

# Total Correctness

Total Correctness =  
Partial Correctness + Termination

Termination must be proved only  
for "while" loops

Often proof of termination is easier  
but not always!

# A FAMOUS EXAMPLE

$\{x \geq 1\}$

while ( $x > 1$ ) {

    if ( $x \% 2 == 0$ )

$x = x/2;$

    else

$x = 3 * x + 1;$

}

$\{x = 1\}$

Partial Correctness: trivial! Termination ???

# Total Correctness for loops

We need to prove that the loop will terminate.  
To do so we should find a variant function  $v$  such that:

- $v$  is an upper bound on the number of loops
- Initially  $v$  evaluates to a positive finite integer

$$\{ \text{Inv} \wedge C \} \Rightarrow v > 0$$

- The value of the variant function decreases each time the loop executes (here  $V$  is a constant)

$$\{ \text{Inv} \wedge C \wedge v = V \} \text{ loop } \{ v < V \}$$

~~loop exit is guaranteed since sooner or later  $v$  becomes  $\leq 0 \Rightarrow C$  must be false!~~

# Guessing $v$

- $v = (N - i)$  if loop always adds a constant to  $i$ .
- $v = (i)$  if loop subtracts a constant from  $i$ .
  - $v \leq 0$  at loop exit
- Other loops
  - Find an expression that is an upper bound on the number of iterations left in the loop.

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# EXERCISES OF SESSION ONE

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# Ex.1

- Square root of a positive integer.

```
{x > 0}
if (x == 1) sqrt := 1
else
    i:=1;
    z:=1;
    while ( z ≤ x )
        i++;
        z:=i*i;
    sqrt := i-1;
{sqrt2 ≤ x < (sqrt + 1)2 }
```

# Ex.1

## ■ Termination proof obligations:

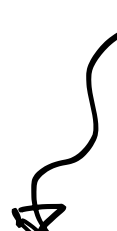
1. Choose a variant  $v$ .
2. Show that it is initially positive:

$$\{\text{Inv} \wedge C\} \Rightarrow v > 0$$

3. Show that it is decreasing:

$$\{\text{Inv} \wedge C \wedge v\} S \{\text{Inv} \wedge v'\}$$

$$v' < v$$


$$v(\cdot) = v$$


$$v(\cdot) = v'$$

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```
while ( z ≤ x )
```

```
    i++;
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```
    z := i * i;
```

$$v(x, i) = x - (i-1)^2$$

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$$C = \{z \leq x\}$$

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$$\{(i-1)^2 \leq x \wedge i^2 \leq x\} \Rightarrow \{x > (i-1)^2\}$$

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3. Show that it is decreasing:

$$\{\text{Inv} \wedge C \wedge v\} \ i++; z := i * i; \{\text{Inv} \wedge v_{\text{new}}\}$$

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$$v \rightarrow v(x_{old}, i_{old}) = x_{old} - (i_{old}-1)^2$$

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## Ex.2

- Least common multiple of two positive integers.

$\{x, y > 0\}$

$z := 1$

While  $(z \% x \neq 0 \vee z \% y \neq 0)$

$z++;$

$\{z \% x = 0 \wedge z \% y = 0 \wedge \forall i (1 \leq i < z \Rightarrow i \% x \neq 0 \vee i \% y \neq 0)\}$

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     $z++$ ;
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While (  $z \% x \neq 0 \vee z \% y \neq 0$  )  
     $z++$ ;
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$$v(x, y, z) = xy - z$$

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$\overline{i \neq x} \quad \overline{i \neq y}$

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$i < x$

$i < y$



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$$i < xy$$

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$$i = z \Rightarrow z < xy$$

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$$v \rightarrow v(x_{\text{old}}, y_{\text{old}}, z_{\text{old}}) = x_{\text{old}}y_{\text{old}} - z_{\text{old}}$$

$$v_{\text{new}} \rightarrow v(x_{\text{new}}, y_{\text{new}}, z_{\text{new}}) = x_{\text{new}}y_{\text{new}} - z_{\text{new}}$$

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