Theory of Differential Equations and Nonlinear Dynamics

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1 Linear systems

2 Phase plane

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4 Bifurcations in 2-D

Strogatz: Chapter 8.1-2,4-7

Bifurcation: when the topological structure changes as a parameter is varied

saddle-node bifurcation As μ changes, fixed points collide when $\mu = \mu_c$. After, there are no more fixed points. Fixed points slide along the unstable manifold.

Example:

$$\dot{x} = \mu - x^2$$
$$\dot{y} = -y$$

transcritical pitchfork 1 fixed point exists for all values of a parameter is never destroyed. A transcritical bifurcation is one in which a fixed point exists for all values of a parameter and is never destroyed. However, such a fixed point interchanges its stability with another fixed point as the parameter is varied.[1] In other words, both before and after the bifurcation, there is one unstable and one stable fixed point. However, their stability is exchanged when they collide. So the unstable fixed point becomes stable and vice versa.

Example:

$$\dot{x} = \mu x - x^2$$
$$\dot{y} = -y$$

supercritical pitchfork 1 stable fixed point turns to saddle and creates two stable fixed points.

Example:

$$\dot{x} = \mu x - x^3$$
$$\dot{y} = -y$$

subcritical pitchfork Start with 2 stable fixed points.

Example:

$$\dot{x} = \mu x + x^3$$
$$\dot{y} = -y$$

Hopf bifurcation Start with two complex conjugate eigenvalues. Start with stable spiral and (subcrit) limit cycle.

supercritical Hopf bifurcation stable spiral turns into unstable spiral surrounded by limit cylce. Limit cycle grows in size. negative real part eigenvalues to positive real part eigenvalues

subcritical Hopf bifurcation After bifurcation, trajectories jump to distant attractors. Start with unstable limit cycle and stable fixed point. Unstable limit cycle shrinks in size to zero, changes fixed point to unstable spiral.

degenerate Hopf bifurcation When stable changes to unstable spiral, but there are no limit cycles.

global birfucation of cycles Ways in which limit cycles are created or destroyed.

saddle-node bifurcation of cycles two limit cycles collide and annihilate. OR stable spiral turns into semi-stable limit cycle turns into unstable and stable limit cycle.

infinite-period bifurcation stable limit cycle turns to fixed point on the circle, which then splits into two fixed points on circle, 1 unstable 1 stable.

homoclinic bifurcation part of a limit cycle moves closer to a saddle point, saddle point plus unstable spiral This is a kind of infinite-period bifurcation.

coupled oscillators Exist on torus. Periodic in both variables.

$$\dot{\theta}_1 = \omega_1 + K_1 \sin(\theta_2 - \theta_1)$$
$$\dot{\theta}_2 = \omega_2 + K_2 \sin(\theta_1 - \theta_2)$$

slope: ω_2/ω_1. If slope is:

- rational: all trajectories are closed orbits on torus
- trajectories are knotted if coprime
- irrational: quasiperiodic, every trajectory goes on forever, each trajectory dense on torus: comes arbitrarily close to any given point on the torus.

Couples, $K_1, K_2 > 0$. saddle-node bifurcation