma_0)^{3/2} - 4\gamma_0 (\varepsilon + 4\gamma_0)^{\frac{1}{2}} - \right.$$

$$\left. - \frac{2}{3} \varepsilon^{3/2} \right) \Big|_0^\infty$$

При $\varepsilon \to \infty$:

$$\frac{2}{3} (\varepsilon + 4\gamma_0)^{3/2} - 4\gamma_0 (\varepsilon + 4\gamma_0)^{\frac{1}{2}} - \frac{2}{3} \varepsilon^{\frac{3}{2}} \approx$$

$$\approx \frac{2}{3} \varepsilon^{3/2} \left(1 + 4 \cdot \frac{3}{2} \frac{\gamma_0}{\varepsilon}\right) - 4\gamma_0 \varepsilon^{\frac{1}{2}} \left(1 + 2\frac{\gamma_0}{\varepsilon}\right)$$

$$- \frac{2}{3} \varepsilon^{3/2} = - 8 \frac{\gamma_0^2}{\varepsilon^{1/2}} \to 0$$

Таким образом,

$$N_{p\neq 0} =$$

$$= -\frac{(2m)^{3/2}}{2}\left(\frac{2}{3}\left(\varepsilon + 4\gamma_0\right)^{\frac{3}{2}} - 4\gamma_0(\varepsilon + 4\gamma_0)^{\frac{1}{2}} - \frac{2}{3}\varepsilon^{3/2}\right)\Bigg|_{\varepsilon=0} =$$

$$= \frac{4}{3}(2m)^{3/2}\gamma_0^{3/2}$$

Забыли всюду $\frac{V}{(2\pi)^2}$. Имеем:

$$N_{p\neq 0} = \frac{V}{(2\pi)^2}\,\frac{4}{3}\,(2m\gamma_0)^{3/2} =$$

$$= \frac{1}{3}\,\pi^{\frac{1}{4}}\,N_0\left(\frac{N_0}{V}\right)^{\frac{1}{2}}\left(mV_0a^3\right)^{3/2}$$

$$\frac{N_{p\neq 0}}{N_0} = \frac{1}{3}\pi^{\frac{1}{4}}\left(\frac{N_0}{V}\right)^{\frac{1}{2}}\left(mV_0a^3\right)^{3/2} \sim$$

$$\sim \left(\frac{N_0}{V}\right)^{1/2}\left(mV_0a^3\right)^{3/2}$$

Требуем:

$$N_{p\neq 0} \ll N_0$$

То есть

$$\left(\frac{N_0}{V}\right)^{\frac{1}{2}}\left(mV_0a^3\right)^{3/2} \ll 1$$