Theory of Functions of a Real Variable 1

University of Houston, Fall 2023

MATH 6320 (Dr. Climenhaga)

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1 Measures

1.1 Definitions

algebra

Let X be a set. An **algebra** is a collection $\mathcal{A} \subset \mathcal{P}(X)$ such that

1. $\emptyset \in \mathcal{A}$ and $X \in \mathcal{A}$
2. if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$
3. if $A_1, A_2, \ldots, A_n \in \mathcal{A}$ then $\displaystyle\bigcup_{i=1}^{n} A_i \in \mathcal{A}$.

σ-algebra

Let X be a set. A **σ-algebra** is a collection $\mathcal{A} \subset \mathcal{P}(X)$ such that

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3. if $A_1, A_2, \ldots \in \mathcal{A}$ then $\displaystyle\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.

measurable space

Let X be a set and $\mathcal{A}$ be a σ-algebra on X. A **measurable space** is a pair $(X, \mathcal{A})$.

measurable set

Let $(X, \mathcal{A})$ be a measurable space. A set A is **measurable** (or **$\mathcal{A}$-measurable**) if $A \in \mathcal{A}$.

algebra generated by $\mathcal{C} \subset \mathcal{P}(X)$

Let $\mathcal{C} \subset \mathcal{P}(X)$ be a collection of subsets of X. The **algebra generated by $\mathcal{C}$** is the intersection of all algebras containing $\mathcal{C}$.

σ-algebra generated by $\mathbf{C} \subset \mathbf{P(X)}$

Let $\mathcal{C} \subset \mathcal{P}(X)$ be a collection of subsets of X. The **σ-algebra generated by** $\mathcal{C}$, denoted σ(C), is the intersection of all σ-algebras containing $\mathcal{C}$, that is,

$$\sigma(\mathcal{C}) := \bigcap\{\mathcal{A}_\alpha \mid \mathcal{A}_\alpha \text{ is a } \sigma\text{-algebra}, \mathcal{C} \subset \mathcal{A}_\alpha\}$$

$\sigma(\mathcal{C})$ is also the the smallest σ-algebra containing $\mathcal{C}$.

1. if $\mathcal{C}_1 \subset \mathcal{C}_2$, then $\sigma(\mathcal{C}_1) \subset \sigma(\mathcal{C}_2)$
2. if $\mathcal{A}$ is a σ-algebra, then $\sigma(\mathcal{A}) = \mathcal{A}$

Borel σ-algebra

Let X be a metric space and let $\mathcal{G}$ be the collection of open subsets of X. Then the σ-algebra generated by $\mathcal{G}$, $\mathcal{B}_X = \sigma(\mathcal{G})$, is the **Borel σ-algebra** on X. If $B \in \mathcal{B}$, it is called a **Borel set** and is said to be **Borel measurable**.

G_δ and F_σ sets

Let X be a topological space.

- A G_δ **set** is a countable intersection of open sets of X. Etymology (German): G for *Gebiet*, open set, and δ for *Durchschnitt*, intersection.
- An F_σ **set** is a countable union of closed sets. Etymology (French): F for *fermé*, closed, and σ for *somme*, sum.

The complement of a G_δ set is an F_σ set.

measure

Let X be a set and $\mathcal{A}$ be a σ-algebra on X. A **measure** on $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \to [0, \infty]$ such that

1. $\mu(\emptyset) = 0$
2. if $A_1, A_2, \ldots \in \mathcal{A}$ are pairwise disjoint, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

countable additivity

Let $\{E_k\}_{k=1}^n$ be a countable disjoint collection of sets. A set function μ possesses **countable additivity** if

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{i=1}^{\infty} \mu(E_k)$$

Measures are countably additive.

measure space

Let X be a set, $\mathcal{A}$ a σ-algebra, and μ a measure on $(X, \mathcal{A})$. Then $(X, \mathcal{A}, \mu)$ is a **measure space**.

finite measure

Let $(X, \mathcal{A}, \mu)$ be a measure space. If $\mu(X) < \infty$, then μ is a **finite measure**.

σ-finite measure

Let $(X, \mathcal{A}, \mu)$ be a measure space. If there exist sets $\{E_i \in \mathcal{A}\}_{i=1}^{\infty}$ such that

1. $\mu(E_i) < \infty$ for each i
2. $X = \bigcup_{i=1}^{\infty} E_i$,

then μ is a **σ-finite measure**.

Let $F_n = \bigcup_{i=1}^{n} E_i$. Then $\mu(F_n) < \infty$ for each n and F_n increases to X. This is an alternate formulation of a **σ-finite measure**.

complete measure

Let $(X, \mathcal{A}, \mu)$ be a measure space and $A \subset X$. If there exists a set $B \in \mathcal{A}$ such that $A \subset B$ and $\mu(B) = 0$, then A is a **null set**. If $\mathcal{A}$ contains all the null sets (all null sets are measurable), then $(X, \mathcal{A}, \mu)$ is a **complete measure space**.

outer measure

Let X be a set. An **outer measure** is a function $\mu^\star : \mathcal{P}(X) \to [0, \infty]$ such that

1. $\mu^\star(\emptyset) = 0$
2. if $A \subset B$, then $\mu^\star(A) \leq \mu^\star(B)$
3. if $A_1, A_2, \ldots,$ are subsets of X, then

$$\mu^\star\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^\star(A_i)$$

$\mu^\star$-measurable set

Let $\mu^\star$ be an outer measure on X. A set $A \subset X$ is **$\mu^\star$-measurable** if for all $E \subset X$,

$$\mu^\star(E) = \mu^\star(E \cap A) + \mu^\star(E \cap A^c).$$

Lebesgue measure on $\mathbb{R}$

Consider $\mathbb{R}$ and let $\mathcal{C} = \{(a, b] \mid -\infty < a < b < \infty\}$. Let $\ell((a, b]) = b - a$ and define the outer measure $m^\star$ by

$$m^\star(E) = \inf\left\{\sum_{i=1}^{\infty} \ell(A_i) \mid A_i \in \mathcal{C} \text{ and } E \subset \bigcup_{i=1}^{\infty} A_i\right\}.$$

Since $m^\star$ is an outer measure, $\mathcal{L} \coloneqq \{A \subset \mathbb{R} \mid A \text{ is } m^\star\text{-measurable}\}$ is the **Lebesgue σ-algebra** and $m \coloneqq m^\star|_{\mathcal{L}})$ is a complete measure, the **Lebesgue measure**.

Lebesgue-Stieltjes measure

Consider $\mathbb{R}$ and let $\mathcal{C} = \{(a,b] \mid -\infty < a < b < \infty\}$. Let $F : \mathbb{R} \to \mathbb{R}$ be a right non-decreasing continuous function. Define $\ell((a,b]) = F(b) - F(a)$ and

$$m^\star(E) = \inf \left\{ \sum_{i=1}^{\infty} \ell(A_i) \mid A_i \in \mathcal{C} \text{ and } E \subset \bigcup_{i=1}^{\infty} A_i \right\}.$$

Since $m^\star$ is an outer measure, $\mathcal{A} = \{A \subset \mathbb{R} \mid A \text{ is } m^\star\text{-measurable}\}$ is a σ-algebra. Then the restriction of $m^\star$ to $\mathcal{A}$, $m := m^\star|_{\mathcal{A}}$ is the **Lebesgue-Stieltjes measure**.

pre-measure on an algebra

Let $\mathcal{A}_0$ be an algebra on a set X. A **pre-measure** on $\mathcal{A}_0$ is a function $\ell : \mathcal{A}_0 \to [0,\infty]$ such that

1. $\ell(\emptyset) = 0$
2. if $A_1, A_2, \ldots \in \mathcal{A}_0$ are pairwise disjoint such that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$, then

$$\ell\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \ell(A_i)$$

2 Integration

2.1 Definitions

measurable function

Let $(X, \mathcal{A})$ be a measurable space. A function $f : X \to \mathbb{R}$ is **measurable** (or **$\mathcal{A}$-measurable**) if $\{x \in X \mid f(x) > a\} \in \mathcal{A}$ for all $a \in \mathbb{R}$. A function $f : X \to \mathbb{C}$ is **measurable** if $\mathrm{Re}\,(f)$ and $\mathrm{Im}\,(f)$ are measurable.

The following are equivalent: for all $a \in \mathbb{R}$,

- $\{x : f(x) > a\} \in \mathcal{A}$
- $\{x : f(x) \geq a\} \in \mathcal{A}$
- $\{x : f(x) < a\} \in \mathcal{A}$
- $\{x : f(x) \leq a\} \in \mathcal{A}$

Borel measurable

Let $(X, \mathcal{B}_X)$ be a metric space with the Borel σ-algebra. A function $f : X \to \mathbb{R}$ is **Borel measurable** if it is measurable with respect to $\mathcal{B}_X$.

Lebesgue measurable

A function $f : \mathbb{R} \to \mathbb{R}$ is **Lebesgue measurable** if it is measurable with respect to the Lebesgue measure.

non-negative Lebesgue measurable functions

Denote the set of all non-negative Lebesgue measurable functions by

$$L^+ \coloneqq \{f : X \to [0, \infty) \mid f \text{ is measurable}\}$$

characteristic function

A **characteristic function** of $E \in \mathcal{A}$ is

$$\mathbb{1}_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

simple function

A function $s : X \to \mathbb{R}$ is **simple** if there exist $c_1, c_2, \ldots, c_n \in \mathbb{R}$ and $E_1, E_2, \ldots, E_n \in \mathcal{A}$ such that

$$s(x) = \sum_{i=1}^{n} c_i \mathbb{1}_{E_i}(x)$$

Lebesgue integral

Let $(X, \mathcal{A}, \mu)$ be a measure space.

1. if $f = \mathbb{1}_E$ is measurable, let
$$\int f \, \mathrm{d}\mu \coloneqq \mu(E)$$

2. if $s = \displaystyle\sum_{i=1}^{n} c_i \mathbb{1}_{E_i}$ is simple and $E_i \in \mathcal{A}$ for all i, let
$$\int s \, \mathrm{d}\mu \coloneqq \sum_{i=1}^{n} c_i \mu(E_i)$$

3. if $f \in L^+$, let
$$\int f \, \mathrm{d}\mu \coloneqq \sup \left\{ \int s \, \mathrm{d}\mu \mid 0 \le s \le f, s \text{ is simple} \right\}$$

4. if $f : X \to \mathbb{R}$, let $f^+(x) = \max(f(x), 0)$ and $f^-(x) = \max(-f(x), 0)$. Thus, $f^+, f^- \in L^+$ and $f = f^+ - f^-$. As long as $\displaystyle\int f^+ \, \mathrm{d}\mu$ and $\displaystyle\int f^- \, \mathrm{d}\mu$ are not both infinite, let
$$\int f \, \mathrm{d}\mu \coloneqq \int f^+ \, \mathrm{d}\mu - \int f^- \, \mathrm{d}\mu$$

5. if $f : X \to \mathbb{C}$ is measurable, let $f = u + v\mathrm{i}$ for $u, v : X \to \mathbb{R}$, let
$$\int f \, \mathrm{d}\mu \coloneqq \int u \, \mathrm{d}\mu + \mathrm{i} \int v \, \mathrm{d}\mu$$

   as long as u, v are not infinite.

almost everywhere equality

Given measurable functions $f, g : X \to \mathbb{R}$, say that $f = g$ **almost everywhere** (a.e. or μ-a.e.) if $\mu(\{x : f(x) \neq g(x)\}) = 0$. Equivalently, $f = g$ **almost everywhere** if there exists $A \subset X$ such that

1. $f|_A = g|_A$
2. $\mu(A^c) = 0$

compactly supported continuous functions on $\mathbb{R}$

The set of compactly supported continuous functions on $\mathbb{R}$ is denoted $C_c(\mathbb{R})$.

convergence almost everywhere

A sequence of functions f_n **converges to f almost everywhere** if there exists $A \in \mathcal{A}$ such that $\mu(A) = 0$ and $f_n(x) \to f(x)$ for all $x \in A^c$.

convergence in measure

A sequence of functions f_n **converges to f in measure** if for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| > \varepsilon\}) = 0$$

convergence in L^p

Let $1 \leq p < \infty$. A sequence of functions f_n **converges to f in L^p** if

$$\lim_{n\to\infty} \int |f - f_n|^p = 0$$

uniform convergence

A sequence of functions f_n **converges to f uniformly** if

$$\lim_{n\to\infty} \sup_{x\in X} (|f_n(x) - f(x)|) = 0$$

convergence in L^∞

A sequence of functions f_n **converges to f in L^∞** if there exists $A \in \mathcal{A}$ such that $\mu(A) = 0$ and

$$\lim_{n\to\infty} \sup_{x\in A^c} (|f_n(x) - f(x)|) = 0$$

measurable rectangle

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces. A **measurable rectangle** is a set of the form $A \times B$, where $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

product σ-algebra

Let $\mathcal{C}_0$ be the collection of finite unions of disjoint measurable rectangles. The **product σ-algebra** is the σ-algebra generated by $\mathcal{C}_0$, $\mathcal{A} \times \mathcal{B} = \sigma(\mathcal{C}_0)$.

product measure

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces. Define the **product measure** $\mu \times \nu : \mathcal{A} \times \mathcal{B} \to [0, \infty]$ by

$$(\mu \times \nu)(E) = \int_X \int_Y \mathbb{1}_E(x, y)\, \mathrm{d}\nu(y)\, \mathrm{d}\mu(x) = \int_Y \int_X \mathbb{1}_E(x, y)\, \mathrm{d}\mu(x)\, \mathrm{d}\nu(y)$$

Lebesgue measure on $\mathbb{R}^n$

Let m denote the Lebesgue measure on $\mathbb{R}$. The Lebesgue measure on $\mathbb{R}^n$ is $m^n = m \times \cdots \times m$.

3 Differentiation

3.1 Definitions

signed measure

Let $\mathcal{A}$ be a σ-algebra. A **signed measure** is a function $\mu : \mathcal{A} \to (-\infty, \infty]$ such that:

1. $\mu(\emptyset) = 0$
2. if $A_1, A_2, \ldots \in \mathcal{A}$ are pairwise disjoint, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

and $\sum_{i=1}^{\infty} \mu(A_i)$ converges absolutely if $\mu\left(\bigcup_{i=1}^{\infty} A_i\right)$ is finite.

positive and negative sets

Let μ be a signed measure.

- A set $A \in \mathcal{A}$ is a **positive set** if $\mu(B) \geq 0$ for all $B \in \mathcal{A}$ such that $B \subset A$.
- A set $A \in \mathcal{A}$ is a **negative set** if $\mu(B) \leq 0$ for all $B \in \mathcal{A}$ such that $B \subset A$.

mutually singular measures

Let μ and ν be two measures on $(X, \mathcal{A})$. The measures μ and ν are **mutually singular** (or ν is singular with respect to μ) if there exist disjoint sets $E, F \in \mathcal{A}$ such that $X = E \cup F$ with $\mu(E) = 0$ and $\nu(F) = 0$. This is denoted as $\mu \perp \nu$.

absolute continuous measure

Let μ and ν be two measures on $(X, \mathcal{A})$. The measure ν is **absolutely continuous with respect to** μ if $\nu(A) = 0$ for all $A \in \mathcal{A}$ for which $\mu(A) = 0$. This is denoted as $\nu \ll \mu$.

total variation measure

Let μ be a signed measure on $(X, \mathcal{A})$ with Jordan decomposition $\mu = \mu^+ - \mu^-$. The **total variation measure** of μ is the measure

$$|\mu| = \mu^+ + \mu^-$$

and $|\mu|(X)$ is the **total variation** of$X$.

Radon-Nikodym derivative

Let μ be a σ-finite measure on $(X, \mathcal{A})$ and ν be a finite positive measure on $(X, \mathcal{A})$ such that ν is absolutely continuous with respect to μ. The **Radon-Nikodym derivativee** is a μ-integrable non-negative function f such that for all $A \in \mathcal{A}$,

$$\nu(A) = \int_A f \, \mathrm{d}\mu$$

averaged function

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. Then the **averaged function** over a ball of radius r is

$$f_r(x) = \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y) \, \mathrm{d}y$$

locally integrable functions

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. The function f is **locally integrable** if for all compact $K \subset \mathbb{R}^n$, $\displaystyle\int_K |f(x)| \, \mathrm{d}x < \infty$. The set of all locally integrable functions is denoted by L^1_{loc}.

Hardy-Littlewood maximal function

Let f be a locally integrable function. The **Hardy-Littlewood maximal function** Mf is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)| \, \mathrm{d}y$$

total variation

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. The **total variation** of f on $[a, b]$ is the supremum over all partitions $a = x_0 < x_1 < \cdots < x_n = b$,

$$V_f[a,b] = \sup \left\{ \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \right\}$$

bounded variation

A function $f : \mathbb{R} \to \mathbb{R}$ is of **bounded variation** on $[a, b]$ if $V_f[a, b] < \infty$, the total variation is finite.

Lipschitz continuous

A function $f : \mathbb{R} \to \mathbb{R}$ is **Lipschitz continuous** if there exists $M > 0$ such that for all $x, y \in \mathbb{R}$,

$$|f(y) - f(x)| \leq M|y - x|$$

absolutely continuous function

A function $f : \mathbb{R} \to \mathbb{R}$ is **absolutely continuous** if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any finite collection of disjoint intervals (a_i, b_i) with $\displaystyle\sum_{i=1}^{k} |b_i - a_i| < \delta$,

$$\sum_{i=1}^{k} |f(b_i) - f(a_i)| < \varepsilon$$