Topology

University of Houston, Fall 2023

MATH 6342 (Dr. Ott)

---

1 General Toplogy

1.1 Definitions

topology

Let X be a set. A **topology** on X is a collection $\mathcal{T}$ of subsets of X such that:

1. $\emptyset$ and X are in $\mathcal{T}$.
2. The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.
3. The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

topological space

A **topological space** is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a topology on X.

open set, closed set

Let $(X, \mathcal{T})$ be a topological space and $U \subset X$. The subset U is **open** if $V \in \mathcal{T}$. The subset U is **closed** if $X \setminus U \in \mathcal{T}$.

basis

Let X be a set. A **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X such that:

1. For every $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.
2. If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cup B_2$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cup B_2$.

topology generated by basis

Let X be a set and $\mathcal{B}$ be a basis. The **topology generated by** $\mathcal{B}$ is defined by:
A subset $U \subset X$ is open in X if for every $x \in U$, there exists a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$.

subspace topology

Let X be a topological space and let $A \subset X$. The **subspace topology** $\mathcal{T}_A$ on A is

$$\mathcal{T}_A = \{V \cap A : V \text{ is open in } X\}$$

interior of a set

Let X be a topological space and $A \subset X$. The **interior** of A, denoted $\mathrm{Int}\,(A)$ is

$$\mathrm{Int}\,(A) = \bigcup_{V \subset A} V, V \text{ is open in } X$$

closure of a set

Let X be a topological space and $A \subset X$. The **closure** of A in X, denoted $\overline{A}$ or $\mathrm{cl_X}\,(A)$ is

$$\overline{A} = \bigcap_{A \subset V} V, V \text{ is closed in } X$$

neighborhood of a point

Let $x \in X$. A **neighborhood** of x is an open set U such that $x \in U$.

accumulation point

Let X be a topological space and $A \subset X$. $x \in X$ is an **accumulation point** of A if every neighborhood of x intersects A in a point other than x. (Every neighborhood contains a point in A other than x).

separation axioms

Let X be a topological space. X is

- T_0: for every pair of distinct points $p, q \in X$, there exists an open set U such that $p \in U$ and $q \notin U$ or there exists an open set W such that $q \in W$ and $p \notin W$.
- T_1: for every pair of distinct points $p, q \in X$, there exists an open set U such that $p \in U$ and $q \notin U$ and there exists an open set W such that $q \in W$ and $p \notin W$.
- T_2 (Hausdorff): for every pair of distinct points $p, q \in X$, there exists an open sets U, V such that $p \in U$, $q \in V$, and $U \cap V = \emptyset$.

convergence of a sequence

Let X be a topological space. A sequence $(x_n)_{n=1}^{\infty}$ **converges** to a point $p \in X$ if for every neighborhood V of p, there exists $N \in \mathbb{N}$ such that $x_n \in V$ whenever $n \geq N$.

continuous function

Let X and Y be topological spaces. A function $f : X \to Y$ is **continuous** if for every open set $V \subset Y$, the pre-image $f^{-1}(V) = \{x \in X : f(x) \in V\}$ is open in X.

sequentially continuous

Let X and Y be topological spaces. A function $f : X \to Y$ is **sequentially continuous** if for every convergent sequence $(x_n)_{n=1}^{\infty}$ in X, the image sequence $(f\,(x_n))_{n=1}^{\infty}$ converges in Y. In other words:

$$f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n)$$

homeomorphism

Let X and Y be topological spaces. The map $f : X \to Y$ is a **homeomorphism** if f is continuous, invertible, and f^{-1} is continuous.

embedding

Let X and Y be topological spaces and $f : X \to Y$ be continuous and one-to-one. If $f(X)$ with the subspace topology is homeomorphic to X, then f is an **embedding**.

product topology

Let X and Y be topological spaces. The **product topology** on $X \times Y$ is the topology generated by the basis

$$\{U \times V : U \subset X \text{ is open in } X, V \subset Y \text{ is open in } Y\}$$

Let $\{X_\alpha : \alpha \in J\}$ be an indexed family of topological spaces. The **product topology** on $\prod_{\alpha \in J} X_\alpha$ is the topology generated by the basis $\{\prod_{\alpha \in J} V_\alpha\}$ where $V_\alpha \subset X_\alpha$ is open for all $\alpha \in J$ and $V_\alpha = X_\alpha$ for all but finitely many values of α. This is equivalent to the product topology on finitely many spaces.

box topology

Let $\{X_\alpha : \alpha \in J\}$ be an indexed family of topological spaces. The **product topology** on $\prod_{\alpha \in J} X_\alpha$ is the topology generated by the basis $\{\prod_{\alpha \in J} V_\alpha\}$ where $V_\alpha \subset X_\alpha$ is open for all $\alpha \in J$. This is equivalent to the product topology on finitely many spaces.

basis and subbasis

open set in a metric space

Let (M, d) be a metric space. A subset $U \subset M$ is **open** if for every $x \in U$ there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset V$.

metric space

metric topology

metrizable

connected

components

Let X be a topological space. The relation on X $p \sim p$ if p and q are in the same conneted subset of X is an equivalence relation on X. The equivalence classes are the **connected components** of X. Connected components are closed.

path connectedness

Let X be a topological space. A **path** is a continuous function $\gamma : [a, b] \to X$. X is **path connected** if for every $p, q \in X$, there exists a path $\gamma : [a, b] \to X$ such that $\gamma(a) = p$ and $\gamma(b) = q$. The relation on X by p related to q if there exists a path in X from p to q is an equivalence relation on X. The equivalence classes are called **path components**.

cover

Let X be a topological space. A **cover** of X is a collection of sets $\hat{E}$ such that $E \subseteq X$ for every$E \in \hat{E}$ and $X = \bigcup_{E \in \hat{E}} E$. $\hat{E}$ is an **open cover** if for all $E \in \hat{E}$, E is open in X.

Let $Y \subset X$. A cover of Y is a collection $\hat{E}$ of subsets of X such that $Y \subset \bigcup_{E \subset \hat{E}} E$. $\hat{E}$ is an **open cover** if for every $E \in \hat{E}$, E is open in X.

compactness

Let X be a topological space. X is compact if for every open cover $\hat{A}$ of X, $\hat{A}$ contains finitely many sets that also cover X.

Every open cover of X has a finite subcover.

sequentially compact

X is sequentially compact if every sequence in X has a convergent subsequence that converges to a limit in X.

complete metric space

X is complete if every Cauchy sequence in X converges to a bound in X.

totally bounded

X is totally bounded if for every $\varepsilon > 0$, X can be covered by finitely many ε-balls.

Cauchy sequence

A sequence $(x_n)_{n=1}^{\infty}$ in X is **Cauchy** if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n, m \geq N$, then $d(x_m, x_n) < \varepsilon$.

locally connected

A topological space X is locally connected at $X \in X$ if for every neighborhood U of x, there is a conected neighborhood V of x such that $V \subset U$.

If X is locally connnected at $x \in X$ for every $x \in X$, then X is **locally connected**.

locally compact

A topological space X is **locally compact** if for every $x \in X$, there exists a compact subspace C of X such that C contains a neighborhood of x.

one-point compactification

Let X be a locally compact Hausdorff space. Let $X^+ = X \cup \{\infty\}$, where ∞ is any point not in X. Define an open set in X^+ to be either an open set in X, or a set $X^+ \setminus C$ such that $C \subset X$ is compact. This defines a topology on X^+ that makes X^+ a compact Hausdorff Space. X^+ is the **one-point compactification** of X. This topology on X^+ is the only topology on X^+ that makes X^+ a compact Hausdorff space with X as a subspace. X^+ extends X uniquely.

normal space

A topological space X is **normal** if for any two disjoin closed sets $E, F \subset X$, there exist disjoint open sets U, V such that $E \subset U$ and $F \subset V$. This is separation axiom T_4.

countable basis

Let X be a topological space. X has a **countable basis** at $x \in X$ if there exsits a countable collection $\hat{B}$ of neighborhoods of x such that each neighborhood of x contains at least one of the elements of $hatB$.

first-countable

X is **first-countable** if it has a countable basis at each $x \in X$.

second-countable

If X has a basis that is countable for its topology, then X is **second-countable**.

dense

A subset A of a topological space X is **dense in** X if $\overline{A} = X$.

directed set

A **directed set** D is a partially orded set such that for any $\alpha, \beta \in D$, there exists $\tau \in D$ such that $\tau \geq \alpha$ and $\tau \geq \beta$.

net

A **net** in X is a directed set D together with a function $\phi : D \to X$.

frequently

Let X be a topological space and let $\phi : D \to X$ be a net. Suppose $A \subset X$. The net ϕ is **frequently in** A if for each $\alpha \in D$, there exists $\beta \geq \alpha$ such that $\phi(\beta) \in A$.

eventually

ϕ is **eventually** in A if there exists $\alpha \in D$ such that for all $\beta \geq \alpha$, $\phi(\beta) \in A$.

net convergence

A net $\phi : D \to X$ in a topological space X converges to $x \in X$ if for every neighborhood $U \subset X$ of x, ϕ is eventually in U.

universal net

Let X be a topological space. A net in X is called **universal** if for every $A \subset X$, the net is either eventually in A or eventually in A^{c}.

final

Let D' and D be directed sets and let $h : D' \to D$ be a function. h is **final** if for every $\delta \in D$, there exists $\delta' \in D'$ such that if $\alpha' \geq \delta'$, then $h(\alpha') \geq \delta$.

subnet

Let X be a topological space. A **subnet** of a net $\mu : D' \to X$ is the composition $\mu \circ h$, $h : D' \to D$ is final.

1.2 Theorems

Theorem 1.1

Let X be a set and $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the collection of all unions of elements of $\mathcal{B}$.

Theorem 1.2

Let (M, d) be a metric space. If $\{V_alpha : \alpha \in I\}$ is any collection of open sets in M, then $\bigcup_{\alpha \in I} V_\alpha$ is open in M.

Theorem 1.3

Let (M, d) be a metric space. If $V_1, V_2, \ldots, V_k$ is a finite collection of open sets in M, then $\bigcap_{i=1}^k V_\alpha$ is open in M.

Proof. Let $p \in \bigcap_{i=1}^k V_\alpha$. For each $1 \leq i \leq k$ there exists $\varepsilon_i > 0$ such that $B(p, \varepsilon_i) \subset V_i$ since each V_i is open. Let $\delta = \min_{1 \leq i \leq k}\{\varepsilon_i\}$. Then $B(p, \delta) \subset V_i$ for all $1 \leq i \leq k$. Thus $B(p, \delta) \subset \bigcap_{i=1}^k V_\alpha$. $\square$

Theorem 1.4

Let X be a topological space.

1. $\emptyset$ and X are closed.
2. The intersection of any family of closed sets is closed.
3. A finite unions of closed sets is closed.

Theorem 1.5

Let X be a topological space and let $A \subset X$.

1. $\operatorname{Int}(A) \subset A \subset \overline{A}$

2. A is open if and only if $A = \text{Int}(A)$
3. A is closed if and only if $A = \overline{A}$

Theorem 1.6

Let X be a topological space and let $A \subset X$. Let $x \in X$. Then $x \in \overline{A}$ if and only if every open set U containing x intersects A.

Proof. By contraposition.

Supposed $x \notin \overline{A}$. Then there exists a closed set K such that $A \subset K$ and $x \notin K$. Let $N = X \setminus K$. Then $N \cap A = \emptyset$.

Suppose there exists a neighborhood N of x such that $N \cap A = \emptyset$. Then $X \setminus N$ is a closed set that contains A and $x \notin X \setminus N$, which is closed. Thus $x \notin \overline{A}$. □

Theorem 1.7

Let X be a topological space and $A \subset X$. Let A' be the set of all accumulation points of A. Then $\overline{A} = A \cup A'$.

Proof. Let $x \in \overline{A}$. If $x \in A$, then done. Suppose $x \notin A$. Then if N is a neighborhood of x then N intersects A at some point other than x. So $x \in A'$.

Let $x \in A$. Then $x \in \overline{A}$. Let $x \in A'$. Then every neighbourhood of x intersects A, so $x \in \overline{A}$. □

Theorem 1.8

Let X be a topological space and $A \subset X$. A is closed if and only if it contains all of its accumulation points.

Proof. A is closed $\Leftrightarrow A = \overline{A} \Leftrightarrow A = A \cup A' \Leftrightarrow A' \subset A$. □

Theorem 1.9

Let X be a topological space. X is T_1 if and only if singletons are closed.

Theorem 1.10

Let X be Hausdorff. Then every sequence $(x_n)_{n=1}^{\infty} \in X$ converges to at most one point in X.

Theorem 1.11

Let X and Y be topological spaces and $f : X \to Y$. The following are equivalent:

1. f is continuous.
2. For every $A \subset X$, $f\left(\overline{A}\right) \subset \overline{f(A)}$.
3. For every closed set $B \subset Y$, the pre-image $f^{-1}(B)$ is closed in X.
4. For every $x \in X$ and every neighborhood V of $f(x)$ in Y, there exists a neighborhood U of x such that $f(U) \subset V$.

boundedness it not a topological property, not preserved by homeomorphisms

Theorem 1.12

Theorem 1.13

If each space X_α is Hausdorff, then $\prod_{\alpha\in J} X_\alpha$ is Hausdorff in the box and product topologies.

closure is closure in product and box topology

Theorem 1.14

product topology continuity

Theorem 1.15

metric generates the same metric

Theorem 1.16

Theorem 1.17

Let X be a topological space. X is connected if and only if for every discrete space D and every continuous map $d : X \to D$, d is constant.

Theorem 1.18

Let X and Y be topological spaces. If X is connected and $f : X \to Y$ is continuous, then $f(X)$ is connected.

Theorem 1.19

If X is connected and X is homeomorphic to Y, then Y is connected.

Theorem 1.20

Suppose A and B are connected spaces and $A \cap B \neq \emptyset$. Then $A \cup B$ is connected.

Furthurmore, let $\{X_\gamma : \gamma \in \Gamma\}$ be a collection of topologcial spaces. If X_γ is connected for all $\gamma \in \Gamma$ and if $\bigcap_{\gamma\in\Gamma} X_\gamma \neq \emptyset$, then $\bigcup_{\gamma\in\Gamma} X_\gamma$ is connected.

Theorem 1.21

Let X be a topological space. Let $A \subset X$. If A is connected, then $\overline{A}$ is connected.

Theorem 1.22

If X is path-connected, then it is connected.

Theorem 1.23

All intervals in $\mathbb{R}$ are connected.

Theorem 1.24

$\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$.

Proof. $\mathbb{R}$ $\{p\}$ is not connected. $\mathbb{R}^2$ $\{p\}$ is path-connected, so it is also connected. $\square$

Theorem 1.25

Let X be a topological space. If X is compact and $Y \subset X$ is closed in Y, then Y is compact.

Theorem 1.26

If X is Hausdorff and $Y \subset X$ is compact, then Y is closed.

Proof. For every $p \in Y$, there exists open sets U_p, V_p of X such that $p \in U_p$ and $x_0 \in V_p$, with $U_p \cap V_p = \emptyset$ and $x_0 \in Y^c$. The collection $\{U_p : p \in Y\}$ is an open cover of Y. Since Y is compact, Y is covered by finitely many of the sets $Y \subset U_{p_1} \cup U_{p_2} \cdots \cup U_{p_N}$. Finally, the set $\bigcap_{k=1}^{N} V_{p_k}$ is an open set that contains x_0 and $\bigcap_{k=1}^{N} V_{p_k} \subset Y^c$.

let $q \in Y$. Then there exists some $j \in \mathbb{N}, 1 \leq j \leq N$ and $q \in U_{p_j}$. Then $q \notin V_{p_j}$, so $q \notin \bigcap_{k=1}^{N} V_{p_k}$.

So $x_0 \in \bigcap_{k=1}^{N} V_{p_k} \subset Y^c$, thus Y^c is open and Y is closed. $\square$

Theorem 1.27

Let X, Y be topological spaces and $f : X \to Y$ be a continuous function. If X is compact, then $f(X)$ is also compact.

Theorem 1.28

If X is compact and X is homeomorphic to Y, then Y is compact.

Theorem 1.29

If X, Y are compact topological spaces, then $X \times Y$ is compact.

Theorem 1.30 (Tube lemma)

Let $x_0 \in X$. let N be an open set in $X \times Y$ that contains the slice $\{x_0\} \times Y$. Then there exists a neighborhood W of x_0 such that $W \times Y \subset N$.

Theorem 1.31 (Heine-Borel compactness)

Let $E \subset \mathbb{R}$. Then E is compact if and only if E is closed and bounded.

Proof. Assume E is compact. Since $\mathbb{R}$ is Hausdorff, E is closed. The collection $\{(-n, n) : n \in \mathbb{N}\}$ is an open cover of E. Since E is compact, choose $n_1, n_2, \ldots, n_L \in \mathbb{N}$ such that $E \subset (-n_1, n_1) \cup (-n_2, n_2) \cup \cdots \cup (-n_L, n_L)$. Thus E is bounded.

Suppse E is closed and bounded. Since E is bounded, there exists $R > 0$ such that $E \subset [-R, R]$. Since $[-R, R]$ is compact and E is closed, E is compact. $\square$

Theorem 1.32

$A \subset \mathbb{R}^n$ is compact if and only if A is closed and bounded.

Theorem 1.33

Let (X, d) be a metric space. The following are equivalent:

1. X is compact.
2. X is sequentially compact.
3. X is complete and totally bounded.

Theorem 1.34

Every metrizable space is normal.

Theorem 1.35

Every compact Hausdorff space is normal.

Theorem 1.36 (Urysohn lemma)

Let X be a normal space. Let A and B be disjoint closed subsets of X. Then there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ for every $x \in A$ and $f(x) = 1$ for every $x \in B$.

lemma X normal A subset X closed, U open set that contains A, then there exists an open set V such that A subset V and V closure of U

Theorem 1.37

Let X be a normal space. Let $A \subset X$ be closed. Suppose U is an open set that contains A. Then there exists an open set V such that $A \subset V$ and $\overline{V} \subset U$.

Theorem 1.38 (Tietze extension theorem)

Let X be a normal space. Suppose $A \subset X$ is a closed subset and $f : A \to [a, b]$ is continuous. Ther there exists a continuous map $g : X \to [a, b]$ such that $g(x) = f(x)$ for all $x \in A$.

Theorem 1.39

Let X be a topological space and $A \subset X$. If there exists a sequence of points in A that converges to x, then $x \in \overline{A}$. The converse holds if X is first-countable.

Theorem 1.40

Let X, Y be topological spaces and $f : X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, $f(x_n) \to f(x)$. The converse holds if X is first-countable.

Theorem 1.41

If X is second-countable, then X contains a countable dense subset.

Theorem 1.42

Second-countable implies first-countable.

Theorem 1.43

A topological space X is Hausdorff if and only if any two limits of any convergent net are equal.

Proof. Suppose X is Hausdorff ☐

Theorem 1.44

A function $f : X \to Y$ between topological spaces is continuous if and only if for every $x \in X$ and every net ϕ in X converging to x, the net $f \circ \phi$ in Y converges to $f(x)$.

Theorem 1.45

Let X be a topological space and let $A \subset X$. Then $\overline{A} = \{p \in X : \text{there exists a net in} A \text{that converges to } p\}$.

Theorem 1.46

Every net has a universal subnet. (This is equivalent to the axiom of choice)

Theorem 1.47

Let X be a topological space. The following are equivalent:

1. X is compact.
2. Every collection of closed subsets of X with the finite intersection property has a nonempty intersection.
3. Every universal net in X converges.
4. Every net in X has a convergent subnet.

Theorem 1.48 (Tychonoff theorem)

Let $\{X_\alpha : \alpha J\}$ be a collection of compact topological spaces. Then $\prod_{\alpha \in J} X_\alpha$ given the product topology, is compact.

Proof. Let $f : D \to \prod_{\alpha \in J} X_\alpha$ be a universal net. For each $\alpha \in J$, $\pi_\alpha \circ f : D \to X_\alpha$ is also a universal net, so by compactness of X_α, f converges to some point x_α. Thus f converges to the point whose α-th coordinate is x_α Thus every universal net converges in $\prod_{\alpha \in J} X_\alpha$. Thus the product is compact. ☐

1.3 Examples

topologies

Let X be a set.

- **trivial topology**: $\mathcal{T} = \{\emptyset, X\}$
- **discrete topology**: $\mathcal{T} = \mathcal{P}(X)$
- **finite complement topology**: $\mathcal{T} = \{U \subset V : X \setminus U \text{ is a finite set or } U = \emptyset\}$

basis for topologies

- Let X be a set. The collection of singletons $\{\{p\} : p \in X\}$ is a basis for the discrete topology on X.
- Let (M, d) be a metric space. Then the collection of metric balls $\{B(p, \varepsilon) : p \in M, \varepsilon > 0\}$ is a basis for the metric topology on M.
- (Non-example): The collection of closed intervals $\{[a, b] : -\infty < a < b < \infty\}$ is not a basis since $[0, 1] \cap [1, 2] = \{1\}$ which does not have a closed interval inside it.

product topologies

- The product topology on $\mathbb{R} \times \mathbb{R}$ is generated from basis of open rectangles, $(a, b) \times (c, d)$.

interior and closure of a set

Consider the usual topology on $\mathbb{R}$. Let $A = (3, 4]$ Then $\operatorname{Int}(A) = (3, 4)$ and $\overline{A} = [3, 4]$.

examples of separation axioms

example sequentially continuous

more example continuous vs sequntially continuous

example homeomorphism

example embedding

example function not continuous with box topology

example metrics

example connectedness

example topologist sine curve

example locally connected

example locally compact

example 1 point compactification

example first countable spaces

example second countable spaces

example dense set

example first countable but not second countable

example Tychonoff theorem

2 Homotopy

2.1 Definitions

homotopy of maps

Let X and Y be topological spaces and let $f : X \to Y$ and $g : X \to Y$ be continuous functions. f is **homotopic** to g if there exists a continuous function $F : X \times I \to Y$ such that for each x, $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ Then the map F is a **homotopy** between f and g, and $f \simeq g$.

homotopy of paths

Let f and g be paths mapping $I = [0, 1]$ into X. f and g are **path homotopic** if they both start at the same point $x_0 \in X$ and end at the same point $x_1 \in X$, and if there is a continuous function $F : I \times I \to X$ such that: for all $s, t \in I$,

1. $F(s, 0) = f(s)$
2. $F(s, 1) = g(s)$
3. $F(0, t) = x_0$
4. $F(1, t) = x_1$

Then F is a **path homotopy** between f and g, and $f \simeq_{\mathrm{p}} g$.

fundamental groupoid

Let X be a topolological space, f, g be paths in X. Define the operation $f * g$ by

$$(f * g)(s) := \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

For all paths $p : I \to X$, write $[p]$ as the equivalence class of p under path homotopy. The operation $\star$ induces an operation on path homotopy classes.

$$[f] \star [g] := [f * g]$$

The operation $\star$ is well-defined. Let $f_1, f_2 \in [f]$ and $g_1, g_2 \in [g]$, where $[f]$ goes from x_0 to x_1 and $[g]$ goes from x_1 to x_2.

Note: if $\star$ is defined, then

1. $\star$ is associative.
$$[f] \star ([g] \star [h]) \simeq ([f] \star [g]) \star [h]$$
2. left and right identities exist. Given $x \in X$ and define $e_x : I \to X$ by $e_x(s) = x$ for all $s \in I$. If f is a path in X from x_0 to x_1, then $[f] \star [e_{x_1}] \simeq [f]$ and $[e_{x_0}] \star [f] \simeq [f]$.
3. left and right inverse exist. If f is a path in X from x_0 to x_1, define the reverse path $\overline{f}$ by $\overline{f}(s) = f(1-s)$. Then $[f] \star \overline{[f]} \simeq [e_{x_0}]$ and $\overline{f} \star [f] \simeq [e_{x_1}]$.

loop

Fix p in X. A path in X that begins and ends at p is called a **loop** based at p.

fundamental group

The set of path-homotopy classes of loops based at p together with the operation $\star$ is the **fundamental group** of X relative to p. It is denoted $\pi_1(X, p)$. prove the group properties. The fundamental group does not depend on base point if X is path connected. They are all the same up to isomorphism.

simply connected

A topological space X is **simply connected** if it is path-connected and if $\pi_1(X, x_0)$ is trivial for all $x_0 \in X$.

covering map

Let $p : E \to B$ be continuous and onto. The open set $U \subset B$ is **evenly covered** by p if $p^{-1}(U) = V_\alpha$ where each V_α is open in E and for each α, p restricted to V_α maps V_α homeomorphically onto U. The collection $\{V_\alpha\}$ is a **partition of** $p^{-1}(U)$ **into slices**. If every point $b \in B$ has a neighbourhood U that is evenly covered by p, then p is a **covering map** and E is a **covering space** of B.

lifting

Let $p : E \to B$ be a function. If $f : X \to B$ is continuous, a **lifting of** f is a map $g : X \to E$ such that $p \circ g = f$. A lifting of f does not necessarily exist.

$$
\begin{tikzcd}
 & E \arrow[d, "p"] \\
X \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & B
\end{tikzcd}
$$

lifting correspondence

Let $p : E \to B$ be a covering map. Let $b_0 \in B$ and choose $e_0 \in E$ such that $p(e_0) = b_0$. Given $[f] \in \pi_1(B, b_0)$, let g be the lifting of f to a path in E that begins at e_0. Let $\phi([f])$ denote that endpoint $\tilde{f}(1)$ of $\tilde{f}$. Then $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ is a well-defined map that is called the **lifting correspondence derived from** p.

retraction

Let $A \subset X$. A **retraction** of X onto A is a continuous map $r : X \to A$ such that $r(a) = a$ for all $a \in A$. If such and r exists, then A is a **retract** of X.

induced homomorphism

Let X, Y be topological spaces. Let $f : X \to Y$ be continuous and suppose that $f(x_0) = y_0$. The **induced homomorphism** is a homomorphism $f_\star : \pi_1(X, x_0) \to \pi_1(Y, y_0)$, $f_\star([\gamma]) = [f \circ \gamma]$.

The induced homomorphism has a functorial property:

- $(g \circ f)_\star = g_\star \circ f_\star$
- $(\mathrm{id})_\star = \mathrm{id}$

homotopy type

Let $f : X \to Y$ and $g : Y \to X$ be continuous. If $g \circ f$ is homotopic to id_X and $f \circ g$ is homotopic to id_Y, then X and Y have the same **homotopy type**. X and Y are **homotopy equivalent**.

free product

Let G_1 and G_2 be disjoint groups. Define the **free produce** of G_1 and G_2 to be the group of reduced words $w_1 w_2 \ldots w_n$ where $w_i \in G_1$ or $w_i \in G_2$ for all i. Denote the free product as $G_1 \star G_2$.

free product with amalgamation

Let A be a group and let $\phi_1 : A \to G_1$ and $\phi_2 : A \to G_2$ be injective homomorphisms. Define the **free product with amalgamation** to be the group $G_1 \star G_2$ with relations given by $\phi_1(a) = \phi_2(a)$ for all $a \in A$.

2.2 Theorems

Theorem 2.1

Homotopy is an equivalence relation.

Theorem 2.2

Let α be a path in X from x_0 to x_1. Define the operation $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ by

$$\hat{\alpha}([f]) = [\overline{\alpha}] \star [f] \star [\alpha]$$

$\hat{\alpha}$ is a group isomorphism. insert proof

if X is path connected, fundamental group does not depend on base point, they are the same up to isomorphism.

Theorem 2.3

Use alpha hat

Theorem 2.4

Let $p : E \to B$ be a covering map and suppose $p(e_0) = b_0$. Then any path $f : [0,1] \to B$ beginning at b_0 has a unique lifting to a path g in E beginning at e_0.

Theorem 2.5

Let $p : E \to B$ be a covering map and suppose $p(e_0) = b_0$. Let $F : I \times I \to B$ be continuous with $F(0,0) = b_0$. Then there exists a unique lifting of F to a continuous map $\tilde{F} : I \times I \to E$ such that $\tilde{F}(0,0) = e_0$. If F is a path homotopy, then so is $\tilde{F}$. **homotopy lifting**

Theorem 2.6

Let $p : E \to B$ be a covering map with $p(e_0) = b_0$. If E is path-connected, then the lifting correspondence is onto. If E is simply connected, then the lifting correpondence is bijective.

Proof. Suppose E is path-connected. Let $e_1 \in p^{-1}(b_0)$. Then there exists a path $\tilde{f}$ from e_0 to e_1 in E. Then $f := p \circ \tilde{f}$ is a loop at b_0 in B and $\phi([f]) = e_1$. Thus every $e_1 \in p^{-1}(b_0)$ can be reached, so ϕ is surjective. Suppose E is simply connected. Then ϕ is onto since path-connected implies simply connected. Let $[f], [g] \in \pi_1(B, b_0)$ such that $\phi([f]) = \phi([g])$. Then let $\tilde{f}, \tilde{g}$ be the liftings of f and g. Then $\tilde{f}(1) = \tilde{g}(1)$. Since E is simply connected, there exists a path homotopy $\tilde{F}$ in E between $\tilde{f}$ and $\tilde{g}$. Thus $F = p \circ \tilde{F}$ is a path homotopy in B from f to g, so $[f] = [g]$. Thus ϕ is injective. □

Theorem 2.7 (Brower fixed point theorem)

Suppose $f : B^2 \to B^2$ is continuous. Then there exists $p \in B^2$ such that $f(p) = p$.

Theorem 2.8

Let $h : S^1 \to X$ be a continuous map. The following are equivalent:

1. h is null-homotopic (homotopic to a constant map)
2. h extends to a continuous map $k : B^2 \to X$

3. $h_\star$ is the trivial homomorphism of fundamental groups

Proof. Assume h is null-homotopic. Then there exists a homotopy $F : S^1 \times I \to X$ such that $F(x, 0) = h(x)$ and $F(x, 1) = y_0$ for some $y_0 \in X$ and for all $x \in S^1$. PICTURE

Suppose h extends to a continuous map $k : B^2 \to X$. Then $h = k \circ j$, j is the inclusion map, so $h_\star = k_\star \circ j_\star$. Since $j_\star : \pi_1\left(S^1, b_0\right) \to \pi_1\left(B^2, b_0\right)$ and $\pi_1\left(B^2, b_0\right)$ is trivial, then since $k_\star$ is a homomorphism, $k_\star$ is trivial homomorphism, and so $h_\star$ is trivial. □

Theorem 2.9 (Fundamental theorem of algebra)

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

$n > 0, a_i \in \mathbb{C}$ has a solution in $\mathbb{C}$.

Theorem 2.10

Suppose X, Y are homeomorphic path-connected topological spaces. Let x_0 be a base point and $y_0 = f(x_0)$, $f : X \to Y$ is a homeomorphism. Then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(Y, y_0)$.

Proof. $f^{-1} \circ f = \mathrm{id}_X$ and $f \circ f^{-1} = \mathrm{id}_Y$. Then $f_\star^{-1} \circ f_\star = (\mathrm{id}_X)_\star = \mathrm{id}_{\pi_1(X,x_0)}$ and $f_\star \circ f_\star^{-1} = (\mathrm{id}_Y)_\star = \mathrm{id}_{\pi_1(Y,y_0)}$.

Thus $f_\star$ is an isomorphism. □

Theorem 2.11

Let $h, k : X \to Y$ be continuous maps, and let $h(x_0) = y_0$ and $k(x_0) = y_1$. If h and k are homotopic, there exists a path $\alpha \in Y$ from y_0 to y_1 such that $k_\star = \hat{\alpha}_\star \circ h_\star$ If $H : X \times I \to Y$ is the homotopy between h and k, then $\alpha(t) = H(x_0, t)$.

Theorem 2.12

Let $f : X \to Y$ be continuous and f a homotopy equivalence. Then $f_\star : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism, where $f(x_0) = y_0$.

Theorem 2.13

$\pi_1\left(S^1\right) \cong \pi_1\left(\mathbb{R}^2 \setminus \{0\}\right)$.

Proof. explain why? □

Seifert Van Kampen

Theorem 2.14

Let $X = U \cup V$ be a topological space such that each $U, V, U \cap V$ are open, non-empty, and path-connected. Choose $x_0 \in U \cap V$. Then $\pi_1(X, x_0) \cong \pi_1(U, x0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0)$.

The induced homomorphism by the inclusions $j : U \cap V \to U$ and $k : U \cap V \to V$ induce an isomorphism.

Then,

$$\pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) = \langle \text{generators of } \pi_1(U), \text{generators of } \pi_1(V) \mid j_\star(c) = k_\star(c) \text{ for all } c \in \pi_1(U \cap V) \rangle$$

2.3 Examples

example fundamental group

example simply connected

covering map

fundamental group of S^1

Define $p : \mathbb{R} \to S^1$ by $p(x) = (\cos(2\pi x), \sin(2\pi x))$. p is a covering map. The lifting correspondence $\phi : \pi_1\left(S^1, (1,0)\right) \to p^{-1}((1,0))$ is a bijection. Note that $p^{-1}((1,0)) = \mathbb{Z}$, so ϕ is a bijection into $\mathbb{Z}$. ϕ can also be seen as a group homomorphism $\phi : \pi_1\left(S^1, (1,0)\right) \to \mathbb{Z}$. Let $[f], [g] \in \pi_1\left(S^1, (1,0)\right)$. Then $\phi([f] \star [g]) = \phi([f * g]) =?$

example same fundamental group but not homeomorphic.

There is no retract of B^2 onto S^1

Proof. Suppose there exists a retraction $r : D^2 \to S^1$. Define $j : S^1 \to B^2$ by $j(p) = p$ for all $p \in S^1$. Then $r \circ j = \mathrm{id}_{S^1}$. Next

$$(\mathrm{id}_{S^1})_\star = (r \circ j)_\star = r_\star \circ j_\star = \mathrm{id}_{\pi_1(S^1, b_0)}$$

Not bijective? □

example homotopy type

Using Seifert Van Kampen

1. wedge sum of two circles (figure 8): $\pi_1(X) \cong \mathbb{Z} * \mathbb{Z}$ The free group on 2 generators.
2. $\mathbb{T}^2 = S^1 \times S^1$ the (2-torus): $\pi_1\left(\mathbb{T}^2\right) \cong \mathbb{Z} \oplus \mathbb{Z}$.

3 Prevalence

3.1 Definitions

nowhere dense set

Let X be a topological space. A set $E \subset X$ is **nowhere dense** if $\mathrm{Int}\left(\overline{A}\right) = \emptyset$.

topologically large and small sets

Let (X, d) be a complete metric space. A set $E \subset X$ is **topologically large** if E is open and dense in X. It is **topologically small** if E is nowhere dense in X.

residual set

Let (X, d) be a complete metric space. A countable intersection of dense open sets in X is a **residual set**.

separable space

The space contains countably many dense subsets.

infinite-dimensional

No finite basis.

Banach space

A normed complete vector space.

prevalence

Let V be a real topological vector space (vector space that is also a topological space with a complete metric) for which addition and scalar multiplication are continuous and a measure defined on the Borel sets of V. A set $S\ subsetV$ is **shy** if there exists a measure μ such that:

1. There exists a compact $U \subset V$ such that $0 \leq \mu(V) < \infty$
2. For all $\mathbf{v} \in V$, $\mu(S + mathbfv)$.

A Borel set in V is **prevalent in** V if its complement is shy.

shy

3.2 Theorems

Theorem 3.1 (Baire category theorem)

Let (X, d) be a complete metric space. Then the intersection of any countable family of dense open sets in X is dense. In other words, every residual set is dense in X.

Theorem 3.2

In an infinite-dimensional separable Banach space with a translation-invariant measure which is not the zero measure, every open set has infinite measure.

Proof. Let X be an infinite-dimensional separable Banach space with a translation-invariant measure. Suppose for some $\varepsilon > 0$ the open ball of radius ε has finite measure. Since X is infinite-dimensional, construct a sequence of pairwise disjoint open balls of radius $\frac{\varepsilon}{4}$ contained in the ball of radius ε. Then each open ball of radius $\frac{\varepsilon}{4}$ has the same measure, so each ball has measure 0. Since X is separable, it can be convered by countably many open balls of radius $\frac{\varepsilon}{4}$. Thus the whole space must have measure 0.

$$X = \bigcup_{n=1} \infty B_n \implies \mu(X) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) \leq \sum_{n=1}^{\infty} \mu\left(B_n\right) \leq 0$$

Thus $\mu(X) = 0$. $\square$

3.3 Examples

Lebesgue measure vs. topology

Let μ denote the Lebesgue measure on $\mathbb{R}$.

- **A topologically large set that has Lebesgue measure 0**.
  Let $\varepsilon > 0$. For each $q_i \in \mathbb{Q}$, consider $B\left(q_i, \frac{\varepsilon}{2^{i+1}}\right)$. Define

  $$A_\varepsilon := \bigcup_{i=1}^{\infty} B\left(q_i, \frac{\varepsilon}{2^{i+1}}\right).$$

  Then,

  $$\begin{aligned}\mu\left(A_\varepsilon\right) &= \mu\left(\bigcup_{i=1}^{\infty} B\left(q_i, \frac{\varepsilon}{2^{i+1}}\right)\right)\\ &\leq \sum_{i=1}^{\infty} \mu\left(B\left(q_i, \frac{\varepsilon}{2^{i+1}}\right)\right)\\ &= \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i}\\ &= \varepsilon\end{aligned}$$

  So A_ε is open and dense in $\mathbb{R}$ but has Lesbesgue measure $\mu\left(A_\varepsilon\right) \leq \varepsilon$. Let $\varepsilon = \dfrac{1}{n}$. By the Baire category theorem, $\bigcap_{n=1}^{\infty} A_{\frac{1}{n}}$ is dense in $\mathbb{R}$ but $\mu\left(\bigcap_{n=1}^{\infty} A_{\frac{1}{n}}\right) = 0$.
- **A topologically small set that has non-zero Lebesgue measure** Let $\alpha \in (0,1)$ and consider the construction of the Cantor set $\mathcal{C}_\alpha$ by deleting the open middle $\frac{\alpha}{3}$. It is possible to make $\mu\left(\mathcal{C}_\alpha\right)$ close to 1 by taking α sufficiently small. Thus $\mathcal{C}_\alpha$ is closed and $\operatorname{Int}\left(\mathcal{C}_\alpha\right) = \emptyset$, so $\mathcal{C}_\alpha$ is topologically small.

exanple prevalence and shyness