

Задание 1.

$$\begin{aligned}
 N_{p \neq 0} &= \langle \psi_z. | \sum_p a_p^\dagger a_p | \psi_z \rangle = \\
 &= \langle \psi_z. \psi_z. | \sum_p u a_p^\dagger u^\dagger u a_p u^\dagger | \psi_z. \psi_z. \rangle \\
 &= \langle \psi_z. \psi_z. | \sum_p (u_p a_p^\dagger + v_p a_{-p}) \cdot (u_p a_p + v_p a_{-p}^\dagger) | \psi_z. \psi_z. \rangle = \sum_p u_p^2
 \end{aligned}$$

$$u_p^2 = \frac{1}{2} \frac{\epsilon_p + 2\gamma_p - \tilde{\epsilon}_p}{\tilde{\epsilon}_p}$$

$$\tilde{\epsilon}_p = \sqrt{\epsilon_p(\epsilon_p + 4\gamma_p)}$$

$$\gamma_p = \frac{N_0}{2V} \cdot u(p) = \frac{N_0}{2V} V_0 a^3 \pi^{3/2} \exp\left(-\left(\frac{pa}{2}\right)^2\right)$$

$$\begin{aligned}
 N_{p \neq 0} &= \frac{V}{(2\pi)^3} \int d^3p \frac{1}{2} \frac{\epsilon_p + 2\gamma_p - \tilde{\epsilon}_p}{\tilde{\epsilon}_p} = \\
 &= \frac{V}{(2\pi)^2} \int_0^\infty p^2 dp \frac{\epsilon_p + 2\gamma_p - \tilde{\epsilon}_p}{\tilde{\epsilon}_p}
 \end{aligned}$$

Расширим логарифмическое выражение.

При малых  $p$ :

$$\gamma_p \approx \frac{N_0}{2V} V_0 a^3 \pi^{3/2}$$

$$\epsilon_p = \frac{p^2}{2m}$$

$$\begin{aligned}\tilde{\epsilon}_p &= \sqrt{\epsilon_p (\epsilon_p + 4\gamma_p)} \approx \sqrt{\epsilon_p} \cdot \sqrt{4\gamma_p} = \\ &= p \sqrt{\frac{\gamma_0}{2m}}\end{aligned}$$

$$p^2 \frac{\epsilon_p + 2\gamma_p - \tilde{\epsilon}_p}{\tilde{\epsilon}_p} \approx p^2 \frac{2\gamma_0}{p \sqrt{\frac{\gamma_0}{2m}}} =$$

$$= 2p \sqrt{2m\gamma_0}$$

Как видно, скорость в нуле обесценивается.

При больших  $p$ :

$$\gamma_p \rightarrow 0$$

$$\begin{aligned}\tilde{\epsilon}_p &= \sqrt{\epsilon_p (\epsilon_p + 4\gamma_p)} = \epsilon_p \left(1 + 4\frac{\gamma_p}{\epsilon_p}\right)^{1/2} \\ &\approx \epsilon_p \left(1 + 2\frac{\gamma_p}{\epsilon_p} - 2\left(\frac{\gamma_p}{\epsilon_p}\right)^2\right) =\end{aligned}$$

$$= \epsilon_p + 2\gamma_p - 2 \frac{\gamma_p^2}{\epsilon_p}$$

$$p^2 \frac{\epsilon_p + 2\gamma_p - \tilde{\epsilon}_p}{\tilde{\epsilon}_p} \approx p^2 \frac{2\gamma_p^2 / \epsilon_p}{\epsilon_p} =$$

$$= 2p^2 \frac{\gamma_p^2}{\epsilon_p^2} = 8m^2 \frac{\gamma_p^2}{p^2}$$

Как видно, и на бесконечности сходимость обеспечена.

Основной вклад дают малые значения  $p$ , поэтому будем считать, что  $\gamma_p \approx \gamma_0$

$$N_{p \neq 0} = \int_0^\infty p^2 dp \frac{\epsilon_p + 2\gamma_p - \tilde{\epsilon}_p}{\tilde{\epsilon}_p} =$$

$$= \int_0^\infty p^2 dp \frac{\epsilon_p + 2\gamma_0 - \sqrt{\epsilon_p(\epsilon_p + 4\gamma_0)}}{\sqrt{\epsilon_p(\epsilon_p + 4\gamma_0)}}$$

$$p^2 dp = 2m\epsilon d\sqrt{2m\epsilon} = (2m)^{3/2} \frac{\sqrt{\epsilon} d\epsilon}{2}$$

Получим:

$$N_{p \neq 0} = (2m)^{3/2} \int_0^\infty \frac{\varepsilon^{1/2} d\varepsilon}{2} \frac{\varepsilon + 2\delta_0 - \sqrt{\varepsilon(\varepsilon + 4\delta_0)}}{\sqrt{\varepsilon(\varepsilon + 4\delta_0)}}$$

$$= \frac{(2m)^{3/2}}{2} \int_0^\infty \varepsilon^{1/2} d\varepsilon \left( \frac{\varepsilon + 4\delta_0}{\sqrt{\varepsilon(\varepsilon + 4\delta_0)}} - \frac{2\delta_0}{\sqrt{\varepsilon(\varepsilon + 4\delta_0)}} - 1 \right)$$

$$= \frac{(2m)^{3/2}}{2} \int_0^\infty d\varepsilon \left( \sqrt{\varepsilon + 4\delta_0} - \frac{2\delta_0}{\sqrt{\varepsilon + 4\delta_0}} - \varepsilon^{1/2} \right)$$

$$= \frac{(2m)^{3/2}}{2} \left( \frac{2}{3} (\varepsilon + 4\delta_0)^{3/2} - 4\delta_0 (\varepsilon + 4\delta_0)^{1/2} - \frac{2}{3} \varepsilon^{3/2} \right) \Big|_0^\infty$$

При  $\varepsilon \rightarrow \infty$  :

$$\frac{2}{3} (\varepsilon + 4\delta_0)^{3/2} - 4\delta_0 (\varepsilon + 4\delta_0)^{1/2} - \frac{2}{3} \varepsilon^{3/2} \approx$$

$$\approx \frac{2}{3} \varepsilon^{3/2} \left( 1 + 4 \cdot \frac{3}{2} \frac{\delta_0}{\varepsilon} \right) - 4\delta_0 \varepsilon^{1/2} \left( 1 + 2 \frac{\delta_0}{\varepsilon} \right)$$

$$- \frac{2}{3} \varepsilon^{3/2} = - 8 \frac{\delta_0^2}{\varepsilon^{1/2}} \rightarrow 0$$

Получим окончательный,

$$N_{p \neq 0} =$$

$$= -\frac{(2m)^{3/2}}{2} \left( \frac{2}{3} (\xi + 4\delta_0)^{3/2} - 4\delta_0(\xi + 4\delta_0)^{1/2} - \frac{2}{3} \xi^{3/2} \right) \Big|_{\xi=0} =$$

$$= \frac{4}{3} (2m)^{3/2} \delta_0^{3/2}$$

Значим в силу  $\frac{V}{(2\pi)^2}$ . Имеем:

$$N_{p \neq 0} = \frac{V}{(2\pi)^2} \frac{4}{3} (2m \delta_0)^{3/2} =$$

$$= \frac{1}{3} \pi^{1/4} N_0 \left( \frac{N_0}{V} \right)^{1/2} (m V_0 a^3)^{3/2}$$

$$\frac{N_{p \neq 0}}{N_0} = \frac{1}{3} \pi^{1/4} \left( \frac{N_0}{V} \right)^{1/2} (m V_0 a^3)^{3/2} \sim$$

$$\sim \left( \frac{N_0}{V} \right)^{1/2} (m V_0 a^3)^{3/2}$$

Предусловие:

$$N_{p \neq 0} \ll N_0$$

Поэтому

$$\left( \frac{N_0}{V} \right)^{1/2} (m V_0 a^3)^{3/2} \ll 1$$