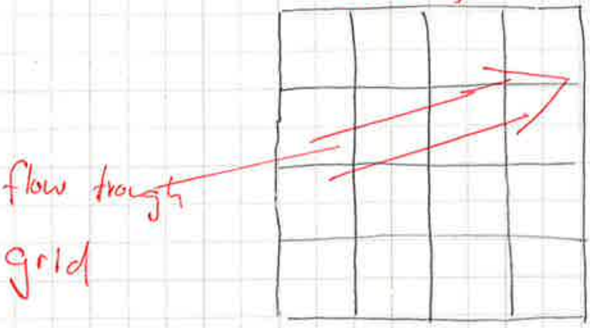
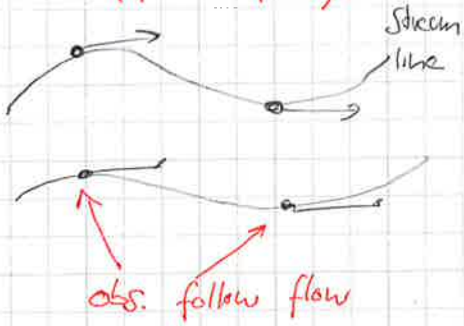
Week 3

Introduction to Smoothed Particle Hydrodynamics (SPH)

1) Coordinate system: Eulerian (fixed) or Lagrangian (follow flow)





Derivative:

$$\frac{\partial}{\partial t} \xrightarrow[\text{derivative}]{\text{Lagrangian}} \frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v}\nabla$$

(total derivative)

flow velocity

temporal deriv. (change with time)

convective deriv (flow change locally)

$$\frac{d\varphi(\vec{x}(t),t)}{dt} = \frac{\partial\varphi}{\partial t} + \frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial\varphi}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial\varphi}{\partial z}\cdot\frac{\partial z}{\partial t} = \frac{\partial\varphi}{\partial t} + \vec{v}\nabla\varphi$$

2) Grid or particle based method

Equation of motion:

Euler equations (inviscid fluid)

$$\frac{D\vec{v}}{Dt} = -\frac{\nabla P}{\rho} \quad \text{(conserv. momentum)}$$

$$\frac{D\rho}{Dt} = -\rho\nabla\vec{v} \quad \text{(" mass)}$$

$$\frac{Du}{Dt} = -\frac{P}{\rho}\nabla\vec{v} \quad \text{(" energy)}$$

6 variables but 5 equations!

↳ discuss later!

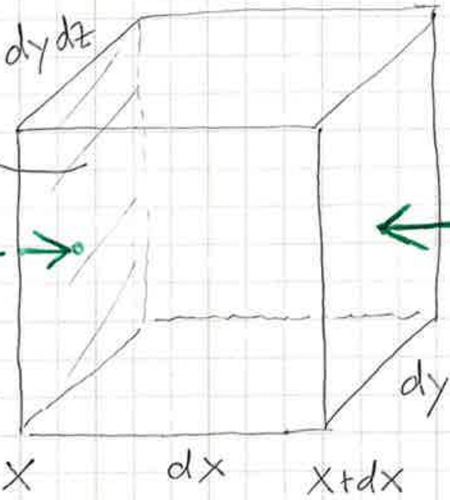
Week 3

1) Momentum equation

$\frac{\partial P}{\partial x}$ press. gradient $\rightarrow$

pressure $\rightarrow P = \frac{F}{A}$ ← force, ← area

$dA_{yz} = dy\,dz$



$F_x = P \cdot dA_{yz} \rightarrow$

$\leftarrow F_{x+dx} = \Big(P + \overbrace{\frac{\partial P}{\partial x} dx}^{dP}\Big) dA_{yz}$

1D: $F_{x,tot} = F_x - F_{x+dx} = P dA_{yz} - \left(P + \frac{\partial P}{\partial x} dx\right) dA_{yz}$

$= P\,dy\,dz - P\,dy\,dz - \frac{\partial P}{\partial x} dx\,dy\,dz = -\frac{\partial P}{\partial x} dV$

3D: $\vec{F} = -\nabla P\, dV \qquad \left(\vec{\nabla} P = \left(\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial P}{\partial z}\right)\right)$

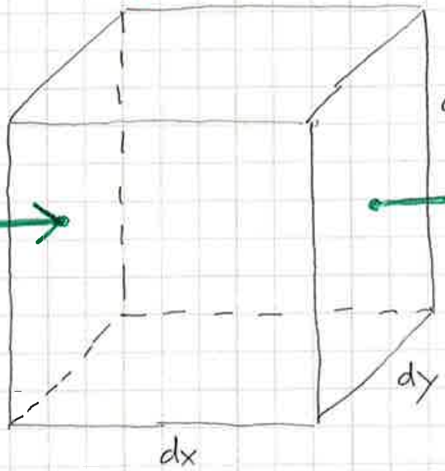
Newton: $\vec{F} = m \cdot \vec{a} = m \frac{D\vec{v}}{Dt} = -\nabla P dx \Rightarrow \boxed{\frac{D\vec{v}}{Dt} = -\frac{\nabla P}{\rho}}$

m ← ρdV

Euler equation

2) Continuity equation

$\frac{\partial}{\partial t}(\rho v)$ mass flux gradient $\rightarrow$



$\dot{m}_{in} = \rho \cdot v\, dA_{yz} \rightarrow$

flux in

$\rightarrow \dot{m}_{out} = \left(\rho v + \frac{\partial(\rho v)}{\partial x} dx\right) dA_{yz}$

flux out

$\dot{m}_{tot} = \rho v dA_{yz} - \left(\rho v + \frac{\partial}{\partial x}(\rho v) \underbrace{dx) dA_{yz}}_{dV} = -\frac{\partial(\rho v)}{\partial x} dV$

$$\dot{m}_{tot} = \frac{\partial \rho}{\partial t} dV \overset{!}{=} - \frac{\partial(\rho v)}{\partial x} dV \Rightarrow \boxed{\frac{\partial \rho}{\partial t} = - \frac{\partial}{\partial x}(\rho v)}$$ Eulerian form

<u>Chain rule:</u> $\frac{\partial}{\partial x}(\rho v) = \rho \frac{\partial v}{\partial x} + v \frac{\partial \rho}{\partial x}$

$$\underbrace{\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x}}_{\frac{D\rho}{Dt}} = \underbrace{- \rho \frac{\partial v}{\partial t}}_{-\rho \nabla \vec{v} \text{ in 3D}}$$

$$\boxed{\frac{D\rho}{Dt} = -\rho \vec{\nabla} \vec{v}}$$ Continuity equation in Lagrangian form

3) Energy equation

1. law of thermodynamics: $du = P dV + \underbrace{T ds}_{=0,\text{ inviscous}}$ (du: spec. int. energy; ds: entropy) $\quad du = -\frac{P}{\rho^2} d\rho$

$$\frac{Du}{Dt} = - \frac{P}{\rho^2} \frac{D\rho}{Dt} = - \frac{P}{\rho^2} \cdot \rho \nabla \vec{u} = - \frac{P}{\rho} \nabla \vec{v}$$

continuity eqn.

$$\boxed{\frac{Du}{Dt} = -\frac{P}{\rho} \nabla \vec{v}}$$ Lagrangian energy equation

$$\underset{\text{total energy}}{E} = \left(\underbrace{\frac{1}{2} \vec{v}^2}_{\text{kinetic}} + \underset{\text{internal}}{u} \right) \rho$$

<u>Problem:</u> We have 5 equations but 6 variables $(\rho, \vec{v}, P, u)$

↪ need one more equation to solve the system!

<u>EOS:</u> e.g. ideal gas (very simple!): $\boxed{P(\rho, u) = \rho u (\gamma - 1)}$ $\quad \gamma = \frac{f+2}{f}$ (d.o.f → f)

mono atomic : $f = 3$ ← translation $\quad \gamma = \frac{5}{3}$ e.g: He, noble gases

$$\boxed{u = \frac{k_B}{\underbrace{(\gamma-1)\mu m_H}_{c_V}} T}$$

μ: mean molecular weight

m_H : mass of hydrogen

e.g: ionized H $\left.\begin{matrix} e^- \\ p^+ \end{matrix}\right\} \mu = 0.5$

SPH : particle based, lagrangian method
↳ use particles to follow the flow and consider various integrals in conservation laws and use interpolants in these!

Two approximations to derive EOM:

1) Integral Interpolants

$$A_I(r) = \int_{V'} A(\vec{r}')\, W(\vec{r}-\vec{r}'; h)\, d\vec{r}'$$

Kernel

1) $\int W d\vec{r}' = 1$ "normalized" — Const. function exact

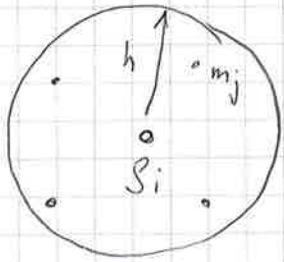
2) $\lim_{h \to 0} W(\vec{r}-\vec{r}'; h) = \delta(\vec{r}-\vec{r}')$ "convergence"

approximate integral by sum over particles

2) Particle approximation

$$A_i = \sum_j^{NN} \frac{m_j}{\underbrace{\rho_j}_{= \Delta V_j}} A_j\, W(\vec{r}_i - \vec{r}_j; h)$$

neighbors → j



Density : $\boxed{\rho_i = \sum_j m_j W(\vec{r}_i - \vec{r}_j; h)}$

Derivative: $\nabla A_i = \sum_j^{NN} \frac{m_j}{\rho_j} A_j \underline{\underline{\nabla W}}(\vec{r}_i - \vec{r}_j; h)$ — apply to kernel

analytic function, no grid needed!

Kernels: Gaussian $\quad w(r;h) = \frac{1}{h\sqrt{\pi}} e^{-\left(\frac{r}{h}\right)^2}$

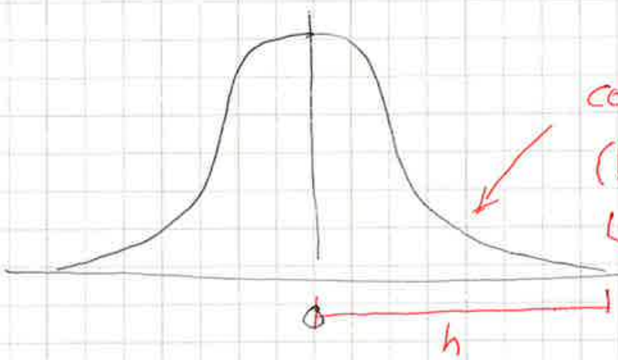
Problem: summation over all particles no matter how far away they are $\mathcal{O}(N^2)$

Monaghan kernel: "cubic-spline" kernel

$$w(r;h) = \frac{\sigma}{h^d} \begin{cases} 6(r/h)^3 - 6(r/h)^2 + 1, & 0 \le r/h < 1/2 \\ 2(1-(r/h))^3 & 1/2 \le r/h \le 1 \\ 0 & \text{else} \end{cases}$$

dimension

Normalising constant $\sigma = \begin{cases} 4/3 & \text{in 1D} \quad (d=1) \\ 40/7\pi & \text{in 2D} \\ 8/\pi & \text{in 3D} \end{cases}$

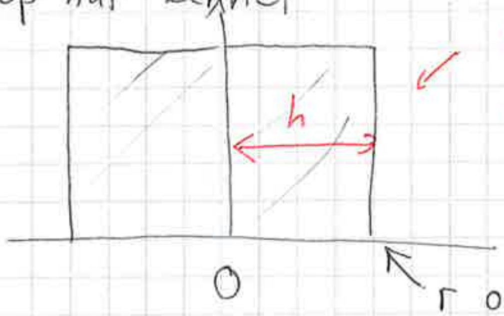


compact support
($h = r$ of 32^{nd} neighbor)
↳ adaptive resolution!!

$$\frac{\partial w(r;h)}{\partial r} = \frac{6\sigma}{h^{d+1}} \begin{cases} 3(r/h)^2 - 2(r/h) & , \quad 0 \le r/h < 1/2 \\ -(1-(r/h))^2 & , \quad 1/2 \le r/h \le 1 \end{cases}$$

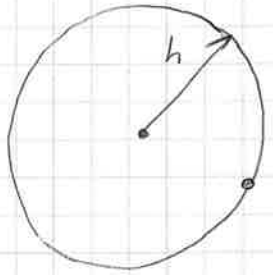
Homework: Calculate ρ for Monaghan kernel and compare to "top-hat" kernel result

top-hat kernel



equal weight for all particles

r of 32^{nd} particle (compact support!)



h = r of 32^{nd} nearest neighbor

$(r/h) = 1$ and $w = 0$!

Tipp
- implement field for particle mass
- k-Nearest Neighbor finding was last week's homework

Q: Does top-hat kernel fulfill all requirements?