# Advanced Multivariable and Vector Calculus

University of Houston, Spring 2023

MATH 3334 (Dr. Jaramillo)

# Contents

## 0.1 Geometry of $\mathbb{R}^n$

**Definition 0.1** (subspace of $\mathbb{R}^n$)**.**
A nonempty subset $V \subset \mathbb{R}^n$ is a **vector subspace** of $\mathbb{R}^n$ if it is closed under addition and scalar multiplication. That is, if $\mathbf{x}, \mathbf{y} \in V$ and $a \in \mathbb{R}$, then

$$\mathbf{x} + \mathbf{y} \in V$$
$$a\mathbf{x} \in V$$

**Definition 0.2** (vector field)**.**
A **vector field** on $\mathbb{R}^n$ is a function $f : \mathbb{R}^n \to \mathbb{R}^n$ which takes in points and outputs vectors emanating from that point.

**Definition 0.3** (dot product)**.**
Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. The **dot product** is defined by

$$\sum_{i=1}^{n} x_i y_i$$

**Properties of the dot product**. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$

- $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$
- $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$
- The **length** of a vector is: $|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \left( \sum_{i=1}^{n} x_i^2 \right)^{1/2}$
- $\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}||\mathbf{y}| \cos \alpha$, where $\alpha$ is the angle between the vectors.

**Definition 0.4.**

**Definition 0.5.**

**Definition 0.6.**

**Theorem 0.1** (Cauchy-Schwarz inequality)**.**

$$|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}||\mathbf{v}|$$

**Theorem 0.2** (Triangle inequality)**.**

$$|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$$

Volume of a parallelepiped: Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$. The volume of a parallelepiped is

$$V = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

definition of vector field, properties of dot and cross product, Schwartz inequality, Triangle inequality (iincluding proof). How to calculate volume of parallelepiped

## 0.2 Topology of $\mathbb{R}^n$

Open and close sets. Interior, boundary and closure of a set. Convergence of sequences, arithmetic of sequences (including proofs). Uniqueness of the limit of a sequence

## 0.3 Functions

Limits of functions, arithmetic of limits (including proofs). Continuity, uniform continuity. How to show that a limit of a function $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ does not exists. How to show that a limit of a function $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ exists.

## 0.4 Compact sets

Definition of a compact set, Bolzano-Weierstrass theorem, supremum, maximum value, infimum, minimum value. Theorem (Existence of max and min). Equivalence of all norms.