Seminar1

Wednesday, March 3, 2021 12:04 PM

1. a) $\int (x^3 - 3x^2 + \sqrt{x})\,dx \quad , x > 0$

$= \int x^3 dx - 3\int x^2 dx + \int x^{\frac{1}{2}} dx$

$= \frac{x^4}{4} - 3 \cdot \frac{x^3}{3} + \frac{x^{\frac{3}{2}}}{\frac{3}{2}}$

$= \frac{x^4}{4} - x^3 + \frac{2}{3} x^{\frac{3}{2}} + \mathcal{C}$

b) $\int \frac{\cos 2x - 3}{\sin^2 x \cdot \cos^2 x}\,dx \ , \ \boxed{x \in (0, \frac{\pi}{2}) - 2}$

$= \int \frac{2\cos^2 x - 1 - 3}{\sin^2 x \cdot \cos^2 x}\,dx = 2\int \frac{\cos^2 x}{\sin^2 x \cdot \cos^2 x}\,dx$

$= 2\left(\int \frac{\cos^2 x}{\sin^2 x \cos^2 x}\,dx - 2\int \frac{1}{\sin^2 x \cos^2 x}\,dx\right)$

$= 2\left(-\operatorname{ctg} x - 2\int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cdot \cos^2 x}\,dx\right)$

$= 2\left(-\overset{tg}{\operatorname{ctg}} x - 2\int \frac{\sin^2 x}{\sin^2 x \cdot \cos^2 x}\,dx - 2\underbrace{\int \frac{\cos^2 x}{\sin^2 x \cdot \cos^2 x}\,dx}_{-\operatorname{ctg} x}\right)$

$= -2\operatorname{ctg} x - 4\operatorname{tg} x + 4\operatorname{ctg} x$

$= -4\operatorname{tg} x + 2\operatorname{ctg} x + \mathcal{C}, \ \mathcal{C} \in \mathbb{R}.$

c) $\int \frac{x^4 + 8x^2 + 17}{x^2 + 4}\,dx$

$= \int \frac{(x^2+4)^2 + 1}{x^2 + 4}\,dx$

$= \int \frac{(x^2+4)^2}{x^2+4}\,dx + \int \frac{1}{x^2+4}\,dx$

$= \int (x^2+4)\,dx + \int \frac{1}{x^2+4}\,dx$

$= \frac{x^3}{3} + 4x + \frac{1}{2}\operatorname{arctg}\frac{x}{2} + \mathcal{C}$

d) $\int \ln(1+x)\,dx = x\ln(1+x) - \int \frac{x}{1+x}\,dx$

$\begin{cases} f(x) = \ln(1+x) \\ g'(x) = 1 \end{cases} \iff \begin{cases} f'(x) = \frac{1}{1+x} \\ g(x) = x \end{cases}$

$= x\ln(1+x) - \int \frac{x+1-1}{1+x}\,dx$

$= x\ln(1+x) - \left(\int dx - \int \frac{1}{1+x}\,dx\right)$

$= x\ln(1+x) - x + \ln(1+x) + \mathcal{C}$

e) $\int \frac{\ln(1+x)}{1+x}\,dx$

$| \Rightarrow \int t\,dt = \frac{t^2}{2} + \mathcal{C}$

$\frac{\quad}{1+x}$

$$\begin{array}{l|l} \ln(1+x) = t & \Rightarrow \int t\,dt = \frac{t^2}{2} + \mathcal{C} \\ \frac{1}{1+x}dx = dt & = \frac{\ln^2(1+x)}{2} + \mathcal{C} \end{array}$$

d) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} ctg^2 x\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos^2 x}{\sin^2 x}dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1-\sin^2 x}{\sin^2 x}dx =$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin^2 x}dx - x\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = -ctg\,x\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} - x\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} =$

$= -\left(ctg\frac{\pi}{2} - ctg\frac{\pi}{4}\right) - \frac{\pi}{2} + \frac{\pi}{4}$

$= 1 - \frac{\pi}{4}$

2. Differentialgleichungen

Suche alle Funktionen $y \in C^1(\mathbb{R})$, s.d.

a) $y' = \sin x$

$y = \int \sin x\,dx = -\cos x + c$, $c \in \mathbb{R}$

b) $y' = 2x \cdot \sin x$

$y' = \int 2x \cdot (-\cos x)'\,dx = -2x\cos x\,dx + \int 2x \cos x\,dx$

$= -2x\cos x + 2\sin x + c$, $c \in \mathbb{R}$

c) $y' = e^{2x} \cdot \cos x$

$y' = \int e^{2x} \cdot \cos x\,dx$

$\int e^{2x} \cdot \cos x\,dx$

$\begin{cases} f(x) = \cos x \\ g'(x) = e^{2x} \end{cases} \Leftrightarrow \begin{cases} f'(x) = -\sin x \\ g(x) = \frac{e^{2x}}{2} \end{cases}$

...

$$= \frac{e^{2x}\cdot\cos x}{2} - \int -\frac{\sin x\cdot e^{2x}}{2}dx$$

$$= \frac{e^{2x}\cdot\cos x}{2} + \frac{1}{2}\underbrace{\int \sin x\cdot e^{2x}}\,dx$$

$$\begin{cases} f(x) = \sin x \\ g'(x) = e^{2x} \end{cases} \iff \begin{cases} f'(x) = \cos x \\ g(x) = \frac{1}{2}e^{2x} \end{cases}$$

$$= \frac{e^{2x}\cdot\cos x}{2} + \frac{1}{2}\left(\frac{1}{2}\sin x e^{2x} - \frac{1}{2}\int e^{2x}\cos x dx\right)$$

$$= \frac{e^{2x}\cdot\cos x}{2} + \frac{1}{4}\sin x e^{2x} - \frac{1}{4}\int e^{2x}\cos x dx$$

$$\Rightarrow \int e^{2x}\cdot\cos x dx + \frac{1}{4}\int e^{2x}\cos x dx = \frac{e^{2x}\cdot\cos x}{2} + \frac{1}{4}\sin x e^{2x} \quad |\cdot 4$$

$$5\int e^{2x}\cdot\cos x dx = 2e^{2x}\cdot\cos x + \sin x\, e^{2x}$$

$$\int e^{2x}\cdot\cos x dx = \frac{2e^{2x}\cdot\cos x + \sin x\, e^{2x}}{5} + C, \quad C\in\mathbb{R}$$

3. Wir betrachten die folgenden DGL

$y' = g(x)$, wobei $g(x) = \begin{cases} 2x-1, & x\in[0,1] \\ x\cdot e^{x-1}, & x\in(1,2] \end{cases}$

$g : [0,2] \to \mathbb{R}$

Bestimme die allgemeine Lösung der DGL

g stetig? $\lim\limits_{\substack{x\to 1\\ x<1}} g(x) = \lim\limits_{\substack{x\to 1\\ x>1}} g(x) = g(1)$ „wahr"

$$y = \int(2x-1)dx = 2\frac{x^2}{2} - x + c_1 = x^2 - x + c_1$$

$$y = \int x\cdot e^{x-1}dx = x\cdot e^{x-1} - \int e^{x-1}dx = x\cdot e^{x-1} - e^{x-1} + c_2$$

g stetig! $\lim\limits_{\substack{x \to 1 \\ x < 1}} g(x) = \lim\limits_{\substack{x \to 1 \\ x > 1}} g(x)$ ...

$y = \int (2x-1)\,dx = 2\frac{x^2}{2} - x + c_1 = x^2 - x + c_1$

$y = \int x \cdot e^{x-1}\,dx = x \cdot e^{x-1} - \int e^{x-1}\,dx = x \cdot e^{x-1} - e^{x-1} + c_2$

$\begin{cases} f(x) = x \\ g'(x) = e^{x-1} \end{cases} \Leftarrow\Rightarrow \begin{cases} f'(x) = 1 \\ g(x) = e^{x-1} \end{cases}$

$= e^{x-1}(x-1) + c_2$

y - stetig sein muss:

$\lim\limits_{\substack{x \to 1 \\ x < 1}} y(x) = \lim\limits_{\substack{x \to 1 \\ x > 1}} y(x)$

$\Rightarrow 1 - 1 + c_1 = e^{1-1}(1-1) + c_2$

$c_1 = c_2$

$y(x) = \begin{cases} x^2 - x + c, & x \in [0,1] \\ e^{x-1}(x-1) + c, & x \in (1,2] \end{cases}$

$c \in \mathbb{R}$

4.

a) Zeige das $y(x) = 2 \cdot e^{3x}$ eine Lösung der DGL $y' = 3y$ ist.

$(2e^{3x})' = 3 \cdot 2 \cdot e^{3x}$

$6e^{3x} = 6e^{3x} \quad | : e^{3x} \neq 0$

$6 = 6 \quad (w)$

$\Rightarrow y(x)$ ist Lösung der DGL

b) Zeige dass $y(x) = e^{-2x} \cdot \cos x$ eine Lösung der DGL $y'' + 4y' + 5y = 0$ ist

$y'(x) = -2e^{-2x} \cdot \cos x - e^{-2x} \sin x$

$= e^{-2x}(-2\cos x - \sin x)$

$y''(x) = -2(-2e^{-2x} \cdot \cos x - e^{-2x} \sin x) + 2e^{-2x} \cdot \sin x - e^{-2x} \cdot \cos x$

$= 4e^{-2x} \cdot \cos x + 2e^{-2x} \cdot \sin x + 2e^{-2x} \sin x - e^{-2x} \cdot \cos x$

$= e^{-2x}(4\cos x + 2\sin x + 2\sin x - \cos x)$

$= e^{-2x}(3\cos x + 4\sin x)$

$\Rightarrow e^{-2x}(3\cos x + 2\sin x) + 4e^{-2x}(-2\cos x - \sin x) + 5 \cdot e^{-2x} \cdot \cos x = 0 \quad | : e^{-2x} \neq 0$

$\Rightarrow 3\cos x + \cancel{4\sin x} - 8\cos x - \cancel{4\sin x} + 5\cos x = 0$

$0 = 0$ „wahr"

5. Finde $a, b \in \mathbb{R}$ s.d. $y(x) = ax + b$ der DGL $y' - 5y = 2x + 3$ genügt

$y'(x) = (ax+b)' = a$

$\Rightarrow a - 5(ax+b) = 2x+3$

$\Leftrightarrow a - 5ax - 5b = 2x+3$

$\begin{cases} -5a = 2 \\ a - 5b = 3 \end{cases} \Rightarrow \begin{cases} a = -\frac{2}{5} \\ b = -\frac{17}{25} \end{cases}$

$y = -\frac{2}{5}x - \frac{17}{25}$