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Empirical Analysis of the Gale-Shapley Deferred Acceptance Algorithm

(Under Uniform and Weighted Preferences)

Author: Himanshu Goyal
Date: May 16, 2025

Overview

In this project, we implement the Gale-Shapley Deferred Acceptance (DA) algorithm for the case of equal numbers of doctors and hospitals with one-to-one matchings. We empirically investigate the behavior of the algorithm under **independent and uniformly random preferences**, replicating and verifying results from the classical paper by Pittel.

Part 1: Average Number of Proposals vs n

Objective

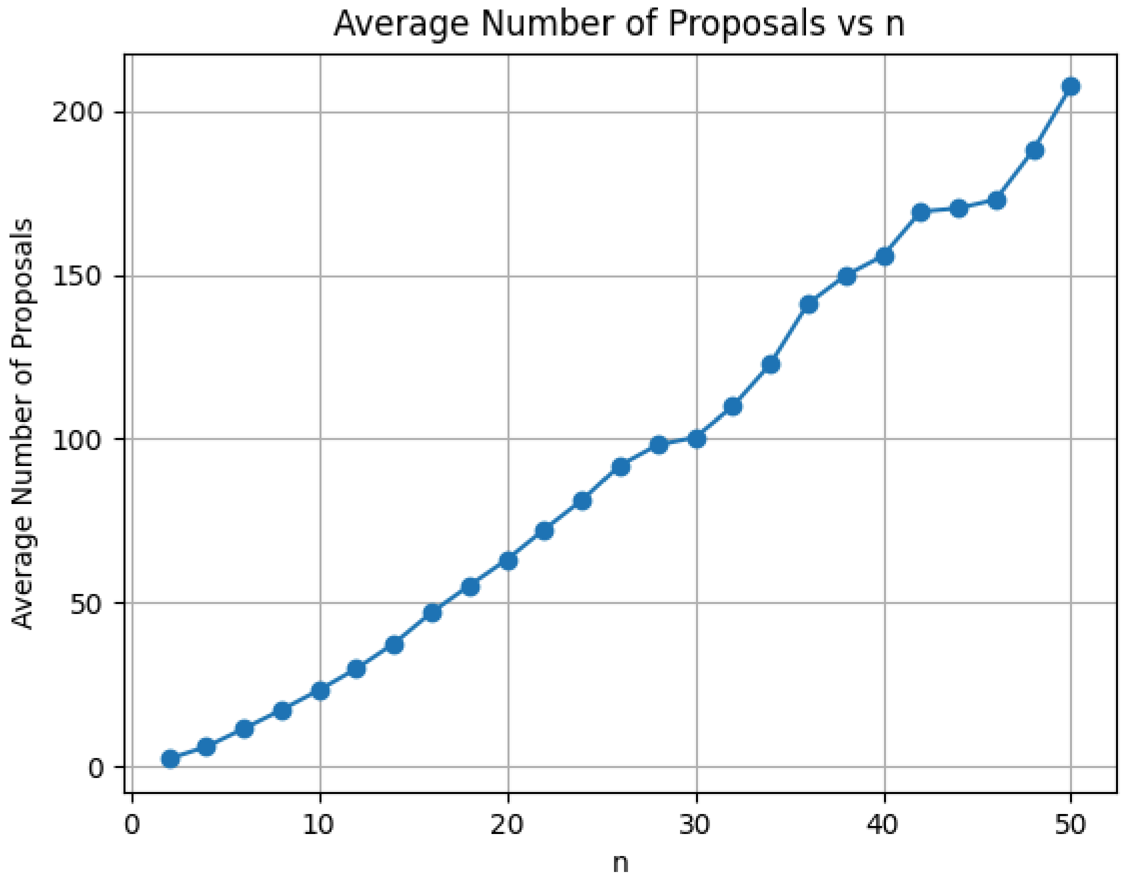
To determine how many proposals are made, **on average**, during a run of **doctor-proposing** DA. We vary `n` (number of agents) and repeat the experiment multiple times for each `n`.

Method

- For each `n`, generate random complete preference lists for all doctors and hospitals.
- Run the DA algorithm and count the number of proposals made until a stable matching is reached.

- Repeat the experiment `T` times and average the number of proposals.

Result



The average number of proposals grows sub-quadratically and appears to follow a trend slightly steeper than linear, consistent with known theoretical bounds.

Part 2: Proposal Distribution for Fixed n

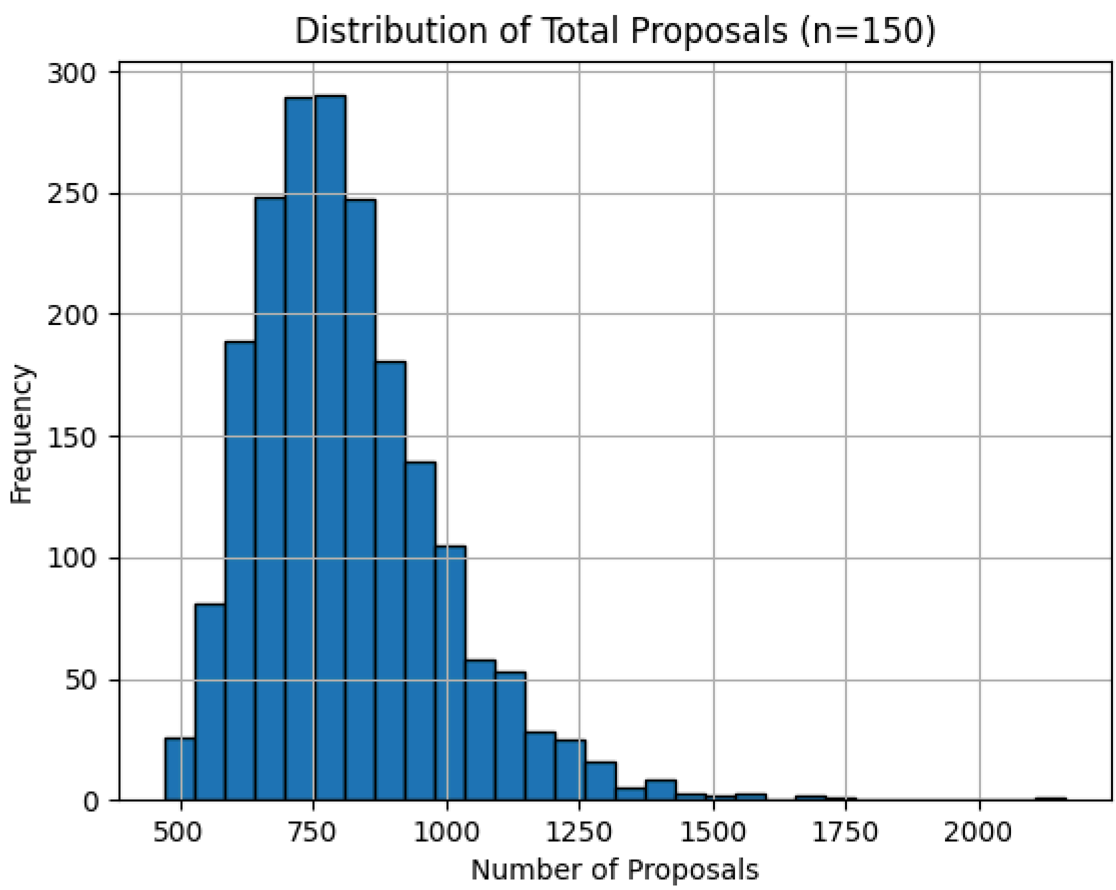
Objective

To understand the **distribution** of the number of proposals made during multiple runs of DA for a fixed `n`.

Method

- Fix a reasonably large value of `n` (e.g., 50 or 100).
- Generate random preference lists and run the DA algorithm for `T = 1000` trials.
- Plot a histogram of the total number of proposals per run.

Result



The histogram shows a concentrated distribution of proposal counts with some variance. The distribution is skewed slightly right, showing the possibility of higher proposal counts in some random instances.

Part 3: Average Rank of Partners vs n

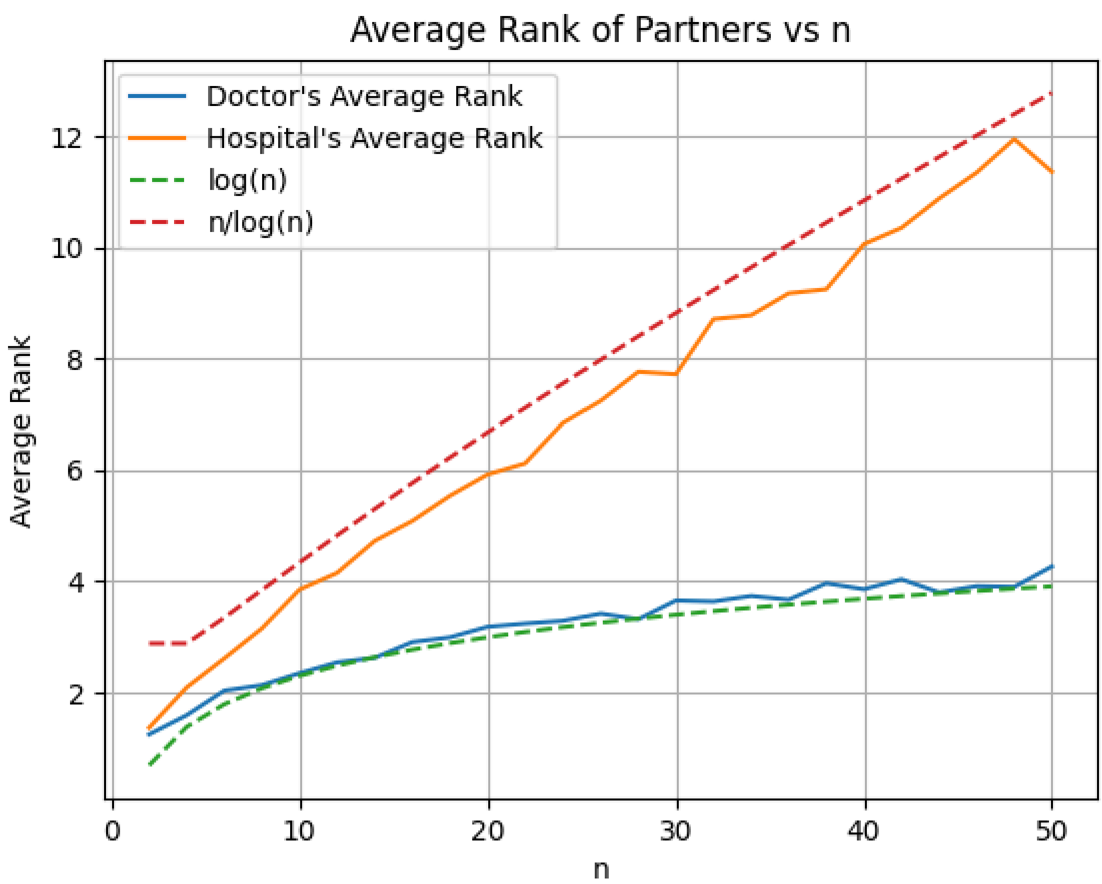
Objective

To evaluate the **quality of matches** from both the doctor's and hospital's perspective, using their **rankings** of matched partners.

Method

- For each n, and across several trials:
  - For each doctor, find the rank of the hospital they were matched with.
  - For each hospital, find the rank of the doctor they were matched with.
- Compute the average rank for each side.
- Plot these averages as functions of n.

Result



- The doctor’s average rank scales as **log(n)** — confirming Pittel’s Theorem 2.
- The hospital’s average rank scales approximately as **n / log(n)** — again matching the theory.
- This confirms the **proposer-advantage** in the DA algorithm.

Part 4 (Optional): Rank Distribution Histograms

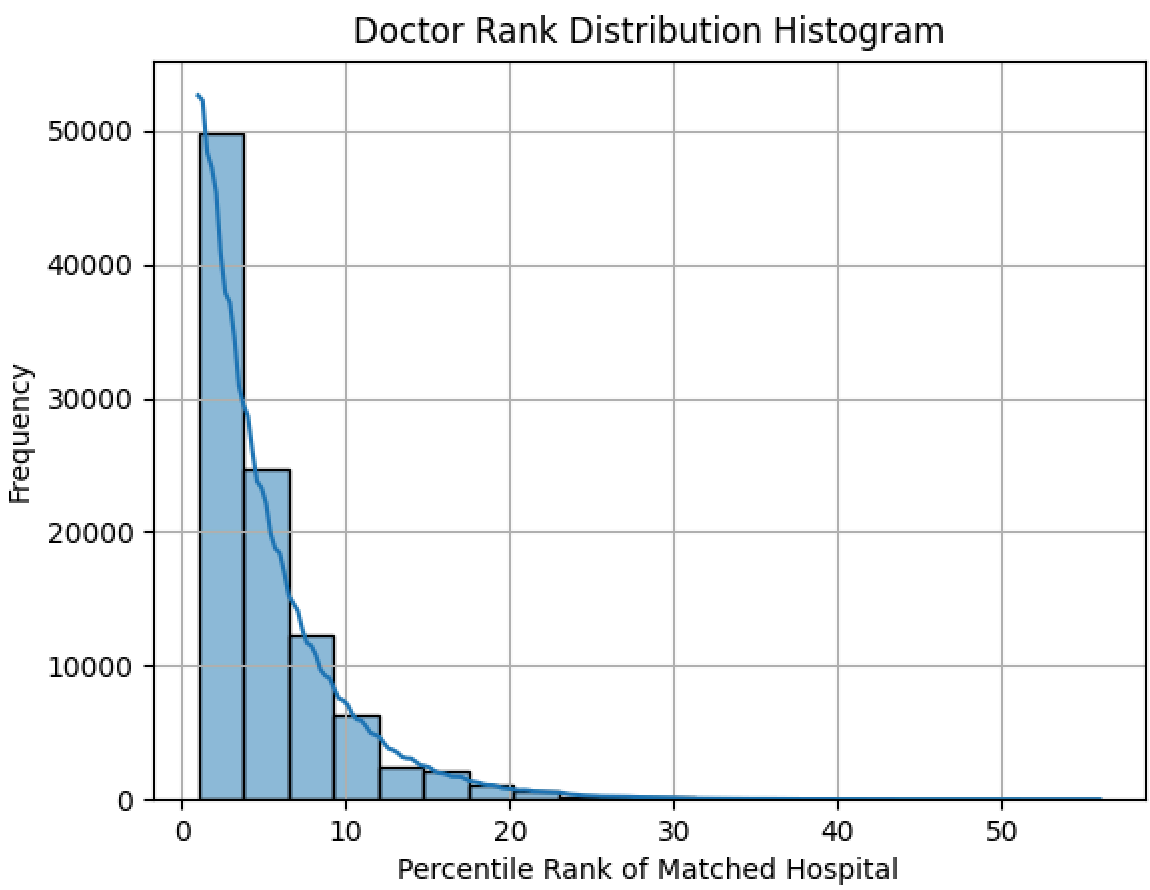
Objective

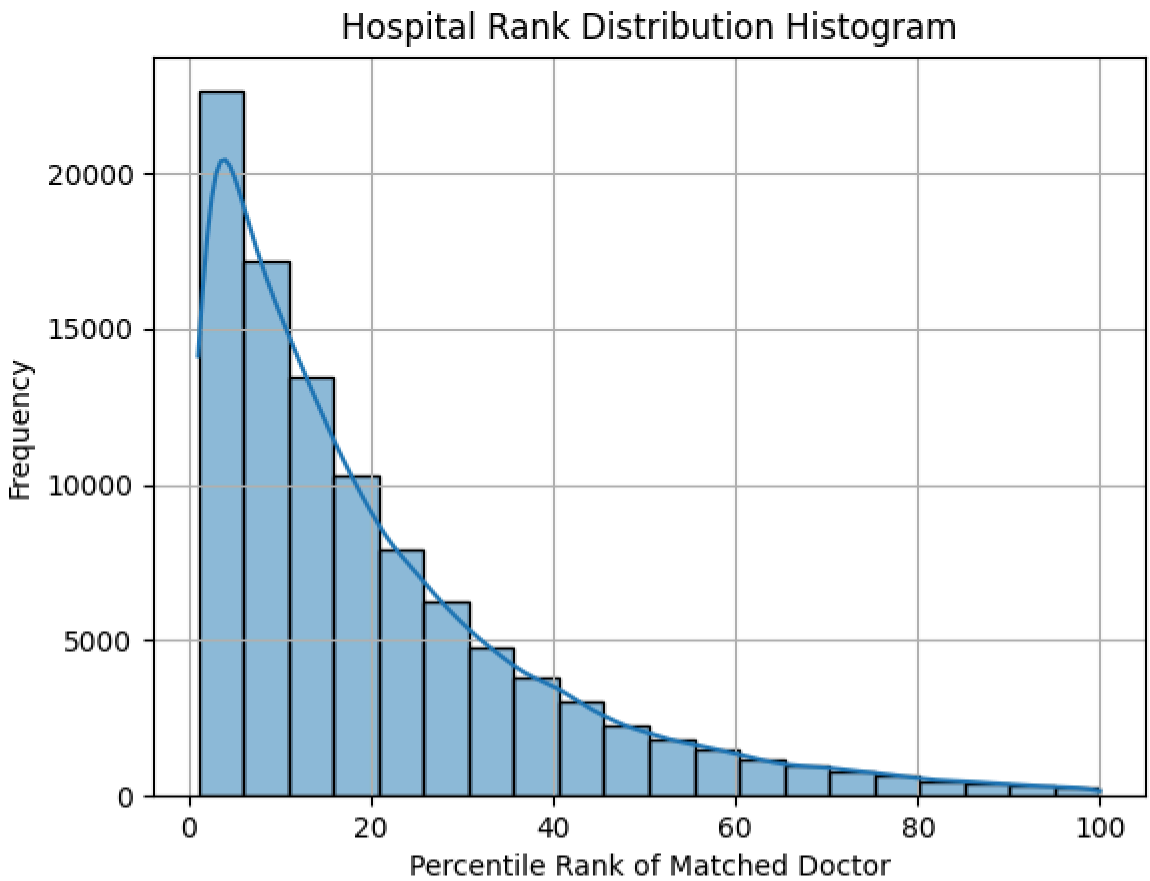
To analyze how many agents (doctors and hospitals) are matched to partners in the **top p%** of their preference lists.

Method

- Fix a value of n (e.g., 100).
- For each agent, calculate the rank of their match as a percentile.
- Accumulate statistics over multiple trials.
- Plot histograms showing the rank distribution for both doctors and hospitals.

Result





The histogram for doctors is skewed towards top preferences, reflecting their advantage. Hospitals, being on the receiving end, exhibit a wider spread in partner quality.

Conclusion

This empirical study confirms key theoretical properties of the Gale-Shapley DA algorithm under uniform random preferences:

- The number of proposals grows sub-quadratically.
- The doctor-proposing DA gives a clear advantage to doctors.
- The expected rank behavior matches asymptotic predictions of **log(n)** and **n/log(n)** respectively.

Extension: Gale-Shapley with Popularity-Based (Weighted) Preferences

Overview

In this section, we explore a generalized model of preference generation based on agent popularity. Each doctor and hospital is assigned a unique popularity score from the set {1, 2, ..., n}. These scores influence the probability that an agent appears earlier in another agent's preference list.

Preference Generation Method

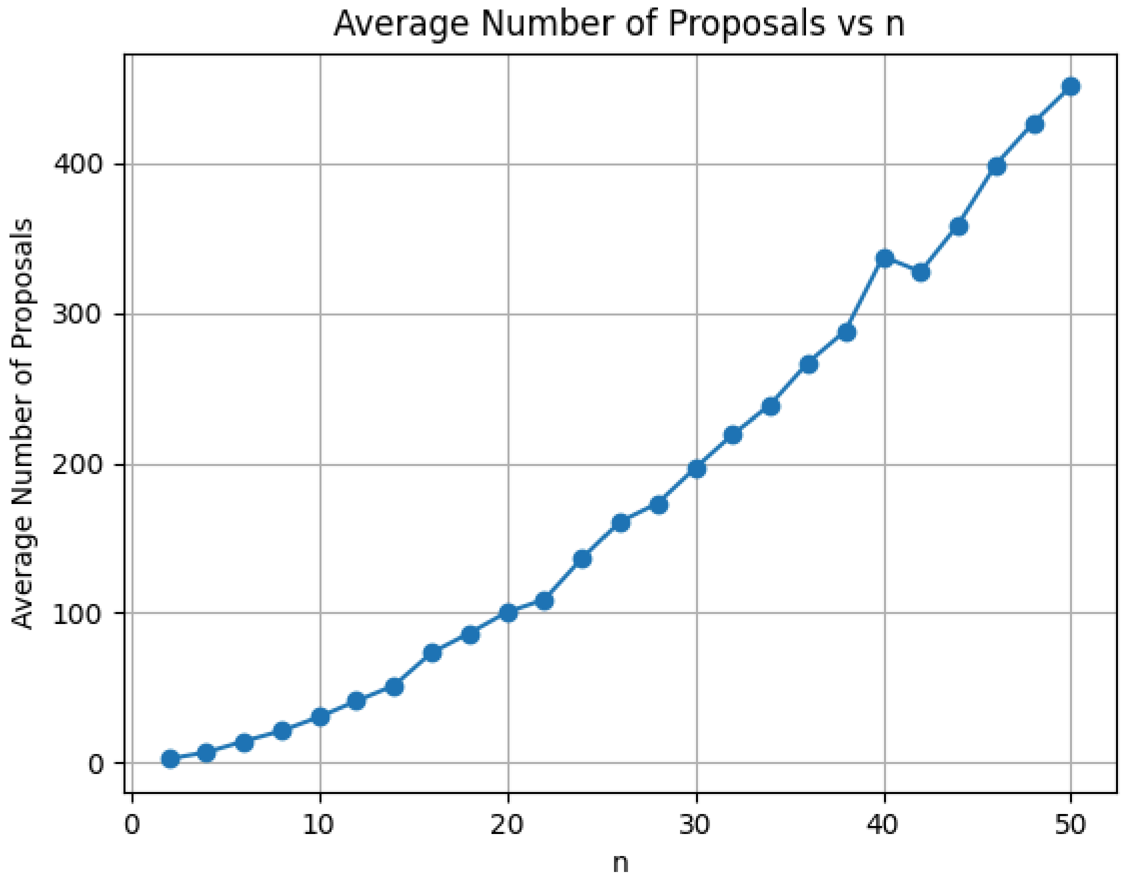
- Let $a_1 < a_2 < ... < a_n$ be n ascending natural numbers.
- Each doctor and hospital receives one of these numbers randomly (permuted).
- A doctor constructs their preference list by repeatedly selecting an unpicked hospital with probability proportional to its popularity score (i.e., hospital *i* is chosen with probability a_i / A, where $A = \sum a_j$ *over unpicked*).
- The same approach is applied for hospital preferences.

> This model generalizes the uniform random preference model (when all popularity scores are equal).

Rank Distribution Analysis (Weighted Model)

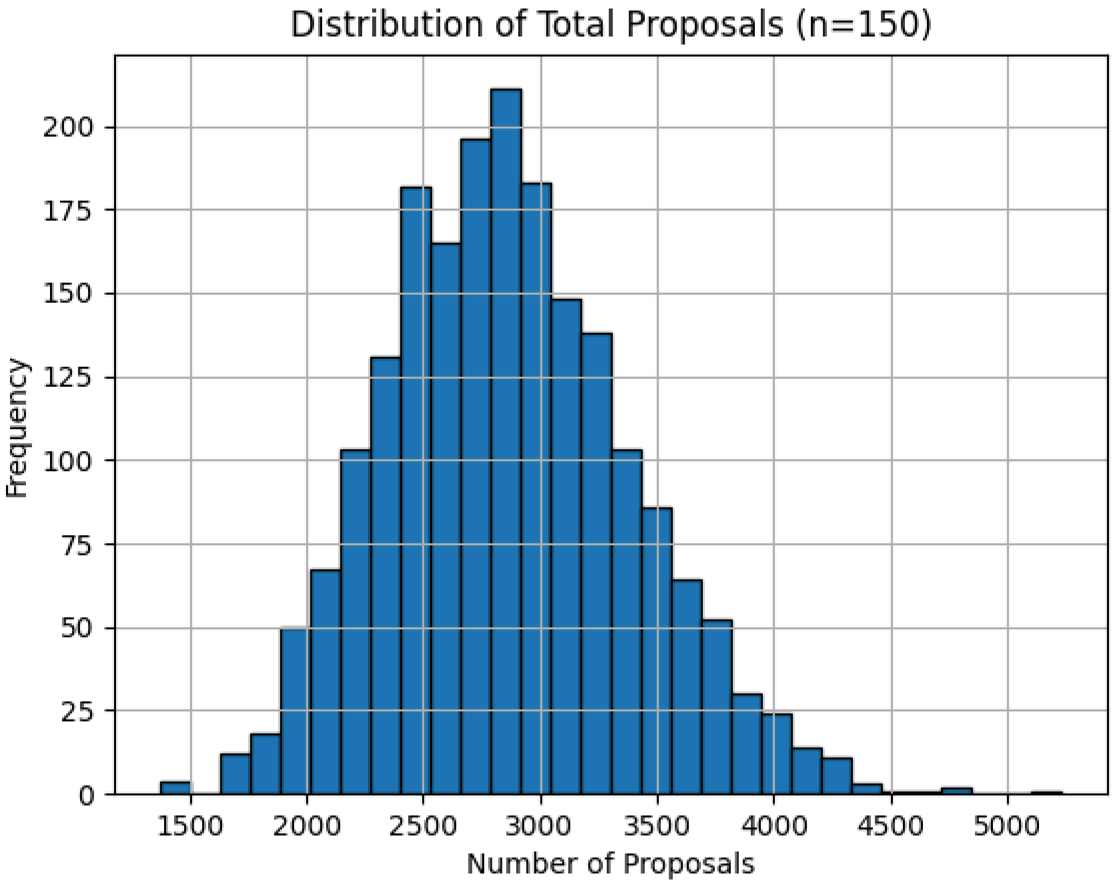
We apply the doctor-proposing DA algorithm under this new preference model and analyze the resulting match quality in terms of percentile ranks.

Part 1: Average Number of Proposals vs n



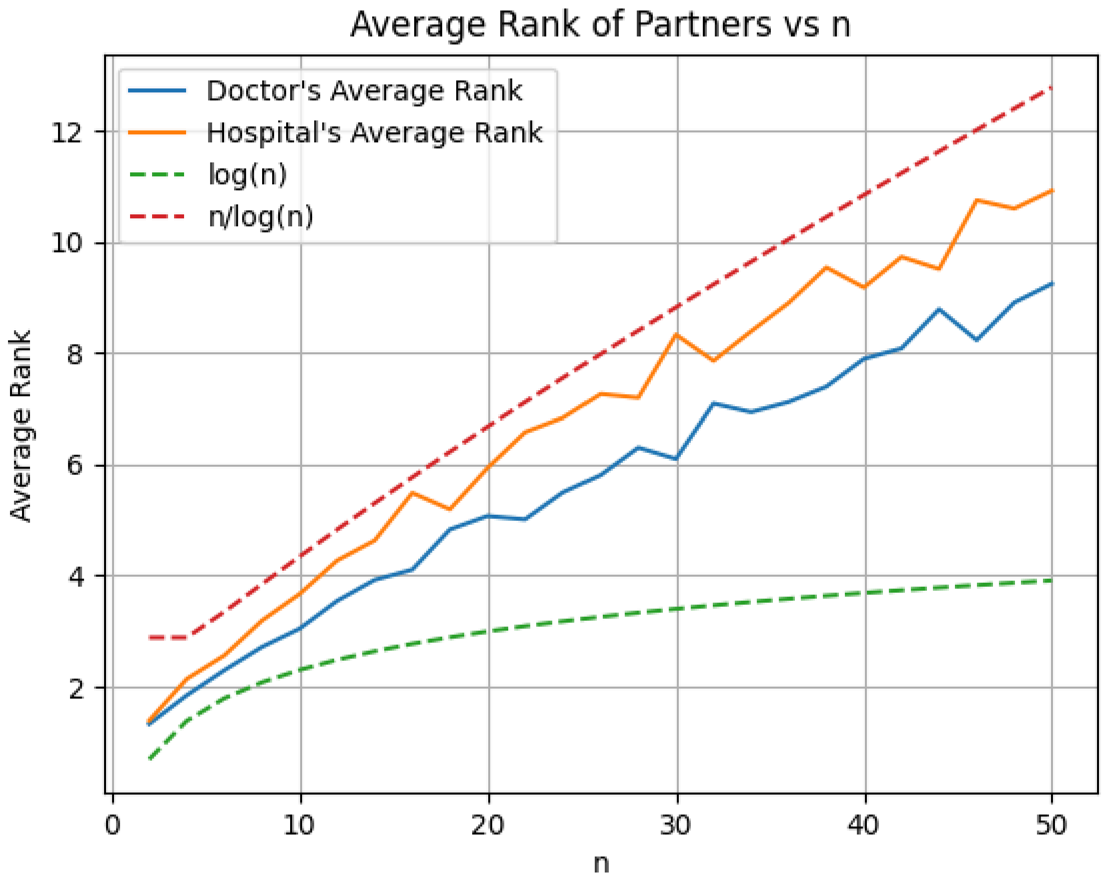
Observation:
The number of proposals increases non-linearly with n, but more sharply than in the uniform model. This suggests that **popularity-weighted preferences introduce more proposal rounds**, likely due to competition for highly popular hospitals.

Part 2: Proposal Distribution for Fixed n



Observation:
The distribution is bell-shaped (roughly normal), centered around a higher mean than the uniform model. This reinforces the idea that **more iterations are needed when preferences are skewed by popularity**.

Part 3: Average Rank of Partners vs n

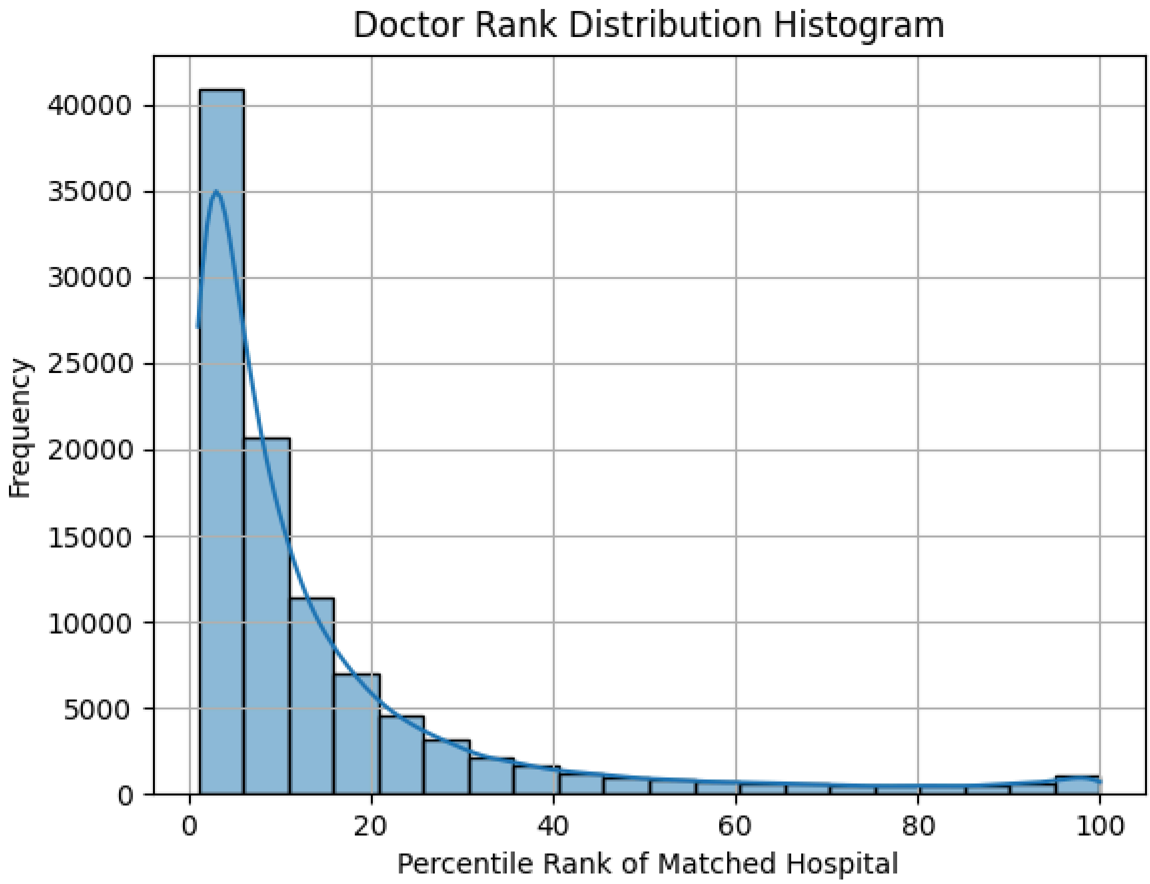


Observation:

- The doctor's average rank increases linearly with n, diverging from the logarithmic behavior seen in the uniform model.
- The hospital's average rank also increases, but slightly more slowly.
- This indicates that **doctors are not able to secure their top choices as easily** when preferences are skewed—**popularity reduces the proposer advantage**.

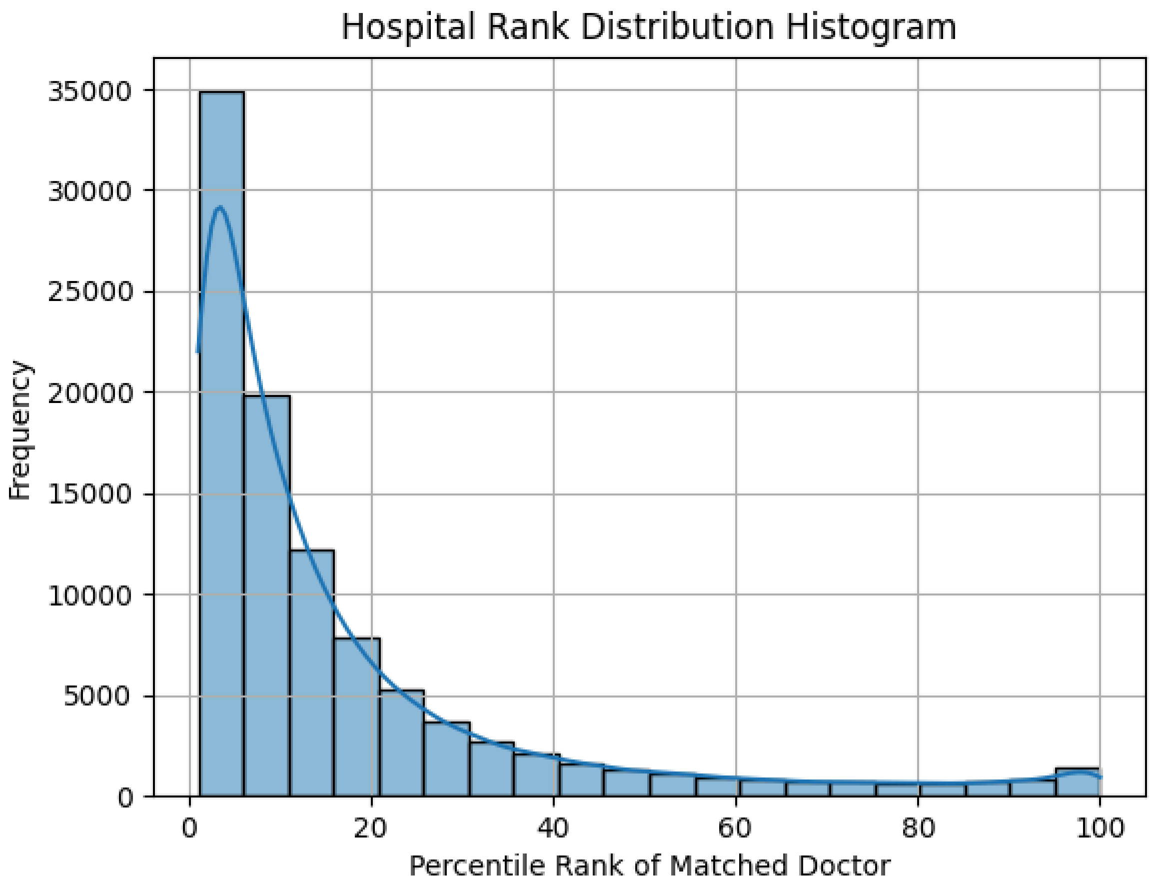
Part 4: Rank Distribution Histograms

Doctor Rank Distribution



Observation:
Doctors are more likely to match with top-percentile hospitals (lower rank values), but the curve is heavier-tailed than in the uniform case. This suggests that **competition for popular hospitals results in a wider spread of match quality** among doctors.

Hospital Rank Distribution



Observation:
The distribution is heavily skewed toward top-percentile doctors, with many hospitals matched to highly ranked individuals. This implies **hospitals benefit more in this model**, perhaps because they are more likely to be highly ranked by popular doctors due to shared bias.

Part 5: Compare Different Distributions

Model	Avg Doctor Rank	Avg Hospital Rank
Equal	4.22	11.67
Linear	8.93	11.25
Exponential	23.99	24.13

Observation:

- As the preference distribution becomes more skewed (from equal to exponential), the **average doctor rank increases significantly**, indicating deteriorating match quality for proposers.
- Interestingly, the **average hospital rank remains relatively stable**, showing that popularity skews help balance power more in favor of hospitals.

Summary of Key Differences from Uniform Model

Metric	Uniform Model	Weighted (Popularity-Based) Model
Preference generation	Equal probability	Skewed by popularity
Doctor match rank distribution	Logarithmic scaling	Linear/log-linear, depends on skew strength
Hospital match rank distribution	n/log n wide spread	Still wide, but favors more popular agents
Proposal count	Sublinear growth	Faster-than-linear due to competition
Distribution fairness	High for doctors	More balanced, hospitals gain advantage

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Conclusion

This weighted preference model highlights how **structured popularity shifts power dynamics** in the Gale-Shapley algorithm:

- **Doctors lose some of the proposer advantage**, particularly in highly skewed distributions.
- **Hospitals benefit more**, securing better matches due to being more desirable.
- The algorithm still terminates in stable matchings, but the **quality and distribution of outcomes vary significantly** with the underlying preference model.