Анализ сетевых данных

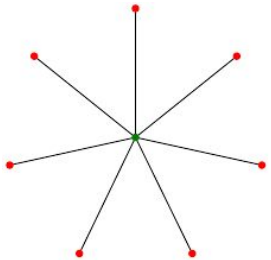
Лекция 2. Random Graph models.

11 Сентября, 2019

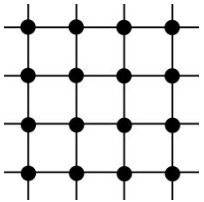
MOOCs

- http://tuvalu.santafe.edu/~aaronc/courses/5352/ **Aaron Clauset**, lecture notes
- https://goo.gl/8CghUx **Leonid Zhukov**, http://www.leonidzhukov.net/, Full course videos + lecture notes.
- http://web.stanford.edu/class/cs224w/ **Jure Leskovec**, lecture notes, bunch of useful materials, *videos are unavailable.*

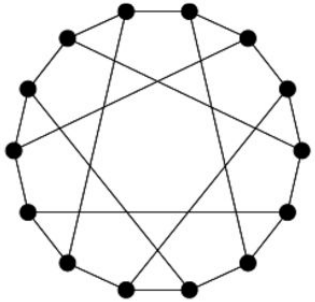
Compare different networks



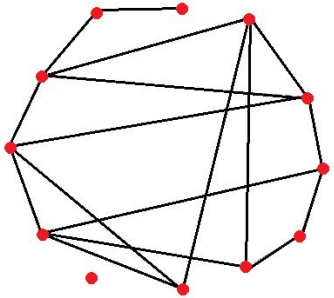
Star graph



Grid graph

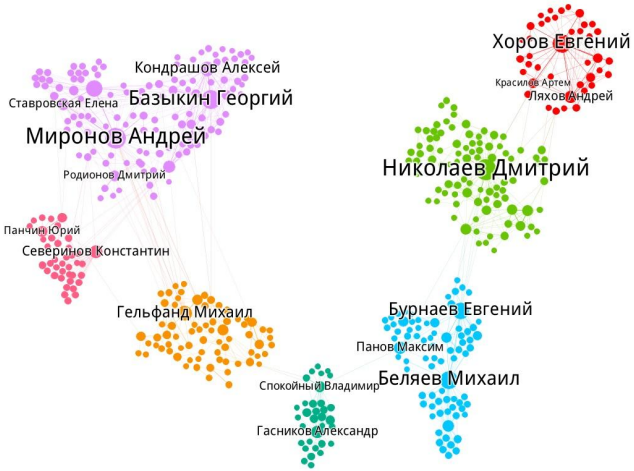


K-regular graph



Random graph

vs



Real network

…?

How to compare different networks?

- # nodes, #edges, density
- (Average) node degree
- (Average) clustering coefficient
- (Average) path length
- Node degree distribution
- Centrality measures (next lecture)
- DL Embeddings/Graph convolutions

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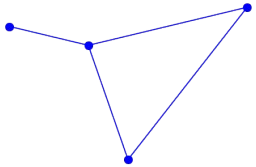

# nodes = 4 (n)
# edges = 4 (m)
density = m / [n(n-1)/2] = 2/3

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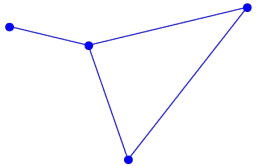

$$k_i = \sum_{j=1}^{n} A_{ij}$$

k = (1 + 3 + 2 + 2) / 4 =

$$= \langle k \rangle = \frac{1}{n} \sum_{i=1}^{n} k_i = \frac{2m}{n}$$

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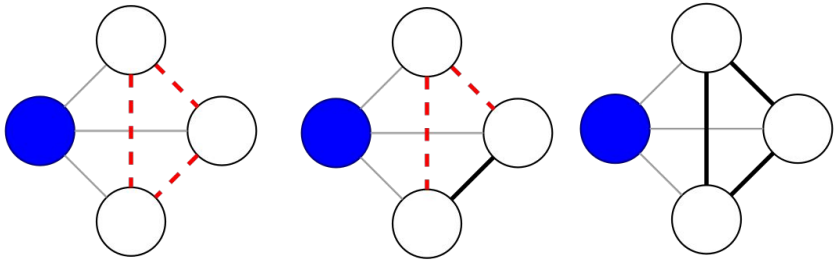
$$C_i = \frac{\text{(number of pairs of neighbors of } i \text{ that are connected)}}{\text{(number of pairs of neighbors of } i\text{)}}$$

$$= \sum_{jk} A_{ij} A_{jk} A_{ki} \Big/ \binom{k_i}{2} = \frac{2\, n_i}{k_i\,(k_i - 1)}$$, n_i = # neighbours of node i



c = 0 c = 1/3 c = 1

$$C = \frac{\text{(number of triangles)} \times 3}{\text{(number of connected triples)}}$$

$$= \sum_{ijk} A_{ij} A_{jk} A_{ki} \Big/ \sum_{ijk} A_{ij} A_{jk} \quad ,$$

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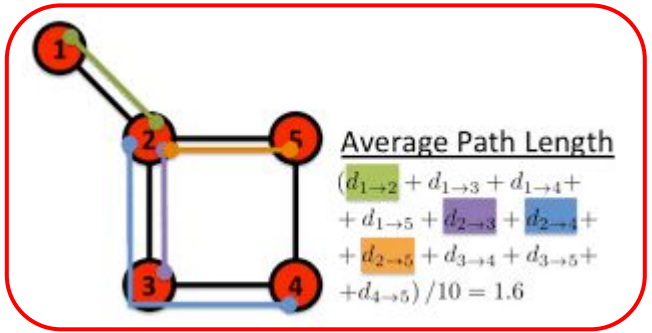

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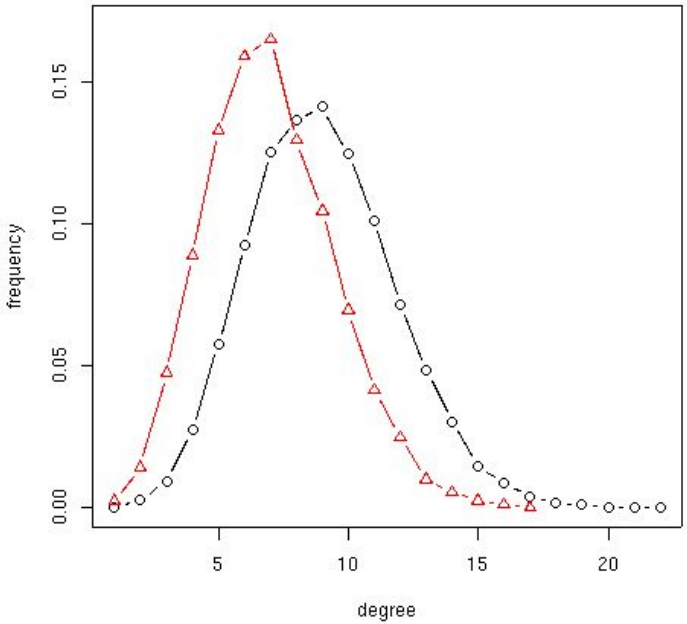

Erdos-Renyi random graph

- $G_{n,p}$ model:

  Graph with *n* nodes and for each pair of nodes the probability of an edge between them is equal to *p*.

- $G_{n,m}$ model:

  A randomly selected graph from the set of C_N^m graphs, with $N = n(n-1)/2$, where *n* = #nodes and *m* = #edges

Random graph model (Erdos & Renyi, 1959)

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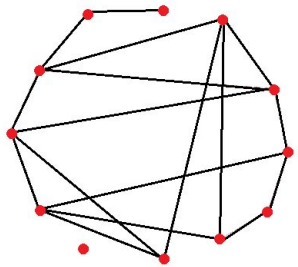
Random graph model (Erdos & Renyi, 1959)

Эти две модели - эквивалентны, Приведем набросок доказательства

Erdos-Renyi properties

$G_{n,p}$ model:

- $\langle m \rangle = p*n*(n-1)/2$
- $\langle k \rangle = (n-1)*p \approx n*p$



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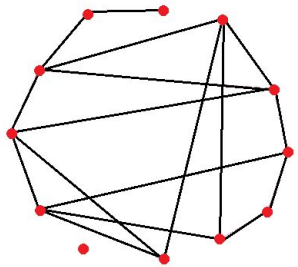
What is node degree distribution?



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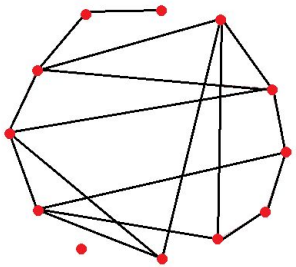

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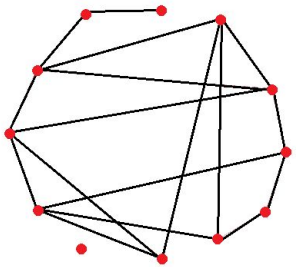

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$$P(k_i = k) = P(k) = C_{n-1}^{k} p^k (1-p)^{n-1-k}$$

(Bernoulli distribution)

Erdos-Renyi degree distribution

Limiting case of Bernoulli distribution (when *n* goes to infinity) - Poisson distribution (with parameter $\lambda = \langle k \rangle = np$)

$$P(k) = \frac{\langle k \rangle^k e^{-\langle k \rangle}}{k!} = \frac{\lambda^k e^{-\lambda}}{k!}$$

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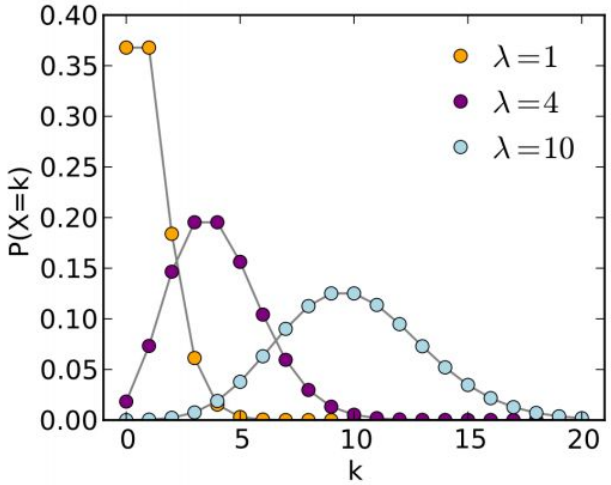

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Erdos-Renyi degree distribution

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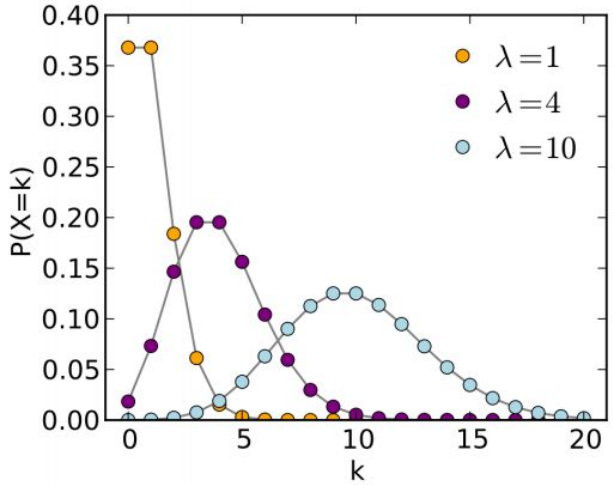

Which in case of np → ∞, goes to Gaussian

$$P(k_i = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad \lambda = pn$$

$G_{n,p}$ vs $G_{n,m}$

Bernoulli distribution

- Mean = $\langle k \rangle = (n-1)p$
- Variance = $\sigma^2 = p(1-p)(n-1)$

With fixed p and $n \to \infty$, distribution becomes *narrow*:

$G_{n,p}$ vs $G_{n,m}$

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thus we are increasingly confident that the degree of a node is equal to $\langle k \rangle$

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$G_{n,p}$ and $G_{n, m}$ are the same

Erdos-Renyi clustering coefficient

$$C_i = \frac{2\,n_i}{k_i\,(k_i - 1)}$$

since edges appear i.i.d. with probability p:

$n_i = p * k_i(k_i - 1) / 2$

then $C_i \approx \langle k \rangle / n$

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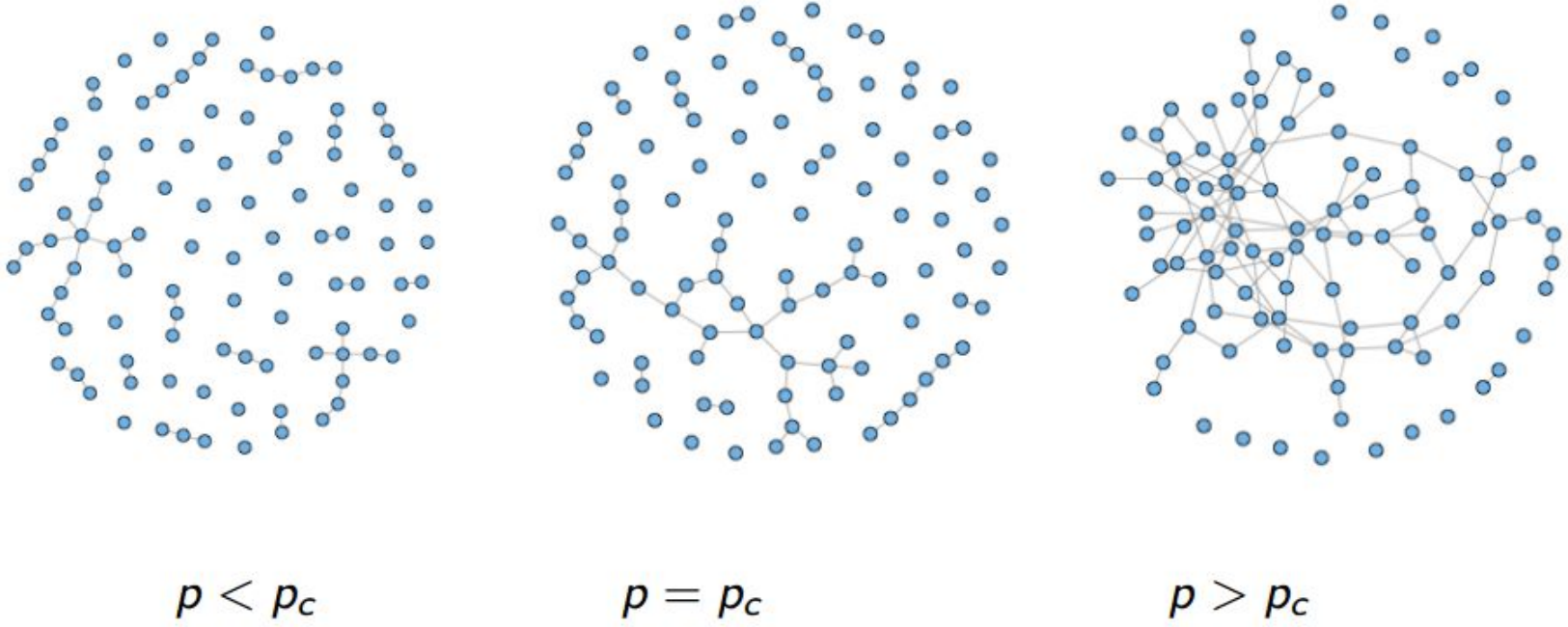
This means that with n goes to infinity clustering coefficient of a random graph goes to 0

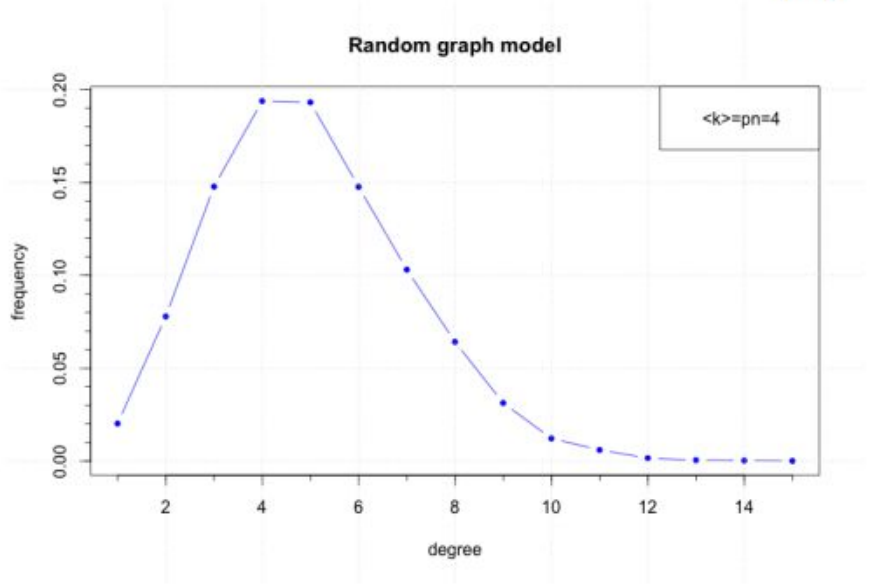
What about connectivity?

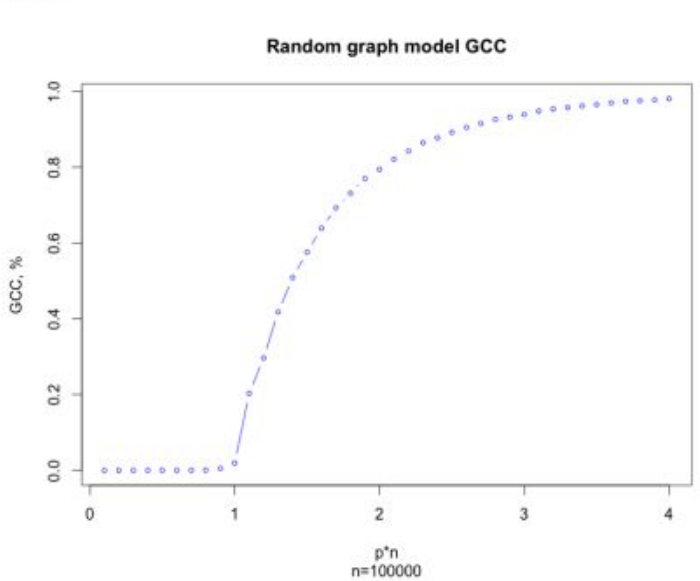


$p < p_c$ $p = p_c$ $p > p_c$

What about connectivity?

$$\langle k \rangle = pn$$





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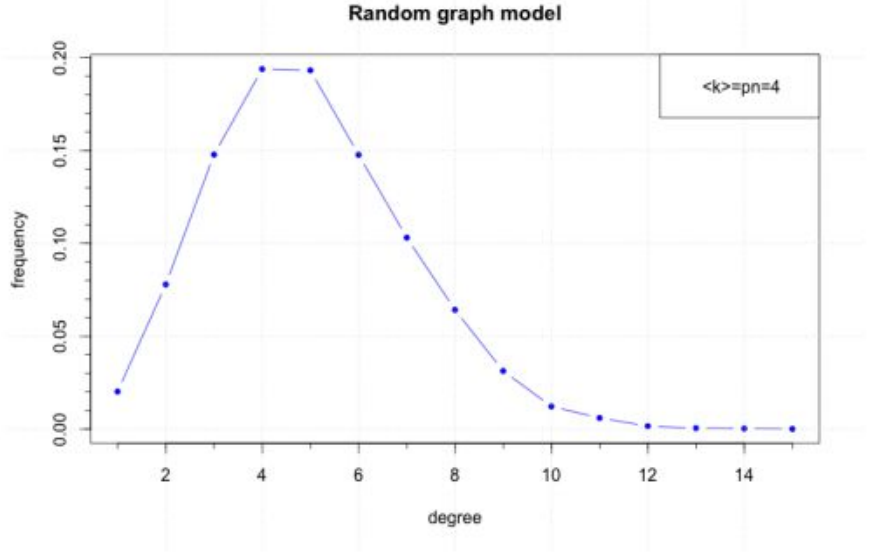
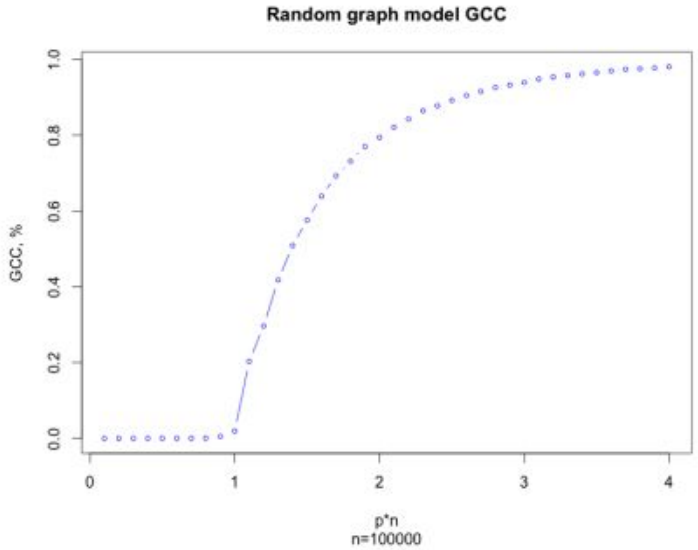
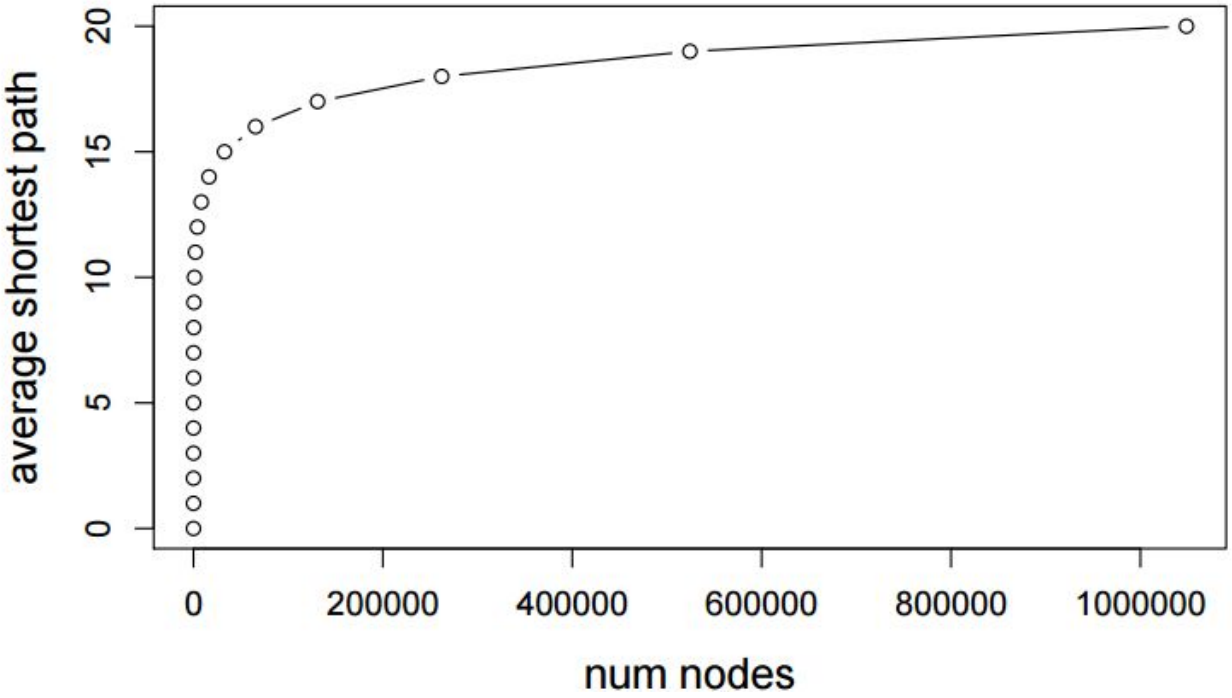




- It could be shown than with <k> = 1 the largest connected component contains $O(n^{2/3})$ nodes.
- With <k> > 1 it quickly has all the nodes.

Average path length



For Erdos-Renyi graph average path length is of order *O(log n)*

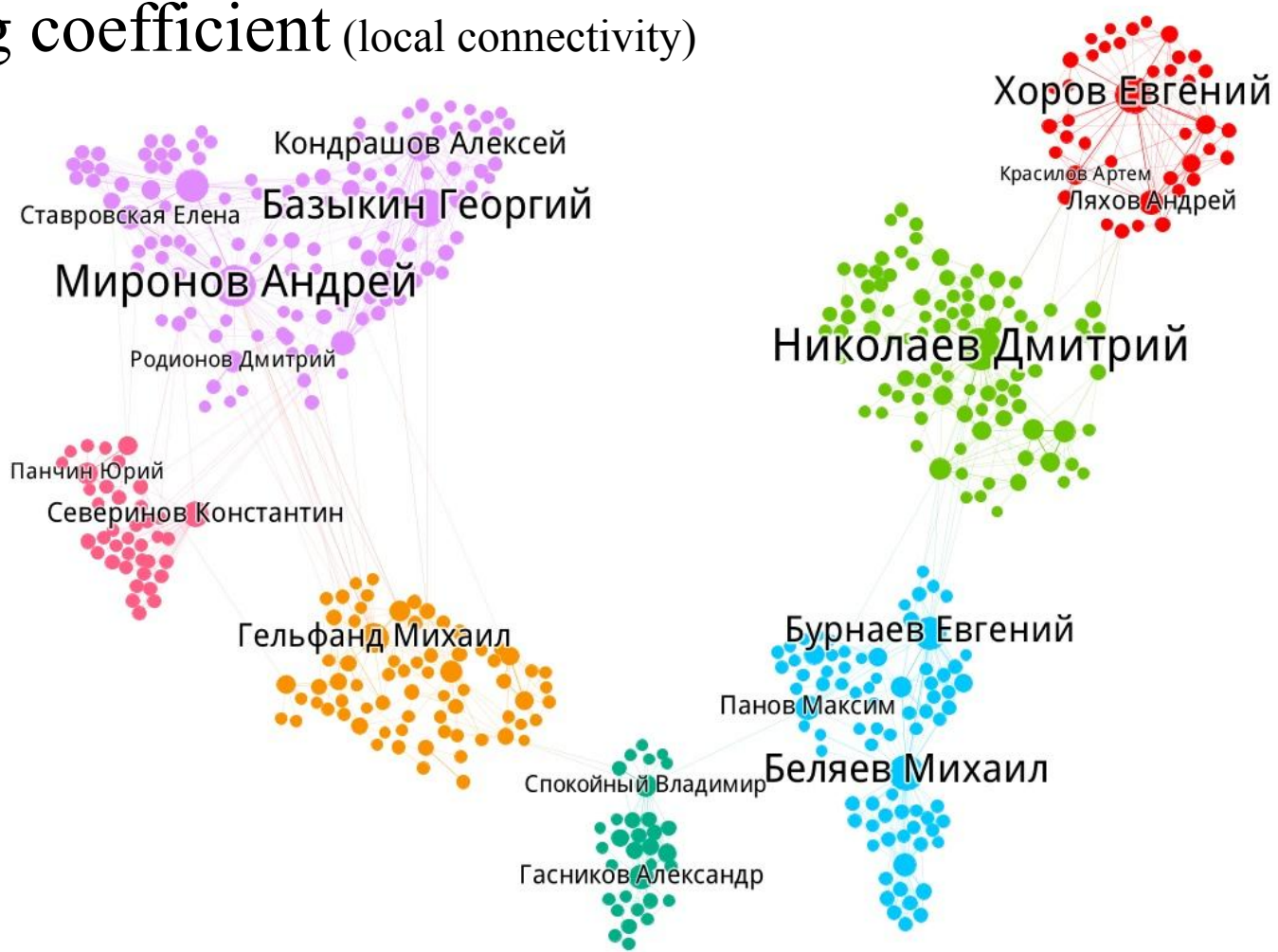
How to compare different networks?

- # nodes, #edges, density
- (Average) node degree
- (Average) clustering coefficient
- (Average) path length
- **Node degree distribution**
- Centrality measures (next lecture)

Key properties

- (Average) clustering coefficient
- (Average) path length
- Node degree distribution

Clustering coefficient (local connectivity)



Key properties

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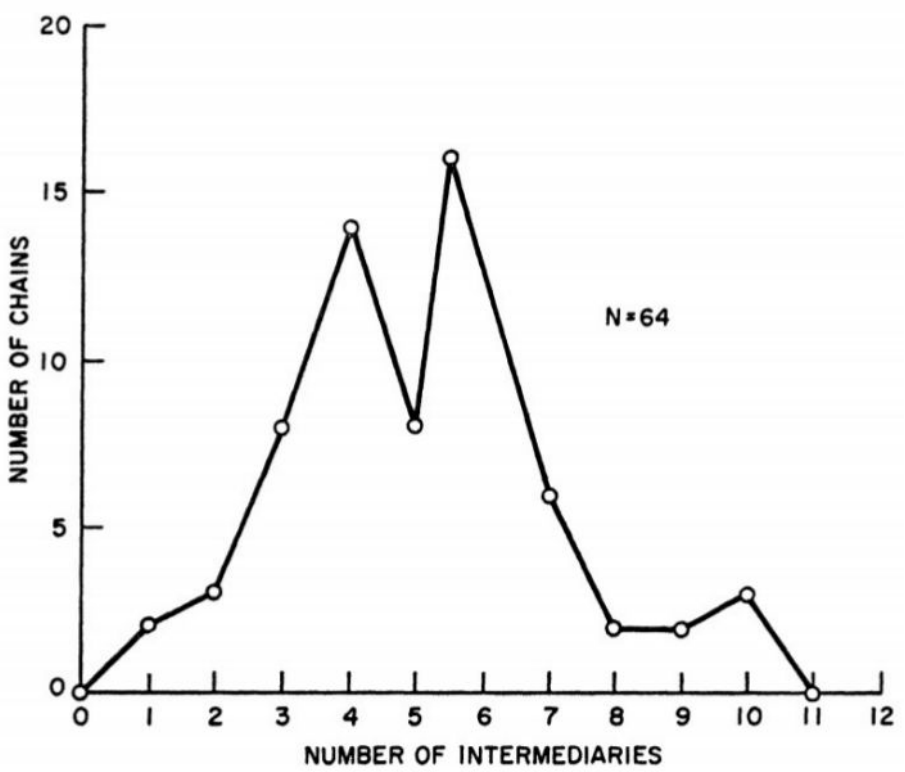
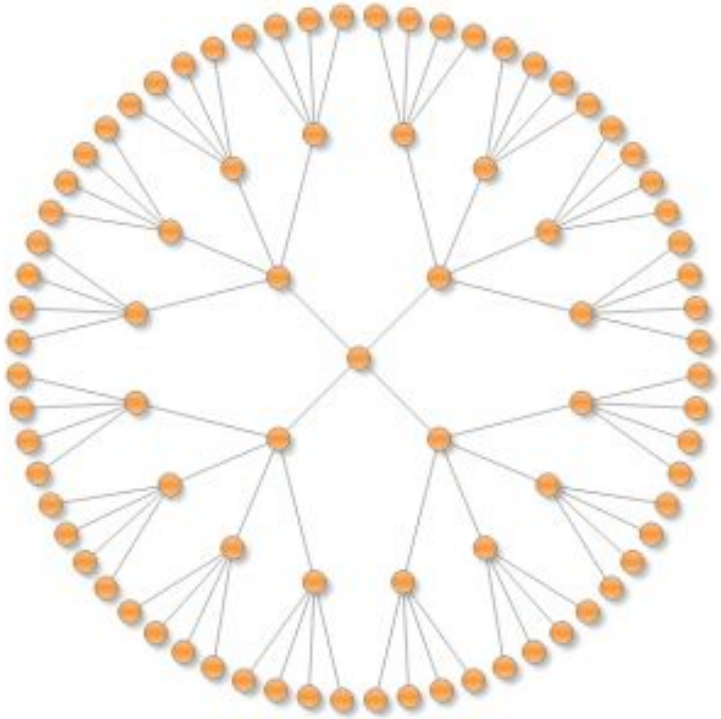
Recall Milgram's experiment



Average path length (idea)



An estimate: $z^d = N$, $d = \log N / \log z$

$N \approx 6.7$ bln, $z = 50$ friends, $d \approx 5.8$.

Consider a simple model:

Each person has the same number of friends *z,* total # of people in the world is N, then what is a diameter?

Diameter = the longest of all the calculated shortest paths in a network

Key properties

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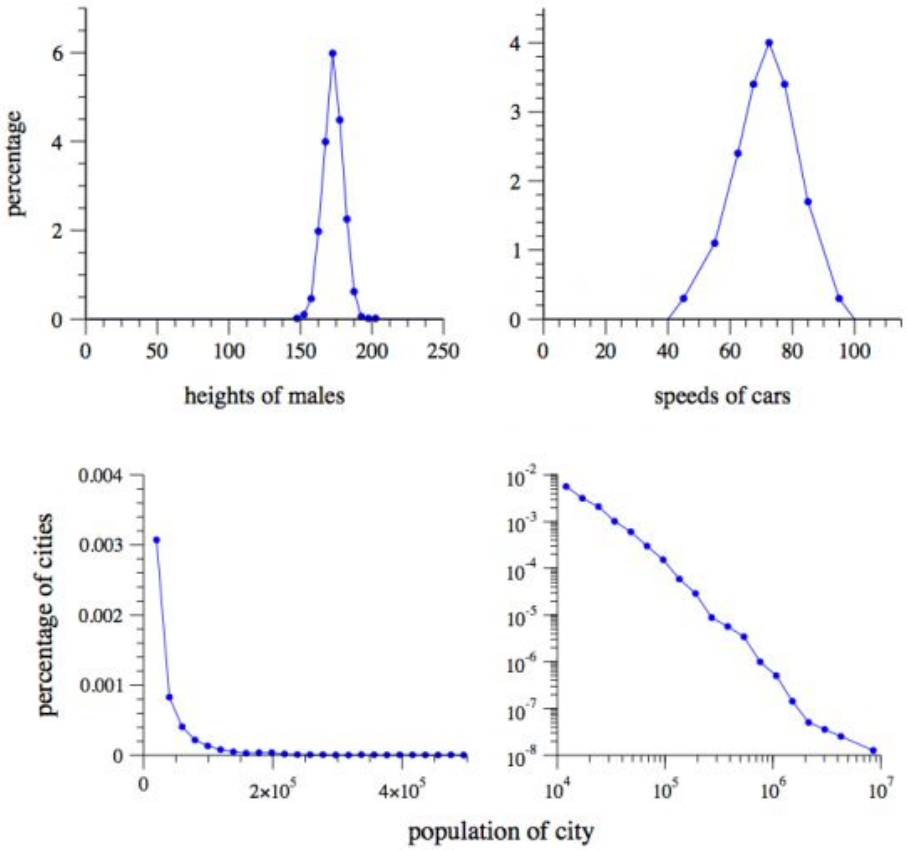
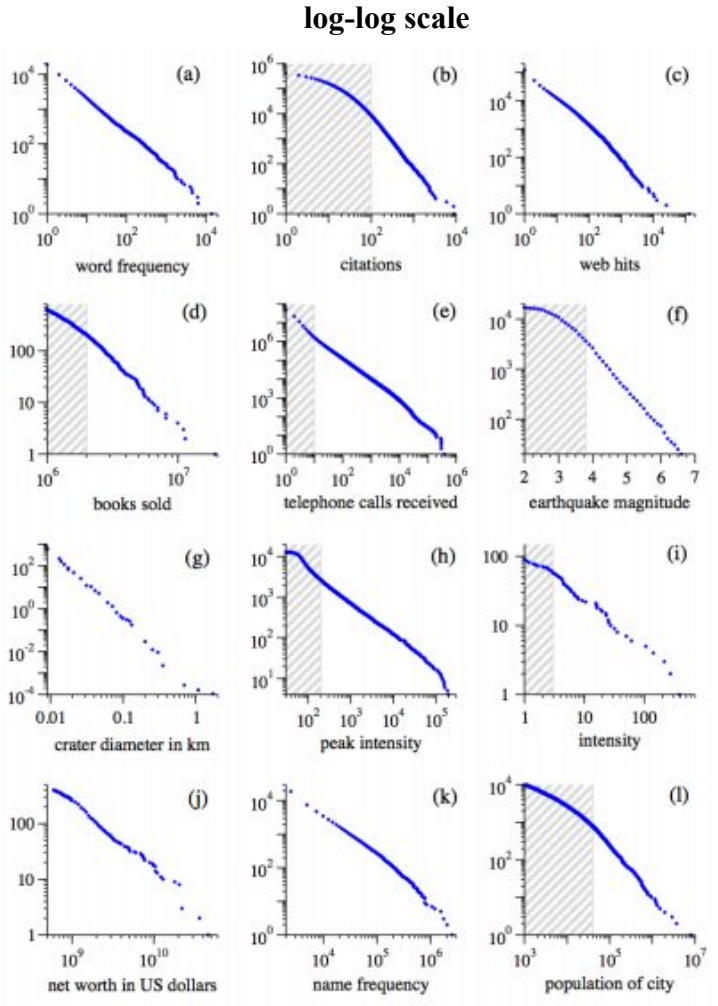
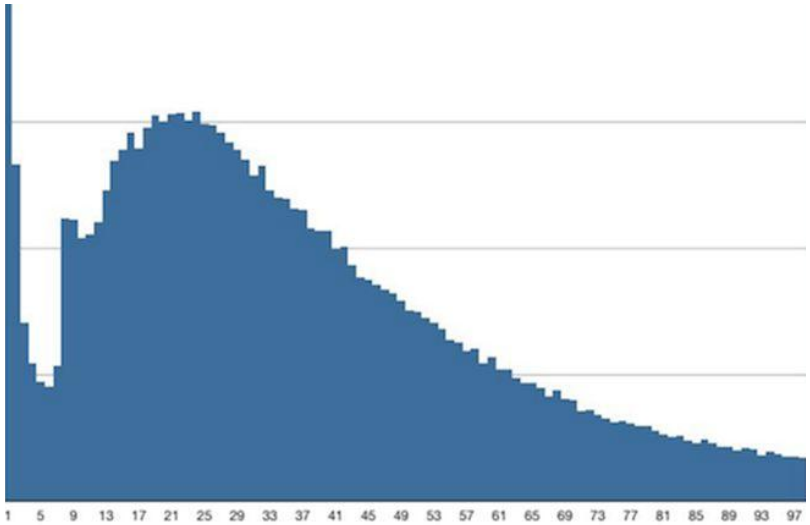
Empirical distributions

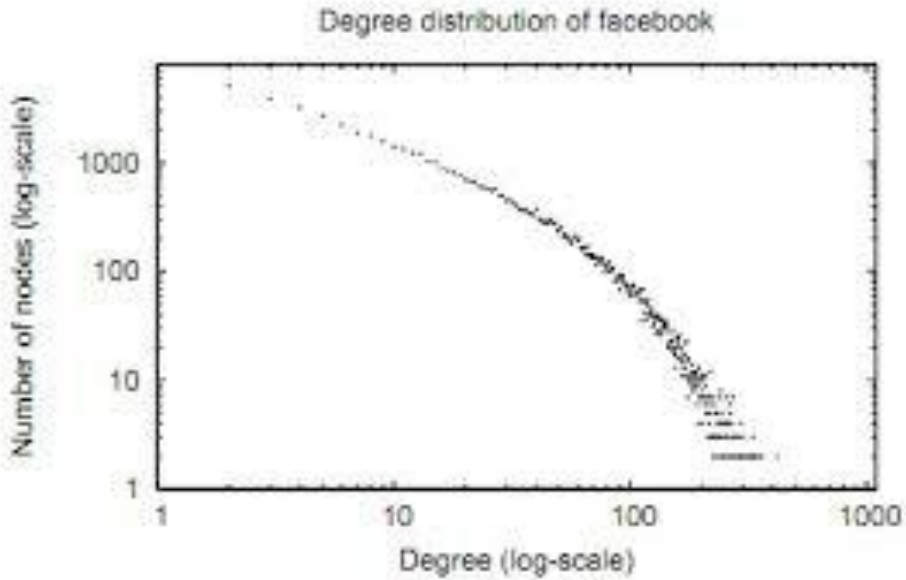




Facebook degree distribution



y = number of people; x = number of friends for those people



Empirical network features

- Power-law (heavy-tailed) degree distribution
- Small average distance (graph diameter)
- Large clustering coefficient (transitivity)
- Giant connected component, hierarchical structure,etc