Seminar4

Wednesday, April 14, 2021 12:02 PM

DGL 2. Ordnung

1. DGL der Form $y'' = f(x)$

a) $y'' = x + \cos x + \sin x \quad | \int$

$$y' = \frac{x^2}{2} + \sin x - \cos x + c_1 \quad | \int$$

$$y = \frac{x^3}{6} - \cos x - \sin x + c_1 + c_2 \quad , c_1, c_2 \in \mathbb{R}$$

b) $y'' = x e^x \quad | \int$

$$y' = \int x e^x dx = \int x \cdot (e^x)' dx = x \cdot e^x - \int e^x dx =$$

$$= x e^x - e^x + c_1 = e^x(x-1) + c_1$$

$$y = \int (x e^x - e^x + c_1) dx = x e^x - e^x - e^x + c_1 x + c_2$$

$$= x e^x - 2e^x + c_1 x + c_2 \quad , c_1, c_2 \in \mathbb{R}$$

2. DGL $y'' = f(x, y')$

Substitution: $y' = z \Rightarrow y'' = z'$

$\Rightarrow z' = f(x, z)$

a) $x y'' + y' + x = 0$

subs: $y' = z \Rightarrow y'' = z'$

$$x \cdot z' + z + x = 0$$

~~$x \cdot z' = -z - x$~~ ~~$| : x$~~

~~dz~~

$$\cancel{x \cdot z' = -z - x}$$

$$\cancel{x \cdot \frac{dz}{dx} = -z - x \quad \Big| : x}$$

$$\cancel{\frac{dz}{dx} = \frac{-z-x}{x}}$$

$$x \cdot z' + z + x = 0$$

$$z' + \frac{z}{x} + 1 = 0$$

$$z' + \frac{1}{x} \cdot z = -1$$

1. Schritt

$$z' + \frac{1}{x} \cdot z = 0$$

$$\frac{dz}{dx} = -\frac{1}{x} \cdot z \quad \Big| \begin{array}{l} : z \neq 0 \\ \cdot dx \end{array}$$

$$\frac{dz}{dx} = -\frac{1}{x} \cdot dx \quad \Big| \int$$

$$\ln|z| = -\ln|x| + \ln c$$

$$\ln|z| = \ln c \cdot x^{-1}$$

$$z_0 = c \cdot x^{-1}$$

2. Schritt

$$z_p = c(x) \cdot x^{-1}$$

$$z_p' + \frac{1}{x} z_p = -1$$

$$c'(x) \cdot x^{-1} \cancel{- c(x) \cdot x^{-2}} + \cancel{\frac{1}{x} \cdot c(x) \cdot x^{-1}} = -1$$

$$c'(x)\cdot x^{-1} + \cancel{c(x)\cdot x^{-2} + \frac{1}{x}\cdot c(x)\cdot x^{-1}} = -1$$

$$c'(x)\cdot x^{-1} = -1 \quad \Rightarrow c'(x) = -x$$

$$\Rightarrow c(x) = -\frac{x^2}{2}$$

$$z_p = -\frac{x^2}{2}\cdot x^{-1} = -\frac{x}{2}$$

3. Schritt

$$z = z_0 + z_p = c_1\cdot x^{-1} - \frac{x}{2}$$

$$y' = z \Rightarrow y = \int \left(c\cdot x^{-1} - \frac{x}{2}\right) dx$$

$$y = c_1\cdot \ln|x| - \frac{x^2}{4} + c_2, \quad c_1, c_2 \in \mathbb{R}$$

b) $x y'' = y' \ln \frac{y'}{x}$

Subst: $y' = z \Rightarrow y'' = z'$

$x\cdot z' = z\cdot \ln\frac{z}{x} \quad |: x \neq 0$

$$z' = \frac{z\cdot \ln\frac{z}{x}}{x}$$

$$z' = \frac{z}{x}\cdot \ln\frac{z}{x}$$

$\boxed{\frac{z}{x} \overset{\text{not}}{=} t} \Rightarrow z = t\cdot x \quad (\,)'$

$$z' = t'\cdot x + t\cdot x'$$

$$z' = t'\cdot x + t$$

$t' \cdot x + t = t \cdot \ln t \;\Big|$

$t' \cdot x = t \ln t - t$

$\frac{dt}{dx} \cdot x = t(\ln t - 1) \quad \Big| : dx \;\; : x \neq 0 \;\; : t(\ln t - 1) \neq 0$

$\frac{dt}{t(\ln t - 1)} = \frac{dx}{x} \;\Big|\; \int$

$\int \frac{(\ln t - 1)'}{\ln t - 1}\, dt = \int \frac{dx}{x}$

$\ln|\ln t - 1| = \ln|x| + \ln c$

$\ln t - 1 = x \cdot c_1$

$\ln t = x \cdot c_1 + 1$

$t = e^{x \cdot c_1 + 1}$

$\frac{z}{x} = t \Rightarrow z = t \cdot x$

$z = x \cdot e^{x \cdot c_1 + 1}$

$y' = z \Rightarrow y = \int x e^{xc_1 + 1}\, dx = \frac{1}{c_1} \int x \left(e^{xc_1+1}\right)' dx$

$= \frac{1}{c_1} \cdot x \cdot e^{xc_1+1} - \frac{1}{c_1} \int e^{xc_1+1}\, dx =$

$= \underbrace{\frac{1}{c_1} x\, e^{xc_1+1} - \frac{1}{c_1^2} \cdot e^{xc_1+1} + c_2, \quad c_1, c_2 \in \mathbb{R}}_{}$

↳ allg. Lös.

$t(\ln t - 1) = 0 \Rightarrow t = 0$

$\ln t - 1 = 0 \Leftrightarrow \ln t = 1 \Rightarrow t = e$

$z = e \cdot x \Rightarrow y = \int ex = e\frac{x^2}{2} + c \rightarrow$ sing. Lös.

3. DGL 2. Ordnung mit konstanten Koeff.

$y'' + ay' + by = f(x), \quad a, b \in \mathbb{R}$

1. a) $y'' + 5y' + 6y = 0$

$r^2 + 5r + 6$

$r_{1,2} = \begin{cases} -2 \rightarrow y_1(x) = e^{-2x} \\ -3 \rightarrow y_2(x) = e^{-3x} \end{cases}$

$y = c_1 \cdot e^{-2x} + c_2 \cdot e^{-3x}, \quad c_1, c_2 \in \mathbb{R}$

b) $y'' - 4y' + 4y = 0$

$r^2 - 4r + 4 = 0$

$r_{1,2} = 2 \begin{cases} \rightarrow y_1(x) = e^{2x} \\ \rightarrow y_2(x) = e^{2x} \cdot x \end{cases}$

$y = c_1 \cdot e^{2x} + c_2 \cdot e^{2x} \cdot x, \quad c_1, c_2 \in \mathbb{R}$

c) $y'' + 9y' = 0$

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$r^2 + 9 = 0$

$r_{1,2} = \pm 3i \quad \begin{matrix} \alpha = 0 \\ \beta = 3 \end{matrix} < \begin{matrix} y_1(x) = e^{\alpha x} \cdot \cos\beta x = \cos 3x \\ y_2(x) = e^{\alpha x} \cdot \sin\beta x = \sin 3x \end{matrix}$

$y = C_1 \cdot \cos 3x + C_2 \cdot \sin 3x \,, \quad C_1, C_2 \in \mathbb{R}$

d) $y'' - 5y' + 6y = \underbrace{6x^2 - 10x + 2}_{f(x)}$

<u>1. Schritt.</u>

$y'' - 5y' + 6y = 0$

$r^2 - 5r + 6 = 0$

$r_{1,2} = < \begin{matrix} 2 \to y_1(x) = e^{2x} \\ 3 \to y_2(x) = e^{3x} \end{matrix}$

$y_0 = C_1 \cdot e^{2x} + C_2 \cdot e^{3x} \,, \quad C_1, C_2 \in \mathbb{R}$

<u>2. Schritt</u>

$f(x) = 6x^2 - 10x + 2 \qquad \text{I } f(x) = P_m(x) = P_2(x)$

"0" nu răd. a ec. caract.

$(\; y_p = ax^2 + bx + c \;)$ [illegible]

$y_p' = 2ax + b$

$y_p'' = 2a$

Einsetzen:

$2a - 5 \cdot (2ax + b) + 6(ax^2 + bx + c) = 6x^2 - 10x + 2$

$2a - 10ax - 5b + 6ax^2 + 6bx + 6c$

$$\begin{cases} 6a = 6 \\ -10a + 6b = -10 \\ 2a - 5b + 6c = 2 \end{cases} \Rightarrow \begin{cases} a = 1 \\ b = 0 \\ c = 0 \end{cases}$$

$y_p = x^2$

3. <u>Schritt</u>

$y = y_0 + y_p = c_1 \cdot e^{2x} + c_2 \cdot e^{3x} + x^2, \quad c_1, c_2 \in \mathbb{R}$

d) $y'' + 2y' + 2y = x \cdot e^{-x}$

1. Schritt

$r^2 + 2r + 2 = 0$

$r_{1,2} = -1 \pm i \quad \alpha = 1 \; ; \beta = 1$

$y_1(x) = e^{x} \cdot \cos x$

$y_2(x) = e^{x} \cdot \sin x$

$y_0 = c_1 \cdot e^{x} \cdot \cos x + c_2 \cdot e^{x} \cdot \sin x, \quad c_1, c_2 \in \mathbb{R}$

2. <u>Schritt</u>

$f(x) = x \cdot e^{-x}$

$P_m(x) = x \; ; \boxed{r = -1}$ $\to$ nu rad a ec. caract.

$y_p = (ax + b) \cdot e^{-x}$

$y_p' = a \cdot e^{-x} - (ax + b) \cdot e^{-x}$

$$y_p'' = -ae^{-x} - [ae^{-x} - (ax+b)\cdot e^{-x}]$$
$$= -2ae^{-x} + (ax+b)\cdot e^{-x}$$

$$-2ae^{-x} + (ax+b)e^{-x} + 2ae^{-x} - 2(ax+b)\cdot e^{-x} + 2(ax+b)e^{-x} = xe^{-x} \,/ : e^{-x} \neq 0$$

$$-2a + ax + b + 2a - 2ax - 2b + 2ax + 2b = x$$

$$\begin{cases} a = 1 \\ b = 0 \end{cases}$$

$$y_p = x \cdot e^{-x}$$

3. Schritt

$$y = c_1 \cdot e^{-x}$$

e) $y'' + y' - 2y = 10 \sin 2x$

$$y_0 = \dots$$

$$f(x) = 10 \cdot \sin 2x$$
$$P_0(x) \cdot e^{0x} \cdot \sin 2x$$

$$y_p = a \cdot \cos 2x + b \cdot \sin 2x$$