Differential equations

University of Houston - MATH 3331

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Textbook: Arnold, Boggess, Polking - Differential Equations, 2nd edition

Chapters covered: 1, 2, 3.1, 4, 5.1-4, 6.1-2, 8, 9

*The contents in these notes do not correspond exactly to the textbook

Contents

1 Introduction

Mathematical modeling tries to imitate reality and then further tries to predict what will happen by creating a **model**. There are two components to a model:

1. data, which reflects the current state. The set of all possible states is called the **state space**.
2. a rule for evolving in time.

Differential equations provide a deterministic rule for modelling continuous state spaces in continuous time. A single state completely determines a future one.

Definition 1.1: General terminology

An **ordinary differential equation** is an equation in an unknown function (the dependent variable) of a single independent variable that includes one or more derivatives.

The **order** of an ordinary differential equation is the order of the highest derivative that appears.

Let n be the highest order, t the independent variable, and y the dependent variable. The **general form** of an ordinary differential equation is

$$f(t, y, y', \dots, y^{(n)}) = 0.$$

The **normal form** of an ordinary differential equation is

$$y^{(n)} = f(t, y, y', \dots, y^{(n-1)}).$$

The **solution** of an ordinary differential equation is a function $y(t)$ that satisfies

$$f(t, y, y', \dots, y^{(n)}) = 0.$$

General solutions to differential equations are a family of functions that may exclude some finite cases. Differential equations may come with an **initial condition**, which can be used to find a **particular solution**. When solving differential equations with an initial condition, the domain of the solution must be carefully examined. The domain must contain the initial value, and if there are any parts (usually asymptotes) where another initial value could influence the particular solution, they must be removed.

Definition 1.2: Classifications of differential equations

Let t be the independent variable and y be the dependent variable.

Let $p_0(t), p_1(t), \ldots, p_n(t)$ and $g(t)$ be arbitraty differentiable functions. A differential equation is **linear** if it is of the form

$$p_0(t)y + p_1(t)y' + \cdots + p_n(t)y^{(n)} + g(t) = 0.$$

It is **autonomous** if it is of the form

$$y^{(n)} = f(y, y', \ldots, y^{(n-1)}).$$

It is **homogenous** if it is of the form

$$f(t, y, y', \ldots, y^{(n)}) = 0.$$

It is **inhomogenous** if it is of the form

$$f(t, y, y', \ldots, y^{(n-1)}) = g(t),$$

where $g(t) \neq 0$ and is called the **forcing term**.

2 First order differential equations

A **first order differential equation** is a differential equation whose highest order derivative is 1.

Theorem 2.1: Existence theorem

Let f be a real number. So,

$$f = f$$

Theorem 2.2: Uniqueness theorem

Let f be a real number. So,

$$f = f$$

2.1 Seperable

Let f and g be nonzero differentiable functions. A first order differential equation is **separable** if it is of the form

$$y' = f(t)g(y).$$

Steps to solving a separable differential equation:

1. Separate the functions with the corresponding differential.

$$\frac{1}{g(y)}\ \mathrm{d}y = f(t)\ \mathrm{d}t$$

2. Integrate both sides.
3. Solve for $y(t)$.

2.2 Integrating factor

If a first order differential equation has the form

$$y' + f(t)y = g(t)$$

2.3 Variation of parameters

Given a solution to the homogenous solution, assume that the solution is a function times homogenous. Then, Solve.

2.4 Exact differential equations

Differential forms.

Steps.

Integrating factor for exact differential equations. If,

$$\mu =$$

If,

$$\mu =$$

2.5 Applications

Radioactive decay

Let $A(t)$ be the mass of a radioactive object and $\lambda > 0$ a constant. Radioactive decay is represented by

$$A' = -\lambda A.$$

Let A_0 be the initial mass. The solution is

$$A(t) = A_0 \mathrm{e}^{-\lambda t}.$$

Half-life is the time it takes for the original mass to decay to half of it's original size.

$$t_{\text{half}} = \frac{\ln 2}{\lambda}.$$

Newton's law of cooling

Let $T(t)$ be the temperature of the object, A the constant ambient temperature, and k a constant. Then, Newton's law of cooling states that

$$T'(t) = -k(T - A).$$

Let T_0 be the initial temperature of the object. The solution is

$$T(t) = A + (T_0 - A)\mathrm{e}^{-kt}.$$

Projectile motion

Projectile motion is governed by Newton's second law and by the following equations.

$$F = ma = mx'' = -mg$$

which simplifies to

$$x'' = -g.$$

This is a separable equation. The resulting solution is

$$x = v_0 t - \frac{1}{2} g t^2.$$

Introducing air resistance. Linear air resistance:

$$F = ma = mx'' = -mg - kv$$

The solution is

Still separable. Quadratic air resistance:

$$F = ma = mx'' = -mg - kv|v| = -mg + kv^2$$

Population malthusian population (exponential)

$$P' = kP$$

The solution is

$$P(t) = C\mathrm{e}^{kt}$$

logistic

$$P' = kP\left(1 - \frac{P}{K}\right)$$

3 Second order differential equations

Definition 3.1: Wronskian

A first order differential equation is **separable** if it is of the form

$$y' = f(t)g(y).$$

theorems

3.1 Solutions of homogenous equations

Given second order linear differential equation

Consider the characteristic equation

$$p(\lambda) = a\lambda^2 + b\lambda + c.$$

Find the roots of the equation. Depending on the roots of the characteristic equation, there are different real-valued solutions.

- two distinct real roots, λ_1, λ_2
$$y = C_1\mathrm{e}^{\lambda_1 t} + C_2\mathrm{e}^{\lambda_2 t}$$
- complex roots, $\lambda = a \pm b\mathrm{i}$
$$\begin{aligned} y &= C_1 \operatorname{Re}\left(\mathrm{e}^{\lambda t}\right) + C_2 \operatorname{Im}\left(\mathrm{e}^{\lambda t}\right) \\ &= \mathrm{e}^{at}\left[C_1\cos(bt) + C_2\sin(bt)\right] \end{aligned}$$

- one real root, λ

$$y = C_1 e^{\lambda t} + C_2 t e^{\lambda t}$$

3.2 Method of undetermined coefficients

Given second order linear differential equation

Steps to solving using the method of undetermined coefficients

- Guess the term.
- Plug into the differential equation.
- Solve for the coefficients.

3.3 Variation of parameters

Given second order linear differential equation

3.4 Applications

Definition 3.2: Harmonic motion

Let m be the mass, μ the damping constant, k the spring constant, and $F(t)$ an external force. Then, the equation for the motion of vibrating spring is

$$mx'' + \mu x' + kx = F(t).$$

Let $c = \dfrac{\mu}{2m}$, $\omega_0 = \sqrt{\dfrac{k}{m}}$ and $f(t) = \dfrac{1}{m}F(t)$, where ω_0 is called the **nautral frequency** of the spring. The resulting equation is

$$x'' + 2cx' + \omega_0^2 x = f(t).$$

- if $f(t) = 0$, it is **unforced**.
- if $f(t) \neq 0$, it is **forced**. The forcing term will be $f(t) = A\cos(\omega t)$ or $f(t) = A\sin(\omega t)$.
- if $\omega = \omega_0$, the spring experiences **resonance**.
- if $c = 0$, it is **undamped**.
- if $c \neq 0$, it is **damped**.

- if $c < \omega_0$, it is **underdamped**.
- if $c > \omega_0$, it is **overdamped**.
- if $c = \omega_0$, it is **critically damped**.

The various kinds of differential equations for harmonic motion can be solved with the technique in the above section.

amplitude, phase angle

Unforced with no damping regular sin/cos curves.

Unforced with damping

Underdamped ($c < \omega_0$).

Overdamped ($c > \omega_0$).

Critically damped ($c = \omega_0$).

Forced with no damping If $\omega \neq \omega_0$, **beats** happens. Physical phenomenon. Superposition of two waves causes interference. Consider intial values $x(0) = x'(0) = 0$

Let **mean fequency** be

$$\overline{\omega} = \frac{1}{2}\left(\omega_0 + \omega\right).$$

Let **half difference** be

$$\delta = \frac{1}{2}\left|\omega_0 - \omega\right|.$$

Then, the particular solution is

$$x(t) = \frac{A}{2\overline{\omega}\delta}\sin(\delta t)\sin(\overline{\omega}t).$$

And the solution has an **envelope**.

$$E(t) = \left|\frac{A}{2\overline{\omega}\delta}\sin(\delta t)\right|.$$

If $\omega = \omega_0$, **resonance** happens.

Forced with damping. The phase angle is

$$\phi = \operatorname{arccot}\left(\frac{\omega_0^2 - \omega^2}{2c\omega}\right)$$

Define **gain** by

$$G = \frac{1}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + (2c\omega)^2}}.$$

Then, the particular solution is

$$x_{\mathrm{p}}(t) = GA\cos(\omega t - \phi).$$

In the general solution, the homogenous solution x_{h} is called the **transient term**, since as $t \to \infty$, $x_{\mathrm{h}} \to 0$ and the particular solution is called the **steady-state term**, since it does not decay.

4 Systems of differential equations

Transforming higher order linear differential equations to a system of first order equations.

4.1 Applications

5 Other considerations

5.1 Laplace transform

Definition 5.1: Exponential order

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. The function is of **exponential order** if there exists $C, a \in \mathbb{R}$ such that for all $t > 0$,

$$|f(t)| \leq C\mathrm{e}^{at}.$$

Definition 5.2: Laplace transform

Let $f : (0, \infty) \to \mathbb{R}$ be a piecewise differentiable function of exponential order. The Laplace transform of f is

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t.$$

Also useful to know is

$$\mathcal{L}\{f^{(n)}(t)\} = f(t)s^n F(s) - s^{(n-1)}f(0) - \cdots - f^{(n-1)}(0).$$

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
t^n	$\dfrac{n!}{s^{n+1}}$
e^{at}	$\dfrac{1}{s-a}$
$\sin(at)$	$\dfrac{a}{s^2+a^2}$
$\cos(at)$	$\dfrac{s}{s^2+a^2}$
$t^n\mathrm{e}^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
$\mathrm{e}^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$
$\mathrm{e}^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$
$f(t)\mathrm{e}^{at}$	$F(s-a)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$

Using the Laplace transform to solve initial value problems:

1. Laplace transform both sides of the equation.
2. Solve for the Laplace transform of the particular solution.
3. Use the inverse Laplace transform to obtain the particular solution.

5.2 Numerical methods

Definition 5.3: Order

The **order** of a numerical method is

5.2.1 Euler's method

Definition 5.4: Euler's method

Consider the differential equation $y' = f(t, y)$ with $y(t_0) = y_0$ and let h be the step size. The recursive formula for **Euler's method** is

$$y_{n+1} = y_n + hf(t_n, y_n).$$

Euler's method is a first order numerical method. $E \propto h$. To compute Euler's method by hand, table.

5.2.2 Runge-Kutta

Definition 5.5: Second order Runge-Kutta method

Consider the differential equation $y' = f(t, y)$ with $y(t_0) = y_0$ and let h be the step size. Define s_1 and s_2 by

$$
\begin{aligned}
s_1 &= f(t_n, y_n) \\
s_2 &= f(t_n + h, y_n + hs_1)
\end{aligned}
$$

The recursive formula for the second order Runge-Kutta method is

$$y_{n+1} = y_n + \frac{1}{2}h(s_1 + s_2).$$

$E \propto h^2$. To compute the second order Runge-Kutta method by hand, table.

Definition 5.6: Fourth order Runge-Kutta method

Consider the differential equation $y' = f(t, y)$ with $y(t_0) = y_0$ and let h be the step size. Define s_1, s_2, s_3 and s_4 by

$$
\begin{aligned}
s_1 &= f(t_n, y_n) \\
s_2 &= f(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hs_1) \\
s_3 &= f(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hs_2) \\
s_4 &= f(t_n + h, y_n + hs_3)
\end{aligned}
$$

The recursive formula for the second order Runge-Kutta method is

$$y_{n+1} = y_n + \frac{1}{6}h(s_1 + 2s_2 + 2s_3 + s_4).$$

$E \propto h^4$. To compute the fourth order Runge-Kutta method by hand, table.