

Задача 2.

$$U = \exp\left(\frac{1}{2} \sum_p (a_p a_{-p} - a_p^\dagger a_{-p}^\dagger) \varphi_p\right)$$

Докажем, что

$$U a_p U^\dagger = \cosh \varphi_p a_p + \sinh \varphi_p a_{-p}$$

$$\begin{aligned} & [a_p a_{-p} - a_p^\dagger a_{-p}^\dagger, a_q a_{-q} - a_q^\dagger a_{-q}^\dagger] = \\ &= -[a_p a_{-p}, a_q^\dagger a_{-q}^\dagger] - [a_p^\dagger a_{-p}^\dagger, a_q a_{-q}] = \\ &= -a_p [a_{-p}, a_q^\dagger a_{-q}^\dagger] - [a_p, a_q^\dagger a_{-q}^\dagger] a_{-p} - \\ &\quad - a_p^\dagger [a_{-p}^\dagger, a_q a_{-q}] - [a_p^\dagger, a_q a_{-q}] a_{-p}^\dagger = \\ &= -a_p a_q^\dagger \delta_{pq} - a_p a_{-q}^\dagger \delta_{-pq} - a_q^\dagger a_{-p} \delta_{-pq} - \\ &\quad - a_{-q}^\dagger a_{-p} \delta_{pq} + a_p^\dagger a_q \delta_{pq} + a_p^\dagger a_{-q} \delta_{-pq} + \\ &\quad + a_q a_{-p}^\dagger \delta_{-pq} + a_{-q} a_{-p}^\dagger \delta_{pq} = \\ &= \delta_{pq} (a_p^\dagger a_p - a_p a_p^\dagger) + \delta_{-pq} (a_p^\dagger a_p - a_p a_p^\dagger) \\ &\quad + \delta_{-pq} (a_{-p} a_{-p}^\dagger - a_{-p}^\dagger a_{-p}) + \delta_{pq} (a_{-p} a_{-p}^\dagger - \\ &\quad - a_{-p}^\dagger a_{-p}) = -\delta_{pq} - \delta_{-pq} + \delta_{-pq} + \delta_{pq} = \\ &= 0 \end{aligned}$$

Отсюда

$$U = \prod_p \exp\left(\frac{\varphi_p}{2}(a_p a_{-p} - a_p^\dagger a_{-p}^\dagger)\right)$$

$\exp\left(\frac{\varphi_q}{2}(a_q a_{-q} - a_q^\dagger a_{-q}^\dagger)\right)$ коммутирует с a_p , если $q \neq \pm p$, и сокращается с $\exp\left(-\frac{\varphi_q}{2}(a_q a_{-q} - a_q^\dagger a_{-q}^\dagger)\right)$.

Поэтому

$$U a_p U^\dagger = \exp(\varphi_p(a_p a_{-p} - a_p^\dagger a_{-p}^\dagger)) \circ a_p \circ \exp(-\varphi_p(a_p a_{-p} - a_p^\dagger a_{-p}^\dagger))$$

Воспользуемся формулой:

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!} [A, [A, B]] + \frac{1}{3!} [A, [A, [A, B]]] + \dots$$

В нашем случае:

$$A = \varphi_p(a_p a_{-p} - a_p^\dagger a_{-p}^\dagger)$$

$$B = a_p$$

$$[A, B] = \varphi_p [a_p a_{-p} - a_p^\dagger a_{-p}^\dagger, a_p] =$$

$$= -\varphi_p [a_p^\dagger a_{-p}^\dagger, a_p] = \varphi_p a_{-p}^\dagger$$

$$[A, [A, B]] = \varphi_p^2 [a_p a_{-p} - a_p^\dagger a_{-p}^\dagger, a_{-p}^\dagger] =$$

$$= \varphi_p^2 [a_p a_{-p}, a_{-p}^\dagger] = \varphi_p^2 a_p$$

и more gauge.

Получим

$$U a_p U^\dagger = a_p + \varphi_p a_{-p}^\dagger + \frac{\varphi_p^2}{2!} a_p +$$

$$+ \frac{\varphi_p^3}{3!} a_{-p}^\dagger + \dots = \operatorname{ch} \varphi_p a_p + \operatorname{sh} \varphi_p a_{-p}^\dagger$$

Получим образы, мы доказали.