Calculus III notes

University of Houston - MATH 2433

Contents

1 Background information

1.1 Cartesian space coordinates

3-dimensional space is denoted by $\mathbb{R}^3$. Points in Cartesian coordinates are described by $P(x, y, z)$.

planes parallel to the

- xy-plane has the form of $z = c$
- yz-plane has the form of $x = c$
- xz-plane has the form of $y = c$

given two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$:

the distance between them is

$$d\left(P_1, P_2\right) = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 + \left(z_2 - z_1\right)^2}$$

the midpoint is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

*intuition: the average between each coordinate

definition 1.1 (n-sphere)**.** The set of points in $(n+1)$-dimensional Euclidean space that are at a fixed distance r from a central point.

the sphere in $\mathbb{R}^3$ with center (a, b, c) and radius r is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

definition 1.2 (open and closed ball)**.** The space bounded by a sphere. A closed ball includes the boundary; an open ball does not.

the open ball in $\mathbb{R}^3$ with center (a, b, c) and radius r is

$$B = \left\{(x, y, z) \mid (x-a)^2 + (y-b)^2 + (z-c)^2 < r^2\right\}$$

the closed ball is

$$B = \left\{(x, y, z) \mid (x-a)^2 + (y-b)^2 + (z-c)^2 \leq r^2\right\}$$

1.2 Vectors

definition 1.3 (vector)**.** An element of a vector space. A Euclidean vector is a geometric object that has direction and magnitude (norm). In $\mathbb{R}^n$ it has n components that describe it.

vector in $\mathbb{R}^3$ from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$ is

$$\overrightarrow{PQ} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

$$\text{*note that } \overrightarrow{QP} = -\overrightarrow{PQ}$$

component notations in $\mathbb{R}^3$

$$\mathbf{v} = (v_x, v_y, v_z) = \langle v_x, v_y, v_z \rangle = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

vector component algebra:

- $\mathbf{a} + \mathbf{b} = (a_x + b_x)\hat{\mathbf{i}} + (a_y + b_y)\hat{\mathbf{j}} + (a_z + b_z)\hat{\mathbf{k}}$
- $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) = [a_x + (-b_x)]\hat{\mathbf{i}} + [a_y + (-b_y)]\hat{\mathbf{j}} + [a_z + (-b_z)]\hat{\mathbf{k}}$
- $k\mathbf{v} = (kv_x)\hat{\mathbf{i}} + (kv_y)\hat{\mathbf{j}} + (kv_z)\hat{\mathbf{k}}$ (scalar multiplication)

the length (norm, magnitude) of a vector in $\mathbb{R}^n$ is

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_1^2 + \cdots v_n^2}$$

properties of vectors

name	symbolic representation
zero vector	$\mathbf{0} = (0, 0, 0)$
commutative	$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
associative	$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$
distributive	$k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$
norm multiplied by scalar	$\|k\mathbf{v}\| = \|k\|\|\mathbf{v}\|$
triangle inequality	$\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$

definition 1.4 (parallel vectors)**.** two vectors are parallel if they are scalar multiples of each other

$$\mathbf{a} \parallel \mathbf{b} \iff \mathbf{a} = k\mathbf{b}$$

if $k > 0$, then they have the same direction

if $k < 0$, then they are antiparallel (parallel but have opposite directions)

definition 1.5 (unit vector)**.** A unit vector has a norm of 1. The unit vector $\hat{\mathbf{v}}$ in the same direction as $\mathbf{v}$ is

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

*tip: unit vectors can be multiplied by a scalar to produce a certain norm.

In Cartesian coordinates, there are 3 unit vectors (basis vectors) that help describe other vectors. Every vector in the vector space $\mathbb{R}^3$ can be written as a linear combination of them.

the basis vectors are

$$\hat{\mathbf{i}} = (1, 0, 0) \qquad \hat{\mathbf{j}} = (0, 1, 0) \qquad \hat{\mathbf{k}} = (0, 0, 1)$$

if a unit vector makes angles $\alpha, \beta,$ and γ with $\hat{\mathbf{i}}, \hat{\mathbf{j}},$ and $\hat{\mathbf{k}}$, then

$$\hat{\mathbf{v}} = (\cos\alpha)\hat{\mathbf{i}} + (\cos\beta)\hat{\mathbf{j}} + (\cos\gamma)\hat{\mathbf{k}}$$

1.3 Dot product

definition 1.6 (dot product)**.** An operation that takes two vectors and returns a scalar.

algebraic definition

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

geometric definition

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos\theta$$

properties of the dot product

name	symbolic representation
magnitude	$\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$
commutative	$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
distributive	$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
Cauchy-Schwarz inequality	$\lvert\mathbf{a} \cdot \mathbf{b}\rvert \leq \lVert\mathbf{a}\rVert \lVert\mathbf{b}\rVert$

the angle between two vectors (from the geometric definition)

$$\theta = \arccos\left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}\right)$$

if $\mathbf{a} \cdot \mathbf{b} > 0$, then $\theta < \dfrac{\pi}{2}$

if $\mathbf{a} \cdot \mathbf{b} < 0$, then $\theta > \dfrac{\pi}{2}$

definition 1.7 (perpedicular vectors)**.** Two vectors are perpendicular if their dot product is 0.

$$\mathbf{a} \perp \mathbf{b} \iff \mathbf{a} \cdot \mathbf{b} = 0 \text{ since} \cos\left(\frac{\pi}{2}\right) = 0$$

definition 1.8 (component)**.** the component is a scalar that measures how much of **a** lies in the direction **b**

$$\operatorname{comp}_{\mathbf{b}} \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|} = \mathbf{a} \cdot \hat{\mathbf{b}} = \|\mathbf{a}\| \cos\theta$$

if $\operatorname{comp}_{\mathbf{b}} \mathbf{a} < 0$, then $\theta \in \left(\dfrac{\pi}{2}, \pi\right]$

if $\operatorname{comp}_{\mathbf{b}} \mathbf{a} > 0$, then $\theta \in \left(0, \dfrac{\pi}{2}\right)$

definition 1.9 (projection)**.** the projection of **a** onto **b** is a vector that describes the component of **a** that lies in the direction of **b**

$$\operatorname{proj}_{\mathbf{b}} \mathbf{a} = (\operatorname{comp}_{\mathbf{b}} \mathbf{a})\, \hat{\mathbf{b}} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2}\right) \mathbf{b}$$

1.4 Cross product

definition 1.10 (cross product)**.** An operation that produces a pseudovector orthogonal (perpendicular, normal) to the plane formed by two vectors in the direction of the thumb of the right hand rule.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

*tip: check that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = 0$

properties of the cross product

- anticommutative: $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
- if $\mathbf{a} \parallel \mathbf{b}$, then $\mathbf{a} \times \mathbf{b} = 0$
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$
- $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta$

definition 1.11 (scalar triple product)**.**

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

*if $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$, then the vectors are coplanar

applications of the cross product

area of a parallelogram formed by $\mathbf{a}$ and $\mathbf{b}$

$$A = \|\mathbf{a} \times \mathbf{b}\|$$

area of a triangle formed by $\mathbf{a}$ and $\mathbf{b}$

$$A = \frac{1}{2}\|\mathbf{a} \times \mathbf{b}\|$$

volume of a parallelepiped formed by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$

$$V = |(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|$$

1.5 Lines

The vector form of a line is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{d}, t \in \mathbb{R}$$

where $\mathbf{r_0}$ is vector and $\mathbf{d}$ is the direction vector of the line. scalar multiples of $\mathbf{d}$ produce the same line.

The component (scalar) form of a line is

$$\mathbf{r}(t) = \begin{cases} x(t) = x_0 + td_1 \\ y(t) = y_0 + td_2 \\ z(t) = z_0 + td_3 \end{cases}$$

The symmetric form of a line is

$$\frac{x(t) - x_0}{d_1} = \frac{y(t) - y_0}{d_2} = \frac{z(t) - z_0}{d_3}$$

If a component of $\mathbf{d}$ is zero, then that component is constant and the line lies on the plane formed by that constant.

Comparing lines $\mathbf{r}(t) = \mathbf{r}_1 + t\mathbf{d}_1$ and $\mathbf{v}(s) = \mathbf{v}_1 + s\mathbf{d}_2$

- $\mathbf{r} \parallel \mathbf{v} \iff \mathbf{d}_1 = k\mathbf{d}_2, k \in \mathbb{R}\,0$
- $\mathbf{r}$ and $\mathbf{v}$ intersect $\iff$ for some t and s such that $\mathbf{r}(t) = \mathbf{v}(s)$
- $\mathbf{r}$ and $\mathbf{v}$ are coincident $\iff$ $\mathbf{r} = \mathbf{v}$. They are parallel and intersecting.
- $\mathbf{r}$ and $\mathbf{v}$ are skew if they do not intersect and are not parallel. However, the planes the lines are on are parallel
- the angle between $\mathbf{r}$ and $\mathbf{v}$ is found by using their direction vectors. use the smallest angle by choosing a positive cosine value.

$$\cos\theta = \left| \frac{\mathbf{d}_1 \cdot \mathbf{d}_2}{\|\mathbf{d}_1\|\|\mathbf{d}_2\|} \right| = \left| \hat{\mathbf{d}}_1 \cdot \hat{\mathbf{d}}_2 \right|$$

the distance from a point to a line is

$$d(P, l) = \frac{\|\overrightarrow{P_0P_1} \times \mathbf{d}\|}{\|\mathbf{d}\|}$$

1.6 Planes

If $\mathbf{N} = A\hat{\mathbf{i}} + B\hat{\mathbf{j}} + C\hat{\mathbf{k}}$ is the normal vector to a plane that contains the point $P(x_0, y_0, z_0)$, then the equation of the plane is

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

The vector equation of a plane is

$$\mathbf{N} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

The intercept form of a plane is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

where $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$ are the intercepts.

Comparing planes $\Pi_1 : \mathbf{N}_1 \cdot (\mathbf{r} - \mathbf{r}_0) = 0$ and $\Pi_2 : \mathbf{N}_2 \cdot (\mathbf{v} - \mathbf{v}_0) = 0$,

- $\Pi_1 \parallel \Pi_2 \iff \mathbf{N}_1 \parallel \mathbf{N}_2$
- if $\Pi_1 \nparallel \Pi_2$, then Π_1 and Π_2 intersect at a line. the direction vector of the line is

  $$\mathbf{N}_1 \times \mathbf{N}_2$$

  choose any point that lies on the line and the plane. If the direction vector is not dependent on a scalar parametric, then do not choose that value for solving.
- the angle between Π_1 and Π_2 is found by using their normal vectors. use the smallest angle by choosing a positive cosine value.

  $$\cos\theta = \left| \frac{\mathbf{N}_1 \cdot \mathbf{N}_2}{\|\mathbf{N}_1\| \|\mathbf{N}_2\|} \right| = \left| \hat{\mathbf{N}}_1 \cdot \hat{\mathbf{N}}_2 \right|$$

distance between a point $P(x_1, y_1, z_1)$ and a plane $Ax + By + Cz + D = 0$ is

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

intersection of a line and plane: substitute parametrized form of the line into the plane and solve.

2 Vector valued functions

2.1 Vector functions

Vector-valued functions in $\mathbb{R}^3$ have the form

$$\mathbf{r}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k} = (f_1(t), f_2(t), f_3(t))$$

where $f_1, f_2,$ and f_3 are called component functions of $\mathbf{r}$

The domain of $\mathbf{r}$ is all values t that the component functions are defined on.

If C is a curve described by $\mathbf{r}$, then C is parametrised by $\mathbf{r}$

$$\lim_{t\to a} \mathbf{r}(t) = \mathbf{L} \iff \lim_{t\to a} \|\mathbf{r}(t) - \mathbf{L}\| = 0$$

$$\lim_{t\to a} \mathbf{r}(t) = \left[\lim_{t\to a} f_1(t)\right]\mathbf{i} + \left[\lim_{t\to a} f_2(t)\right]\mathbf{j} + \left[\lim_{t\to a} f_3(t)\right]\mathbf{k}$$

continuity: $\mathbf{r}(t)$ is continuous at $t = a$ if

1. $\mathbf{r}(a)$ exists
2. $\lim_{t\to a} \mathbf{r}(t)$ exists
3. $\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a)$

derivatives of vector functions

$$\mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r} = \mathbf{r}'(t) = f_1'(t)\mathbf{i} + f_2'(t)\mathbf{j} + f_3'(t)\mathbf{k}$$

if a component derivative does not exist, then the vector function is not differentiable at that point.

$$\int \mathbf{r}(t)\,\mathrm{d}t = \left[\int f_1(t)\,\mathrm{d}t\right]\mathbf{i} + \left[\int f_2(t)\,\mathrm{d}t\right]\mathbf{j} + \left[\int f_3(t)\,\mathrm{d}t\right]\mathbf{k}$$

2.2 Differentiation properties

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[k\mathbf{r}(t)] = k\mathbf{r}'(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[f(t)\mathbf{r}(t)] = f'(t)\mathbf{r}(t) + f(t)\mathbf{r}'(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}'(t) \cdot \mathbf{u}(t) + \mathbf{r}(t) \cdot \mathbf{u}'(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}'(t) \times \mathbf{u}(t) + \mathbf{r}(t) \times \mathbf{u}'(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}[f(t)] = f'(t)\mathbf{r}'[f(t)]$$

if $\mathbf{r} \cdot \mathbf{r} = \|\mathbf{r}(t)\|^2 = k$, then $\mathbf{r}' \cdot \mathbf{r} = 0$. $\mathbf{r}(t) \perp \mathbf{r}'(t)$

2.3 Curves

A curve $C : \mathbf{r}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$ is smooth on an interval I if $f_1'(t), f_2'(t)$ and $f_3'(t)$ are continuous on I and $\mathbf{r}'(t) \neq 0 \forall t \in I$

unit tangent vector

$$\mathbf{T} = \frac{\mathbf{r}'}{\|\mathbf{r}'\|}$$

If $\mathbf{T} = 0$, then the $\mathbf{T}$ does not change direction.

tangent line passes though point parallel to tangent vector.

principal unit normal vector

$$\mathbf{N} = \frac{\mathbf{T}'}{\|\mathbf{T}'\|}$$

$$\mathbf{T} \perp \mathbf{N}$$

the normal vector points in the direction of the center of the osculating circle.

binormal vector

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

determines the osculating plane tangent to the curve.

the binormal vector is a unit vector

osculating circle formed on the plane formed by tangent and normal. radius of osculating circle is $1/\kappa$

two curves $C_1 : \mathbf{r}_1(t)$ and $C_2 : \mathbf{r}_2(s)$ intersect $\iff \exists\, t \,\text{and}\, s$ such that $r_1(t) = r_2(s)$

$$\cos\theta = \left|\frac{\mathbf{r}_1' \cdot \mathbf{r}_2'}{\|\mathbf{r}_1\|\|\mathbf{r}_2\|}\right| = |\mathbf{T}_1 \cdot \mathbf{T}_2|$$

2.4 Arc length

for a vector valued function $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, t \in [a, b]$

$$s = \int_a^b \|\mathbf{r}'(t)\| \,\mathrm{d}t = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} \,\mathrm{d}t$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \|\mathbf{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}$$

the derivative of of arc length is speed

arc length parameterisation: changing the function $\mathbf{r}(t)$ to $\mathbf{r}(s)$. the distance traveled along the curve is equal to the parameter s

to get the arc length parameterisation, substitute s into $\mathbf{r}(t)$

2.5 Curvature and curvilinear motion

curvature: how sharply a smooth curve turns.

- a circle has constant curvature $\kappa = 1/r$
- a line has constant zero curvature
- curvature can be used to inscribe a circle "tangent" to a curve and measure the curvature of that circle.
- the radius of curvature is $r = 1/\kappa$
- the center of curvature is the point the distance of the radius in the direction of the principal normal

$$\kappa = \|\mathbf{T}'(s)\| = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$$

for 3-dimensional curves,

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

for planar parametric functions $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$

pro tip:

when $y = f(x)$, then

$$x' = 1 \qquad\qquad y' = y$$
$$x'' = 0 \qquad\qquad y'' = y''$$

and

$$\kappa = \frac{|x'y'' - y'x''|}{\left[(x')^2 + (y')^2\right]^{3/2}} = \frac{|y''|}{[1 + (y')^2]^{3/2}}$$

the top part of the fraction is:

$$\begin{vmatrix} x' & x'' \\ y' & y'' \end{vmatrix}$$

kinematic relations:

$$\mathbf{r}'(t) = \mathbf{v}(t) \qquad\qquad \mathbf{r}''(t) = \mathbf{a}(t)$$

$$\mathbf{v} = \|\mathbf{r}\|\mathbf{T}$$

acceleration

$$\mathbf{a} = \mathbf{v}' = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}$$

$$\mathbf{a} = \|\mathbf{r}'\|'\mathbf{T} + \kappa \left(\|\mathbf{r}'\|\right)^2 \mathbf{N}$$

tangential acceleration - the change in size of the velocity.

normal acceleration - the change in direction of the velocity.

tangential acceleration

$$a_{\mathbf{T}} = \frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{r}'\| = \mathbf{T} \cdot \mathbf{a} = \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|} = \frac{\mathbf{r}' \cdot \mathbf{r}''}{\|\mathbf{r}'\|}$$

normal acceleration

$$a_{\mathbf{N}} = \kappa \left(\|\mathbf{r}'\|\right)^2 = \|\mathbf{T} \times \mathbf{a}\| = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|} = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|}$$

3 Multivariable functions

finding domain

- if $1/u$, then $u \neq 0$
- if $sqrt[2k]u, k \in \mathbb{Z}^+$, then $u \geq 0$
- if $\log_a(u), a \in \mathbb{R}$, then $u > 0$

check the range: negative, zero, any special numbers.

try to find the most negative number and how to get the most positive number

fix a variable to some number to check

projections:

if two surfaces $f(x, y$ and $g(x, y$ intersect in a curve, then the projection of the intersection onto the xy - plane is the set of all points $(x, y, 0)$ that satisfy $f = g$

level curves show the curves on a function that have the same value.

$$f(x, y) = c$$

3.1 Limits and continuity

definitions:

- the neighborhood of a point $\mathbf{x}_0$

$$\{\mathbf{x} : \|\mathbf{x} - \mathbf{x}_0\| < \delta\} \,\text{where}\, \delta > 0$$

- an interior point $\mathbf{x}_0$ is if a set contains a neighbourhood of $\mathbf{x}_0$. the set of all interior points is the interior of the set
- a boundary point $\mathbf{x}_0$ of a set is if all neighbourhoods of $\mathbf{x}_0$ contains points not in the set
- the boundary of a set is ∂S
- a set is closed if it contains its boundary
- a set is open if it contains a neighbourhood of all its interior points (if all its points are interior points or it contains no boundary points)

definition of a limit:

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = L$$

$$\text{if}\, \forall\, \varepsilon > 0\, \exists\, \delta > 0$$

$$\text{if}\, 0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta \,\text{then}\, |f(x) - L| < \varepsilon$$

if any two paths to limit does not equal each other, then the limit does not exists

curves to use:

- the axis ($y = 0$, $x = 0$)
- lines ($y = kx$)
- parabola $y = x^2$

plug a path into the limit to turn it into a 1 dimensional limit

3.2 Partial derivatives

partial derivatives are found by setting all other variables constant then differentiating with respect to one variable.

the notation is:

$$\frac{\partial}{\partial x} f = \partial_x f = f_x$$

higher order partials

notation:

$$\frac{\partial^2}{\partial x^2} f = f_{xx}$$

if a function and its partial derivatives are continous, then the mixed partials will be equal

equation of tangent line on a plane by setting a variable constant.

slope is partial derivative with respect to the other variables

4 gradient

f is differentiable at $\mathbf{x}$ if $\exists\, \mathbf{y}$ such that

$$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = \nabla f \cdot \mathbf{h} + o(\mathbf{h})$$

where o is the little o

gradient

$$\nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}$$

properties of the gradient

$$\begin{aligned} \nabla[f(\mathbf{x}) + g(\mathbf{x})] &= \nabla f(\mathbf{x}) + \nabla g(\mathbf{x}) \\ \nabla[\alpha f(\mathbf{x})] &= \alpha \nabla f(\mathbf{x}) \\ \nabla[f(\mathbf{x}) g(\mathbf{x})] &= f(\mathbf{x}) \nabla g(\mathbf{x}) + g(\mathbf{x}) \nabla f(\mathbf{x}) \end{aligned}$$

directional derivatives

a directional derivative gives the derivative of f in any direction $\hat{\mathbf{u}}$. it can also be described as the component of the gradient vector in the direction.

$$f'_{\mathbf{u}}(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \hat{\mathbf{u}} = f'_{\mathbf{u}}(\mathbf{x}) = \mathrm{comp}_{\mathbf{u}} \nabla f(\mathbf{x})$$

a function increases most rapidly in the direction of the gradient (rate of change is $\|\nabla f(\mathbf{x})\|$) and decreases most rapidly in the direction opposite of the gradient (rate of change is $-\|\nabla f(\mathbf{x})\|$).

definition 4.1 (mean value theorem of several variables)**.** Let $D \subset \mathbb{R}^n$ be open, $f : D \to \mathbb{R}$ is differentiable, and $\overline{\mathbf{ab}} \in D$. Then $\exists\, \mathbf{c} \in [\mathbf{a}, \mathbf{b}]$ such that

$$f(\mathbf{b}) - f(\mathbf{a}) = \nabla f(\mathbf{c}) \cdot (\mathbf{b} - \mathbf{a})$$

chain rule along a curve

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[f(\mathbf{r}(t))\right] = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$$

4.1 Applications of derivatives

the gradient of a function is normal to the level curve.

suppose $\mathbf{r}(t)$ parameterizes a level curve of f.

then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{r}(t)) &= 0 \\ \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= 0 \end{aligned}$$

as a result,

$$\nabla f(\mathbf{r}(t)) \perp \mathbf{r}'(t)$$

the tangent vector is then

$$\mathbf{t} = \frac{\partial f}{\partial y}\hat{\mathbf{i}} - \frac{\partial f}{\partial x}\hat{\mathbf{i}}$$

the tangent line is

$$\nabla f \cdot \mathbf{d} = 0$$

$$y - y_0 = \frac{\partial f/\partial y}{\partial f/\partial x}(x - x_0)$$

the normal line is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\nabla f(\mathbf{r}_0)$$

the tangent plane to a surface:

$$\nabla f \cdot (\mathbf{r} - \mathbf{r}_0) = f_x(x - x_0) + f_y(y - y_0) + f_z(z - z_0) = 0$$

horizontal tangent planes:

set $\nabla f(x, y) = \mathbf{0}$ and solve the system

$$\begin{aligned} f_x &= 0 \\ f_y &= 0 \end{aligned}$$

the normal vector of this is

the equation of the plane is of the form $z = k, k \in \mathbb{R}$

angle of intersection between a $f : \mathbb{R}^n \to \mathbb{R}$ and a curve

$$\cos\theta = \frac{\mathbf{r}'(t) \cdot \nabla f}{\|\mathbf{r}'(t)\|\|\nabla f\|}$$

definitions:

for a function $f : \mathbb{R}^n \to \mathbb{R}^m$

Jacobian matrix: all the first partials of f arranged in a matrix.

$$\mathbf{J}_{ij} = \frac{\partial f_i}{\partial x_j} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Hessian matrix: all the second partials of f arranged in a matrix.

$$\mathbf{H}_{i,j} = \frac{\partial^2 f}{\partial x_i \partial x_j} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

definition 4.2 (local extrema)**.** Let $f : D \to \mathbb{R}, D \subset \mathbb{R}^n, \mathbf{x}_0 \in D$.

f attains an *local maximum* at $\mathbf{x}_0$ if

$$f(\mathbf{x}_0) \geq f(\mathbf{x}) \,\forall\, \mathbf{x} \,\text{in a neighborhood of}\, \mathbf{x}_0$$

f attains an *local minimum* at $\mathbf{x}_0$ if

$$f(\mathbf{x}_0) \leq f(\mathbf{x}) \,\forall\, \mathbf{x} \,\text{in a neighborhood of}\, \mathbf{x}_0$$

theorem 4.1. *If f has a local extreme value at $\mathbf{x}_0$, then*

$$\nabla f(\mathbf{x}_0) = 0 \; or \, \nabla f(\mathbf{x}_0) \; does\ not\ exist$$

- stationary point - where $\nabla f = 0$
- saddle point - a stationary point where there is no local extrema

second partial derivative test:

Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $\nabla f(\mathbf{x}_0) = 0$ (or does not exist),

- $f(\mathbf{x}_0)$ is a local minimum if

$$\det \mathbf{H}(\mathbf{x}_0) > 0 \,\text{and}\, f_{xx}(\mathbf{x}_0) \,\text{or}\, f_{yy}(\mathbf{x}_0) > 0$$

- $f(\mathbf{x}_0)$ is a local maximum if

$$\det \mathbf{H}(\mathbf{x}_0) > 0 \,\text{and}\, f_{xx}(\mathbf{x}_0) \,\text{or}\, f_{yy}(\mathbf{x}_0) < 0$$

- $f(\mathbf{x}_0)$ is a saddle point if

$$\det \mathbf{H}(\mathbf{x}_0) < 0$$

- the test is inconclusive if

$$\det \mathbf{H}(\mathbf{x}_0) = 0$$

definition 4.3 (absolute extrema)**.** Let $f : D \to \mathbb{R}, D \subset \mathbb{R}^n$.

f attains an *absolute maximum* at $\mathbf{x}_0$ if

$$f(\mathbf{x}_0) \geq f(\mathbf{x}) \, \forall \, \mathbf{x} \in D$$

f attains an *absolute minimum* at $\mathbf{x}_0$ if

$$f(\mathbf{x}_0) \leq f(\mathbf{x}) \, \forall \, \mathbf{x} \in D$$

common parameterisations:

- the line segment from (x_0, y_0) to (x_1, y_1)

$$\begin{aligned} x &= x_0 + t(x_1 - x_0) \\ y &= y_0 + t(y_1 - y_0), t \in [0, 1] \end{aligned}$$

- the circle $\left(x^2 + y^2 = r^2\right)$

$$\begin{aligned} x &= r \cos t \\ y &= r \sin t, t \in [0, 2\pi] \end{aligned}$$

- the ellipse $\left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1\right)$

$$\begin{aligned} x &= a \cos t \\ y &= b \sin t, t \in [0, 2\pi] \end{aligned}$$

- functions of the form $y = f(x)$ from $f(a)$ to $f(b)$

$$\begin{aligned} x &= t \\ y &= f(t), t \in [a, b] \end{aligned}$$

definition 4.4 (bounded set)**.** A set S is *bounded* if $\exists \, R \in \mathbb{R}^+$ such that $\|\mathbf{x}\| \leq R \, \forall \, \mathbf{x} \in S$

in other words, if the set can be surrounded by an n-sphere, then it is bounded.

theorem 4.2 (extreme value theorem)**.** *If f is continuous on a bounded closed set D, then f takes on an absolute maximum value and an absolute minimum value.*

steps to find absolute extrema:

1. find the critical points of f in the interior of D
2. find the endpoints of the boundary of D
3. find the critical points of f on the boundary of D by parametrizing D
4. evaluate f at those points

4.1.1 Lagrange multipliers

the method of Lagrange multipliers is a strategy for finding local extrema of a function subject to constraint.

steps to using Lagrange multipliers

1. set the constraint $g = 0$
2. solve the system for either λ or in terms of λ

$$\begin{aligned} \nabla f &= \lambda \nabla g \\ g &= 0 \end{aligned}$$

3. test points from the solutions of the system

tips:

- when dividing by a variable, consider that the variable may be equal to 0
- when choosing signs of solutions, be sure they match for all λ

4.1.2 differentials and approximation

the differential of f is

$$\mathrm{d}f = \nabla f \cdot \mathbf{h}$$

where $\mathbf{h}$ is a small increment.

to approximate f starting at the point $\mathbf{x}_0$ by an increment $\mathbf{h}$

$$f(\mathbf{x}_0 + \mathbf{h}) = f(\mathbf{x}_0) + \mathrm{d}f(\mathbf{x}_0)$$

4.2 potential functions

given a conservative ector field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$, there is a scalar potential function f that satisfies

$$\nabla f = \mathbf{F}$$

theorem 4.3. *If $\Omega \subset \mathbb{R}^2$, $P : \mathbb{R}^2 \to \mathbb{R}$ and $Q : \mathbb{R}^2 \to \mathbb{R}$ differentiable on a simply connected open region Ω, then*

$$P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} = \nabla f \iff \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \,\forall\, (x, y) \in \Omega$$

steps to find a potential function f of a gradient field $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$

1. integrate with respect to x, keeping y constant. the "constant of integration" is a function solely dependent on y
2. differentiate the result with respect to y
3. equate the result with $Q(x, y)$ and solve for $h(y)$

$$f = \int P(x, y)\,\mathrm{d}x + h(y) + C$$

5 Multiple integrals

5.1 Approximations

double Riemann sum

$$\iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right)\Delta x_i \Delta y_j$$

lower sum:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\min[f(x_{ij}, y_{ij})]\Delta x_i \Delta y_j$$

upper sum:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\max[f(x_{ij}, y_{ij})]\Delta x_i \Delta y_j$$

5.2 Double integrals

theorem 5.1 (Fubini's theorem)**.** *If A and B are complete measure spaces, $f(x,y)$ is $A\times B$ measurable, and $\int_{A\times B}|f(x,y)|\,\mathrm{d}(x,y)<\infty$, then*

$$\int_A\left(\int_B f(x,y)\,\mathrm{d}y\right)\mathrm{d}x = \int_B\left(\int_A f(x,y)\,\mathrm{d}x\right)\mathrm{d}y = \int_{A\times B} f(x,y)\,\mathrm{d}(x,y)$$

Fubini's theorem allows the order of integration to be changed in iterated integrals.

Type I region:

$$\Omega = \{(x,y) : a \le x \le b, g(x) \le y \le h(x)\}$$

then

$$\iint_\Omega f(x,y)\,\mathrm{d}A = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

Type II region:

$$\Omega = \{(x,y) : a \leq y \leq b, g(y) \leq y \leq h(y)\}$$

then

$$\iint_\Omega f(x,y)\,\mathrm{d}A = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

area of a region R

$$A = \iint_R 1\,\mathrm{d}A$$

volume over a region R

$$V = \iint_R f(x,y)\,\mathrm{d}A$$

tip: when finding the volume of a tetrahedron, use u-substitution. for the entire function.

5.3 Triple integrals

5.4 Integrals in other coordinates

area:

$$\iint_\Gamma r\,\mathrm{d}r\,\mathrm{d}\theta$$

volume:

$$\iint_\Gamma F(r,\theta) r\,\mathrm{d}r\,\mathrm{d}\theta$$

mass:

$$M = \iint_\Omega \rho(x,y)\,\mathrm{d}A$$

center of mass:

$$x_{cm} = \frac{1}{M} \iint_\Omega x\rho(x,y)\,\mathrm{d}A$$

$$y_{cm} = \frac{1}{M} \iint_\Omega y\rho(x,y)\,\mathrm{d}A$$

cartesian to spherical

$$\begin{aligned} \rho^2 &= x^2 + y^2 + z^2 & & \\ z &= \rho\cos\phi & x &= r\cos\theta \\ r &= \rho\sin\phi & y &= r\sin\theta \end{aligned}$$

Changing the volume element:

suppose the function $f : \mathbb{R}^n \to \mathbb{R}^n$

then

Using the Jacobian to change coordinates:

the element is

$$|\det(\mathbf{J})|\,\mathrm{d}u\,\mathrm{d}v$$

properties of double integrals:

linearity

$$\iint_R f(x,y) + g(x,y)\,\mathrm{d}A = \iint_R f(x,y)\,\mathrm{d}A + \iint_R g(x,y)\,\mathrm{d}A$$

$$\iint_R cf(x,y)\,\mathrm{d}A = c\iint_R f(x,y)\,\mathrm{d}A$$

if $R = S \cup T$, S and T are disjoint (except boundary overlaps), then

$$\iint_R f(x,y)\,\mathrm{d}A = \iint_S f(x,y)\,\mathrm{d}A + \iint_T f(x,y)\,\mathrm{d}A$$

if $m \le f(x,y) \le M$, then

$$m \times A(R) \le \iint_R f(x,y)\,\mathrm{d}A \le M \times A(R)$$

if $f(x,y) = g(x) \cdot h(y)$, then

$$\int_R f(x,y)\,\mathrm{d}A = \left(\int_a^b g(x)\,\mathrm{d}x\right)\left(\int_c^d h(y)\,\mathrm{d}y\right)$$

Specific area and volume elements

coordinates	area ($\mathrm{d}A$)	volume ($\mathrm{d}V$)
Cartesian	$\mathrm{d}x\,\mathrm{d}y$	$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$
polar	$\mathrm{d}r\,\mathrm{d}\theta$	
cylindrical		$r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$
spherical		$\rho^2 \sin\phi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi$

Mean-value theorem for 2 variable functions

$$\mu_f = \frac{1}{A(R)} \iint_R f(x,y)\,\mathrm{d}A$$

5.5 Applications of multiple integrals

6 Lines and surface integrals

A line integral is defined as:

For a parameterised curve, $\mathbf{r}(t)$, and a vector field $\mathbf{F}$,

$$\int_C \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}[\mathbf{r}(t)] \cdot \mathbf{r}'(t)\,\mathrm{d}t$$

line integral of a scalar function: C has a parameterisation $\mathbf{r}(u)$

$$\int_C f(x,y,z)\,\mathrm{d}s = \int_a^b f(\mathbf{r})\|\mathbf{r}\|\,\mathrm{d}u$$

theorem 6.1 (gradient theorem)**.** *Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$ be continuouly differentiable and C any curve in D that starts at $\mathbf{a}$ and ends at $\mathbf{b}$. then*

$$\int_C \nabla f(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = f(\mathbf{b}) - f(\mathbf{a})$$

potential (conservative) function of a field

for closed curves $\mathbf{a} = \mathbf{b}$, then

$$\int_C \nabla f(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = 0$$

alternate notation for line integrals

$$\int_C \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \int_C P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y$$

definition 6.1 (closed curve)**.** A curve with no endpoints and which completely encloses an area

definition 6.2 (Jordan curve)**.** the image C of an injective continuous map of a circle into the plane

$$\phi : S^1 \to \mathbb{R}^2$$

notes:

- a curve that does not cross itself
- also called a *simple closed curve*
- the positive direction is counterclockwise

definition 6.3 (Jordan region)**.** a region bounded by a Jordan curve

theorem 6.2 (Green's theorem)**.**

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \oint_C P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y = \iint_D (\partial_x Q - \partial_y P)\,\mathrm{d}x\,\mathrm{d}y$$

corollary: if partials are equal, then integral is 0.

area with Green's theorem

$$\oint_C -y\,\mathrm{d}x = \oint_C x\,\mathrm{d}y = \frac{1}{2}\oint_C -y\,\mathrm{d}x + x\,\mathrm{d}y$$

surface integrals

fundamental vector product

$$\mathbf{N} = \mathbf{r}'_u \times \mathbf{r}'_v$$

if $z = f(x, y)$, then

$$\|\mathbf{N}(x, y)\| = \sqrt{(f_x)^2 + (f_y)^2 + 1}$$

surface area:

$$\iint_\Omega \|\mathbf{N}(u, v)\| \, \mathrm{d}u \, \mathrm{d}v$$

$$\iint_\Omega \sqrt{(f_x)^2 + (f_y)^2 + 1} \, \mathrm{d}x \, \mathrm{d}y$$

surface integral

$$\iint_S H(x, y, z) \, \mathrm{d}\sigma$$

surface element:

$$\mathrm{d}\sigma = \|\mathbf{N}(u, v)\| \, \mathrm{d}u \, \mathrm{d}v$$

6.1 divergence and curl

$$\mathbf{F} = f_1 \hat{\mathbf{i}} + f_2 \hat{\mathbf{j}} + f_3 \hat{\mathbf{k}}$$

$$\nabla \cdot \mathbf{F} = \partial_x f_1 + \partial_y f_2 + \partial_z f_3$$

curl

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

divergence: how much a vector field points away from / to a source.

curl: how much rotation a vector field is doing. follows right hand rule

divergence theorem: extension of Green's theorem into 3D

theorem 6.3 (divergence theorem)**.** *Suppose $V \subset \mathbb{R}^n$ is compact and has a piecewise smooth boundary S. If $\mathbf{F}$ is a continuously differentiable vector field defined on a neighborhood of V, then*

$$\iiint_V \nabla \cdot \mathbf{F}\,\mathrm{d}V = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S$$

physical meaning: flux of $\mathbf{F}$ through S, the amount of $\mathbf{F}$ pointing out of S.

the double integral is easier

theorem 6.4 (Stokes' theorem)**.** *Suppose $V \subset \mathbb{R}^n$ is compact and has a piecewise smooth boundary S. If $\mathbf{F}$ is a continuously differentiable vector field defined on a neighborhood of V, then*

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\,\mathrm{d}\mathbf{S}$$

do the line integral