Seminar2

Wednesday, March 17, 2021 12:05 PM

1. DGL mit getrennten Variablen.

$$\left.\begin{array}{l} y' = f(x)\cdot g(y) \\ y' = \frac{dy}{dx} \end{array}\right\} \Rightarrow \frac{dy}{dx} = f(x)\cdot g(y) \;\Big|\; :g(y)\neq 0,\ \cdot dx$$

$$\frac{dy}{g(y)} = f(x)\cdot dx \;\Big|\; \int$$

$$\int \frac{dy}{g(y)} = \int f(x)\,dx$$

$$\boxed{G(y) = F(x) + c,\ c\in\mathbb{R}}$$

→ die allg. Lösung in impliziter Form.

$$\boxed{y = G^{-1}(F(x)+c),\ c\in\mathbb{R}}$$

→ die allg. Lös. im expliziten Form

II. $g(y) = 0$

$\exists\, y_0 \in \mathbb{R}\ .\ g(y_0) = 0 \Rightarrow y = y_0 \Rightarrow$ eine singuläre (Konstante) Lösung

Ex:

1. $y' = x\cdot y$

$f(x) = x,\ g(y) = y$

$y' = \frac{dy}{dx}$

I $g(y)\neq 0$

$$\frac{dy}{dx} = x\cdot y \;\Big|\; :g(y)\neq 0,\ \cdot dx$$

$$\frac{dy}{y} = x\,dx \;\Big|\; \int$$

$$\int \frac{1}{y}\,dy = \int x\,dx$$

$$\ln|y| = \frac{x^2}{2} + c,\ c\in\mathbb{R}$$

$$y = \pm e^{\frac{x^2}{2}}\, e^{c}$$

$$y = c_1\cdot e^{\frac{x^2}{2}},\ c_1\in\mathbb{R}^*$$

II $g(y) = 0$

$y = 0$
$\overset{\sim}{y_0}$

$\Rightarrow \boxed{y = c \cdot e^{\frac{x^2}{2}}, \ c \in \mathbb{R}}$ → die allgm. Lösung

2. $(x^2-1) \cdot y' + 2x \cdot y^2 = 0$

- rezolvat de mine -

3. $x \cdot y' = y^3 + y$

$y' = \frac{y^3+y}{x}$

$f(x) = \frac{1}{x}; \ g(y) = y^3 + y$

$y' = \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{x}(y^3+y) \quad \Big| \ : y^3+y \neq 0, \ \cdot dx$

I $g(y) \neq 0$

$\frac{dy}{y^3+y} = \frac{1}{x} dx$

$I = \int \frac{1}{y^3+y} dy = \int \frac{1}{y(y^2+1)} dy$

$= \int \frac{y^2+1-y^2}{y(y^2+1)} dy = \int \frac{1}{y} dy - \int \frac{y}{y^2+1} dy$

$$- \quad y(y^2+1)$$

$$= \ln|y| - \frac{1}{2}\ln y^2+1 + C$$

$$\ln \frac{\ln|y|}{\sqrt{y^2+1}} = \underbrace{\ln|X| + \underset{\ln c}{\overset{\shortparallel}{C}}}_{\ln|X| \cdot c}$$

$$\ln \frac{|y|}{\sqrt{y^2+1}} = \ln|X| \cdot c$$

$$\frac{|y|}{\sqrt{y^2+1}} = |X| \cdot c$$

$$\boxed{\frac{y}{\sqrt{y^2+1}} = X \cdot c, c \in \mathbb{R}}$$

$$\frac{y^2+1-1}{y^2+1} = x^2 c^2$$

$$1 - \frac{1}{y^2+1} = x^2 \cdot c^2$$

$$\frac{1}{y^2+1} = 1 - x^2 \cdot c^2$$

1

$y^{\prime}{}^{"+"}$

$$y^2+1=\frac{1}{1-x^2c^2}$$

$$y^2=\frac{1}{1-x^2\cdot c^2}-1$$

$$=\frac{1-1+x^2\cdot c^2}{1-x^2\cdot c^2}$$

$$=\frac{x^2\cdot c^2}{1-x^2\cdot c^2}$$

$$\boxed{y=\pm\sqrt{\frac{x^2\cdot c_1}{1-x^2\cdot c_1}}} \qquad c^2=c_1$$

$\rightarrow$ die allgem. Lösung.

$$y(\underbrace{1}_{x_0})=\underbrace{-5}_{y_0} \overset{<0}{\Rightarrow} c$$

$$y(x)=-\sqrt{\frac{x^2\cdot c_1}{1-x^2\cdot c_1}}$$

$$+5=+\sqrt{\frac{1\cdot c_1}{1-c_1}}\Rightarrow$$

$$\Rightarrow 25=\frac{c_1}{1-c_1}\Rightarrow 25-25c_1=c_1$$

$$26c_1=25$$

$c_1 = \frac{25}{26}$

$$y(x) = -\sqrt{\frac{x^2 \cdot \frac{25}{26}}{1 - x^2 \cdot \frac{25}{26}}}$$

$\rightarrow$ Lösung des Cauchypb. mit der Anfangsbed. $y(1) = -5$

II $g(y) = 0$

$y^3 + y = 0 \Leftrightarrow y(y^2+1) = 0 \Rightarrow y = 0 \rightarrow$ sing. Lösung

2. <u>Eulersche homogene DGL.</u>

$$y' = f\left(\frac{y}{x}\right)$$

$$z = \frac{y}{x}, \quad z = z(x)$$

$$y = z \cdot x \,|' \Rightarrow y' = z' \cdot x + z$$

$$z' \cdot x + z = f(z)$$

$$z' \cdot x = f(z) - z$$

$\rightarrow$ DGL. mit getrennten Variable

1) $2x^2 y' = x^2 + y^2$

$$y' = \frac{x^2 + y^2}{2x^2}$$

$$y' = \frac{1}{2} + \frac{1}{2} \cdot \frac{y^2}{x^2}$$

$$y' = \underbrace{\frac{1}{2} + \frac{1}{2} \cdot \left(\frac{y}{x}\right)^2}_{f\left(\frac{y}{x}\right)}$$

Subst: $z = \frac{y}{x} \Rightarrow y = z \cdot x \Rightarrow y' = z' \cdot x + z$

$$z' \cdot x + z = \frac{1}{2} + \frac{1}{2} \cdot z^2$$

$$z' \cdot x = \frac{1}{2} + \frac{1}{2} \cdot z^2 - z$$

$$z' \cdot x = \frac{(z-1)^2}{2} \quad \Big| : x \neq 0$$

$$z' = \frac{1}{x} \cdot \frac{(z-1)^2}{2}$$

$$f(x) = \frac{1}{x} ; \quad g(z) = \frac{(z-1)^2}{2}$$

$$f : \mathbb{R}^* \to \mathbb{R}$$

I. $g(z) \neq 0$

$$\frac{dz}{dx} = \frac{1}{x} \cdot \frac{(z-1)^2}{2} \quad \Big| \begin{array}{l} : (z-1)^2 \\ \cdot dx \end{array}$$

$$\frac{dz}{(z-1)^2} = \frac{dx}{2x} \Big| \int$$

$$\int \frac{dz}{(z-1)^2} = \frac{1}{2} \int \frac{1}{x}\, dx$$

$$\int (z-1)^{-2}\, dz = \frac{1}{2} \cdot \ln|x| + c$$

$$\frac{-1}{z-1} = \frac{1}{2} \ln|x| + c$$

$$z - 1 = \frac{-1}{\frac{1}{2} \ln|x| + c}$$

$$z = \frac{-1}{\frac{1}{2} \ln|x| + c} + 1, \quad c \in \mathbb{R}$$

$z = \frac{y}{x}$ $$y = z \cdot x = \frac{-x}{\frac{1}{2} \ln|x| + c} + x, \quad c \in \mathbb{R}$$

II $g(z) = 0$

$(z-1)^2 = 0 \Rightarrow z = 1 \rightarrow$ sol. sing. a ec. in z

$y = x \rightarrow$ sing. Lösung für die DGL in y

$$2.\quad y' = e^{\frac{y}{x}} + \frac{y}{x}$$

$$y' = f\left(\frac{y}{x}\right)$$

$$z = \frac{y}{x} \Leftrightarrow y = z \cdot x \Rightarrow y' = z' \cdot x + z$$

Einsetzen:

$$z' \cdot x + \cancel{z} = e^{z} + \cancel{z}$$

$$z' \cdot x = e^{z}$$

$$z' = \frac{e^{z}}{x} \qquad f(x) = \frac{1}{x}\ ;\ f: \mathbb{R}^{*} \to \mathbb{R}$$

$$g(z) = e^{z}\ ;\ g: \mathbb{R} \to \mathbb{R}$$

$$\frac{dz}{dx} = \frac{e^{z}}{x} \quad \Big|\ : e^{z} \neq 0,\ \cdot dx$$

$$\frac{dz}{e^{z}} = \frac{dx}{x} \quad \Big|\ \int$$

$$\int \frac{dz}{e^{z}} = \int \frac{dx}{x}$$

$$-e^{-z} = \ln|x| + c$$

$$e^{-z} = -\ln|x| + c$$

$$e^{-z} = -\ln|x| + c$$

$$-z = \ln(-\ln|x| + c)$$

$$z = -\ln(-\ln|x| + c)$$

$$\boxed{y = z \cdot x = -x \ln(-\ln|x| + c), \quad c \in \mathbb{R}}$$

3. $y' = \dfrac{y}{x+y}, \quad x, y > 0$

$$y' = \frac{1}{\frac{x+y}{y}} = \frac{1}{\frac{x}{y} + 1}$$

Subst: $\boxed{z = \dfrac{y}{x}} \Rightarrow y = z \cdot x$

$$y' = z' \cdot x + z$$

$$z' \cdot x + z = \frac{1}{\frac{1}{z} + 1}$$

$$z' \cdot x = \frac{1}{\frac{1}{z} + 1} - z$$

$$z' = \frac{\frac{1}{\frac{1}{z} + 1} - z}{x} \qquad f(x) = \frac{1}{x}; \quad g(x) = \frac{1}{\frac{1}{z} + 1} - z$$

$$\frac{dz}{dx} = \frac{\frac{1}{\frac{1}{z} + 1} - z}{x} \quad \Bigg| : \frac{1}{\frac{1}{z} + 1} - z \neq 0, \ \cdot dx$$

$$\frac{dz}{\frac{1}{\frac{1}{z}+1} - z} = \frac{dx}{x} \quad \Big| \int$$

$$\boxed{\frac{1}{z} - \ln|z| = \ln|x| + c} \quad c \in \mathbb{R}$$

$$\frac{1}{y} - \ln\left|\frac{y}{x}\right| = \ln|x| + c$$

$$\frac{x}{y} - \ln\left|\frac{y}{x}\right| = \ln|x| + c \quad , \quad c \in \mathbb{R}$$

$$\boxed{\frac{x}{y} - \ln\frac{y}{x} = \ln x + c \,, \, c \in \mathbb{R}}$$