ME6406 – Machine Vision – Homework #3

Problem 1

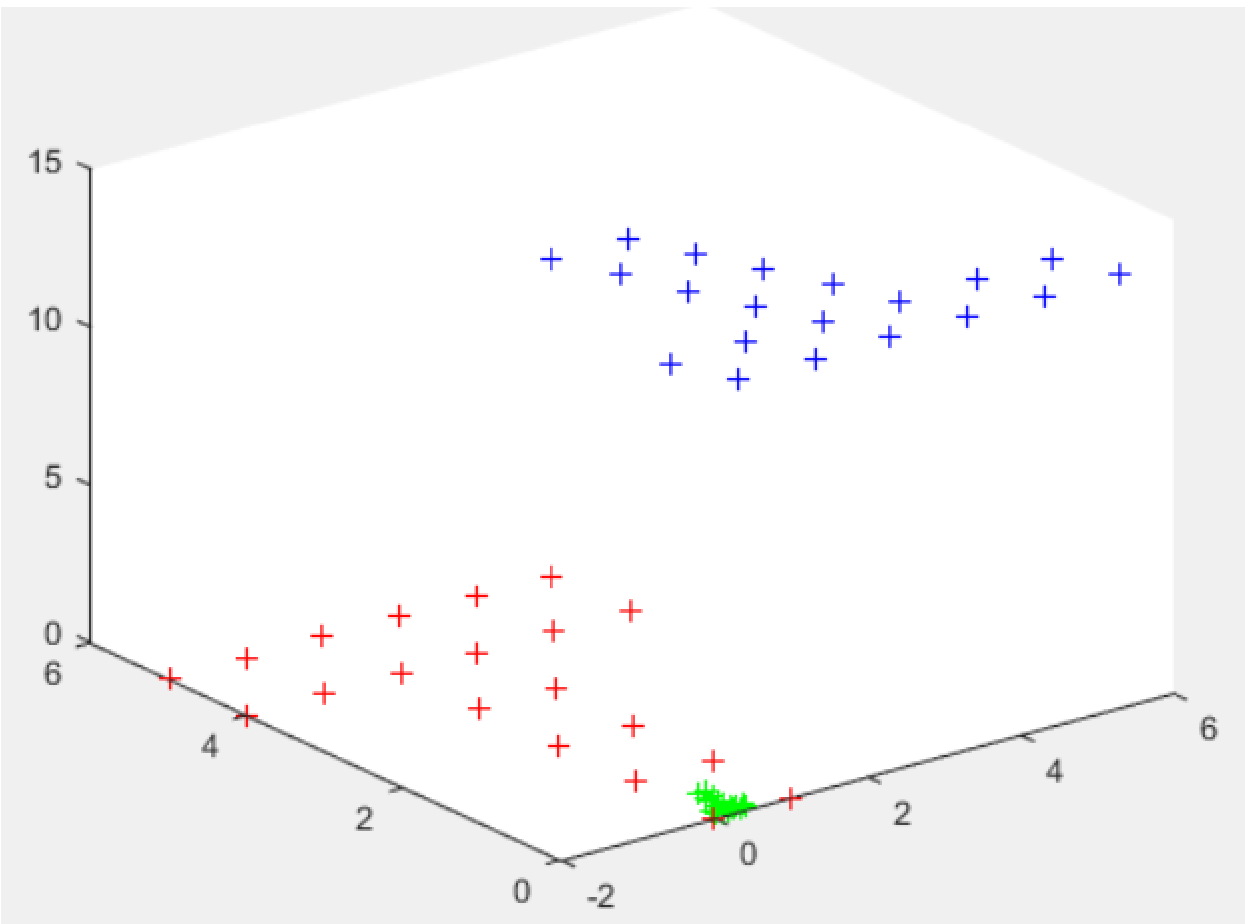
a)We compute the points (x,y,z) in the camera coordinate from the global points (Xw,Yw,Zw) by using the following expression (one rotation then one translation):

$$\mathbf{P_{camera}} = [\mathrm{R_x}]\mathbf{P_{global}} + \mathbf{T}$$

Then, we get the image points such that:

$$u = \frac{fx}{z} \quad \& \quad v = \frac{fy}{z}$$

Finally, you can see in red the global points, transformed with the first equation in the camera points in blue which are next transformed in the green points in the image coordinates.



So, we have followed the step 1 and step 2 of Tsai's Camera Model to transform the global points into image points.

b) We save all the points in the file “HW3_Pierre_Oucif_pb_1_camera_calibration_data.mat”.

Global Points			Camera Points			Image Points	
Xw	Yw	Zw	x	y	z	u	v
-2	5	0	1.000	0.670	12.500	0.080	0.054
-1	5	0	2.000	0.670	12.500	0.160	0.054
0	5	0	3.000	0.670	12.500	0.240	0.054
1	5	0	4.000	0.670	12.500	0.320	0.054
2	5	0	5.000	0.670	12.500	0.400	0.054
3	5	0	6.000	0.670	12.500	0.480	0.054
-2	4	0	1.000	1.536	12.000	0.083	0.128
-1	4	0	2.000	1.536	12.000	0.167	0.128
0	4	0	3.000	1.536	12.000	0.250	0.128
1	4	0	4.000	1.536	12.000	0.333	0.128
2	4	0	5.000	1.536	12.000	0.417	0.128
3	4	0	6.000	1.536	12.000	0.500	0.128
0	3	0	3.000	2.402	11.500	0.261	0.209
0	2	0	3.000	3.268	11.000	0.273	0.297
0	1	0	3.000	4.134	10.500	0.286	0.394
0	0	0	3.000	5.000	10.000	0.300	0.500
1	3	0	4.000	2.402	11.500	0.348	0.209
1	2	0	4.000	3.268	11.000	0.364	0.297
1	1	0	4.000	4.134	10.500	0.381	0.394
1	0	0	4.000	5.000	10.000	0.400	0.500

c) Then, we follow the Two-stage approach to compute f, [R] and **T** from the previous data.

First stage, we compute each terms of the equation $[A]\boldsymbol{\mu} = \mathbf{b}$ that we saw in class, then we solve this equation to get $\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}$ & μ_5.

Moreover, we can express T_y^2 and the components of [R] in function of the components of $\boldsymbol{\mu}$ and **T**:

-we have to check if $\mu_{11}\mu_{22} = \mu_{21}\mu_{12}$ to get T_y^2.

-we choose the sign of T_y and then we have to check if this one is compatible or not. If not, we change the sign of T_y.

-we have to check the sign of the components of the rotation matrix (we use the equations seen in class).

Second step, we solve $[A']\mathbf{x}' = \mathbf{b}'$ where [A'], **x'** & **b'** were defined in class, such that we get relationships between x,u,f,Tz,X,Y and R (there is no distortion).

Finally, we find f, T and R equal to the values used on the first question.

f =	1		
	3		
T =	5		
	10		
	1	0	0
[R] =	0	-0.866	-0.5
	0	0.5	-0.866

Problem 2

We follow the demarche presented in the paper of Tsai to realize Eye-on-hand Calibration.

a)We load the data from the given file. Then, we find H_{c12} and H_{c23} by using the following formulas:

$H_{c12} = H_{c2}.H_{c1}^{-1}$ and $H_{c23} = H_{c3}.H_{c2}^{-1}$

Then, to get R_{cij} and T_{cij}, we decompose H_{cij} such that: $H_{cij} = \begin{bmatrix} R_{cij} & T_{cij} \\ 0 & 1 \end{bmatrix}$ (4x4 matrix)

We use the same computations to get R_{gij} and T_{gij}.

Rc12			Rc23			Tc12	Tc23
-0.0958	0.9346	-0.3425	0.0041	-0.2779	0.9606	4.8930	-4.4892
-0.8060	-0.2747	-0.5243	0.4665	0.8502	0.2440	4.8245	-1.2969
-0.5841	0.2258	0.7797	-0.8845	0.4471	0.1331	-1.9306	8.5167

b) From the class note, we know that each rotation matrix can be written in a general form such that:

$$[R_k(\theta)] = \begin{bmatrix} k_x k_x V_\theta + \cos\theta & k_x k_y V_\theta - k_z \sin\theta & k_x k_z V_\theta + k_y \sin\theta \\ k_x k_y V_\theta + k_z \sin\theta & k_y k_y V_\theta + \cos\theta & k_y k_z V_\theta - k_x \sin\theta \\ k_x k_z V_\theta - k_y \sin\theta & k_y k_z V_\theta + k_x \sin\theta & k_z k_z V_\theta + \cos\theta \end{bmatrix}$$

Where, $V_\theta = 1 - \cos\theta$

So, we get: $\quad \mathrm{Trace}\big(R_k(\theta)\big) = 1 + 2\cos\theta \;\Rightarrow\; \theta = \cos^{-1}\left(\frac{r_{11}+r_{22}+r_{33}-1}{2}\right)$

Then, we can compute the vector **k** by using the following relationship:

$$\begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} = \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} . \frac{1}{2\sin\theta} \qquad \text{where } \theta \neq n\pi$$

We have to check the sign of k_x because of the fact that $\cos\theta = \cos(-\theta)$ & $\sin\theta = -\sin(-\theta)$. So, we know that $\mathrm{sign}(k_x) = \mathrm{sign}(r_{32} - r_{23})$.

Finally, we find that:

$$\mathbf{n} = \frac{\mathbf{k}}{\|\mathbf{k}\|}$$

And we can now compute each vectors **P**, such that:

$$\mathbf{Pc_{ij}} = 2\sin\left(\frac{\theta_{cij}}{2}\right)\mathbf{n_{cij}} \quad \& \quad \mathbf{Pg_{ij}} = 2\sin\left(\frac{\theta_{cij}}{2}\right)\mathbf{n_{gij}}$$

We find:

nc12	theta_c12 (rad)	Pc12	nc23	theta_c23 (rad)	Pc23
-0.3926	-1.8707	0.6319	-0.1016	-1.5771	0.1441
-0.1265		0.2035	-0.9226		1.3088
0.9110		-1.4663	-0.3722		0.5281

ng12	theta_g12 (rad)	Pg12	ng23	theta_g23 (rad)	Pg23
0.1268	-1.5770	-0.2041	0.9225	-1.8705	-1.3087
-0.3927		0.6320	-0.1014		0.1438
0.9109		-1.4661	-0.3723		0.5282

Moreover, we can check these results by computing the $[R_{g12}]$ and $[R_{g23}]$ with the equation [8] and [10] from Tsai's paper, so we get two different methods to check these solutions.

<u>First method:</u> we can write the rotation matrix [R] as following (from equation [8] of Tsai's paper)

$$[R_n(\theta)] = \begin{bmatrix} n_1^2 + (1-n_1^2)\cos\theta & n_1n_2(1-\cos\theta) - n_3\sin\theta & n_1n_3(1-\cos\theta) + n_2\sin\theta \\ n_1n_2(1-\cos\theta) + n_3\sin\theta & n_2^2 + (1-n_2^2)\cos\theta & n_2n_3(1-\cos\theta) - n_1\sin\theta \\ n_1n_3(1-\cos\theta) - n_2\sin\theta & n_2n_3(1-\cos\theta) + n_1\sin\theta & n_3^2 + (1-n_3^2)\cos\theta \end{bmatrix}$$

Where, $\mathbf{n} = [n_1 \quad n_2 \quad n_3]^T$ (that we found on the previous questions)

<u>Second method:</u> we can use the vector **Pr** as following (from equation [10] of Tsai's paper)

$$[R] = \left(1 - \frac{|\mathbf{Pr}|^2}{2}\right).[I] + \frac{1}{2}\left(\mathbf{Pr}.\mathbf{Pr^T} + \alpha.\mathrm{Skew}(\mathbf{Pr})\right)$$

Where, $\alpha = \sqrt{4 - |\mathbf{Pr}|^2}$ and $\mathrm{Skew}(\mathbf{Pr}) = \begin{bmatrix} 0 & -Pr3 & Pr2 \\ Pr3 & 0 & -Pr1 \\ -Pr2 & Pr1 & 0 \end{bmatrix}$

Finally, we get the same matrices for the two calculations and they are equal to the initial rotation matrix.

Rg23_equation_8			Rg23_equation_10		
0.8502	-0.4664	-0.2442	0.8502	-0.4664	-0.2442
0.2782	0.0041	0.9605	0.2782	0.0041	0.9605
-0.4470	-0.8845	0.1333	-0.4470	-0.8845	0.1333

Rc12_equation_8			Rc12_equation_10		
-0.2744	0.8058	0.5248	-0.2744	0.8058	0.5248
-0.9348	-0.0955	-0.3421	-0.9348	-0.0955	-0.3421
-0.2256	-0.5845	0.7794	-0.2256	-0.5845	0.7794

c) We follow the procedure presented in the paper of Tsai to compute $\mathbf{P_{cg}}$, $[\mathrm{R_{cg}}]$ & $\mathbf{T_{cg}}$.

Step 1: we compute $\mathbf{P_{cg}}'$ such that,

$$\mathrm{Skew}(\mathbf{P_{gij}} + \mathbf{P_{cij}}).\mathbf{P'_{cg}} = \mathbf{P_{cij}} - \mathbf{P_{gij}}$$

We use the 1 by 6 matrices defined respectively by the concatenation of the two pairs of $\mathbf{P_{gij}} + \mathbf{P_{cij}}$ and $\mathbf{P_{cij}} - \mathbf{P_{gij}}$ to solve the previous equation and to get $\mathbf{P_{cg}}'$

Step 2: we use the vector $\mathbf{P_{cg}}'$ to compute $\theta_{\mathrm{R_{gc}}}$ such that,

$$\theta_{\mathrm{R_{cg}}} = 2\tan^{-1}|\mathbf{P'_{cg}}|$$

Step 3: and we can directly compute $\mathbf{P_{cg}}$ which is defined by,

$$\mathbf{P_{cg}} = \frac{\mathbf{P'_{cg}}}{\sqrt{1 + |\mathbf{P'_{cg}}|^2}}$$

Moreover, we also know from the previous derivation, that $\mathbf{n_{cg}} = \frac{\mathbf{P_{cg}}}{2\sin\left(\frac{\theta_{cg}}{2}\right)}$

Then, we use the general expression of [R] in function of **n** and θ to find $[\mathrm{R_{cg}}]$.

Finally, we use the procedure for computing $\mathbf{T_{cg}}$ which is presented in the paper of Tsai, such that,

$$(\mathbf{R_{gij}} - [\mathrm{I}])\mathbf{T_{cg}} = \mathbf{R_{cg}}\mathbf{T_{cij}} - \mathbf{T_{gij}}$$

As for the computation of $\mathbf{P'_{cg}}$, we have to use the concatenation of two pairs of $\mathbf{R_{gij}} - [\mathrm{I}]$ and $\mathbf{R_{cg}}\mathbf{T_{cij}} - \mathbf{T_{gij}}$ to be able to solve the previous equation.

We find: $\mathbf{Pcg} = [0.0003 \quad 0.0001 \quad 1.4143]^{\mathrm{T}}$, $[\mathrm{R_{cg}}] = \begin{bmatrix} -0.0002 & -1 & 0.0003 \\ 1 & -0.0002 & -0.0001 \\ 0.0001 & 0.0003 & 1 \end{bmatrix}$

And, $\mathbf{T_{cg}} = [-0.6070 \quad -0.8034 \quad -0.5022]^{\mathrm{T}}$

Problem 3

a) We want to find the matrix C such that the equation $\mathbf{x_c^T}[\mathbf{C}]\mathbf{x_c} = 0$ is equal to the equation (1). The matrix C must be a 3x3 matrix in order to satisfy the previous equation. Then, we use a general form of C to make the calculation:

$$C = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Moreover, we know that: $u = \frac{fx_c}{z_c}$ and $v = \frac{fy_c}{z_c}$

By replacing u and v by the previous relationships in the equation (1), we get:

$$\frac{Ax_c^2}{z_c^2}f^2 + \frac{2Bx_cy_c}{z_c^2}f^2 + \frac{Cy_c^2}{z_c^2}f^2 + \frac{2Dx_c}{z_c}f + \frac{2Ey_c}{z_c}f + F = 0$$

Then, we find:

$$Ax_c^2 + 2Bx_cy_c + Cy_c^2 + \frac{2Dz_cx_c}{f} + \frac{2Ey_cz_c}{f} + \frac{Fz_c^2}{f^2} = 0$$

Moreover, if we compute $\mathbf{x_c^T}[\mathbf{C}]\mathbf{x_c} = 0$, then we find:

$$ax_c^2 + x_cy_c(d+b) + x_cz_c(g+c) + z_cy_c(h+f) + ey_c^2 + iz_c^2 = 0$$

By identification term by term, we have:

$a = A$, $d + b = 2B$, $e = C$, $i = \frac{F}{f^2}$, $g + c = \frac{2D}{f}$ & $h + f = \frac{2E}{f}$

Finally, we get:

$$C = \begin{bmatrix} A & B & \frac{D}{f} \\ B & C & \frac{E}{f} \\ \frac{D}{f} & \frac{E}{f} & \frac{F}{f^2} \end{bmatrix}$$

b) A point P(x,y,z) is on a plane, if its coordinates satisfy the following equation: $ax + by + cz + d = 0$

Where the vector $\mathbf{n} = [a\ b\ c]^T$ is the vector normal to this plane. In our case, the vector normal to the object plane is $\mathbf{z_0}$ and it is obtained by the rotation of the camera coordinate frame by an angle θ, such that we have:

$$Rx = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \ \& \ [x_c\ \ y_c\ \ z_c]^T = Rx.[x_0\ \ y_0\ \ z_0]$$

By using the projection of $\boldsymbol{z_0}$ in the z_cy_c plane, we get: $\boldsymbol{z_0} = [0\ \ cos\theta\ \ sin\theta]^T$

So, the plane perpendicular to $\boldsymbol{z_0}$ has the following expression:

$$cos\theta y_c + sin\theta z_c + d = 0$$

Such that, $z_c = \alpha x_c + \beta y_c + \gamma$ where $\alpha = 0,\ \beta = -tan\theta\ \ \&\ \gamma = -d$

Finally, the object plane is described by all point P in the camera frame respecting the following equation:

$$z_c = -tan\theta \mathrm{y}_c + \gamma$$

c) We use the general equation of an ellipse: $\left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2 = 1$

and by using the relationships between u, v & x_c, y_c, z_c, we get:

$$\frac{f^2 x_c^2}{z_c^2 a^2} + \frac{f^2 y_c^2}{z_c^2 b^2} = 1$$

Then, we find:

$$\frac{x_c^2}{a^2} + \frac{y_c^2}{b^2} = \frac{z_c^2}{f^2}$$

Moreover, by using the equation of the object plane, we can replace z_c such that:

$$\frac{x_c^2}{a^2} + \frac{y_c^2}{b^2} = \frac{(-y_c tan\theta + \gamma)^2}{f^2}$$

And, we finally end up with the equation of the ellipse on the camera coordinate frame:

$$\frac{x_c^2}{a^2} + y_c^2\left(\frac{1}{b^2} - \frac{tan^2\,\theta}{f^2}\right) - y_c\left(\frac{2\gamma tan\theta}{f^2}\right) = \frac{\gamma^2}{f^2}$$

d) Then, we rotate the camera frame by $-\theta$ around the axe **x** and we get the following relationships between the old camera coordinates (xc,yc,zc) and the new camera coordinates (x,y,z):

$$[x_c\ \ y_c\ \ z_c]^T = [R_x(-\theta)].[x\ \ y\ \ z]^T \quad \Rightarrow \quad \begin{cases} x = x_c \\ y_c = cos(-\theta)\,y - sin(-\theta)\,z \\ z_c = sin(-\theta)\,y + cos(-\theta)\,z \end{cases}$$

Moreover, the distance between the object plane and the plane defined by xy is equal to $\gamma cos\theta$, because these two planes are parallel. Then, we have $z = \gamma cos\theta$ and so $y_c = ycos\theta + \gamma cos\theta sin\theta$

So, we get by replacing this expression of y_c in the previous equation of the ellipse:

$$\frac{x_c^2}{a^2} + (ycos\theta + \gamma cos\theta sin\theta)^2\left(\frac{1}{b^2} - \frac{tan^2\,\theta}{f^2}\right) - (ycos\theta + \gamma cos\theta sin\theta)\left(\frac{2\gamma tan\theta}{f^2}\right) = \frac{\gamma^2}{f^2}$$

And, after some derivations, we get:

$$\frac{x_c^2}{a^2} + y^2\left(\frac{cos^2\,\theta}{b^2} - \frac{sin^2\,\theta}{f^2}\right) + 2\gamma y\sin\theta\left(\frac{\cos^2\theta}{\mathrm{b}^2} - \frac{\sin^2\theta}{\mathrm{f}^2} + \frac{1}{\mathrm{f}^2}\right) + \frac{\gamma^2}{\mathrm{b}^2}\sin^2\theta\cos^2\theta - \frac{\gamma^2\sin^4\theta}{\mathrm{f}^2} + \frac{2\gamma^2\sin^2\theta}{\mathrm{f}^2} = \frac{\gamma^2}{\mathrm{f}^2}$$

Then, we are looking for a θ such that this equation is a circle, it means that:

$$\frac{1}{a^2} = \frac{\cos^2\theta}{b^2} - \frac{\sin^2\theta}{f^2}$$

After some derivations, we get:

$$\sin\theta = \frac{f}{a}\sqrt{\frac{a^2 - b^2}{f^2 + b^2}} \qquad \& \qquad \cos\theta = \frac{b}{a}\sqrt{\frac{a^2 + f^2}{b^2 + f^2}}$$

Finally, we have:

$$\theta = \arcsin\left(\frac{f}{a}\sqrt{\frac{a^2 - b^2}{f^2 + b^2}}\right) \text{ and } \theta = \arccos\left(\frac{b}{a}\sqrt{\frac{a^2 + f^2}{b^2 + f^2}}\right)$$

We use the two equations to determine the sign of θ.