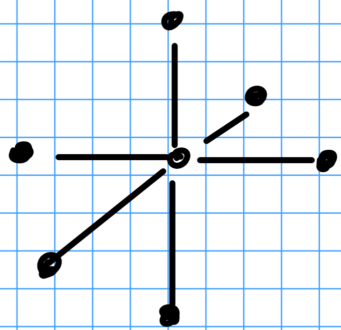
Задание 8.



$$A^{xx}(\pm a, 0, 0) = -\gamma_1$$
$$A^{xx}(0, \pm a, 0) = -\gamma_2$$
$$A^{xx}(0, 0, \pm a) = -\gamma_2$$

$$A^{yy}(\pm a, 0, 0) = -\gamma_2$$
$$A^{yy}(0, \pm a, 0) = -\gamma_2$$
$$A^{yy}(0, 0, \pm a) = -\gamma_1$$

$$A^{zz}(\pm a, 0, 0) = -\gamma_2$$
$$A^{zz}(0, \pm a, 0) = -\gamma_2$$
$$A^{zz}(0, 0, \pm a) = -\gamma_1$$

$$\sum_{\vec{n}} A^{\alpha\beta}(\vec{n}) = 0$$

$$A^{xx}(0,0,0) + A^{xx}(a,0,0) + A^{xx}(-a,0,0)$$
$$+ A^{xx}(0,a,0) + A^{xx}(0,-a,0) + A^{xx}(0,0,a)$$
$$+ A^{xx}(0,0,-a) = 0$$

$$A^{xx}(0,0,0) = 4\gamma_2 + 2\gamma_1$$

Аналогично

$$A^{yy}(0,0,0) = 4\gamma_2 + 2\gamma_1$$

$$A^{zz}(0,0,0) = 4\gamma_2 + 2\gamma_1$$

$$C^{\alpha\beta} = \frac{1}{m}\sum_{\vec{n}} A^{\alpha\beta}(\vec{n})\, e^{-i\vec{n}\cdot\vec{f}}$$

$$C^{xx} = \frac{1}{m}\Big(A^{xx}(0,0,0) + A^{xx}(a,0,0)\,e^{-iaf_x}$$

$$+ A^{xx}(-a,0,0)\,e^{if_x a} + A^{xx}(0,a,0)\,e^{-iaf_y}$$

$$+ A^{xx}(0,-a,0)\,e^{iaf_y} +$$

$$+ A^{xx}(0,0,a)\,e^{-iaf_z} + A^{xx}(0,0,a)\,e^{iaf_z}\Big)$$

$$= \frac{1}{m}\Big(2\gamma_1 - 2\gamma_1\cos(f_x a) + 2\gamma_2 - 2\gamma_2\cos(f_y a)$$

$$+ 2\gamma_2 - 2\gamma_2\cos(f_z a)\Big) =$$

$$= \frac{4}{m}\left(\gamma_1 \sin^2\frac{f_x a}{2} + \gamma_2 \sin^2\frac{f_y a}{2} + \gamma_2 \sin^2\frac{f_z a}{2}\right)$$

C^{yy}, C^{zz} находим аналогично.

Так C диагональна, то

$$\omega_1^2 = C^{xx} = \frac{4}{m}\Big(\gamma_1 \sin^2\frac{f_x a}{2} + $$
$$+ \gamma_2 \sin^2\frac{f_y a}{2} + \gamma_2 \sin^2\frac{f_z a}{2}\Big)$$

$$\omega_2^2 = C^{yy} = \frac{4}{m}\Big(\gamma_2 \sin^2\frac{f_x a}{2} + $$
$$+ \gamma_1 \sin^2\frac{f_y a}{2} + \gamma_2 \sin^2\frac{f_z a}{2}\Big)$$

$$\omega_3^2 = C^{zz} = \frac{4}{m}\Big(\gamma_2 \sin^2\frac{f_x a}{2} + $$
$$+ \gamma_2 \sin^2\frac{f_y a}{2} + \gamma_1 \sin^2\frac{f_z a}{2}\Big)$$

Сразу перейдем к безразмерным единицам. Длину измерим в a, т.е. f в $\frac{1}{a}$. Массу в m. Время в $\sqrt{\frac{m}{4\gamma_1}}$, то есть ω в $\sqrt{\frac{4\gamma_1}{m}}$. Получаем

$$\omega_1^2 = \sin^2\frac{f_x}{2} + \frac{\gamma_2}{\gamma_1}\sin^2\frac{f_y}{2} + \frac{\gamma_2}{\gamma_1}\sin^2\frac{f_z}{2}$$

$$\omega_2^2 = \frac{\gamma_2}{\gamma_1}\sin^2\frac{f_x}{2} + \sin^2\frac{f_y}{2} + \frac{\gamma_2}{\gamma_1}\sin^2\frac{f_z}{2}$$

$$\omega_3^2 = \frac{\gamma_2}{\gamma_1}\sin^2\frac{f_x}{2} + \frac{\gamma_2}{\gamma_1}\sin^2\frac{f_y}{2} + \sin^2\frac{f_z}{2}$$

$$g(\omega) = \frac{1}{3N}\sum_{\xi} \delta(\omega - \omega_\xi) =$$

$$= \frac{1}{3N}\sum_{\xi} \frac{\delta(\omega^2 - \omega_\xi^2)}{2\omega} =$$

$$= 2\omega \cdot \frac{1}{3N}\sum_{\xi} \delta(\omega^2 - \omega_\xi^2) = 2\omega G(\omega^2)$$

$$G(\omega^2) = \frac{1}{3N}\sum_{\xi} \delta(\omega^2 - \omega_\xi^2)$$

$$\delta(\omega^2 - \omega_1^2) = \delta\Big(\omega^2 - \frac{1}{2} + \frac{1}{2}\cos f_x -$$

$$- \frac{1}{2}\frac{\gamma_2}{\gamma_1} + \frac{1}{2}\frac{\gamma_2}{\gamma_1}\cos f_y - \frac{1}{2}\frac{\gamma_2}{\gamma_1} + \frac{1}{2}\frac{\gamma_2}{\gamma_1}\cos f_z$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dp\, \exp\left(ip\left(\omega^2 - \frac{1}{2} - \frac{\gamma_2}{\gamma_1}\right)\right)\cdot$$

$$\cdot \exp\left(i\frac{p}{2}\cos f_x\right)\exp\left(i\frac{p}{2}\frac{\gamma_2}{\gamma_1}\cos f_y\right)\cdot$$

$$\cdot \exp\left(i\frac{p}{2}\frac{\gamma_2}{\gamma_1}\cos f_z\right)$$

Рассмотрим часть $G(\omega^2)$, отвечающую за частоты ω_1^2. Затем умножим на 3, ведь все ветви вносят одинаковый вклад.

$$\frac{1}{2\pi}\;\frac{1}{N}\cdot\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} dp\,\exp\left(ip\left(\omega^2-\frac{1}{2}-\frac{\gamma_2}{\gamma_1}\right)\right)\cdot$$

$$\cdot\int_{-\pi}^{\pi} df_x\,\exp\left(i\frac{p}{2}\cos f_x\right)\int_{-\pi}^{\pi} df_y\,\exp\left(i\frac{p}{2}\frac{\gamma_2}{\gamma_1}\cos f_y\right)$$

$$\cdot\int_{-\pi}^{\pi} df_z\,\exp\left(i\frac{p}{2}\frac{\gamma_2}{\gamma_1}\cos f_z\right)=$$

$$=\frac{1}{2\pi}\;\frac{1}{N}\int_{-\infty}^{\infty} dp\,\exp\left(ip\left(\omega^2-\frac{1}{2}-\frac{\gamma_2}{\gamma_1}\right)\right)\cdot$$

$$\cdot\, J_0\left(\frac{p}{2}\right)J_0^2\left(\frac{p}{2}\frac{\gamma_2}{\gamma_1}\right)=$$

$$=\frac{1}{\pi N}\int_0^{\infty} dp\,\cos\left(p\left(\omega^2-\frac{1}{2}-\frac{\gamma_2}{\gamma_1}\right)\right)J_0\left(\frac{p}{2}\right)J_0^2\left(\frac{p}{2}\frac{\gamma_2}{\gamma_1}\right)$$

$$g(\omega)=\frac{2\omega}{\pi N}\int_0^{\infty} dp\,\cos\left(p\left(\omega^2-\frac{1}{2}-\frac{\gamma_2}{\gamma_1}\right)\right)J_0\left(\frac{p}{2}\right)J_0^2\left(\frac{p}{2}\frac{\gamma_2}{\gamma_1}\right)$$

Численно строим график:

$$g(\omega)=\frac{2\omega}{\pi}\int_0^{\infty} dp\,\cos\left(p\left(\omega^2-\frac{1}{2}-\frac{\gamma_2}{\gamma_1}\right)\right)J_0\left(\frac{p}{2}\right)\cdot$$

$$\cdot\, J_0^2\left(\frac{p}{2}\frac{\gamma_2}{\gamma_1}\right)$$

Для построения используем python, пакеты matplotlib, numpy, scipy.

Графики можно найти в папке graphs8. Названия файлов имеют вид graph8_xxx.png, где на месте xxx стоит отношение $\frac{\gamma_1}{\gamma_2}$.

Дебаевское приближение.

$$\omega_1^2 = \frac{1}{4}\left(f_x^2 + \frac{\gamma_2}{\gamma_1} f_y^2 + \frac{\gamma_2}{\gamma_1} f_z^2 \right)$$

$$\omega_2^2 = \frac{1}{4}\left(\frac{\gamma_2}{\gamma_1} f_x^2 + f_y^2 + \frac{\gamma_2}{\gamma_1} f_z^2 \right)$$

$$\omega_3^2 = \frac{1}{4}\left(\frac{\gamma_2}{\gamma_1} f_x^2 + \frac{\gamma_2}{\gamma_1} f_y^2 + f_z^2 \right)$$

Очевидно, все ветви дают одинаковый вклад. Поэтому возьмём ω_3^2.

$$g(\omega) = \frac{1}{3N} \sum_{\xi} \delta(\omega - \omega_\xi) =$$

$$= \frac{1}{N} \int \frac{d^3 f}{(2\pi)^3} \delta(\omega - c(\vec{n}) f) =$$

$$= \frac{1}{(2\pi)^3 N} \int d\Omega_{\vec{n}} \int f^2 \, df \, \delta(\omega - C(\vec{n}) f)$$

$$= \frac{1}{(2\pi)^3 N} \int \frac{d\Omega_{\vec{n}}}{(C(\vec{n}))^3} \int f^2 \, df \, \delta(\omega - f) =$$

$$= \frac{\omega^2}{(2\pi)^3 N} \int d\Omega \, \frac{1}{C^3}$$

$$\omega_3^2 = \frac{1}{4} \left(\frac{\gamma_2}{\gamma_1} f_x^2 + \frac{\gamma_2}{\gamma_1} f_y^2 + f_z^2 \right) =$$

$$= \frac{1}{4} \left(\frac{\gamma_2}{\gamma_1} f^2 \sin^2\theta + f^2 \cos^2\theta \right) =$$

$$= \frac{1}{4} \left(\frac{\gamma_2}{\gamma_1} f^2 + \left(1 - \frac{\gamma_2}{\gamma_1}\right) f^2 \cos^2\theta \right)$$

$$C(\vec{n}) = \frac{1}{2} \sqrt{\frac{\gamma_2}{\gamma_1} + \left(1 - \frac{\gamma_2}{\gamma_1}\right) \cos^2\theta}$$

$$\int d\Omega \, \frac{1}{C^3(\vec{n})} = 2\pi \cdot 8 \int \frac{\sin\theta \, d\theta}{\left(\frac{\gamma_2}{\gamma_1} + \left(1 - \frac{\gamma_2}{\gamma_1}\right) \cos^2\theta \right)^{3/2}}$$

$$= 2\pi \cdot 8 \int_{-1}^{1} \frac{dx}{\left(\frac{\gamma_2}{\gamma_1} + \left(1 - \frac{\gamma_2}{\gamma_1}\right) x \right)^{3/2}} =$$

$$= 2\pi \cdot 8 \left(\frac{\gamma_1}{\gamma_2}\right)^{3/2} \int_{-1}^{1} \frac{dx}{\left(1 + \left(\frac{\gamma_1}{\gamma_2} - 1\right) x^2\right)^{3/2}}$$

$$= 2\pi \cdot 8 \left(\frac{\gamma_1}{\gamma_2}\right)^{3/2} \frac{1}{\sqrt{\frac{\gamma_1}{\gamma_2} - 1}} \int_{-\sqrt{\frac{\gamma_1}{\gamma_2}-1}}^{\sqrt{\frac{\gamma_1}{\gamma_2}-1}} \frac{dx}{\left(1 + x^2\right)^{3/2}}$$

$$x = \operatorname{sh}\xi \,, \quad 1 + x^2 = \operatorname{ch}^2\xi$$

Получаем:

$$\int \frac{dx}{(1+x^2)^{3/2}} = \int \frac{\operatorname{ch}\xi \, d\xi}{\operatorname{ch}^2\xi} = \int \frac{d\xi}{\operatorname{ch}^2\xi} =$$

$$= \operatorname{th}\xi = \frac{\operatorname{sh}\xi}{\sqrt{1 + \operatorname{sh}^2\xi}} = \frac{x}{\sqrt{x^2+1}}$$

Получаем:

$$\int_{-\sqrt{\frac{\gamma_1}{\gamma_2}-1}}^{\sqrt{\frac{\gamma_1}{\gamma_2}-1}} \frac{dx}{(1+x^2)^{3/2}} = 2 \cdot \frac{\sqrt{\frac{\gamma_1}{\gamma_2} - 1}}{\sqrt{\frac{\gamma_1}{\gamma_2} - 1 + 1}}$$

$$= 2\sqrt{1-\frac{\gamma_2}{\gamma_1}}$$

$$\int \frac{d\Omega_{\vec{n}}}{C^3(\vec{n})} = 2\pi \cdot 8 \cdot \left(\frac{\gamma_1}{\gamma_2}\right)^{3/2} \frac{1}{\sqrt{\frac{\gamma_1}{\gamma_2}-1}} \cdot 2\sqrt{1-\frac{\gamma_2}{\gamma_1}}$$

$$= 4\pi \cdot 8\, \frac{\gamma_1}{\gamma_2}$$

$$(\bar{C})^{-3} = \frac{1}{4\pi}\int \frac{d\Omega_{\vec{n}}}{C^3(\vec{n})} = 8\, \frac{\gamma_1}{\gamma_2}$$

$$\bar{C} = \frac{1}{2}\left(\frac{\gamma_2}{\gamma_1}\right)^{1/3}$$

$$\left(\frac{2\pi}{a}\right)^3 = \frac{4\pi}{3} \cdot f_D^3$$

$$f_D = 2\pi\left(\frac{3}{4\pi}\right)^{1/3} - \text{в}$$

безразмерных единицах

$$\omega_D = f_D \cdot \bar{C} = \pi\left(\frac{3}{4\pi}\right)^{1/3} \cdot \left(\frac{\gamma_2}{\gamma_1}\right)^{1/3}$$

При $\omega > \omega_D$ $g(\omega) = 0$

Таким образом можно вычислить дебаевскую функцию плотности.

$$g(\omega) = \frac{\omega^2}{(2\pi)^3 N} 32\pi \frac{\gamma_1}{\gamma_2} \quad \text{при}$$

$$\omega < \pi \left(\frac{3}{4\pi}\right)^{\frac{1}{3}} \left(\frac{\gamma_2}{\gamma_1}\right)^{\frac{1}{3}}$$

$$g(\omega) = 0 \quad \text{при} \quad \omega > \pi \left(\frac{3}{4\pi}\right)^{\frac{1}{3}} \left(\frac{\gamma_2}{\gamma_1}\right)^{\frac{1}{3}}$$

Графики можно найти там же.

Последний (5-ый) пункт задания.

$$\omega_1 = \sqrt{\sin^2\frac{f_x}{2} + \frac{\gamma_2}{\gamma_1}\sin^2\frac{f_y}{2} + \frac{\gamma_2}{\gamma_1}\sin^2\frac{f_z}{2}}$$

$$\nabla_{\vec{f}}\,\omega_1 = \frac{1}{4\omega_1}\left(\sin f_x,\ \frac{\gamma_2}{\gamma_1}\sin f_y,\ \frac{\gamma_2}{\gamma_1}\sin f_z\right)$$

$$\nabla_{\vec{f}}\,\omega_1 = 0 \quad \text{при} \quad f_\alpha = 0, \pm\pi$$

Рассмотрим $\vec{f} = \left(0, \frac{\pi}{a}, \frac{\pi}{a}\right)$

Пусть $\vec{f} \approx \left(0, \frac{\pi}{a}, \frac{\pi}{a}\right)$

$$f_x = \delta_x \,, \quad f_z = \frac{\pi}{a} + \delta_z$$

$$f_y = \frac{\pi}{a} + \delta_y$$

$$\sin^2 \frac{f_x}{2} \approx \frac{1}{4}\delta_x^2$$

$$\frac{\gamma_2}{\gamma_1} \sin^2 \frac{f_y}{2} \approx \frac{\gamma_2}{\gamma_1} - \frac{1}{2}\frac{\gamma_2}{\gamma_1}\delta_y^2$$

$$\frac{\gamma_2}{\gamma_1} \sin^2 \frac{f_z}{2} \approx \frac{\gamma_2}{\gamma_1} - \frac{1}{2}\frac{\gamma_2}{\gamma_1}\delta_z^2$$

$$\omega_1 \approx \sqrt{\frac{1}{4}\delta_x^2 + \frac{\gamma_2}{\gamma_1} - \frac{1}{2}\frac{\gamma_2}{\gamma_1}\delta_y^2 + \frac{\gamma_2}{\gamma_1} - \frac{1}{2}\frac{\gamma_2}{\gamma_1}\delta_z^2}$$

$$= \sqrt{2\frac{\gamma_2}{\gamma_1} + \frac{1}{4}\delta_x^2 - \frac{1}{2}\frac{\gamma_2}{\gamma_1}\delta_y^2 - \frac{1}{2}\frac{\gamma_2}{\gamma_1}\delta_z^2}$$

Переопределив $\delta_x, \delta_y, \delta_z$ получим

$$\omega_1 \sim \sqrt{2\frac{\gamma_2}{\gamma_1}} + \delta_x^2 - \delta_y^2 - \delta_z^2$$

$$g(\omega) \sim \int d^3\varepsilon\, \delta\left(\omega - \sqrt{2\frac{\gamma_2}{\gamma_1}} - \delta_x^2 + \right.$$

$$\left. + \delta_y^2 + \delta_z^2\right)$$

$$g(\omega) \sim \int \delta_\perp d\delta_\perp \int d\delta_x \, \delta\left(\omega - \sqrt{2\frac{\gamma_2}{\gamma_1}} - \delta_x^2 + \delta_\perp^2\right)$$

$$\sim \int d\delta_\perp^2 \int \frac{d\delta_x^2}{\sqrt{\delta_x^2}} \, \delta\left(\left|\omega - \sqrt{2\frac{\gamma_2}{\gamma_1}}\right| - \delta_x^2 + \delta_\perp^2\right)$$

$$= \int d\delta_\perp^2 \frac{1}{\sqrt{\left|\omega - \sqrt{2\frac{\gamma_2}{\gamma_1}}\right| + \delta_\perp^2}}$$

$$\sim \sqrt{\delta_\perp^2 + \left|\omega - \sqrt{2\frac{\gamma_2}{\gamma_1}}\right|}\;\Bigg|_0^{\delta_0} \sim$$

$$\sim \sqrt{\left|\omega - \sqrt{2\frac{\gamma_2}{\gamma_1}}\right|}$$

Итак, получаем корневую зависимости с изломом в точке $\sqrt{2\frac{\gamma_2}{\gamma_1}}$