

Modern Algebra 2

University of Houston, Spring 2024

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1 Background

1.1 Definitions

1.1.1 Vector spaces

norm

Let V be a vector space over \mathbb{R} . A **norm** on V is a function $\|\cdot\| : V \rightarrow [0, \infty)$ such that:

1. $\|x\| = 0$ if and only if $x = 0$ for all $x \in V$
2. $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in V, \alpha \in \mathbb{R}$
3. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$

normed vector space

A **normed vector space** is a vector space equipped with a norm.

norm metric

Given a normed space $(V, \|\cdot\|)$, define the **norm metric** $d : V \times V \rightarrow \mathbb{R}$ by $d(x, y) := \|x - y\|$.

complete metric space

A **complete metric space** is a metric space in which every Cauchy sequence converges.

algebra

normed algebra

A **normed algebra** \mathcal{A} is an algebra over a field which has a sub-multiplicative norm, that is, for all $x, y \in \mathcal{A}$, $\|xy\| \leq \|x\|\|y\|$.

non-degenerate bilinear form

A bilinear map $T : V \times V \rightarrow V$ on a vector space V is **non-degenerate** if and only if for all $x \in V$, if $T(x, y) = 0$ for all $y \in V$, then $x = 0$.

1.1.2 Matrices

operator norm

The **operator norm** on $M_n(\mathbb{R})$ is defined by

$$\|A\| := \sup\{\|Av\|_2 : v \in \mathbb{R}^n \text{ and } \|v\|_2 \leq 1\}$$

The operator norm satisfies $\|AB\| \leq \|A\|\|B\|$ for all $A, B \in M_n$.

Proof. The operator norm is a norm.

1. $\|A\| = 0$ means that $\|Av\|_2 = 0$ for all $\|v\|_2 \leq 1$, so $A = 0$.

2. By properties of the $\|\cdot\|_2$ norm and the supremum,

$$\|\alpha A\| = \sup\{\|(\alpha A)v\|_2\} = \sup\{|\alpha|\|Av\|_2\} = |\alpha| \sup\{\|Av\|_2\} = |\alpha|\|A\|$$

3. Again, by properties of the $\|\cdot\|_2$ norm and the supremum,

$$\|A + B\| = \sup\{\|Av + Bv\|_2\} \leq \sup\{\|Av\|_2 + \|Bv\|_2\} \leq \|A\| + \|B\|$$

The operator norm is submultiplicative.

First, by the definition of the operator norm, $\|A\| \geq \|Au\|_2$ where $u \neq 0$ is a unit vector.

Since u is a unit vector, write $u = \frac{v}{\|v\|_2}$. Finally, $\|A\| \geq \left\|A \frac{v}{\|v\|_2}\right\|_2 = \frac{1}{\|v\|_2} \|Av\|_2$ and so $\|Av\|_2 \leq \|A\| \|v\|_2$. Then,

$$\|AB\| = \sup\{\|A(Bv)\|_2\} \leq \sup\{\|A\| \|Bv\|_2\} \leq \sup\{\|A\| \|B\| \|v\|_2\} \leq \|A\| \|B\|$$

□

general linear group

Let $n \in \mathbb{N}$. The **general linear group** over \mathbb{R} is the set of all $n \times n$ invertible real matrices.

$$\text{GL}_n(\mathbb{R}) := \{A \in \text{M}_n(\mathbb{R}) \mid A \text{ is invertible}\}$$

special linear group

Let $n \in \mathbb{N}$. The **special linear group** over \mathbb{R} is the set of all $n \times n$ invertible real matrices with determinant 1.

$$\text{SL}_n(\mathbb{R}) = \{A \in \text{M}_n(\mathbb{R}) \mid \det(A) = 1\}$$

unitary group

Let $n \in \mathbb{N}$. The **unitary group** is the set of all $n \times n$ invertible complex matrices whose inverse is equal to its conjugate transpose.

$$\text{U}(n) := \{A \in \text{M}_n(\mathbb{C}) \mid A^* A = I\}$$

Properties of unitary matrices:

- $\|Ux\|_2 = \|x\|_2$ for all $x \in \mathbb{C}^n$.

$$\|Ux\|_2^2 = \langle Ux, Ux \rangle = (Ux)^* \cdot Ux = x^* U^* \cdot Ux = x^* \cdot x = \|x\|_2^2$$

- $\|UX\| = \|X\|$ for all $X \in \text{M}_n(\mathbb{C})$.

$$\|UX\| = \sup\{\|UXv\|_2 : \|v\|_2 \leq 1\} = \sup\{\|U(Xv)\|_2\} = \sup\{\|Xv\|_2\} = \|X\|$$

orthogonal group

Let $n \in \mathbb{N}$. The **orthogonal group** is the set of all $n \times n$ invertible real matrices whose inverse is equal to its transpose.

$$\text{O}(n) := \{A \in \text{M}_n(\mathbb{R}) \mid A^\top A = I\}$$

nilpotent matrix

A matrix $T \in M_n(\mathbb{C})$ is **nilpotent** if there exists $k \in \mathbb{N}$ such that $T^k = 0$. Every nilpotent matrix has trace 0.

Jordan block

A **Jordan block** is a square matrix of the form

$$\begin{bmatrix} \lambda_1 & 1 & 0 \\ & \lambda_2 & \\ & & \ddots \\ 0 & & & \lambda_n \end{bmatrix}$$

Jordan normal form

A **Jordan form** is a block diagonal matrix with each block on the diagonal being a **Jordan block**.

Facts:

- Every complex square matrix can be put in Jordan form in some basis, called its **Jordan normal form**.
- **Jordan decomposition**: A matrix can be written as $A = D + N$ where D is diagonal and N is upper triangular with 0 on the diagonal and $DN = ND$. N is nilpotent.
- The Jordan normal form is unique.
- There exists a polynomial such that $p(A) = D$.

generalized eigenspaces

Let V be a finite dimensional vector space and $T : V \rightarrow V$ be a linear map with eigenvalue λ . A **generalized eigenvector** is a $v \in V$ such that $(T - \lambda I)^n v = 0$ for some $n \in \mathbb{N}$. The set of all such generalized eigenvectors corresponding to λ forms the **generalized eigenspace** of V .

simultaneous diagonalizability

1.1.3 Matrix functions

exponential map

For every $A \in M_n(\mathbb{R})$ the sequence of partial sums $\left\{ \sum_{n=0}^N \frac{A^n}{n!} \right\}_{N=1}^{\infty}$ converges. The limit of this sequence is denoted e^A .

$$e^A := \lim_{N \rightarrow \infty} \left(\sum_{n=0}^N \frac{A^n}{n!} \right)$$

The **exponential map** is $\exp : M_n(\mathbb{R}) \rightarrow M_n(\mathbb{R})$ defined by $\exp(A) := e^A$.

Proof. Since all norms on $M_n(\mathbb{R})$ are equivalent, use the operator norm. Then,

$$\sum_{n=0}^{\infty} \left\| \frac{A^n}{n!} \right\| = \sum_{n=0}^{\infty} \frac{\|A^n\|}{n!} \leq \sum_{n=0}^{\infty} \frac{\|A\|^n}{n!} < \infty$$

Thus the sequence of partial sums $\left\{ \sum_{n=0}^N \frac{A^n}{n!} \right\}_{N=1}^{\infty}$ is Cauchy, and since $M_n(\mathbb{R})$ is a complete metric space, the sequence converges. \square

Properties of the exponential map:

1. $e^0 = I$
2. if $A, B \in M_n(\mathbb{R})$ commute ($AB = BA$), then $e^{A+B} = e^A \cdot e^B$.
Since A and B commute,

$$e^A e^B = \left(\sum_{m=0}^{\infty} \frac{A^m}{m!} \right) \left(\sum_{n=0}^{\infty} \frac{B^n}{n!} \right) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{A^m B^n}{m! n!}$$

Next, since the series converges absolutely, let $k = m + n$. Then

$$\begin{aligned} &= \sum_{k=0}^{\infty} \sum_{n=0}^k \frac{A^{k-n} B^n}{(k-n)! n!} = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{n=0}^k \frac{k!}{(k-n)! n!} A^{k-n} B^n \\ &= \sum_{k=0}^{\infty} \frac{\sum_{n=0}^k \binom{k}{n} A^{k-n} B^n}{k!} = \sum_{k=0}^{\infty} \frac{(A+B)^k}{k!} \\ &= e^{A+B} \end{aligned}$$

3. e^A is invertible for any A . The inverse is e^{-A} .
Since A commutes with $-A$, $e^A \cdot e^{-A} = e^{-A} \cdot e^A = e^0 = I$.

logarithmic map

Given $A \in M_n(\mathbb{C})$, define

$$\log(A) := \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(A-I)^n}{n}$$

whenever it is defined.

If $\|A - I\| < 1$, then

$$\sum_{n=1}^{\infty} \left\| (-1)^{n+1} \frac{(A-I)^n}{n} \right\| < \sum_{n=1}^{\infty} \frac{\|A-I\|^n}{n} < \infty$$

hence, $\log(A)$ is defined. Note that even if $\|A - I\| \geq 1$, $\log(A)$ can still converge. For example, if $A - I$ is nilpotent (**unipotent**), then $(A - I)^n$ is eventually 0, so the sum is finite.

1.2 Theorems

1.2.1 Vector spaces

Theorem 1.1

All norms on a finite-dimensional vector space are equivalent.

In other words, if $\|\cdot\|$ and $\|\cdot\|'$ are two norms on a finite-dimensional vector space V , then there are constants $c, d > 0$ such that $\|x\| \leq c\|x\|'$ and $\|x\|' \leq d\|x\|$ for all $x \in V$.

Theorem 1.2

Any two norms on a finite-dimensional vector space give the same topology.

Proof. Suppose $\|\cdot\|$ and $\|\cdot\|'$ are two norms. Let (x_n) be a sequence that converges to x in the $\|\cdot\|$ norm. Then the sequence $\|x_n - x\|$ converges to 0. Since any two norms on a finite-dimensional vector space are equivalent, the sequence $\|x_n - x\|'$ also converges to 0. Thus (x_n) converges to x in the $\|\cdot\|'$ norm.

A similar argument shows that any sequence that converges in the $\|\cdot\|'$ norm also converges in the $\|\cdot\|$ norm.

Since the two norm metrics have the same convergent sequences, they generate the same topology. \square

Theorem 1.3

The topology of $M_n(\mathbb{R})$ is defined by any norm on $M_n(\mathbb{R})$.

Proof. Since $M_n(\mathbb{R})$ is a finite-dimensional vector space, all norms are equivalent and thus give the same topology. \square

Theorem 1.4

Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in a normed vector space. If $\sum_{n \in \mathbb{N}} \|x_n\| < \infty$, then the sequence of partial sums $\left\{ \sum_{n=1}^N x_n \right\}_{N=1}^{\infty}$ is Cauchy.

Proof. By the triangle inequality, for all $N_1 < N_2 \in \mathbb{N}$,

$$\left\| \sum_{n=1}^{N_2} x_n - \sum_{n=1}^{N_1} x_n \right\| = \left\| \sum_{n=N_1+1}^{N_2} x_n \right\| \leq \sum_{n=N_1+1}^{N_2} \|x_n\|$$

Since the sum is finite, $\lim_{n \rightarrow \infty} \|x_n\| = 0$, so the right hand side can be made arbitrarily small. \square

Theorem 1.5

$GL_n(\mathbb{R})$ is open in $M_n(\mathbb{R})$. In fact, in any unital complete normed algebra, the set of all invertible elements is open.

Proof. **Using the determinant**

$$\det(A) = \sum_{\sigma \in S_n} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^n a_{i, \sigma(i)} \right)$$

The determinant is a polynomial, thus is continuous. Since $\det : \operatorname{GL}_n(\mathbb{R}) \rightarrow \mathbb{R} \setminus \{0\}$ is a continuous function and $\mathbb{R} \setminus \{0\}$ is open in \mathbb{R} , $\operatorname{GL}_n(\mathbb{R})$ is open.

Using open sets

Recall the series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ which converges if $|x| < 1$. Similarly if $\|A - I\| < 1$, then

$$A^{-1} = \sum_{n=0}^{\infty} (A - I)^n.$$

Let $X \in \operatorname{GL}_n(\mathbb{R})$ and consider the open ball

$$B = \left\{ Y \in M_n(\mathbb{R}) : \|X - Y\| < \frac{1}{\|X^{-1}\|} \right\}$$

If $Y \in B$, then by sub-multiplicativity of the norm,

$$1 > \|X - Y\| \|X^{-1}\| \geq \|(X - Y)X^{-1}\| = \|I - YX^{-1}\|$$

So YX^{-1} is invertible. Let Z be its inverse. Then $(YX^{-1})Z = I$ implies that $Y = Z^{-1}X$. Thus Y is also invertible, and hence $B \subset \operatorname{GL}_n(\mathbb{R})$. Since every invertible element has an open neighborhood contained in $\operatorname{GL}_n(\mathbb{R})$, $\operatorname{GL}_n(\mathbb{R})$ is open. \square

1.2.2 Matrices

Theorem 1.6

Let $A \in M_n(\mathbb{C})$ be a diagonal matrix

$$A = \begin{bmatrix} \alpha_1 & & & 0 \\ & \alpha_2 & & \\ & & \ddots & \\ 0 & & & \alpha_n \end{bmatrix}$$

Then

$$\|A\| = \max_{1 \leq k \leq n} |\alpha_k|$$

Proof.

$$\begin{aligned} \|A\| &= \sup \{ \|Ax\|_2 \} = \sup \left\{ \left(\sum_{k=1}^n (\alpha_k x_k)^2 \right)^{1/2} \right\} \\ &\leq \sup \left\{ \left(\max_{1 \leq k \leq n} |\alpha_k| \right) \left(\sum_{k=1}^n (x_k)^2 \right)^{1/2} \right\} \\ &= \max_{1 \leq k \leq n} |\alpha_k| \end{aligned}$$

The reverse inequality is given by definition of the operator norm. \square

Theorem 1.7 (Schur decomposition)

Every complex square matrix is unitarily equivalent to an upper triangular matrix.

Theorem 1.8

The set of diagonalizable matrices is dense in $M_n(\mathbb{C})$.

Proof. Let $X \in M_n(\mathbb{C})$ and U be a unitary matrix such that UXU^{-1} is upper triangular. Let

$$A = UXU^{-1} = \begin{bmatrix} \alpha_1 & & & \star \\ & \alpha_2 & & \\ & & \ddots & \\ 0 & & & \alpha_n \end{bmatrix}$$

Let $\varepsilon > 0$ be given. For each $k = 1, 2, \dots, n$, choose $0 \leq \varepsilon_k < \varepsilon$ such that $\alpha_1 + \varepsilon_1, \alpha_2 + \varepsilon_2, \dots, \alpha_n + \varepsilon_n$ are distinct. Let

$$B = \begin{bmatrix} \varepsilon_1 & & & 0 \\ & \varepsilon_2 & & \\ & & \ddots & \\ 0 & & & \varepsilon_n \end{bmatrix}$$

and let $C = A + B$. Then $\|C - A\| = \|B\| < \varepsilon$ and

$$C = \begin{bmatrix} \alpha_1 + \varepsilon_1 & & & \star \\ & \alpha_2 + \varepsilon_2 & & \\ & & \ddots & \\ 0 & & & \alpha_n + \varepsilon_n \end{bmatrix}$$

which is diagonalizable. So $U^{-1}CU$ is also diagonalizable and

$$\|X - U^{-1}CU\| = \|U^{-1}(UXU^{-1} - C)U\| = \|U^{-1}(A - C)U\| = \|A - C\| < \varepsilon$$

□

1.2.3 Matrix functions**Theorem 1.9**

SURJECTIVITY OF $\exp : M_n(\mathbb{C}) \rightarrow GL_n(\mathbb{C})$.

Theorem 1.10

Let $n \in \mathbb{N}$. For every $m \in \mathbb{N}$, the map $f : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ defined by $f(X) = X^m$ is continuous.

Proof. By induction.

Let $m = 1$ and $\varepsilon > 0$ be arbitrary. Choose $\delta = \varepsilon$. Then $\|X - Y\| < \delta$ implies that $\|f(X) - f(Y)\| = \|X - Y\| < \delta = \varepsilon$. So $f(X) = X$ is continuous.

Assume that for any $m - 1$, $f(X) = X^{m-1}$ is continuous. Thus there exists δ_0 such that if $\|X - Y\| < \delta_0$, then $\|X^{m-1} - Y^{m-1}\| < \frac{\varepsilon}{\|Y\|}$.

Next, let $\varepsilon > 0$ be arbitrary. Choose $\delta = \min \left\{ \delta_0, \frac{\varepsilon - \|X^{m-1} - Y^{m-1}\| \|Y\|}{\|X^{m-1}\|} \right\}$. By the triangle inequality and properties of the operator norm, if $\|X - Y\| < \delta$, then

$$\begin{aligned} \|X^m - Y^m\| &= \|X^m - X^{m-1}Y + X^{m-1}Y - Y^m\| \\ &= \|X^{m-1}(X - Y) + (X^{m-1} - Y^{m-1})Y\| \\ &< \|X^{m-1}\| \|X - Y\| + \|X^{m-1} - Y^{m-1}\| \|Y\| \\ &< \|X^{m-1}\| \left(\frac{\varepsilon - \|X^{m-1} - Y^{m-1}\| \|Y\|}{\|X^{m-1}\|} \right) + \|X^{m-1} - Y^{m-1}\| \|Y\| \\ &< \varepsilon \end{aligned}$$

□

Theorem 1.11

THE CONTINUITY OF EXP FUNCTION?

Theorem 1.12

The function \log is continuous on the open set $B_1(I) := \{X \in M_n(\mathbb{C}) : \|X - I\| < 1\}$.

Proof. Let $\varepsilon > 0$ be arbitrary and $X_0 \in B_1(I)$. Then, $\|X_0\| < 2$. Next, define α such that $1 \leq \|X_0\| < \alpha < 2$ and let $N \in \mathbb{N}$ be such that

$$\sum_{k=N}^{\infty} (-1)^{k+1} \frac{(\alpha - 1)^k}{k} < \frac{\varepsilon}{2}$$

Since the map $g : B_1(I) \rightarrow M_n(\mathbb{C})$ defined by

$$g(X) = \sum_{k=1}^{N-1} (-1)^{k+1} \frac{X^k}{k}$$

is continuous, there exists $\delta_0 > 0$ such that if $Y \in B_1(I)$ with $\|X_0 - Y\| < \delta_0$, then $\|g(X_0) - g(Y)\| < \frac{\varepsilon}{2}$. Let $\delta = \min \{\delta_0, \alpha - \|X_0\|\}$. Then, if $Y \in B_1(I)$ and $\|X_0 - Y\| < \delta$,

$$\|\log(X_0) - \log(Y)\| \leq \left\| \sum_{k=1}^{N-1} (-1)^{k+1} \frac{X_0^k - Y^k}{k} \right\| + \sum_{k=N}^{\infty} \frac{\|X_0\|^k + \|Y\|^k}{k} < \frac{3\varepsilon}{2}$$

Additionally, $\log : B_1(I) \rightarrow M_n(\mathbb{C})$ is uniformly continuous. It follows from taking $\delta = 2$ since $\delta \leq \alpha - \|X_0\| < 2$. □

Theorem 1.13

Let $U \in M_n(\mathbb{C})$ be invertible and let $A \in M_n(\mathbb{C})$. Then,

1. $e^{UAU^{-1}} = Ue^AU^{-1}$.
2. if $\|A - I\| < 1$ and $\|UAU^{-1} - I\| < 1$, then $\log(UAU^{-1}) = U \log(A) U^{-1}$.

This can apply to any function defined by a convergent power series. In general, if $f : D \subset M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ such that $f(X) := \sum_{n=0}^{\infty} \alpha^n X^n$, then $f(UXU^{-1}) = Uf(X)U^{-1}$.

Proof.

$$e^{UAU^{-1}} = \sum_{k=0}^{\infty} \frac{(UAU^{-1})^k}{k!} = \sum_{k=0}^{\infty} \frac{UAU^{-1} \cdot UAU^{-1} \cdots}{k!} = \sum_{k=0}^{\infty} \frac{UA^k U^{-1}}{k!} = Ue^A U^{-1}$$

If $\|A - I\| < 1$ and $\|UAU^{-1} - 1\| < 1$, then

$$\begin{aligned} \log(UAU^{-1}) &= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(UAU^{-1} - I)^k}{k} = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{U(A - I)U^{-1} \cdot U(A - I)U^{-1} \cdots}{k} \\ &= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{U(A - I)^k U^{-1}}{k} = U \log(A) U^{-1} \end{aligned}$$

□

Theorem 1.14

The exponential and logarithmic maps are inverses of each other. Let $A \in M_n(\mathbb{C})$. Then,

1. If $\|A - I\| < 1$, then $e^{\log(A)} = A$
2. If $\|A\| < \log(2)$, then $\log(e^A)$ is defined and is equal to A .

Proof. First, note that if $A \in M_n(\mathbb{C})$ is a diagonal matrix

$$A = \begin{bmatrix} \alpha_1 & & 0 \\ & \alpha_2 & \\ & & \ddots \\ 0 & & & \alpha_n \end{bmatrix}$$

then by the Taylor series definition of $\exp : \mathbb{R} \rightarrow \mathbb{R}$,

$$e^A = \begin{bmatrix} e^{\alpha_1} & & 0 \\ & e^{\alpha_2} & \\ & & \ddots \\ 0 & & & e^{\alpha_n} \end{bmatrix}$$

and if $\|A - I\| < 1$, then $|\alpha_k - 1| < 1$ for all $k = 1, 2, \dots, n$ and again by the Taylor series of $\log : (0, 2) \rightarrow \mathbb{R}$,

$$\log(A) = \begin{bmatrix} \log(\alpha_1) & & 0 \\ & \log(\alpha_2) & \\ & & \ddots \\ 0 & & & \log(\alpha_n) \end{bmatrix}$$

Let $A \in M_n(\mathbb{C})$ be diagonalizable and $\|A - I\| < 1$. Let $U \in M_n(\mathbb{C})$ be a unitary matrix such that $B = UAU^{-1}$ is diagonal,

$$B = \begin{bmatrix} \alpha_1 & & 0 \\ & \alpha_2 & \\ & & \ddots \\ 0 & & & \alpha_n \end{bmatrix}$$

Then $\|B - I\| = \|UAU^{-1} - I\| = \|A - I\| < 1$, so $|\alpha_k - 1| < 1$ for all $k = 1, 2, \dots, n$, and

$$\log(B) = \begin{bmatrix} \log(\alpha_1) & & 0 \\ & \log(\alpha_2) & \\ & & \ddots \\ 0 & & & \log(\alpha_n) \end{bmatrix}$$

So

$$e^{\log(B)} = \begin{bmatrix} e^{\log(\alpha_1)} & & 0 \\ & e^{\log(\alpha_2)} & \\ & & \ddots \\ 0 & & & e^{\log(\alpha_n)} \end{bmatrix} = B$$

Hence,

$$e^{\log(A)} = e^{\log(U^{-1}BU)} = e^{U^{-1}\log(B)U} = U^{-1}e^{\log(B)}U = U^{-1}BU = A$$

Thus, for every diagonalizable matrix A with $\|A - I\| < 1$, $e^{\log(A)} = A$. Since the exponential and logarithmic maps are continuous and diagonalizable matrices are dense in $M_n(\mathbb{C})$, $e^{\log(A)} = A$ for all $A \in M_n(\mathbb{C})$ with $\|A - I\| < 1$.

For $\log(e^A)$ to be defined, $\|e^A - I\| < 1$, so

$$\|e^A - I\| = \left\| A + \frac{A^2}{2!} + \dots \right\| \leq \|A\| + \frac{\|A\|^2}{2} + \dots = e^{\|A\|} - 1 < 1$$

hence $e^{\|A\|} < 2$.

Let $A \in M_n(\mathbb{C})$ be diagonalizable and $\|A\| < \log(2)$. Let $U \in M_n(\mathbb{C})$ be a unitary matrix such that $B = UAU^{-1}$ is diagonal,

$$B = \begin{bmatrix} \alpha_1 & & 0 \\ & \alpha_2 & \\ & & \ddots \\ 0 & & & \alpha_n \end{bmatrix}$$

Then

$$e^B = \begin{bmatrix} e^{\alpha_1} & & 0 \\ & e^{\alpha_2} & \\ & & \ddots \\ 0 & & & e^{\alpha_n} \end{bmatrix}$$

with $\|e^B - I\| = \|e^{UAU^{-1}} - I\| = \|Ue^A U^{-1} - I\| = \|e^A - I\| < 1$. So $|e^{\alpha_k} - 1| < 1$ for all $k = 1, 2, \dots, n$, and hence $\log(e^B)$ is defined. Then

$$\log(e^B) = \begin{bmatrix} \log(e^{\alpha_1}) & & 0 \\ & \log(e^{\alpha_2}) & \\ & & \ddots \\ 0 & & & \log(e^{\alpha_k}) \end{bmatrix} = B$$

Hence,

$$\log(e^A) = \log(e^{U^{-1}BU}) = \log(U^{-1}e^B U) = U^{-1}\log(e^B)U = U^{-1}BU = A$$

Thus, for every diagonalizable matrix A with $\|A\| < \log(2)$, $\log(e^A) = A$. Since the exponential and logarithmic maps are continuous and diagonalizable matrices are dense in $M_n(\mathbb{C})$, $\log(e^A) = A$ for all $A \in M_n(\mathbb{C})$ with $\|A\| < \log(2)$. \square

Theorem 1.15

Let $A \in M_n(\mathbb{C})$ and define $\omega : \mathbb{R} \rightarrow GL_n(\mathbb{C})$ by $\omega(t) := e^{tA}$. This a group homomorphism. Moreover, the map is differentiable.

Proof. Using the definition of the derivative, prove for $t = 0$.

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{e^{tA} - I}{t - 0} &= \lim_{t \rightarrow 0} \frac{1}{t} \left(tA + \frac{t^2 A^2}{2!} + \frac{t^3 A^3}{3!} + \dots \right) \\ &= \lim_{t \rightarrow 0} \left(A + \frac{t A^2}{2!} + \frac{t^2 A^3}{3!} + \dots \right) \\ &= A + \lim_{t \rightarrow 0} t \left(\frac{A^2}{2!} + \frac{t A^3}{3!} + \dots \right) \end{aligned}$$

The norm of $\lim_{t \rightarrow 0} \left(\frac{A^2}{2!} + \frac{t A^3}{3!} + \dots \right)$ is bounded.

$$\left\| \lim_{t \rightarrow 0} \left(\frac{A^2}{2!} + \frac{t A^3}{3!} + \dots \right) \right\| \leq \sum_{n=2}^{\infty} \frac{|t|^{n-2} \|A\|^n}{n!} \leq e^{\|A\|}$$

Hence, the limit exists and is 0, ω is differentiable at $t = 0$, and $\omega'(0) = A$. \square

Theorem 1.16

Let $\phi : (\mathbb{R}, +) \rightarrow (\mathbb{R}^{>0}, \cdot)$ be a continuous group homomorphism. There exists $a \in \mathbb{R}$ such that $\phi(t) = e^{ta}$ for all $t \in \mathbb{R}$.

Proof. Let $\phi(1) = b$. Since ϕ is a group homomorphism, for all $k \in \mathbb{N}$, $\phi(k) = b^k$ and $\phi(-k) = b^{-k}$. Next, consider $\phi\left(\frac{1}{2}\right)$.

$$\phi(1) = \phi\left(\frac{1}{2} + \frac{1}{2}\right) = \phi\left(\frac{1}{2}\right) \cdot \phi\left(\frac{1}{2}\right) = b$$

So b has a unique square root. Similarly, for all $\frac{k}{m} \in \mathbb{Q}$, $\phi\left(\frac{k}{m}\right) = b^{k/m}$. Since \mathbb{Q} is dense in \mathbb{R} and ϕ is continuous, for all $t \in \mathbb{R}$, $\phi(r) = b^t$. Changing the base, $\phi(t) = e^{t \log(b)}$, thus $a = \log(b)$. \square

Theorem 1.17

Let $n \in \mathbb{N}$. For every $m \in \mathbb{N}$, there exists $\varepsilon > 0$ such that if $A \in M_n(\mathbb{C})$ with $\|A\| < \varepsilon$, then $B = e^{\frac{1}{m}A}$ is the unique element in $\exp(B_\varepsilon(0))$ such that $B^m = e^A$.

Proof. Let $0 < \varepsilon < \frac{\log(2)}{m}$. Suppose $C \in \exp(B_\varepsilon(0)) := \{e^X \in M_n(\mathbb{C}) : \|X\| < \varepsilon\}$ such that $C^m = e^A$. Then $\|C\| < e^\varepsilon < e^{\frac{\log(2)}{m}} = \sqrt[m]{2}$, and since \exp and \log are inverses, there exists $D \in B_\varepsilon(0)$ such that $e^D = C$. Raising both sides by m , $e^{mD} = e^A$. Since $\|mD\| < \log(2)$ and $\|A\| < \log(2)$, again the functions are inverses of each other, so $\log(e^{mD}) = \log(e^A)$ gets $mD = A$ which implies that $D = \frac{1}{m}A$. So $C = e^{\frac{1}{m}A}$. \square

Theorem 1.18

Let $n \in \mathbb{N}$ and suppose $\omega : \mathbb{R} \rightarrow GL_n(\mathbb{C})$ is a continuous group homomorphism. Then there exists a unique matrix $A \in M_n(\mathbb{C})$ such that $\omega(t) = e^{tA}$ for all $t \in \mathbb{R}$.

Proof. Let $0 < \varepsilon < \frac{\log(2)}{2} < 1$ and $U = \exp(B_\varepsilon(0))$. By continuity of ω , there exists $t_0 > 0$ such that if $|t - 0| \leq t_0$, then $\|\omega(t) - \omega(0)\| < \varepsilon$. Since ω is a group homomorphism, $\omega(0) = I$.

Consider $\omega(t_0) \in U$ and let A be such that $t_0 A = \log(\omega(t_0))$. Note that $t_0 A \in \log(U) = B_\varepsilon(0)$ and hence $\|t_0 A\| < \varepsilon$. Such an A exists since $\|\omega(t_0) - I\| < \varepsilon < 1$. Then $e^{t_0 A} = \omega(t_0)$.

Next, consider $\omega\left(\frac{t_0}{2}\right)$. Since ω is a group homomorphism,

$$\omega(t_0) = \omega\left(\frac{t_0}{2} + \frac{t_0}{2}\right) = \omega\left(\frac{t_0}{2}\right)^2$$

By the previous theorem, since $\|t_0 A\| < \varepsilon$, there exists a unique element $Y \in U$ such that $Y^2 = e^{t_0 A} = \omega(t_0)$. Thus, $Y = \omega\left(\frac{t_0}{2}\right)$.

Repeat the above process for $t_k = \frac{t_0}{2^k}$, $k = \mathbb{N}$. Note that for all $k \in \mathbb{N}$, $\omega\left(\frac{t_0}{2^k}\right) = \omega\left(\frac{t_0}{2^{k+1}} + \frac{t_0}{2^{k+1}}\right) = \omega\left(\frac{t_0}{2^{k+1}}\right)^2$ and that since $\left\|\frac{t_0}{2^k} A\right\| = \frac{1}{2^k} \|t_0 A\| < \varepsilon$, $Y_k = e^{\frac{1}{2}\left(\frac{t_0}{2^k} A\right)} = e^{\frac{t_0}{2^{k+1}} A}$ is the unique element in U such that $Y_k^2 = e^{\frac{t_0}{2^k} A}$.

So for all $k = 0, 1, 2, \dots$, $\omega\left(\frac{t_0}{2^k}\right)$ is defined by $\omega\left(\frac{t_0}{2^k}\right) = e^{\frac{t_0}{2^k} X}$. Since ω is a group homomorphism:

- For each $m \in \mathbb{N}$, define

$$\omega\left(\frac{mt_0}{2^k}\right) = \omega\left(\frac{t_0}{2^k}\right)^m = e^{\frac{mt_0}{2^k} X}$$

- For inverses, if $y = \frac{mt_0}{2^k}$ for some k and m , then

$$\omega(-y) = (\omega(y))^{-1} = (e^{yX})^{-1} = (e^{-yX}) = e^{\frac{-mt_0}{2^k} X}$$

Thus for all $y \in \left\{\frac{mt_0}{2^k}, m \in \mathbb{Z}, k = 0, 1, 2, \dots\right\}$, $\omega(y) = e^{yX}$. Finally, since the set of all such y is dense in \mathbb{R} and ω is a continuous map, for all $t \in \mathbb{R}$, $\omega(t) = e^{tX}$. \square

Theorem 1.19

For every $A \in M_n(\mathbb{C})$, $\det(e^A) = e^{\text{tr}(A)}$.

Proof. Let $X \in M_n(\mathbb{C})$ be diagonal. FINISH \square

Theorem 1.20 (Lie product, Trotter product)

Let $n \in \mathbb{N}$ and $X, Y \in M_n(\mathbb{C})$. Then

$$e^{X+Y} = \lim_{k \rightarrow \infty} \left(e^{\frac{X}{k}} e^{\frac{Y}{k}}\right)^k$$

Proof. For all $k \in \mathbb{N}$,

$$e^{\frac{X}{k}} e^{\frac{Y}{k}} = \left(I + \frac{X}{k} + \frac{X^2}{k^2 2!} + \dots\right) \left(I + \frac{Y}{k} + \frac{Y^2}{k^2 2!} + \dots\right) = I + \frac{X}{k} + \frac{Y}{k} + \frac{1}{k^2} C_k$$

where there exists $c > 0$ such that $\|C_k\| < c$ for all k . Why? Observe that by submultiplicativity and the triangle inequality,

$$\|e^{\frac{X}{k}} e^{\frac{Y}{k}}\| \leq e^{\frac{\|X\|}{k}} e^{\frac{\|Y\|}{k}} = e^{\frac{\|X\| + \|Y\|}{k}}$$

Since the exponential map is continuous, $\lim_{k \rightarrow \infty} e^{\frac{X}{k}} e^{\frac{Y}{k}} = I$ so for large enough k , $\|e^{\frac{X}{k}} e^{\frac{Y}{k}} - I\| < 1$ so $\log\left(e^{\frac{X}{k}} e^{\frac{Y}{k}}\right)$ is defined. Next,

$$\begin{aligned} \log\left(e^{\frac{X}{k}} e^{\frac{Y}{k}}\right) &= \log\left(I + \frac{X}{k} + \frac{Y}{k} + \frac{1}{k^2}C_k\right) \\ &= \left(\frac{X}{k} + \frac{Y}{k} + \frac{1}{k^2}C_k\right) - \frac{1}{2}\left(\frac{X}{k} + \frac{Y}{k} + \frac{1}{k^2}C_k\right)^2 + \frac{1}{3}\left(\frac{X}{k} + \frac{Y}{k} + \frac{1}{k^2}C_k\right)^3 - \dots \\ &= \frac{X}{k} + \frac{Y}{k} + \frac{1}{k^2}D_k \end{aligned}$$

And D_k is uniformly bounded in norm, so there exists $d > 0$ such that $\|D_k\| < d$ for all k large enough. Thus, multiply both sides by k and exponentiating,

$$\begin{aligned} k \log\left(e^{\frac{X}{k}} e^{\frac{Y}{k}}\right) &= k\left(\frac{X}{k} + \frac{Y}{k} + \frac{1}{k^2}D_k\right) \\ \left(e^{\frac{X}{k}} e^{\frac{Y}{k}}\right)^k &= e^{X+Y+\frac{1}{k}D_k} \end{aligned}$$

Thus, since the norm of D_k is bounded uniformly, that is, $\|D_k\| < \infty$ for large enough k ,

$$\lim_{k \rightarrow \infty} \left(e^{\frac{X}{k}} e^{\frac{Y}{k}}\right)^k = e^{X+Y}$$

□

Theorem 1.21

Let $\phi : \mathbb{R} \rightarrow M_n(\mathbb{C})$ and $\psi : \mathbb{R} \rightarrow M_n(\mathbb{C})$ be differentiable functions. Then for all $s \in \mathbb{R}$,

$$(\phi\psi)'(s) = \phi'(s)\psi(s) + \phi(s)\psi'(s)$$

Proof.

$$\begin{aligned} (\phi\psi)'(s) &= \lim_{x \rightarrow s} \frac{\phi(x)\psi(x) - \phi(s)\psi(s)}{x - s} \\ &= \lim_{x \rightarrow s} \frac{\phi(x)\psi(x) - \phi(s)\psi(s) + \phi(s)\psi(x) - \phi(s)\psi(x)}{x - s} \\ &= \lim_{x \rightarrow s} \frac{\phi(x)\psi(x) - \phi(s)\psi(s) + \phi(s)\psi(x) - \phi(s)\psi(x)}{x - s} \\ &= \lim_{x \rightarrow s} \left(\frac{(\phi(x) - \phi(s))\psi(x)}{x - s} + \frac{\phi(s)(\psi(x) - \psi(s))}{x - s} \right) \\ &= \phi'(s)\psi(s) + \phi(s)\psi'(s) \end{aligned}$$

□

Theorem 1.22

Let $\phi : \mathbb{R} \rightarrow M_n(\mathbb{C})$ be differentiable and $A, B \in M_n(\mathbb{C})$. Let $\psi : \mathbb{R} \rightarrow M_n(\mathbb{C})$ be defined by $\psi(t) = A\phi(t)B$. Then ψ is also differentiable and $\psi'(t) = A\phi'(t)B$.

Proof.

$$\begin{aligned}
 \psi'(t) &= \lim_{x \rightarrow t} \frac{\psi(x) - \psi(t)}{x - t} \\
 &= \lim_{x \rightarrow t} \frac{A\phi(x)B - A\phi(t)B}{x - t} \\
 &= \lim_{x \rightarrow t} \frac{A(\phi(x) - \phi(t))B}{x - t} \\
 &= A \left(\lim_{x \rightarrow t} \frac{\phi(x) - \phi(t)}{x - t} \right) B \\
 &= A\phi'(t)B
 \end{aligned}$$

□

1.3 Examples

1.3.1 Vector spaces

normed vector space

Let $V = \mathbb{R}^m$ and let $1 \leq p < \infty$. The a norm on \mathbb{R}^m is

$$\|(a_1, a_2, \dots, a_m)\|_p = \left(\sum_{i=1}^m |a_i|^p \right)^{1/p}$$

complete metric space

$M_n(\mathbb{R})$ is a complete metric space since $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$.

normed algebra

- $M_n(\mathbb{R})$ with the operator norm is a normed algebra. In fact, since $M_n(\mathbb{R})$ is finite-dimensional, every norm on it will induce a normed algebra.
- The set of continuous functions on a closed interval $f : I \rightarrow \mathbb{R}$ equipped with the sup norm $\|f\|_\infty = \sup_{x \in I} \{|f(x)|\}$ is a normed algebra.
- Not every norm on an infinite-dimensional vector space induces a normed algebra.

1.3.2 Matrices

Jordan form

2 Matrix Lie groups

2.1 Definitions

manifold

An n -dimension **manifold** is a second-countable Hausdorff topological space X such that for every $x \in X$, there is an open set $x \in V \subseteq X$ and a homeomorphism ϕ from V onto an open neighborhood of $0 \in \mathbb{R}^n$ or \mathbb{C}^n (real or complex manifold).

smooth manifold

A manifold X is **smooth** if and only if for all $x, y \in X$ and homeomorphisms $\phi : V_x \rightarrow \phi(V_x) \subseteq \mathbb{R}^n$ and $\psi : V_y \rightarrow \psi(V_y) \subseteq \mathbb{R}^n$, the map $\psi \circ \phi^{-1} : \phi(V_x \cap V_y) \rightarrow \psi(V_x \cap V_y)$ is smooth (infinitely differentiable).

Lie group

A **Lie group** is a group G that is a smooth manifold where the group operation $G \times G \rightarrow G, (g, h) \mapsto gh$ and the inverse operation $G \rightarrow G, g \mapsto g^{-1}$ are smooth maps.

matrix Lie group

A **matrix Lie group** is a closed subgroup of $GL_n(\mathbb{R})$ (or \mathbb{C}) for some $n \in \mathbb{N}$.

Lie algebra of a matrix Lie group

Let $G \subseteq GL_n(\mathbb{C})$ be a matrix Lie group. The **Lie algebra** of G is the set

$$\mathfrak{g} := \{X \in M_n(\mathbb{C}) \mid e^{tX} \in G \text{ for all } t \in \mathbb{R}\}$$

with the **Lie bracket** $[X, Y] = XY - YX$.

complex matrix Lie group

A matrix Lie group $G \subseteq GL_n(\mathbb{C})$ is **complex** if its Lie algebra \mathfrak{g} is a complex subspace of $M_n(\mathbb{C})$. In other words, if for all $X \in \mathfrak{g}$, $iX \in \mathfrak{g}$.

representation

Let G be a group and V a vector space. A **representation** of G on V is a group homomorphism $\pi : G \rightarrow \text{Lin}(V)$ where $\text{Lin}(V) := \{T : V \rightarrow V \mid T \text{ is linear and bijective}\}$, the set of all linear and bijective maps on V .

adjoint representation of a matrix Lie group

Let G be a matrix Lie group and \mathfrak{g} its Lie algebra. For all $g \in G$, define $\text{Ad}_g : \mathfrak{g} \rightarrow \mathfrak{g}$ for all $X \in \mathfrak{g}$ by

$$\text{Ad}_g(X) = gXg^{-1}$$

This gives a representation of G on \mathfrak{g} called the **adjoint representation** of G .

analytical subgroup

Let G be a matrix Lie group and H a subgroup of G that is not necessarily closed. H is an **analytical subgroup** if

1. The set $L = \{X \in \mathfrak{g} \mid e^{tX} \in H \text{ for all } t \in \mathbb{R}\}$.
2. H is generated by the set $\{e^X : X \in L\}$.

universal cover

Let G be a matrix Lie group. A **universal cover** for G is a simply connected Lie group \tilde{G} such that there exists a continuous group homomorphism $\phi : \tilde{G} \rightarrow G$ such that $d\phi : \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$ is an isomorphism.

2.2 Theorems

Theorem 2.1

Let $G \subseteq \text{GL}_n(\mathbb{C})$ be a matrix Lie group. Then the Lie algebra \mathfrak{g} of G is a real subspace of $M_n(\mathbb{C})$.

Proof. First, note that $0 \in \mathfrak{g}$. Next, let $X \in \mathfrak{g}$ and $\alpha \in \mathbb{R}$. Then $e^{t(\alpha X)} = e^{t\alpha X} \in G$ for all $t \in \mathbb{R}$, hence $\alpha X \in \mathfrak{g}$.

Let $X, Y \in \mathfrak{g}$. Then for every $k \in \mathbb{N}$ and $t \in \mathbb{R}$, $e^{\frac{tX}{k}} \cdot e^{\frac{tY}{k}} \in G$. So $\left(e^{\frac{tX}{k}} \cdot e^{\frac{tY}{k}}\right)^k$ is also in G . Since G is closed,

$$\lim_{k \rightarrow \infty} \left(e^{\frac{tX}{k}} \cdot e^{\frac{tY}{k}}\right)^k = e^{tX+tY} = e^{t(X+Y)} \in G$$

Hence $X + Y \in \mathfrak{g}$. □

Theorem 2.2

If $g \in G$ and $X \in \mathfrak{g}$, then $gXg^{-1} \in \mathfrak{g}$. In other words, a matrix Lie group G acts on its Lie algebra \mathfrak{g} by conjugation.

Proof. Since g is invertible,

$$e^{t(gXg^{-1})} = ge^{tX}g^{-1}$$

□

Theorem 2.3

For all $X, Y \in \mathfrak{g}$, $XY - YX \in \mathfrak{g}$.

Proof. Define $\phi(t) := e^{tX}Ye^{-tX}$. Note that for all $t \in \mathbb{R}$, $\phi(t) \in \mathfrak{g}$ since $e^{tX} \in G$. Since \mathfrak{g} is a real subspace of $M_n(\mathbb{C})$, for all $t \neq 0 \in \mathbb{R}$,

$$\frac{\phi(t) - \phi(0)}{t - 0} \in \mathfrak{g}$$

Moreover, \mathfrak{g} is a finite-dimensional normed vector subspace of $M_n(\mathbb{C})$, thus it is also complete and a closed subspace. Thus

$$\phi'(0) = \lim_{t \rightarrow 0} \frac{\phi(t) - \phi(0)}{t - 0} \in \mathfrak{g}$$

Finally, by the product rule,

$$\phi'(0) = \left. \frac{d}{dt} (e^{tX}) \right|_{t=0} \cdot Ye^{-tX} + e^{tX} \cdot \left. \frac{d}{dt} (Ye^{-tX}) \right|_{t=0} = XY - YX$$

□

Theorem 2.4

Let G be a matrix Lie group and \mathfrak{g} its Lie algebra. If G is abelian, then \mathfrak{g} is abelian.

Proof. Let $X, Y \in \mathfrak{g}$. Then

$$\begin{aligned}
[X, Y] &= \left. \frac{d}{dt} (e^{tX} Y e^{-tX}) \right|_{t=0} \\
&= \left. \frac{d}{dt} \left(e^{tX} \left(\left. \frac{d}{ds} e^{sY} \right|_{s=0} e^{-tX} \right) \right) \right|_{t=0} \\
&= \left. \frac{d}{dt} \left(\left. \frac{d}{ds} (e^{tX} e^{sY} e^{-tX}) \right|_{s=0} \right) \right|_{t=0} \\
&= \left. \frac{d}{dt} (Y) \right|_{t=0} \\
&= 0
\end{aligned}$$

□

Theorem 2.5

Let G and H be matrix Lie groups and $\phi : G \rightarrow H$ be a continuous group homomorphism. Then there exists a unique Lie algebra homomorphism $T : \mathfrak{g} \rightarrow \mathfrak{h}$ such that $\phi(e^X) = e^{T(X)}$ for all $X \in \mathfrak{g}$.

Proof. Let $X \in \mathfrak{g}$ and define $\omega : \mathbb{R} \rightarrow H$ by $\omega(t) = \phi(e^{tX})$. Then ω is a continuous group homomorphism, hence there exists a unique $Y \in M_n(\mathbb{C})$ such that $\omega(t) = e^{tY}$. Since $e^{tY} = \omega(t) = \phi(e^{tX}) \in H$ for all $t \in \mathbb{R}$, $Y \in \mathfrak{h}$.

Define $T : \mathfrak{g} \rightarrow \mathfrak{h}$ such that $T(X)$ is the unique element Y from above. Thus, by construction, $\phi(e^X) = e^{T(X)}$ for all $X \in \mathfrak{g}$.

Next, let $X \in \mathfrak{g}$ and $s \in \mathbb{R}$.

$$T(sX) = e^{tT(sX)} = \phi(e^{tsX}) = e^{tsT(X)}$$

□

2.3 Examples

Lie group

Consider $GL_n(\mathbb{R}) \subset M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$. For each $X \in GL_n(\mathbb{R})$, take the open set around X to be all of $GL_n(\mathbb{R})$ and the identity homomorphism from $GL_n(\mathbb{R})$ to \mathbb{R}^{n^2} . Thus, $GL_n(\mathbb{R})$ is a manifold. The identity map, matrix multiplication, and matrix inverse operations are all smooth, so $GL_n(\mathbb{R})$ is a smooth manifold and a Lie group.

matrix Lie group

Recall that the determinant $\det : GL_n(\mathbb{C}) \rightarrow \mathbb{C} \setminus \{0\}$ is a continuous function.

- $GL_n(\mathbb{R})$: It is closed in itself.
- $SL_n(\mathbb{R})$: $\det(A) = 1$, so $\det : SL_n(\mathbb{R}) \rightarrow \{1\}$, and $\{1\}$ is closed in \mathbb{R} .
- $U(n)$: $\det(A^*A) = \overline{\det(A)} \det(A) = |\det(A)| = 1$, so $\det : U(n) \rightarrow \{z \in \mathbb{C} : |z| = 1\}$, which is closed in \mathbb{C} .
- $O(n)$: $\det(A^T A) = (\det(A))^2 = 1$, so $\det : O(n) \rightarrow \{-1, 1\}$, and $\{-1, 1\}$ is closed in \mathbb{R} .

Lie algebras

- $\mathfrak{gl}_n(\mathbb{C}) = M_n(\mathbb{C})$. If $X \in M_n(\mathbb{C})$, then e^{tX} is invertible.
- $\mathfrak{sl}_n(\mathbb{C}) = \{X \in M_n(\mathbb{C}) \mid \text{tr}(X) = 0\}$

$$\det(e^{tA}) = e^{t \cdot \text{tr}(A)} = 1$$

implies that $\text{tr}(A) = 0$ for all $t \in \mathbb{R}$.

- $\mathfrak{u}(n) = \{X \in M_n(\mathbb{C}) \mid X^* = -X\}$

$$(e^{tX})(e^{tX})^* = (e^{tX})(e^{tX^*}) = e^{t(X+X^*)} = I$$

This implies that $X + X^* = 0$ so $X^* = -X$. These are called **skew-symmetric matrices**.

- $\mathfrak{sl}_n(\mathbb{R}) = \{X \in M_n(\mathbb{R}) \mid X^T = -X\}$

$$(e^{tX})(e^{tX})^T = (e^{tX})(e^{tX^T}) = e^{t(X+X^T)} = I$$

This implies that $X + X^T = 0$ so $X^T = -X$.

complex matrix Lie group

- $\text{GL}_n(\mathbb{C})$
- $\text{GL}_2(\mathbb{R})$ is not a complex matrix Lie group. Let $X = \begin{bmatrix} 1 & 0 \\ 0 & \frac{\pi}{2} \end{bmatrix}$. For all $t \in \mathbb{R}$, $e^{tX} = \begin{bmatrix} e^t & 0 \\ 0 & e^{t\frac{\pi}{2}} \end{bmatrix} \in \text{GL}_2(\mathbb{R})$, so $X \in \mathfrak{g}$. However, for $t = 1$, $e^{itX} = \begin{bmatrix} e^{it} & 0 \\ 0 & e^{it\frac{\pi}{2}} \end{bmatrix} \notin \text{GL}_2(\mathbb{R})$, so $iX \notin \mathfrak{g}$.

Lie algebra homomorphism

3 Lie algebras

3.1 Definitions

Lie algebra

A **Lie algebra** is a vector space V endowed with a binary operation $[\cdot, \cdot] : V \times V \rightarrow V$ called the **Lie bracket** that satisfies for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$,

1. $[\cdot, \cdot]$ is bilinear.
2. $[\mathbf{x}, \mathbf{y}] = -[\mathbf{y}, \mathbf{x}]$
3. $[\mathbf{x}, [\mathbf{y}, \mathbf{z}]] + [\mathbf{y}, [\mathbf{z}, \mathbf{x}]] + [\mathbf{z}, [\mathbf{x}, \mathbf{y}]] = 0$. (the **Jacobi identity**).

The **Lie bracket** is anticommutative, not associative, and satisfies the Jacobi identity. For all $A, B, X, Y, Z \in M_n(\mathbb{C})$,

- $[A, B] = AB - BA = -BA + AB = -(BA - AB) = -[B, A]$
- $[[X, Y], Z] = [XY - YX, Z] = XYZ - YXZ - ZXY + ZYX$ but $[X, [Y, Z]] = [X, YZ - ZY] = XYZ - XZY - YZX + ZYX$
- $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = [X, YZ - ZY] + [Y, ZX - XZ] + [Z, XY - YX] = XYZ - XZY - YZX + ZYX + YZX - YXZ - ZXY + XZY + ZXY - ZYX - XYZ + YXZ = 0$

commutative Lie algebra

A Lie algebra V is **commutative** if and only if for all $x, y \in V$, $[x, y] = 0$.

Lie algebra homomorphism

Let \mathfrak{g} and \mathfrak{h} be Lie algebras. A map $T : \mathfrak{g} \rightarrow \mathfrak{h}$ is a **homomorphism** if and only if T is linear and

$$T([X, Y]) = [T(X), T(Y)]$$

Lie subalgebra

Let L be a Lie algebra. A **Lie subalgebra** is a subset L' such that L' is a subspace of L and for all $x, y \in L'$, $[x, y] \in L'$.

ideal

Let L be a Lie algebra. An ideal of L is a subspace $I \subseteq L$ such that for all $x \in I$ and $y \in L$, $[x, y] \in I$.

quotient Lie algebra

The quotient Lie algebra is L/I where I is an ideal of L equipped with the Lie bracket $[x + I, y + I] := [x, y] + I$.

The Lie bracket is well-defined:

Proof.

$$\begin{aligned} [x, y] - [x' - y'] &\in I \\ [x, y] + [x + y'] - [x, y'] &= [x' - y'] \\ [x, y - y'] - [x - x', y'] &\in I \end{aligned}$$

□

direct sum of Lie algebras

Let L_1, L_2 be Lie algebras. The **direct sum** $L_1 \oplus L_2$ is the direct sum of vector spaces L_1 and L_2 with Lie bracket

$$[(x_1, x_2), (y_1, y_2)] := ([x_1, y_1], [x_2, y_2])$$

The Lie bracket is a Lie bracket.

Proof. $[(y, y'), (x, x')] = ([x, y], [y', x']) = (-[x, y], -[x', y']) = -[(x, y), (x', y')]$. $[(\alpha x_1 + x_2, \alpha x'_1 + x'_2), (y, y')] = ([\alpha x_1 + x_2, y], [\alpha x'_1 + x'_2, y']) = (\alpha[x_1, y] + [x_2, y], \alpha[x'_1, y'] + [x'_2, y'])$.

$$\begin{aligned} [(x, x'), [(y, y'), (z, z')]] &+ [(y, y'), [(z, z'), (x, x')]] + [(z, z'), [(x, x'), (y, y')]] \\ &= [(x, x'), ([y, z], [y', z'])] + \dots = \\ &= ([x, [y, z]], [x', [y', z']]) + \dots \end{aligned}$$

□

algebra

Let \mathbb{F} be \mathbb{R} or \mathbb{C} . An **algebra** is a vector space \mathcal{A} with a multiplication that turns it into a ring, and for all $\alpha \in \mathbb{F}$, $a, b \in \mathcal{A}$,

$$\alpha(a \cdot b) = (\alpha a) \cdot b = a \cdot (\alpha b)$$

A Lie algebra can be constructed from an algebra by defining the Lie bracket $[\cdot, \cdot] : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ by $[x, y] := xy - yx$.

derivation

Let \mathcal{A} be an algebra. A **derivation** on \mathcal{A} is a linear map $D : \mathcal{A} \rightarrow \mathcal{A}$ such that $D(xy) = D(x) \cdot y + x \cdot D(y)$.

If D_1 and D_2 are derivations on \mathcal{A} , then $D_1 \circ D_2$ is not a derivation on \mathcal{A} , however, $\alpha D_1 + D_2$ is a derivation on \mathcal{A} for all $\alpha \in \mathbb{F}$. Thus, the set $\mathcal{D}(\mathcal{A})$ of all derivations $D : \mathcal{A} \rightarrow \mathcal{A}$ is a subspace of $\text{Lin}(\mathcal{A})$, the set of all linear maps from \mathcal{A} to \mathcal{A} .

representation of a Lie algebra

Let L be a Lie algebra. A **representation** of L on a vector space V is a Lie algebra homomorphism $\pi : L \rightarrow \text{Lin}(V)$ (the set of all linear maps on V) equipping $\text{Lin}(V)$ with $[T, S] := T \circ S - S \circ T$.

adjoint representation of a Lie algebra

Let L be a Lie algebra. For every $x \in L$, define $\text{ad}_x : L \rightarrow L$ for all $y \in L$ by

$$\text{ad}_x(y) = [x, y]$$

This is the **adjoint representation** of L on $\text{Lin}(L)$ $\text{ad} : L \rightarrow \text{Lin}(L)$.

Proof.

□

Killing form

Let L be a Lie algebra. The **Killing form** of L is a bilinear map $B : L \times L \rightarrow \mathbb{F}$ by $B(x, y) := \text{tr}(\text{ad}_x \circ \text{ad}_y)$.

derived Lie subalgebra

Let L be a Lie algebra. The **derived Lie subalgebra** of L is

$$L' := [L, L] = \text{span} \{[a, b] \mid a \in L, b \in L\}$$

derived series

Denote the derived Lie subalgebra by $L^{(1)} := L'$ and $L^{(2)} := (L^{(1)})'$, and for every $r \in \mathbb{N}$, $L^{(r+1)} := (L^{(r)})'$. Then

$$L \supseteq L' \supseteq L^{(2)} \supseteq \dots$$

which is the **derived series** of L .

solvable Lie algebra

A Lie algebra is **solvable** if there exists $r \in \mathbb{N}$ such that $L^{(r)} = \{0\}$.

center

The **center** of a Lie algebra L is

$$Z(L) := \{x \in L \mid [x, y] = 0 \text{ for all } y \in L\}$$

simple Lie algebra

A Lie algebra L is **simple** if $\dim(L) > 1$ and L has no proper non-zero ideals.

radical

The **radical** of a finite dimensional Lie algebra is the unique maximal solvable ideal. If I is any solvable ideal of L , then $I \subseteq r$.

reductive Lie algebra

A **reductive Lie algebra** is one that is the direct sum of a simple and a one-dimensional Lie algebra.

commutator

$$\mathcal{C} := \{x \in L \mid [x, A] = 0 \text{ for all } A \in \mathcal{A}\}$$

3.2 Theorems

Theorem 3.1

If D and D' are derivations on an algebra \mathcal{A} , then $D \circ D' - D' \circ D$ is also a derivation. $\mathcal{D}(\mathcal{A})$ with this product $[D, D']$ is a Lie algebra.

Proof. Let $x, y \in A$.

$$\begin{aligned} (D \circ D' - D' \circ D)(xy) &= D \circ D'(xy) - (D' \circ D)(xy) \\ &= D \circ D'(x) \cdot y + D'(x)D(y) + D(x)D'(y) + \\ &\quad x \cdot (D \circ D')(y) - (D' \circ D)(x) \cdot y \end{aligned}$$

FINISH

□

Theorem 3.2 (first isomorphism theorem)

Let $\phi : L \rightarrow L'$ be a Lie algebra homomorphism. The $\ker \phi$ is an ideal of L and $L / \ker \phi \cong \text{Im}(\phi)$.

Proof. $\ker \phi$ is an ideal.

Since ϕ is a Lie algebra homomorphism, it is linear, so $\ker \phi$ is a vector space. Let $x \in \ker \phi$ and $y \in L$. Then $\phi([x, y]) - [\phi(x), y] = [0, y] = 0$ by bilinearity. So $\ker \phi$ is closed under the Lie bracket, hence it is an ideal.

Define $\psi(x + \ker \phi) = \phi(x)$. The map is well-defined:FINISH Linear map.... bijective,

□

Theorem 3.3

If $\phi : L \rightarrow P$ is a Lie algebra homomorphism, then $\phi(L') \subseteq P'$. Moreover, if ϕ is surjective, then $\phi(L') = P'$. This implies that for all $r \in \mathbb{N}$, $\phi(L^{(r)}) \subseteq P^{(r)}$.

Proof. First, $\phi(L') = \phi([L, L]) = [\phi(L), \phi(L)] \subseteq [P, P] = P'$. If ϕ is surjective, then $\phi(L) = P$, so $\phi(L') = P'$. Note that $\phi : L' \rightarrow P'$ is a homomorphism of Lie subalgebras since it inherits linearity and the bracket operation. So by repeatedly deriving L , the theorem applies for all $L^{(r)}$, $r \in \mathbb{N}$. \square

Theorem 3.4

Let I be an ideal of L . L is solvable if and only if both I and L/I are solvable.

Proof. Suppose L is solvable. Since $I^{(r)} \subseteq L^{(r)}$, I is solvable. By the above theorem, $(L/I)^{(r)} = \pi(L^{(r)})$ for all $r \in \mathbb{N}$ where $\pi : L \rightarrow L/I$ is the canonical projection. So L/I is solvable.

Suppose L/I and I are solvable. Consider again the canonical projection $\pi : L \rightarrow L/I$. Then $(L^{(r)}) = (L/I)^{(r)} = \{0\}$ for some $r \in \mathbb{N}$. So $L^r \subseteq I$ for some $r \in \mathbb{N}$. Then for every $s \in \mathbb{N}$, $(L^r)^s = L^{(r+s)} \subseteq I^{(s)}$. Since I is solvable, $L^k = \{0\}$ for some $k \in \mathbb{N}$. \square

Theorem 3.5 (Second isomorphism theorem)

Let L be a finite-dimensional Lie algebra and I, J be ideals of L . Then $(I + J)/I \cong J/(I \cap J)$.

Proof. \square

Theorem 3.6

If I, J are solvable ideals of L , then so is $I + J$.

Proof. By the second isomorphism theorem, $(I + J)/I \cong J/(I \cap J)$. Since $J/(I \cap J)$ is an ideal of J and J is solvable, $J/(I \cap J)$ is solvable and hence so is $(I + J)/I$. Finally, since I and $(I + J)/I$ are solvable, $I + J$ is solvable. \square

Theorem 3.7

Every finite dimensional Lie algebra contains a unique maximal solvable ideal.

Proof. Suppose r is a solvable ideal of L with the largest dimension. Let I be a solvable ideal of L . Then $r + I$ is also solvable and since $r \subseteq r + I$, $\dim(r) \leq \dim(r + I)$. However, $\dim r$ is maximal, so $\dim(r) = \dim(r + I)$ and $r = r + I$. So $I \subseteq r$ and r is unique. \square

3.3 Examples

Lie algebra

The Lie algebra of a matrix Lie group is a Lie algebra. It satisfies the Jacobi identity. For all $X, Y, Z \in M_n(\mathbb{C})$,

$$\begin{aligned} [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] &= [X, YZ - ZY] + [Y, ZX - XZ] + [Z, XY - YX] \\ &= X(YZ - ZY) - (YZ - ZY)X \\ &\quad + Y(ZX - XZ) - (ZX - XZ)Y \\ &\quad + Z(XY - YX) - (XY - YX)Z \\ &= XYZ - XZY - YZX + ZYX \\ &\quad + YZX - YXZ - ZXY + XZY \\ &\quad + ZXY - ZYX - XYZ + YXZ \\ &= 0 \end{aligned}$$

ideal

Let $G = GL_n(\mathbb{R}) \subset M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$. Take any point in G . Then, take the open set to be all of $GL_n(\mathbb{R})$ and the homeomorphism to be the identity. Thus, G satisfies all the requirements to be a Lie group.

algebra

- $GL_n(\mathbb{R})$: It is closed in itself.
- $SL_n(\mathbb{R})$: The determinant is a continuous function and $\det : SL_n(\mathbb{R}) \rightarrow \{1\}$, and $\{1\}$ is closed in \mathbb{R} .
- $U(n)$: The determinant is a continuous function and for all unitary matrices A , $|\det(A)| = 1$, so $\det : U(n) \rightarrow \{z \in \mathbb{C} : |z| = 1\}$, which is closed in \mathbb{C} .
- $O(n)$: The determinant is a continuous function and $\det : O(n) \rightarrow \{-1, 1\}$, and $\{-1, 1\}$ is closed in \mathbb{R} .

adjoint representation

- $\mathfrak{gl}_n(\mathbb{C}) = M_n(\mathbb{C})$. If $X \in M_n(\mathbb{C})$, then e^{tX} is invertible.
- $\mathfrak{sl}_n(\mathbb{C}) = \{X \in M_n(\mathbb{C}) \mid \text{tr}(X) = 0\}$ PROOOOOOF
- $\mathfrak{u}(n) = \{X \in M_n(\mathbb{C}) \mid X^* = -X\}$ PROOOOOOF
- $\mathfrak{o}(n) = \{X \in M_n(\mathbb{R}) \mid X^T = -X\}$ PROOOOOOF

adjoint representation part 2

- $GL_n(\mathbb{C})$
- $GL_2(\mathbb{R})$ is not a complex matrix Lie group. Let $X = \begin{bmatrix} 1 & 0 \\ 0 & \frac{\pi}{2} \end{bmatrix}$. For all $t \in \mathbb{R}$, $e^{tX} = \begin{bmatrix} e^t & 0 \\ 0 & e^{t\frac{\pi}{2}} \end{bmatrix} \in GL_2(\mathbb{R})$, so $X \in \mathfrak{g}$. However, for $t = 1$, $e^{iX} = \begin{bmatrix} e^{it} & 0 \\ 0 & e^{it\frac{\pi}{2}} \end{bmatrix} \notin GL_2(\mathbb{R})$, so $iX \notin \mathfrak{g}$.

Killing form

derived series

4 Semisimple Lie algebras

4.1 Definitions

semisimple Lie algebra

A Lie algebra is **semisimple** if it satisfies THEOREM NUMBER.

toral subalgebra

Let L be a finite dimensional Lie algebra. A subalgebra \mathcal{A} of L is called **toral** if for all $A \in \mathcal{A}$, $\text{ad}_A : L \rightarrow L$ is diagonalizable.

Cartan subalgebra

Let L be a finite dimensional complex semisimple Lie algebra. A **Cartan subalgebra** of L is a maximal abelian subalgebra \mathcal{A} such that $\text{ad}_x : L \rightarrow L$ is diagonalizable for all $x \in \mathcal{A}$.

root

Let L be a finite dimensional complex Lie algebra and \mathcal{A} be a Cartan subalgebra of L . A non-zero $\phi \in \mathcal{A}^*$ is called a **root** of L with respect to \mathcal{A} if and only if the space $E_\phi := \bigcap_{A \in \mathcal{A}} \ker(\text{ad}_A - \phi(A)I)$ is non-trivial. The set of all roots is denoted by Δ .

root decomposition

Let $\Delta = \{\phi_1, \phi_2, \dots, \phi_r\}$ be the set of all roots of L with respect to \mathcal{A} . The **root decomposition** of L is

$$L = \mathcal{A} \oplus E_{\phi_1} \oplus E_{\phi_2} \oplus \dots \oplus E_{\phi_r}$$

rank of a Lie algebra

Cartan integer

Weyl group

root system

Let V be a finite dimensional inner product space. A **root system** in V is a finite subset $\Delta \subseteq V \setminus \{0\}$ such that

1. if $w \in \Delta$, then $-w \in \Delta$
2. for all $w, u \in \Delta$, $\frac{2\langle w, u \rangle}{\langle u, u \rangle} \in \mathbb{Z}$
3. for all $w, u \in \Delta$, $w - \frac{2\langle w, u \rangle}{\langle u, u \rangle} u \in \Delta$

A **reduced root system** additionally satisfies that if $c \in \mathbb{F}$, then $cw \in \Delta$ if and only if $c = \pm 1$.

4.2 Theorems

Theorem 4.1 (Characterization of semisimple Lie algebras)

Let L be a non-zero finite dimensional Lie algebra. The following are equivalent:

1. L has no non-zero abelian ideals.
2. L has no non-zero solvable ideals.
3. The radical of L is zero.
4. The Killing form of L is non-degenerate.
5. L is the direct sum of simple Lie algebras.

A **semisimple Lie algebra** is a Lie algebra that satisfies the theorem.

Proof. 1 \implies 2:

2 \implies 1:

If the only solvable ideal of L is zero,

2 \iff 3:

If L has no non-zero solvable ideals, then the maximal solvable ideal must be zero, which is the radical of L . If the radical of L is zero, then there are no non-zero solvable ideals since any other solvable ideal I must satisfy $I \subseteq \{0\}$.

4 \implies 1:

Assume L has a non-zero abelian ideal I . Then for every $0 \neq x \in I$ and $y, z \in L$,

$$(\text{ad}(x) \circ \text{ad}(y) \circ \text{ad}(x))(z) = [x, [y, [x, z]]] = 0$$

So $(\text{ad}(x) \circ \text{ad}(y))^2 = 0$ which implies that $\text{tr}(\text{ad}(x) \circ \text{ad}(y)) = 0$. Thus $B(x, y) = 0$ for all $y \in L$, so the Killing form B is degenerate.

2 \implies 4:

Assume the Killing form B_L of L is degenerate. Let $I = \{x \in L \mid B_L(x, y) = 0 \text{ for all } y \in L\}$. Then I is a non-zero ideal of L EXPLAIN WHYYYYYYYY. Moreover, B_L restricts to zero on I . By 2 theorems above (restriction and killing form is zero), I is solvable.

□

Theorem 4.2

If the Killing form of L is non-degenerate, then for every linear map $f : L \rightarrow \mathbb{F}$ there exists a unique $z_0 \in L$ such that $f(x) = B(x, z_0)$.

Proof. Since L is finite dimensional, it has the same dimension as its dual L^* . Since the Killing form is a bilinear form, fixing z_0 makes $B(x, z_0)$ a linear functional.

For every $z \in L$, define $f_z : L \rightarrow \mathbb{F}$ by $f_z(x) = B(x, z)$ for all $x \in L$. Then the map $F : L \rightarrow L^*$ defined by $F(z) = f_z$ is linear. Then,

$$\ker(F) = \{z \mid f_z \equiv 0\} = \{z \mid B(x, z) = 0 \text{ for all } x \in L\}$$

From the assumption that B is non-degenerate, $\ker(F) = \{0\}$. Hence F is injective. Since $\dim(L^*) = \dim(L)$, F is bijective and so z_0 exists and is unique. □

Theorem 4.3

Let L be a finite dimensional Lie algebra and I be an ideal of L . The restriction of the Killing form of L to I is the Killing form of I .

Proof. Suppose $\dim(L) = n$ and $\dim(I) = k < n$. Choose a basis $\{e_1, e_2, \dots, e_n\}$ for L that extends a basis $\{e_1, e_2, \dots, e_k\}$ of I . Then for all $x \in I$, $\text{ad}(x) : L \rightarrow L$ has the matrix form

$$\text{ad}(x) = \begin{bmatrix} A_{k \times k} & * \\ & 0 \end{bmatrix}$$

where A is a $k \times k$ matrix representation of $\text{ad}(x) : I \rightarrow I$ with respect to the basis $\{e_1, e_2, \dots, e_k\}$.

So for $x, y \in I$ with

$$\begin{aligned} \text{ad}(x) &= \begin{bmatrix} X_{k \times k} & * \\ & 0 \end{bmatrix} & \text{ad}(y) &= \begin{bmatrix} Y_{k \times k} & * \\ & 0 \end{bmatrix} \\ \text{ad}(x) \circ \text{ad}(y) &= \begin{bmatrix} XY & * \\ * & 0 \\ & * & 0 \end{bmatrix} \end{aligned}$$

So $B_L(x, y) = \text{tr}(\text{ad}(x) \circ \text{ad}(y)) = \text{tr}(XY) = B_I(x, y)$. □

Theorem 4.4

Let L be a finite dimensional Lie algebra. If the Killing form of L is 0, then L is solvable.

Proof. Claim: For all $x \in L'$, $\text{ad}(x)$ is nilpotent.

Let $x = [y, z] \in L'$ and $T = \text{ad}(x) : L \rightarrow L$. Write $T = D + N$ in its Jordan-Chevalley decomposition, where D is the diagonal matrix and N is the nilpotent matrix.

Let D^* be the complex conjugate of D . Via Lagrange interpolating polynomials,

$$p(x) = \sum_{i=1}^n \prod_{j \neq i} \frac{(x - x_j)}{x_i - x_j} y_i$$

there exists a polynomial $p(x)$ such that $p(D) = D^*$. Thus, $D^*N = ND^*$ (D^* commutes with N).

Since $\text{ad} : \text{Lin}(L) \rightarrow \text{Lin}(\text{Lin}(L))$ is a Lie algebra homomorphism, so $\text{ad}(N)$ commutes with both $\text{ad}(D)$ and $\text{ad}(D^*)$. $0 = \text{ad}([D, N]) = [\text{ad}(D), \text{ad}(N)] = 0$.

Since D is diagonal, $\text{ad}(D)$ is also diagonal. A basis for $\text{Lin}(L)$ is $E_{ij} = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & 1_{ij} & \vdots \\ 0 & \dots & \vdots \end{bmatrix}$. Then

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} E_{ij} - E_{ij} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = (\lambda_i - \lambda_j) E_{ij}$$

So E_{ij} is an eigenvector of $\text{ad}(D)$ with eigenvalue $\lambda_i - \lambda_j$, which means that $\text{ad}(D)$ is diagonalizable with respect to the basis of E_{ij} . The values on the diagonal are then $\lambda_i - \lambda_j$. Moreover, $\text{ad}(D^*)$ is diagonalizable and in particular, $(\text{ad}(D))^* = \text{ad}(D^*)$.

Additionally, since N is nilpotent, $\text{ad}(N)$ is nilpotent. Assume that $N^j = 0$.

$$\text{ad}^k(N)(x) = \sum_{r=0}^k (-1)^r \binom{k}{r} n^{k-r} x n^r$$

Thus, choose $k = 2j - 1$. Then for $0 \leq r < j$, the first term satisfies $k - r > j$ and thus the first term is 0. For $j \leq r \leq 2j - 1$, the second term satisfies $r > j$ and thus is 0 as well. So all of the terms in the summation is 0 thus, for $m \geq 2j - 1$, $\text{ad}^m(n) = 0$, so ad is nilpotent.

Thus from the above, and the uniqueness of the Jordan normal form, $\text{ad}(T) = \text{ad}(D) + \text{ad}(N)$.

Thus, $\text{ad}(D)$ is a polynomial of $\text{ad}(T)$. Additionally, $\text{ad}(D^*)$ is a polynomial of $\text{ad}(T)$. However, D^* is not necessarily in L , but by what just happened, for all $W_1 \in L$, $[D^*, W_1] = \text{ad}(W_2) \in \text{ad}(L)$. Finally,

$$\text{tr}(D^*T) = \text{tr}(D^* \text{ad}[Y, Z])$$

FINISH So there exists $A \in L$ such that $[D^*, \text{ad}(y)] = \text{ad} A$. So $\text{tr}(\text{ad}(y) \text{ad}(z)) = B(\text{ad}(y), \text{ad}(z)) = 0$ by the initial assumption. Since n is nilpotent and D^* commutes with N , D^*N is nilpotent, so $\text{tr}(D^*N) = 0$.

Blah blah blah blah blah. Thus for all $x \in L'$, $\text{ad}(x)$ is nilpotent.

□

Theorem 4.5

Let L be a finite dimensional Lie algebra and $\rho : L \rightarrow \text{Lin}(V)$ be a representation of L on a non-zero finite dimensional vector space V . If $\rho(X)$ is a nilpotent matrix for all $X \in L$, then there exists a non-zero $w \in V$ such that $\rho(X)(w) = 0$ for all $X \in L$.

In other words, $\cap_{X \in L} \ker(\rho(X)) \neq \{0\}$

Proof. Proof by induction on $\dim(L)$.

Base:

If $L = \{0\}$, then every $w \in V$ satisfies the conclusion. Since V is non-zero, such a w exists.

Induction:

Assume the claim hold for any finite dimensional Lie algebra P and any nilpotent representation $\phi : P \rightarrow \text{Lin}(W)$ with $\dim(P) < n$. Suppose L is a Lie algebra with $\dim(L) = n$ and $\rho : L \rightarrow \text{Lin}(V)$.

If $\ker(\rho) \neq \{0\}$ then $\dim(L/\ker(\rho)) < n$ and ρ gives a representation $\rho : L/\ker(\rho) \rightarrow \text{Lin}(V)$ so by the induction hypothesis, there exists $w \neq 0 \in V$ such that $\rho(x)(w) = \rho(x + \ker(\rho))(w) = 0$ for all $x \in L$.

If $\ker(\rho) = \{0\}$, by the first isomorphism theorem, $L \cong \rho(L)$ which is a subalgebra of $\text{Lin}(V)$. So L is a Lie subalgebra of $\text{Lin}(V)$ with Lie bracket inherited from $\rho(L)$.

Since $\dim(L)$ is finite, choose L_0 such that it is a proper Lie subalgebra of L with the maximum dimension. L_0 is the maximal proper Lie subalgebra of L . Since L is Lie subalgebra, $[L_0, L_0] \leq L_0$, thus there is a nilpotent representation $\phi : L_0 \rightarrow \text{Lin}(L/L_0)$ defined by $\phi(x)(z + L_0) := [x, z] + L_0$ where L/L_0 is only the quotient vector space.

CHECK THAT THIS IS INDEED A REPRESENTATION.

By the induction hypothesis, there exists $z_0 + L_0 \neq \{0\} \in L/L_0$ such that $[x, z_0] \in L_0$ for all $x \in L_0$. In particular, the subspace $\text{span}\{L_0 \cup \{z_0\}\}$ is a Lie subalgebra of L . Since $z_0 \notin L_0$ and L_0 was chosen to be a maximal proper Lie subalgebra, $\text{span}\{L_0 \cup \{z_0\}\} = L$.

Apply the induction hypothesis to L_0 with nilpotent representation $\rho : L \rightarrow \text{Lin}(V)$. The restriction of ρ to L_0 is still a nilpotent representation.

Consider the space $V_0 = \{w \in V \mid X(w) = 0 \text{ for all } x \in L_0\}$. The space is non-zero. Let $w \in V_0$. Then for all $x \in L_0$, $xz_0(w) = z_0x(w) + [x, z_0](w) = 0$ which implies that $z_0(V_0) : V_0 \rightarrow V_0 \subseteq V_0$ is nilpotent. Since z_0 is nilpotent, it is not injective (it has a non-zero kernel), so there exists $0 \neq w_0 \in V_0$ such that $z_0(w_0) = 0$. SO w_0 gets vanished in L_0 and it got vanished by z_0 . So $X(w_0) = 0$ for all $x \in \text{span}\{L_0 \cup \{z_0\}\} = L$.

NOT DONE

□

Theorem 4.6

There is no two dimensional complex simple Lie algebra.

Proof. Let L be a two dimensional Lie algebra with basis $\{x, y\}$. Then consider $[x, y] = ax + by$ for some $a, b \in \mathbb{C}$.

- If $a = b = 0$, then $[x, y] = 0$ which would mean L is abelian, and hence not simple.
- If $a = 0$ and $b \neq 0$, then the ideal $I = \{ky : k \in \mathbb{C}\}$ is proper ideal of L , hence not simple.
- Suppose $a, b \neq 0$. Then $[x, \frac{1}{a}y] = x + \frac{b}{a}y$. Let $z = x + \frac{b}{a}y$. By bilinearity of the Lie bracket, $z = [z - \frac{b}{a}y, \frac{1}{a}y] = [z, \frac{1}{a}y] - [\frac{b}{a}y, \frac{1}{a}y] = [z, \frac{1}{a}y]$. So $I = \{kz : k \in \mathbb{C}\}$ is a proper ideal of L , hence not simple.

□

Theorem 4.7

If L is a finite dimensional complex semisimple Lie algebra, then $\text{ad} : L \rightarrow \text{Lin}(L)$ is injective. So $L \cong \text{ad}(L) \leq \text{Lin}(L)$.

Theorem 4.8

For all $\phi \in \mathcal{A}^*$, there exists a unique $A_\phi \in \mathcal{A}$ such that $\phi(x) = B(A_\phi, x)$ for all $x \in \mathcal{A}$.

Theorem 4.9

$\text{span}\{\Delta\} = \mathcal{A}^*$.

Theorem 4.10

If $\phi \in \Delta$ then $-\phi \in \Delta$.

Theorem 4.11

Let $\phi \in \Delta$, $x \in L_\phi$, $y \in L_{-\phi}$. Then $[x, y] = B(x, y)A_\phi$. So $[L_\phi, L_{-\phi}]$ is a 1-dimensional subspace.

Theorem 4.12

Every L contains a copy of $\mathfrak{sl}_2(\mathbb{C})$.

Theorem 4.13

For every $\phi \in \Delta$, $\dim(L_\phi) = 1$.

Theorem 4.14

Every L has a Cartan subalgebra \mathcal{A} and

$$L = \mathcal{A} \oplus_{\phi \in \Delta} L_\phi$$

Theorem 4.15

Let $\mathcal{A}_1, \mathcal{A}_2$ be two Cartan subalgebras of L . Then there exists a Lie algebra isomorphism $\sigma : L \rightarrow L$ such that $\sigma(\mathcal{A}_1) = \mathcal{A}_2$.

Theorem 4.16

Let L be a finite dimensional semisimple complex Lie algebra. Then the restriction of the Killing form of L to \mathcal{A} is positive definite.

Define $\langle \cdot, \cdot \rangle$ on \mathcal{A}^* be $\langle \phi, \psi \rangle := B(A_\phi, A_\psi)$.

4.3 Examples

$\mathfrak{sl}_3(\mathbb{C})$

- Consider $\mathfrak{sl}_2(\mathbb{C})$ with Cartan subalgebra $\mathcal{A} = \left\{ \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} : a \in \mathbb{C} \right\}$.

A basis for \mathcal{A} is

$$\{A\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\} = \{E_{1,1} - E_{2,2}\}$$

So $\dim(\mathcal{A}) = \dim(\mathcal{A}^*) = 1$. Every linear functional $\phi \in \mathcal{A}^*$ is completely determined by what it does to the basis elements of \mathcal{A} .

A basis for $\mathfrak{sl}_2(\mathbb{C})$ is

$$\{A, E_{1,2}, E_{2,1}\}$$

Let $X \in \mathfrak{sl}_2(\mathbb{C})$. Since X is a linear combination of the basis, check what $\text{ad}(A)$ does to each basis element of $\mathfrak{sl}_2(\mathbb{C})$. Then,

$$\begin{aligned} [A, E_{1,2}] &= 2E_{1,2} \\ [A, E_{2,1}] &= -2E_{2,1} \end{aligned}$$

So the roots must be $\phi_2 \left(\begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} \right) = 2a$ and $\phi_{-2} \left(\begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} \right) = -2a$.

- Consider $\mathfrak{sl}_3(\mathbb{C})$ with Cartan subalgebra $\mathcal{A} = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & -a+b & 0 \\ 0 & 0 & -b \end{bmatrix} : a, b \in \mathbb{C} \right\}$.

A basis for \mathcal{A} is

$$\{A_1, A_2\} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\} = \{E_{1,1} - E_{2,2}, E_{2,2} - E_{3,3}\}$$

So $\dim(\mathcal{A}) = \dim(\mathcal{A}^*) = 2$. Every linear functional $\phi \in \mathcal{A}^*$ is completely determined by what it does to the basis elements of \mathcal{A} .

A basis for $\mathfrak{sl}_3(\mathbb{C})$ is

$$\{A_1, A_2, E_{1,2}, E_{1,3}, E_{2,1}, E_{2,3}, E_{3,1}, E_{3,2}\}$$

Let $X \in \mathfrak{sl}_3(\mathbb{C})$. Since X is a linear combination of the basis, check what $\text{ad}(A_1)$ and $\text{ad}(A_2)$ does to each basis element of $\mathfrak{sl}_3(\mathbb{C})$. Then,

$$\begin{array}{ll} [A_1, E_{1,2}] = 2E_{1,2} & [A_2, E_{1,2}] = -E_{1,2} \\ [A_1, E_{1,3}] = E_{1,3} & [A_2, E_{1,3}] = E_{1,3} \\ [A_1, E_{2,1}] = -2E_{2,1} & [A_2, E_{2,1}] = E_{2,1} \\ [A_1, E_{2,3}] = -E_{2,3} & [A_2, E_{2,3}] = 2E_{2,3} \\ [A_1, E_{3,1}] = -E_{3,1} & [A_2, E_{3,1}] = -E_{3,1} \\ [A_1, E_{3,2}] = E_{3,2} & [A_2, E_{3,2}] = -2E_{3,2} \end{array}$$

For all linear functionals $\phi \in \mathcal{A}^*$, define them by what they do to each basis element of \mathcal{A} . Denote $\phi_{\alpha,\beta}(A_1) = \alpha$ and $\phi_{\alpha,\beta}(A_2) = \beta$.

Since roots are defined by $[A, X] = \phi(A)X$, match the pairs of E together to get the roots

$$\Delta = \{\phi_{2,-1}, \phi_{1,1}, \phi_{-2,1}, \phi_{-1,2}, \phi_{-1,-1}, \phi_{1,-2}\}$$

Observe that Δ spans \mathcal{A}^* , but it is NOT a basis since some of the functionals are linear combinations of the other.

$$\begin{aligned} \phi_{-1,2} &= -\phi_{2,1} + \phi_{1,1} \\ \phi_{1,1} &= -\phi_{-2,1} + \phi_{-1,2} \end{aligned}$$

root system

- Lie algebras are reduced root systems.
- Rank 1 Lie algebras. If $|\Delta| = 2$, then $\Delta = \{w, -w\}$. So there is a unique semisimple Lie algebra with rank 1.
- Rank 2 Lie algebras.