Seminar3

Wednesday, March 31, 2021 12:02 PM

Lineare DGl 1. Ordnung

1. a) $y' + \frac{1}{x}\cdot y = 3x \;,\; x > 0$

1. Schritt

$y' + \frac{1}{x}\cdot y = 0$
$y' = \frac{dy}{dx}$
$\Rightarrow \frac{dy}{dx} = -\frac{1}{x}\cdot y \;\Big|\; : y \neq 0,\ \cdot dx$

$\frac{dy}{y} = -\frac{1}{x}dx \;\Big|\; \int$

$\int \frac{dy}{y} = -\int \frac{1}{x}\,dx$

$\ln|y| = -\ln|x| + c$

$y = e^{-\ln x + c} = e^{-\ln x}\cdot c_1 =$

$= e^{\ln x^{-1}}\cdot c_1 = x^{-1}\cdot c_1 \;,\; c_1 \in \mathbb{R}$

2. Schritt

$y_p = x^{-1}\cdot c(x)$

$y'_p + \frac{1}{x}\cdot y_p = 3x$

$y'_p = -x^{-2}\cdot c(x) + x^{-1}\cdot c'(x)$

Einsetzen

$\cancel{-x^{-2}\cdot c(x)} + x^{-1}\cdot c'(x) + \cancel{\frac{1}{x}\cdot x^{-1} c(x)} = 3x$

$x^{-1}\cdot c'(x) = 3x \Rightarrow c'(x) = 3x^2$

$\Rightarrow c(x) = \int 3x^2 dx = \cancel{3}\cdot\frac{x^3}{\cancel{3}}$

$y_p = x^{-1}\cdot x^3 = x^2$

3. Schritt

$y = y_0 + y_p = x^{-1}\cdot c + x^2 \;,\; c \in \mathbb{R}$

b) $y' + y\cdot tg x = \frac{1}{\cos x}$

Schritt I:

$y' + y\cdot tg x = 0$
$y' = \frac{dy}{dx}$

$\frac{dy}{dx} = -y\cdot tg x \;\Big|\; : y \neq 0,\ \cdot dx$

$$\frac{dy}{dx} = -y \cdot tg\, x$$

$$\frac{dy}{y} = -tg\,x\,dx \quad \Big|\int$$

$$\int \frac{dy}{y} = -\int tg\,x\,dx$$

$$\ln|y| = +\ln|\cos x| + c$$

$$y = e^{\ln|\cos x| + c}$$

$$y_0 = c_1 \cdot \cos x, \; c_1 \in \mathbb{R}$$

Bernoullische DGL

1.a) $y' - \frac{1}{x} \cdot y = -2x \cdot y^2 \quad \Big| : y^2 \neq 0 \quad \alpha = 2$

$$y' \cdot y^{-2} - \frac{1}{x} \cdot y^{-1} = -2x$$

$$y^{-1} = z \Rightarrow -y^{-2} \cdot y' = z' \Rightarrow$$

$$\Rightarrow y' = -\frac{z'}{y^{-2}}$$

$$-\frac{z'}{y^{-2}} \cdot y^{-2} - \frac{1}{x} \cdot z = -2x$$

$$+z' + \frac{1}{x} \cdot z = +2x$$

LDGL 1. Ordnung

1. Schritt

$$z' + \frac{1}{x} \cdot z = 0$$

$$\frac{dz}{dx} = -\frac{1}{x} \cdot z \quad \Big| \begin{array}{l} :z \\ \cdot dx \end{array}$$

$$\frac{dz}{z} = -\frac{1}{x}\,dx \quad \Big|\int$$

$$\ln|z| = -\ln|x| + c$$

$$z_0 = e^{-\ln|x|} \, e^c$$

$$z_0 = c_1 \cdot x^{-1}, \; c_1 \in \mathbb{R}$$

2. Schritt

$z_p = c(x) \cdot x^{-1}$

$z'_p = -x^{-2} \cdot c(x) + c'(x) \cdot x^{-1}$

$-x^{-2} \cdot c(x) + c'(x) \cdot x^{-1} + \cancel{\frac{1}{x} \cdot c(x) \cdot x^{-1}} = +2x$

$c'(x) \cdot x^{-1} = 2x$

$c'(x) = 2x^2 \Rightarrow c(x) = \frac{2}{3} x^3$

$z_p = \frac{2}{3} x^2$

3. Schritt

$z = c_1 \cdot x^{-1} + \frac{2}{3} \cdot x^2$.

$y = z^{-1} = \frac{1}{c_1 \cdot x^{-1} + \frac{2}{3} x^2}, \quad c_1 \in \mathbb{R}$

5) $x y' = 2x^2 \sqrt{y} + 4y$

$x y' - 4y = 2x^2 \cdot y^{\frac{1}{2}} \quad \Big| : y^{\frac{1}{2}} \neq 0$
$\quad : x \neq 0$

$y' \cdot y^{-\frac{1}{2}} - 4 y^{\frac{1}{2}} \cdot \frac{1}{x} = 2x$

z

$z = y^{\frac{1}{2}} \Rightarrow z' = \frac{1}{2} y^{-\frac{1}{2}} \cdot y'$

$y' = \frac{z'}{\frac{1}{2} \cdot y^{-\frac{1}{2}}}$

$\frac{2z'}{\cancel{y^{-\frac{1}{2}}}} \cdot \cancel{y^{-\frac{1}{2}}} - 4z \cdot \frac{1}{x} = 2x \,/ :2$

$z' - 2\frac{z}{x} = x$

1. Schritt

$z' - 2 \cdot \frac{z}{x} = 0$

$z' = dz$

$\Rightarrow \frac{dz}{dx} = \frac{2z}{x} \quad \Big| : z \neq 0$
$\quad \cdot dx$

$dz \qquad 2dx$

$z' = \ldots \frac{}{x}$ | $\frac{}{dx} \quad \ldots x \ldots dx$

$z' = \frac{dz}{dx}$ | $\frac{dz}{z} = \frac{2dx}{x}$

$\ln|z| = 2\ln|x| + \ln c$

$z_0 = c \cdot x^2$

2. Schritt

$z_p = c(x) \cdot x^2$

$z_p' = c'(x) \cdot x^2 + 2x \cdot c(x)$

$c'(x) \cdot x^2 + \cancel{2x \cdot c(x)} - \cancel{2 \frac{1}{x} \cdot c(x) \cdot x^2} = x$

$c'(x) = \frac{1}{x} \Rightarrow c(x) = \ln|x|$

$z_p = \ln|x| \cdot x^2$

3. Schritt

$z = c \cdot x^2 + \ln|x| \cdot x^2$

$y = z^2 = (c \cdot x^2 + x^2 \ln|x|)^2, \quad c \in \mathbb{R}$

Exakte DGL

3. a) $3x^2 y^2 + 4x^3 + 2x^3 \cdot y \cdot y' = 0$

$y' = \frac{dy}{dx}$ $\Rightarrow$

$\Rightarrow 3x^2 \cdot y^2 + 4x^3 + 2x^3 y \cdot \frac{dy}{dx} = 0 \mid dx$

$(3x^2 \cdot y^2 + 4x^3)dx + 2x^3 \cdot y \cdot dy = 0$

$3x^2 y^2 + 4x^3$

$P(x,y) = 3x^2y^2 + 4x^3$

$Q(x,y) = 2x^3y$

Die Bedingung für die Exaktheit

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$$

$$\left.\begin{array}{l}(3x^2y^2+4x^3)'_y = 6x^2y \\ (2x^3y)'_x = 6x^2y\end{array}\right) =$$

$$\begin{cases}\frac{\partial u}{\partial x} = P(x,y) \\ \frac{\partial u}{\partial y} = Q(x,y)\end{cases} \Leftrightarrow \begin{cases}\frac{\partial u}{\partial x} = 3x^2y^2+4x^3 \\ \frac{\partial u}{\partial y} = 2x^3y\end{cases}$$

$\Rightarrow u(x,y) = \int (3x^2y^2+4x^3)\,dx =$

$= x^3y^2 + x^4 + c(y) \quad |'_y$

$$\left.\begin{array}{l}\frac{\partial u}{\partial y} = 2x^3y + c'(y) \\ \frac{\partial u}{\partial y} = 2x^3y\end{array}\right\} \Rightarrow \begin{array}{l}c'(y) = 0 \\ c(y) = c_1\end{array}$$

$$\left.\begin{array}{l}u(x,y) = x^3y^2 + x^4 + c_1 \\ u(x,y) = c\end{array}\right\} \Rightarrow$$

$\Rightarrow x^3y^2 + x^4 + c_1 = c$

$$\boxed{x^3y^2 + x^4 = C_2 \; ; \; C_2 = C - C_1}$$

die allg. Lösung

b) $(3x-y)\,dx - (x+3y)\,dy = 0$

$$\left.\begin{array}{l} P(x,y) = 3x-y \Rightarrow \frac{\partial P}{\partial y} = -1 \\ Q(x,y) = -x-3y \Rightarrow \frac{\partial Q}{\partial x} = -1 \end{array}\right] =$$

$$\begin{cases} \frac{\partial u}{\partial x} = P(x,y) \\ \frac{\partial u}{\partial y} = Q(x,y) \end{cases} \Leftrightarrow \begin{cases} \frac{\partial u}{\partial x} = 3x-y \\ \frac{\partial u}{\partial y} = -x-3y \end{cases}$$

$$\Rightarrow \begin{cases} \frac{\partial u}{\partial x} = 3x-y \Rightarrow u = 3\frac{x^2}{2} - yx + c(y) \Big|'_y = -x + c'(y) = -x-3y \\ \frac{\partial u}{\partial y} = -x-3y \end{cases}$$

$\Rightarrow c'(y) = -3y$

$\Rightarrow c(y) = -\frac{3y^2}{2}$

$$u = -3\frac{x^2}{2} - yx - \frac{3y^2}{2}$$

$u(x,y) = c$

$$\Rightarrow \boxed{\frac{3x^2}{2} - xy - \frac{3y^2}{2} = c, \quad c \in \mathbb{R}}$$

c) $(2xy^2 + 6\cdot x^2y)\,dx + x^2(x+2y)\,dy = 0$

Bestimme b s.d. die DGL exakt ist und schreibe die allg. Lösung