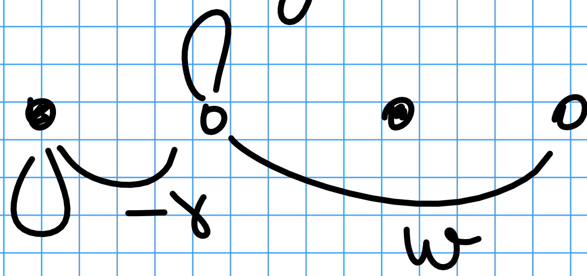
Контрольная работа по теоретической физике твёрдого тела



Масса иона: M

Масса электрона: m

$$A_{12}(0) = -\gamma, \quad A_{12}(-a) = -\gamma$$

$$A_{21}(0) = -\gamma, \quad A_{21}(a) = -\gamma$$

$$A_{22}(a) = w, \quad A_{22}(-a) = w$$

Известно, что $\sum\limits_{jn} A_{ji}(n) = 0$

$$\sum_{jn} A_{j1}(n) = \sum_{n} A_{11}(n) + \sum_{n} A_{21}(n) = 0$$

$$A_{11}(0) + A_{21}(0) + A_{21}(a) = 0$$

$$A_{11}(0) = -A_{21}(0) - A_{21}(a) = 2\gamma$$

$$\sum_{jn} A_{j2}(n) = \sum_{n} A_{12}(n) + \sum_{n} A_{22}(n) =$$

$$= 0$$

$$A_{12}(0) + A_{12}(-a) + A_{22}(0) + A_{22}(a) +$$
$$+ A_{22}(-a) = 0$$
$$A_{22}(0) = 2\gamma - 2w$$
$$C_{ij} = \frac{1}{\sqrt{M_i M_j}} \sum_n A_{ij}(n) e^{-inf}$$
$$C_{11} = \frac{1}{M} \cdot A_{11}(0) = \frac{2\gamma}{M}$$
$$C_{22} = \frac{1}{m} \left(A_{22}(0) + A_{22}(a) e^{-iaf} + A_{22}(-a) e^{ifa} \right)$$
$$= \frac{1}{m} (2\gamma - 2w + 2w \cos fa) =$$
$$= \frac{2\gamma}{m} \left(1 - 2\alpha \sin^2 \frac{fa}{2} \right),$$
где $\alpha = \frac{w}{\gamma}$
$$C_{12} = \frac{1}{\sqrt{Mm}} \left(A_{12}(0) + A_{12}(-a) e^{ifa} \right) =$$
$$= \frac{1}{\sqrt{Mm}} (-\gamma - \gamma e^{ifa}) = -\frac{\gamma}{\sqrt{Mm}} (1 + e^{ifa})$$
$$C_{21} = \frac{1}{\sqrt{Mm}} \left(A_{21}(0) + A_{21}(a) e^{-ifa} \right) =$$
$$= -\frac{\gamma}{\sqrt{Mm}} (1 + e^{-ifa})$$
Таким образом,
$$C_{11} = \frac{2\gamma}{M}, \quad C_{22} = \frac{2\gamma}{m} \left(1 - 2\alpha \sin^2 \frac{fa}{2} \right)$$
$$C_{12} = -\frac{\gamma}{\sqrt{Mm}} (1 + e^{ifa}), \quad C_{21} = -\frac{\gamma}{\sqrt{Mm}} (1 + e^{-ifa})$$

Найдем собственные числа матрицы C, то есть ω^2:

$$\begin{vmatrix} C_{11}-\omega^2 & C_{12} \\ C_{21} & C_{22}-\omega^2 \end{vmatrix} = (C_{11}-\omega^2)(C_{22}-\omega^2)$$

$$-C_{12}C_{21} = \omega^4 - \omega^2(C_{11}+C_{22}) +$$

$$+ C_{11}C_{22} - C_{12}C_{21} = 0$$

Получим:

$$\omega^2 = \frac{C_{11}+C_{22}}{2} \pm \sqrt{\left(\frac{C_{11}+C_{22}}{2}\right)^2 - (C_{11}C_{22}-C_{12}C_{21})}$$

Подставим C_{ij}.

$$C_{12}C_{21} = \frac{1}{Mm}\gamma^2(1+e^{ifa})(1+e^{-ifa}) =$$

$$= \frac{1}{Mm}\gamma^2(2+e^{ifa}+e^{-ifa}) = \frac{1}{Mm}\gamma^2(2+$$

$$+2\cos fa) = \frac{4\gamma^2}{Mm}\cos^2\frac{fa}{2}$$

$$C_{11}C_{22} = \frac{4\gamma^2}{Mm}\left(1-2\alpha\sin^2\frac{fa}{2}\right)$$

$$C_{11}C_{22} - C_{12}C_{21} = \frac{4\gamma^2}{Mm}\left(1-2\alpha\sin^2\frac{fa}{2}-\cos^2\frac{fa}{2}\right)$$

$$= \frac{4\gamma^2}{Mm}(1-2\alpha)\sin^2\frac{fa}{2}$$

$$\frac{C_{11}+C_{22}}{2} = \frac{\gamma}{M} + \frac{\gamma}{m} - \frac{2\alpha\gamma}{m}\sin^2\frac{fa}{2} =$$

$$= \frac{\gamma}{m}\left(\frac{m}{M} + 1 - 2\alpha \sin^2\frac{fa}{2}\right)$$

Итак,

$$\omega^2 = \frac{\gamma}{m}\left(\frac{m}{M} + 1 - 2\alpha\sin^2\frac{fa}{2}\right) \pm$$

$$\pm \sqrt{\gamma^2\left(\frac{m}{M} + 1 - 2\alpha\sin^2\frac{fa}{2}\right)^2 - \frac{4\gamma^2}{Mm}(1-2\alpha)\sin^2\frac{fa}{2}}$$

Можно записать в таком форме:

$$\boxed{\omega^2 = \frac{\gamma}{m}\left(\frac{m}{M} + 1 - 2\alpha\sin^2\frac{fa}{2}\right) \cdot \left(1 \pm \sqrt{1 - 4\frac{m/M\,(1-2\alpha)\sin^2\frac{fa}{2}}{\left(m/M + 1 - 2\alpha\sin^2\frac{fa}{2}\right)^2}}\right)}$$

Приступим ко второй части задания

Пусть $fa \ll 1$. Имеем:

$$\sin^2\frac{fa}{2} \approx \left(\frac{fa}{2}\right)^2$$

$$\omega^2 \approx \frac{\gamma}{m}\left(\frac{m}{M} + 1 - 2\alpha\left(\frac{fa}{2}\right)^2\right) \cdot$$

$$\cdot \left(1 \pm \sqrt{1 - 4\frac{(m/M)(1-2\alpha)\left(\frac{fa}{2}\right)^2}{\left(\frac{m}{M}+1\right)^2}}\right) \approx$$

$$\approx \frac{\gamma}{m}\left(\frac{m}{M}+1-2\alpha\left(\frac{fa}{2}\right)^2\right)\cdot$$

$$\cdot\left(1\pm\left(1-\frac{1}{2}(1-2\alpha)\frac{mM}{(m+M)^2}(fa)^2\right)\right)$$

$$\omega_1^2 = \frac{\gamma}{m}\left(\frac{m}{M}+1\right)\cdot\frac{1}{2}(1-2\alpha)\frac{mM}{(m+M)^2}(fa)^2 =$$

$$= (1-2\alpha)\frac{\gamma}{2(m+M)}(fa)^2$$

$$\omega_2^2 \approx \frac{\gamma}{m}\left(\frac{m}{M}+1-2\alpha\left(\frac{fa}{2}\right)^2\right)\cdot$$

$$\cdot\left(2-\frac{1}{2}(1-2\alpha)\frac{mM}{(m+M)^2}(fa)^2\right)\approx$$

$$\approx \frac{\gamma}{m}\left(2\frac{m}{M}+2-\alpha(fa)^2-\left(\frac{m}{M}+1\right)\cdot\right.$$

$$\left.\cdot\frac{1}{2}(1-2\alpha)\frac{mM}{(m+M)^2}(fa)^2\right)=$$

$$= \frac{\gamma}{m}\left(2\frac{m+M}{M}-\alpha(fa)^2-\frac{m}{m+M}\left(\frac{1}{2}-\alpha\right)(fa)^2\right)$$

$$= \frac{\gamma}{m}\left(2\frac{m+M}{M}-\frac{M\alpha+m/2}{m+M}(fa)^2\right)=$$

$$= 2\gamma\frac{m+M}{mM}\left(1-\frac{M\left(M\alpha+\frac{m}{2}\right)}{2(m+M)^2}(fa)^2\right)$$

$$\omega_2 = \sqrt{2\gamma\frac{m+M}{mM}}\left(1-\frac{1}{4}\frac{M(M\alpha+m/2)}{(m+M)^2}(fa)^2\right)$$

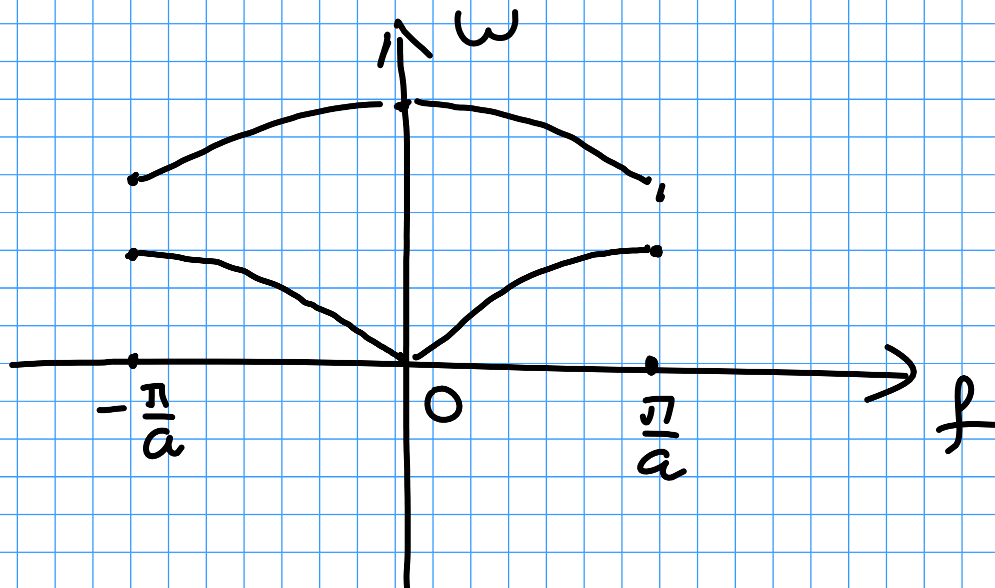
Итак, получили:

$$\boxed{\begin{aligned}\omega_1 &\approx \sqrt{(1-2\alpha)\frac{\gamma}{2(m+M)}}\cdot|fa| \\ \omega_2 &\approx \sqrt{2\gamma\frac{m+M}{mM}}\left(1-\frac{1}{4}\frac{M\left(M\alpha+\frac{m}{2}\right)}{(m+M)^2}(fa)^2\right)\end{aligned}}$$

Пусть $f=\frac{\pi}{a}$.

$$\sin^2\frac{fa}{2}=1$$

$$\boxed{\omega_{1,2}^2=\frac{\gamma}{m}\left(\frac{m}{M}+1-2\alpha\right)\cdot\left(1\pm\sqrt{1-4\frac{mM(1-2\alpha)}{(m+M-2\alpha M)^2}}\right)}$$



Определим условие устойчивости.

$\omega^2 < 0$, если $\frac{m}{M} + 1 - 2\alpha \sin^2 \frac{fa}{2} < 0$

Получаем условие устойчивости:

$$\frac{m}{M} + 1 \geq 2\alpha \sin^2 \frac{fa}{2}$$

при любых $f < \frac{\pi}{a}$

Отсюда

$$\frac{m}{M} + 1 \geq 2\alpha$$

Получаем в итоге:

$$\boxed{\alpha \leq \frac{1}{2} \frac{m+M}{M}}$$