H11 - RAZVAN POSTESCU

Friday, December 18, 2020 12:09 AM

H20. $f(x,y) = 2y^3 + x^2y + x^2 + 5y^2$

① Sei $(x,y) \in \mathbb{R}^2$

$$\Rightarrow \frac{\partial f}{\partial x}(x,y) = 2xy + 2x$$

$$\frac{\partial f}{\partial y}(x,y) = 6y^2 + x^2 + 10y$$

② Die stationären Punkte sind die Lösungen des systems:

$$\begin{cases} 2xy + 2x = 0 \\ 6y^2 + x^2 + 10y = 0 \end{cases} \Leftrightarrow \begin{cases} xy + x = 0 \\ 6y^2 + x^2 + 10y = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x(y+1) = 0 \overset{\Rightarrow}{} y = -1 \quad (1) \\ \overset{(1)}{\Rightarrow} +6 + x^2 - 10 = 0 \Rightarrow x^2 = 4 \end{cases}$$

$$\Rightarrow x = \pm 2$$

$$\Rightarrow \begin{cases} (2,0) \\ (-2,0) \end{cases}$$ sind die einzige stationären Punkte

③ $$\frac{\partial^2 f}{\partial x^2}(x,y) = 2y + 2 \quad ; \quad \frac{\partial^2 f}{\partial y \partial x}(x,y) = 2x$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = 2x \quad ; \quad \frac{\partial^2 f}{\partial y^2}(x,y) = 12y + 10$$

$$\Rightarrow Hf(x,y) = \begin{pmatrix} 2y+2 & 2x \\ 2x & 12y+10 \end{pmatrix}$$

④ $(2,0)$:

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$$Hf(2,0) = \begin{pmatrix} 2 & 4 \\ 4 & 10 \end{pmatrix}$$

$$\begin{vmatrix} 2 & 4 \\ & 10 \end{vmatrix} = 20 - 16 = 4 > 0 \Rightarrow Hf \text{ - ist positiv def.}$$

$\downarrow$

$$\begin{vmatrix} 2 & 4 \\ 4 & 10 \end{vmatrix} = 20 - 16 = 4 > 0 \underset{\text{Th3}}{\Rightarrow} Hf \text{ - ist positiv def.}$$

$\Rightarrow (2,0)$ - minimalstelle von f

$(-2,0)$:

$$Hf(-2,0) = \begin{array}{c|cc} & h_1 & h_2 \\ \hline h_1 & 2 & -4 \\ h_2 & -4 & 10 \end{array} ; \quad \begin{vmatrix} 2 & -4 \\ -4 & 10 \end{vmatrix} = 20 - 16 = 4$$

Aber $|-4| = -4 < 0 \Rightarrow Hf$ - ist nicht pos und auch nicht neg. def.

$$\Phi_{Hf}(h_1, h_2) = 2h_1^2 - 4h_1h_2 - 4h_1h_2 + 10h_2^2$$
$$= 2h_1^2 - 8h_1h_2 + 10h_2^2$$

$$\Phi_{Hf}(1,2) = 2 - 16 + 40 = 26$$

$\Phi_{Hf} \geq 0 \Rightarrow Hf(-2,0) \rightarrow$ positiv semidefinit

H20.

$f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x^3 + y^3 + z^3 + 12xy - 3z$

a) $\nabla f(x,y,z)$, $Hf(x,y,z)$

$\frac{\partial f}{\partial x}(x,y,z) = 3x^2 + 12y$

$\frac{\partial f}{\partial y}(x,y,z) = 3y^2 + 12x$

$\frac{\partial f}{\partial z}(x,y,z) = 3z^2 - 3$

$\Rightarrow \nabla f(x,y,z) = (3x^2 + 12y, 3y^2 + 12x, 3z^2 - 3)$

$\frac{\partial^2 f}{\partial x^2}(x,y,z) = 6x$; $\frac{\partial^2 f}{\partial y \partial x}(x,y,z) = 12$; $\frac{\partial^2 f}{\partial z \partial x}(x,y,z) = 6z$

$\partial^2 f \quad = 6y$; $\partial^2 f \,(x,y,z) = 0$

$\frac{\partial^2 f}{\partial x \partial y}(x,y,z) = 12$; $\frac{\partial^2 f}{\partial y^2}(x,y,z) = 6y$; $\frac{\partial^2 f}{\partial z \partial y}(x,y\ldots$

$\frac{\partial^2 f}{\partial x \partial z}(x,y,z) = 0$; $\frac{\partial^2 f}{\partial y \partial z}(x,y,z) = 0$; $\frac{\partial^2 f}{\partial z^2}(x,y,z) = 6z$

$$\Rightarrow Hf(x,y,z) = \begin{pmatrix} 6x & 12 & 6z \\ 12 & 6y & 0 \\ 0 & 0 & 6z \end{pmatrix}$$

b) $u := \nabla f(\overset{x}{0}, \overset{y}{1}, \overset{z}{0})$

$v := \nabla f(-1,0,0)$

$\langle u,v \rangle$

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$$\boxed{\nabla f(x,y,z) = (3x^2+12y,\ 3y^2+12x,\ 3z^2-3)}$$

$u := (12, 3, -3)$

$v := (3, -12, -3)$

$\langle u,v \rangle = 36 - 36 + 9 = 9$

c) ①
$$\begin{cases} 3x^2+12y = 0 \\ 3y^2+12x = 0 \\ 3z^2-3 = 0 \end{cases} \Leftrightarrow \begin{cases} x^2+4y=0 \\ y^2+4x=0 \\ z^2-1=0 \Rightarrow z = \pm 1 \end{cases} \Rightarrow \begin{cases} x=0 \\ \text{oder} \\ y=0 \end{cases} \text{oder} \begin{cases} x=-4 \\ y=-4 \end{cases}$$

$\Rightarrow$ stationäre Punkte : $(0,0,1)$, $(0,0,-1)$, $(-4,-4,1)$, $(-4,-4,-1)$

② $(0,0,1)$, $(-4,-4,1)$

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$$Hf(0,0,1) = \begin{array}{c|ccc} & h_1 & h_2 & h_3 \\ \hline h_1 & 0 & 12 & 6 \\ h_2 & 12 & 0 & 0 \\ h_3 & 0 & 0 & 6 \end{array} \;;\; \Delta_2 \begin{vmatrix} 0 & 12 \\ 12 & 0 \end{vmatrix} = -24 < 0$$

$$\Delta_3 = \begin{vmatrix} 0 & 12 & 6 \\ 12 & 0 & 0 \\ 0 & 0 & 6 \end{vmatrix} = 864 > 0$$

$$\Phi_{Hf}(h_1,h_2,h_3) = 24h_1h_2 + 6h_1h_3 + 6h_3^2$$

$\Phi_{Hf}(h_1, h_2, h_3) = 24h_1h_2 + 6h_1h_3 + 6h_3^2$

$\Phi_{Hf}(-1, 1, 1) = -24 - 6 + 6 = -24 < 0$

$\Phi_{Hf}(1, 1, 1) = 24 + 6 + 6 = 36 > 0$

> indefinit

=> (0,0,1) - ist keine lokale Extremstelle

$(-4, -4, -1)$

$\rightarrow$

$$\boxed{\Rightarrow Hf(x,y,z) = \begin{pmatrix} 6x & 12 & 6z \\ 12 & 6y & 0 \\ 0 & 0 & 6z \end{pmatrix}}$$

$$Hf(-4,-4,-1) = \begin{pmatrix} -24 & 12 & -6 \\ 12 & -24 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

$|-24| < 0$

$|12| > 0$; $\begin{vmatrix} -24 & 12 \\ 12 & -24 \end{vmatrix} > 0$

$$\begin{array}{c|ccc|} & h_1 & h_2 & h_3 \\ h_1 & -24 & 12 & -6 \\ h_2 & 12 & -24 & 0 \\ h_3 & 0 & 0 & -6 \end{array} = -24 \cdot 24 \cdot 6 + 6 \cdot 12 \cdot 12 < 0$$

$\Phi_{Hf}(h_1, h_2, h_3) = -24h_1^2 + 24h_1h_2 - 6h_1h_3 - 24h_2^2 - 6h_3^2$

$\Phi_{Hf}(1, 1, 1) = -\cancel{24} + \cancel{24} - 6 - 24 - 6 = -36$

.... $\Phi_{Hf} \leq 0$

=> negativ semidefinit