Задача 4.

$$\Gamma_{\alpha\beta} = \langle u_\alpha(\vec{r}_1)\, u_\beta(\vec{r}_2) \rangle, \quad \vec{r}_1 \neq \vec{r}_2$$

$$u_\alpha(\vec{r}_1) = \sum_\xi \sqrt{\frac{\hbar}{2\rho V \omega_\xi}} \left\{ l_{\alpha\xi} a_\xi e^{i\vec{f}\cdot\vec{r}_1 - i\omega t} + l^*_{\alpha\xi} a^+_\xi e^{-i\vec{f}\cdot\vec{r}_1 + i\omega t} \right\}$$

$$u_\beta(\vec{r}_2) = \sum_\zeta \sqrt{\frac{\hbar}{2\rho V \omega_\zeta}} \left\{ l_{\beta\zeta} a_\zeta e^{i\vec{f}\cdot\vec{r}_2 - i\omega t} + l^*_{\beta\zeta} a^+_\zeta e^{-i\vec{f}\cdot\vec{r}_1 + i\omega t} \right\}$$

Учитываем, что

$$\langle a_\xi a_\zeta \rangle = 0$$

$$\langle a^+_\xi a^+_\zeta \rangle = 0$$

$$\langle a^+_\xi a_\zeta \rangle = \bar{n}_\xi \delta_{\xi\zeta}$$

$$\langle a_\xi a^+_\zeta \rangle = (\bar{n}_\xi + 1)\delta_{\xi\zeta}$$

$$\vec{R} = \vec{r}_1 - \vec{r}_2 .$$

Получим:

$$\Gamma_{\alpha\beta}(\vec{R}) = \sum_{\xi} \frac{\hbar}{2\rho V \omega_\xi} \Big(l^*_{\alpha\xi} l_{\beta\xi} \bar{n}_\xi e^{-i\vec{f}\cdot\vec{R}}$$

$$+ \; l_{\alpha\xi} l^*_{\beta\xi} (\bar{n}_\xi + 1) e^{i\vec{f}\cdot\vec{R}} \Big)$$

На семинаре мы выяснили, что характерное расстояние есть $R_0 = \frac{c\hbar}{\pi T}$, где c – характерная скорость звука.

Пусть $\omega_1 = c_l \cdot f$, $\omega_2, \omega_3 = c_t \cdot f$. Вычислим $\Gamma_{\alpha\beta}(\vec{R})$, полагая, что

$$R \gg R_0$$

$$\Gamma_{\alpha\beta} = \sum_s \frac{V}{(2\pi)^3} \int d^3 f \frac{\hbar}{2\rho V c_s f} \Big(l^*_{\alpha s\vec{f}} \, l_{\beta s\vec{f}} \cdot$$

$$\cdot \, \bar{n}_{sf} e^{-i\vec{f}\cdot\vec{R}} + l_{\alpha s\vec{f}} \, l^*_{\beta s\vec{f}} (\bar{n}_{sf} + 1) e^{i\vec{f}\cdot\vec{R}} \Big) =$$

$$= \sum_s \frac{V}{(2\pi)^3} \int d^3 f \, \frac{\hbar}{\rho V c_s f} \, l^*_{\alpha s\vec{f}} \, l_{\beta s\vec{f}} \, \cdot$$

$$\cdot \Big(\bar{n}_{sf} + \frac{1}{2} \Big) e^{-i\vec{f}\cdot\vec{R}} =$$

$$= \sum_s{}' \frac{V}{(2\pi)^3} \int d^3 f \, \frac{\hbar}{\rho V c_s f} \, e^*_{\alpha s \vec{f}} \, e_{\beta s \vec{f}} \cdot$$

$$\cdot \left(\frac{1}{\exp\left(\frac{\hbar c_s f}{T}\right) - 1} + \frac{1}{2} \right) e^{-i \vec{f} \cdot \vec{R}}$$

Считаем, что интеграл набирается при не очень больших $\vec{f}$.

$$R \gg R_0 = \frac{c_s \hbar}{\pi T}, \quad f R_0 \ll 1$$

$$\frac{1}{\exp\left(\frac{\hbar c_s f}{T}\right) - 1} + \frac{1}{2} \approx \frac{T}{\hbar c_s f} + \frac{1}{2} \approx$$

$$\approx \frac{T}{\hbar c_s f}$$

Получаем:

$$\Gamma_{\alpha\beta}(\vec{R}) = \sum_s \frac{V}{(2\pi)^3} \int d^3 f \, \frac{\hbar}{\rho V c_s f} \cdot \frac{T}{\hbar c_s f} \cdot$$

$$\cdot \, e^*_{\alpha s \vec{f}} \, e_{\beta s \vec{f}} \, e^{-i \vec{f} \cdot \vec{R}} =$$

$$= \frac{V}{(2\pi)^3} \int d^3 f \, \frac{T}{\rho V f^2} \, e^{-i \vec{f} \cdot \vec{R}} \sum_s{}' e^*_{\alpha s \vec{f}} \, e_{\beta s \vec{f}} / c_s^2$$

$$\sum_s{}' \frac{l^*_{\alpha s \vec{f}}\, l_{\beta s \vec{f}}}{C_s^2} = \frac{1}{C_l^2} \overset{*}{l}{}^{(l)}_{\alpha}\, l^{(l)}_{\beta} +$$

$$+ \frac{1}{C_t^2} l^{*(t)}_{\alpha 1}\, l^{(t)}_{\beta 1} + \frac{1}{C_t^2} \overset{*}{l}{}^{(t)}_{\alpha 2}\, l^{(t)}_{\beta 2} +$$

$$+ \frac{1}{C_t^2} l^{*(l)}_{\alpha}\, l^{(l)}_{\beta} - \frac{1}{C_t^2} l^{*(l)}_{\alpha}\, l^{(l)}_{\beta} =$$

$$= \frac{1}{C_t^2} \sum_s{}' l^*_{\alpha s}\, l_{\beta s} + \left(\frac{1}{C_l^2} - \frac{1}{C_t^2}\right) \overset{*}{l}{}^{(l)}_{\alpha}\, l^{(l)}_{\beta} =$$

$$= \frac{1}{C_t^2} \delta_{\alpha\beta} + \left(\frac{1}{C_l^2} - \frac{1}{C_t^2}\right) \frac{f_\alpha f_\beta}{f^2}$$

$$\Gamma_{\alpha\beta}(R) = \frac{V}{(2\pi)^3} \int d^3 f \frac{T}{\rho V f^2} e^{-i\vec{f}\cdot\vec{R}} \Bigg($$

$$\frac{1}{C_t^2} \delta_{\alpha\beta} + \left(\frac{1}{C_l^2} - \frac{1}{C_t^2}\right) \frac{f_\alpha f_\beta}{f^2}\Bigg) =$$

$$= \frac{V}{(2\pi)^3} \frac{T}{\rho V C_t^2} \delta_{\alpha\beta} \int d^3 f \frac{e^{-i\vec{f}\cdot\vec{R}}}{f^2} +$$

$$+ \left(\frac{1}{C_l^2} - \frac{1}{C_t^2}\right) \frac{V}{(2\pi)^3} \frac{T}{\rho V} \int d^3 f \frac{e^{-i\vec{f}\cdot\vec{R}}}{f^4} f_\alpha f_\beta$$

$$\int d^3 f \frac{e^{-i\vec{f}\cdot\vec{R}}}{f^2} = \int_0^\infty 2\pi f^2 df \frac{e^{-ifR} - e^{ifR}}{-ifR\, f^2}$$

$$= 4\pi \int_0^\infty \frac{\sin(fR)}{fR}\, df = \frac{4\pi}{R}\int_0^\infty \frac{\sin\xi}{\xi}\, d\xi =$$

$$= \frac{2\pi^2}{R}$$

$$\int d^3\vec{f}\ \frac{e^{-i\vec{f}\cdot\vec{R}}}{f^4}\, f_\alpha f_\beta = A(R)\,\delta_{\alpha\beta} +$$

$$+ \frac{B(R)}{R^2} R_\alpha R_\beta$$

Свернем:

$$\int d^3\vec{f}\ \frac{e^{-i\vec{f}\cdot\vec{R}}}{f^4}\cdot f^2 = 3A(R) + \frac{B(R)}{R^2}\cdot R^2$$

$$\frac{2\pi^2}{R} = 3A(R) + B(R)$$

Домножим на $R_\alpha R_\beta$ и свернём:

$$\int d^3f\ \frac{e^{-i\vec{f}\cdot\vec{R}}}{f^4}\left(\vec{f}\cdot\vec{R}\right)^2 = A(R)R^2 + B(R)R^2$$

$$\int d^3f\ \frac{e^{-i\vec{f}\cdot\vec{R}}}{f^4}\left(\vec{f}\cdot\vec{R}\right)^2 = 2\pi\int_0^\infty f^2 df\ \frac{f^2R^2}{f^4}\ \cdot$$

$$\cdot \int \sin\theta\, d\theta\ e^{-ifR\cos\theta}\cos^2\theta =$$

$$= 2\pi R^2 \int_0^\infty df \int_{-1}^{1} d\xi\, e^{-ifR\xi}\, \xi^2 =$$

$$= 2\pi R^2 \int_{-1}^{1} d\xi\, \xi^2 \int_0^\infty df\, e^{-ifR\xi} =$$

$$= 2\pi R^2 \int_{-1}^{1} d\xi\, \xi^2 \frac{1}{\xi} \int d\eta\, e^{-iR\eta} =$$

$$= 0$$

$$A(R) + B(R) = 0$$

$$3A(R) + B(R) = \frac{2\pi^2}{R}$$

$$A(R) = \frac{\pi^2}{R}, \quad B(R) = -\frac{\pi^2}{R}$$

Таким образом, получаем

$$\Gamma_{\alpha\beta}(R) = \frac{T\delta_{\alpha\beta}}{(2\pi)^3 \rho C_t^2} \frac{2\pi^2}{R} +$$

$$+ \frac{T}{(2\pi)^3 \rho} \left(\frac{1}{C_c^2} - \frac{1}{C_t^2} \right) \frac{\pi^2}{R} \left(\delta_{\alpha\beta} - \frac{R_\alpha R_\beta}{R^2} \right)$$