



Review

- $\lim_{x \rightarrow x_0} f(x) = A \in [-\infty, +\infty]$ 的定义与几何意义

$$\lim_{x \rightarrow x_0^+} f(x), \quad \lim_{x \rightarrow \infty} f(x), \quad \lim_{x \rightarrow \pm\infty} f(x)$$

- 极限的性质

唯一性, 局部有界性, 保序性, 四则运算,
夹挤原理, 单调收敛原理, 复合函数的极限

- 重要不等式

$$|\sin x| \leq |x|, \forall x \in \mathbb{R}. \quad |x| \leq |\tan x|, \forall x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$



●重要极限

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e,$$

$$\lim_{x \rightarrow 0} (1+x)^{1/x} = e,$$

$$\lim_{x \rightarrow +\infty} \frac{\log_a x}{x^b} = 0, \lim_{x \rightarrow 0^+} x^b \log_a x = 0 \quad (a > 1, b > 0),$$

$$\lim_{x \rightarrow +\infty} \frac{x^b}{a^x} = 0 \quad (a > 1, b \in \mathbb{R}), \quad \lim_{x \rightarrow +\infty} \frac{a^x}{x^x} = 0 \quad (a > 0, a \neq 1),$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1, \quad \lim_{x \rightarrow x_0} e^x = e^{x_0}, \quad \lim_{x \rightarrow x_0} \ln x = \ln x_0,$$

$$\lim_{x \rightarrow x_0} u(x)^{v(x)} = \left(\lim_{x \rightarrow x_0} u(x) \right)^{\lim_{x \rightarrow x_0} v(x)} \quad (\text{成立的条件?})$$



• **Thm.** f 在 $U(x_0, \rho)$ 中有定义, 则以下命题等价:

(1) $\forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in U(x_0, \delta)$, 有 $|f(x) - f(y)| < \varepsilon$;

(2) $\exists A \in \mathbb{R}$, 对 $U(x_0, \rho)$ 中任意收敛到 x_0 的点列 $\{x_n\}$, 有

$$\lim_{n \rightarrow \infty} f(x_n) = A;$$

$$(3) \lim_{x \rightarrow x_0} f(x) = A.$$

Remark. (1) \Leftrightarrow (3) (函数极限的Cauchy收敛原理)

Remark. (2) \Leftrightarrow (3) (用数列的极限来研究函数的极限)



§ 4. 无穷小量与无穷大量

Def. (无穷小量与无穷大量)

(1) 若 $\lim_{x \rightarrow x_0} f(x) = 0$, 则称 $x \rightarrow x_0$ 时, $f(x)$ 是无穷小量, 记作

$$f(x) \rightarrow 0 (x \rightarrow x_0);$$

(2) 若 $\lim_{x \rightarrow x_0} f(x) = \infty$, 则称 $x \rightarrow x_0$ 时, $f(x)$ 是无穷大量, 记作

$$f(x) \rightarrow \infty (x \rightarrow x_0);$$

(3) 若 $\lim_{x \rightarrow x_0} f(x) = \pm\infty$, 则称 $x \rightarrow x_0$ 时, $f(x)$ 是正(负)无穷大量, 记作 $f(x) \rightarrow \pm\infty (x \rightarrow x_0)$.



Def. 设 $x \rightarrow x_0$ 时, $f(x)$ 与 $g(x)$ 都是无穷小量.

(1) 若 $\lim_{x \rightarrow x_0} f(x)/g(x) = 0$, 则称 $x \rightarrow x_0$ 时, $f(x)$ 是 $g(x)$ 的 **高阶**

无穷小量, 记作 $f(x) = o(g(x)) \ (x \rightarrow x_0)$;

(2) 若 $\lim_{x \rightarrow x_0} f(x)/g(x) = c \neq 0$, 则称 $x \rightarrow x_0$ 时, $f(x)$ 与 $g(x)$ 是 **同**

阶无穷小量; 特别地, 当 $c = 1$ 时, 称 $x \rightarrow x_0$ 时, $f(x)$ 与 $g(x)$

是 **等价无穷小量**, 记作 $f(x) \sim g(x) \ (x \rightarrow x_0)$;

(3) 若 $\exists M > 0, \delta > 0$, 当 $0 < |x - x_0| < \delta$ 时, 有 $|f(x)/g(x)| < M$,
则记为 $f(x) = O(g(x)) \ (x \rightarrow x_0)$.

(4) 若 $\lim_{x \rightarrow x_0} \frac{f(x)}{(x - x_0)^k} = c \neq 0$, 称 $x \rightarrow x_0$ 时, $f(x)$ 是 **k 阶无穷小量**.



Def. 设 $x \rightarrow x_0$ 时, $f(x)$ 与 $g(x)$ 都是无穷大量.

(1) 若 $\lim_{x \rightarrow x_0} f(x)/g(x) = 0$, 则称 $x \rightarrow x_0$ 时, $f(x)$ 是 $g(x)$ 的 **低阶**

无穷大量, 记作 $f(x) = o(g(x)) \ (x \rightarrow x_0)$;

(2) 若 $\lim_{x \rightarrow x_0} f(x)/g(x) = c \neq 0$, 则称 $x \rightarrow x_0$ 时, $f(x)$ 与 $g(x)$ 是 **同**

阶无穷大量; 特别地, 当 $c = 1$ 时, 称 $x \rightarrow x_0$ 时, $f(x)$ 与 $g(x)$ 是 **等价无穷大量**, 记作 $f(x) \sim g(x) \ (x \rightarrow x_0)$;

(3) 若 $\exists M > 0, \delta > 0$, 当 $0 < |x - x_0| < \delta$ 时, 有 $|f(x)/g(x)| < M$, 则记为 $f(x) = O(g(x)) \ (x \rightarrow x_0)$.



Prop. $x \rightarrow x_0$ 时, $f(x) = o(1)$, $g(x) = O(1)$, 则 $f(x)g(x) = o(1)$.

Thm. 当 $x \rightarrow 0$ 时:

$$(1) \sin x \sim \tan x \sim x; \quad (2) 1 - \cos x \sim \frac{1}{2}x^2; \quad (3) \ln(1+x) \sim x;$$

$$(4) e^x - 1 \sim x, \quad a^x - 1 \sim x \ln a (a > 0); \quad (5) (1+x)^\alpha - 1 \sim \alpha x.$$

Proof. (1) $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1, \quad \lim_{x \rightarrow 0} \frac{\tan x}{x} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0} \frac{1}{\cos x} = 1.$

$$(2) \lim_{x \rightarrow 0} \frac{1 - \cos x}{\frac{1}{2}x^2} = \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x}{2}}{\frac{1}{2}x^2} = \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 = 1.$$



$$(3) \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = \lim_{x \rightarrow 0} \ln(1+x)^{\frac{1}{x}} = \ln e = 1.$$

(4) 令 $u = e^x - 1$, 则 $x = \ln(1+u)$, $x \rightarrow 0$ 等价于 $u \rightarrow 0$,

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \lim_{u \rightarrow 0} \frac{u}{\ln(1+u)} = 1.$$

$$a^x - 1 = e^{x \ln a} - 1 \sim x \ln a \quad (x \rightarrow 0).$$

$$(5) \frac{(1+x)^\alpha - 1}{\alpha x} = \frac{e^{\alpha \ln(1+x)} - 1}{\alpha \ln(1+x)} \cdot \frac{\ln(1+x)}{x} \rightarrow 1 \quad (x \rightarrow 0).$$



Remark. 设 $x \rightarrow x_0$ 时, $f(x)$ 与 $g(x)$ 是等价无穷小量, 则

$$f(x) = g(x) + o(g(x)), \quad x \rightarrow x_0.$$

Proof. 记 $h(x) = f(x) - g(x)$. $\lim_{x \rightarrow x_0} f(x)/g(x) = 1$, 则

$$\lim_{x \rightarrow x_0} \frac{h(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f(x) - g(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} - 1 = 0.$$

即

$$h(x) = o(g(x)), \quad x \rightarrow x_0.$$

也即

$$f(x) = g(x) + o(g(x)), \quad x \rightarrow x_0. \quad \square$$



Remark.

$$\sin x \sim x(x \rightarrow 0) \quad \Rightarrow \quad \sin x = x + o(x)(x \rightarrow 0)$$

$$\tan x \sim x(x \rightarrow 0) \quad \Rightarrow \quad \tan x = x + o(x)(x \rightarrow 0)$$

$$1 - \cos x \sim \frac{1}{2}x^2(x \rightarrow 0) \quad \Rightarrow \quad 1 - \cos x = \frac{1}{2}x^2 + o(x^2)(x \rightarrow 0)$$

$$\ln(1+x) \sim x \quad \Rightarrow \quad \ln(1+x) = x + o(x)(x \rightarrow 0)$$

$$e^x - 1 \sim x(x \rightarrow 0) \quad \Rightarrow \quad e^x - 1 = x + o(x)(x \rightarrow 0)$$

$$a^x - 1 \sim x \ln a(x \rightarrow 0) \quad \Rightarrow \quad a^x - 1 = x \ln a + o(x)(x \rightarrow 0)$$

$$(1+x)^\alpha - 1 \sim \alpha x(x \rightarrow 0) \quad \Rightarrow \quad (1+x)^\alpha - 1 = \alpha x + o(x)(x \rightarrow 0)$$



Question. $f(x) \rightarrow 0 (x \rightarrow x_0)$, 是否一定存在 $k > 0, s.t.$

$x \rightarrow x_0$ 时, $f(x)$ 为 k 阶无穷小量? **否**

试考虑 $f(x) = x \sin \frac{1}{x}$.

Ex. $\lim_{x \rightarrow 0^+} (\sin x)^{1/\ln x}$ (0^0 型)

$$= \lim_{x \rightarrow 0^+} e^{(\ln \sin x)/\ln x} = \lim_{x \rightarrow 0^+} e^{\left(\ln \frac{\sin x}{x} + \ln x \right) / \ln x} = e. \square$$

Remark. 1) 指数-对数变换. 2) 利用极限典式.

$$\text{Ex. } \lim_{x \rightarrow 0^+} x^{\sin x} = \lim_{x \rightarrow 0^+} e^{\sin x \ln x} = \lim_{x \rightarrow 0^+} e^{\frac{\sin x}{x} x \ln x} = e^0 = 1. \square$$

(0^0 型)



Ex. $\lim_{x \rightarrow 0^+} (e^x + 2x)^{1/x}$ (1^∞ 型)

$$= \lim_{x \rightarrow 0^+} (1 + e^x + 2x - 1)^{\frac{1}{e^x + 2x - 1}} \cdot \frac{e^x + 2x - 1}{x}$$

$$= e^{\lim_{x \rightarrow 0^+} \frac{e^x + 2x - 1}{x}}$$

$$= e^{2 + \lim_{x \rightarrow 0^+} \frac{e^x - 1}{x}}$$

$$= e^3. \square$$

Remark. 极限运算中 $o(\cdot)$ 的运用有时能简化计算.



Ex. $\lim_{x \rightarrow 1} \left(\frac{m}{1-x^m} - \frac{n}{1-x^n} \right)$, m, n 为正整数.

解: 令 $x = 1+t$, 则 $x \rightarrow 1$ 等价于 $t \rightarrow 0$.

$$\begin{aligned} \frac{m}{1-x^m} - \frac{n}{1-x^n} &= \frac{n}{(1+t)^n - 1} - \frac{m}{(1+t)^m - 1} \\ &= \frac{n}{nt + \frac{n(n-1)}{2}t^2 + o(t^2)} - \frac{m}{mt + \frac{m(m-1)}{2}t^2 + o(t^2)} \\ &= \frac{\frac{mn(m-n)}{2}t^2 + o(t^2)}{mnt^2 + o(t^2)} \rightarrow \frac{m-n}{2} (t \rightarrow 0) \square \end{aligned}$$



Ex. $\lim_{x \rightarrow 1} \frac{\sqrt[m]{x} - 1}{\sqrt[n]{x} - 1}$

解：令 $x = 1 + t$, 则 $x \rightarrow 1$ 等价于 $t \rightarrow 0$.

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{\sqrt[m]{x} - 1}{\sqrt[n]{x} - 1} &= \lim_{t \rightarrow 0} \frac{(1+t)^{1/m} - 1}{(1+t)^{1/n} - 1} \\ &= \lim_{t \rightarrow 0} \frac{(1+t)^{1/m} - 1}{t/m} \cdot \lim_{t \rightarrow 0} \frac{t/n}{(1+t)^{1/n} - 1} \cdot \lim_{t \rightarrow 0} \frac{t/m}{t/n} \\ &= \lim_{t \rightarrow 0} \frac{t/m}{t/n} = \frac{n}{m}. \end{aligned}$$

Remark. 等价因子替换法. $t \rightarrow 0$ 时, $(1+t)^{1/m} - 1 \sim \frac{t}{m}$,

$(1+t)^{1/n} - 1 \sim \frac{t}{n}$, 则 $\lim_{t \rightarrow 0} \frac{(1+t)^{1/m} - 1}{(1+t)^{1/n} - 1} = \lim_{t \rightarrow 0} \frac{t/m}{t/n}.$



Ex. $\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^2 \ln(1+x)}$

解法一: $x \rightarrow 0$ 时, $\tan x \sim x$,
 $\sin x \sim x, \ln(1+x) \sim x$.

$$\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^2 \ln(1+x)} = \lim_{x \rightarrow 0} \frac{x - x}{x^2 \cdot x} = 0. \text{ 是否正确? } \times$$

解法二:

$$\begin{aligned} \frac{\tan x - \sin x}{x^2 \ln(1+x)} &= \frac{\sin x(1 - \cos x)}{x^2 \cos x \ln(1+x)} \\ &= \frac{\sin x}{x} \cdot \frac{1 - \cos x}{x^2 / 2} \cdot \frac{x}{\ln(1+x)} \cdot \frac{1}{2 \cos x} \rightarrow \frac{1}{2} \quad (x \rightarrow 0). \end{aligned}$$

解法三:

$$\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^2 \ln(1+x)} = \lim_{x \rightarrow 0} \frac{\sin x(1 - \cos x)}{x^2 \cos x \ln(1+x)} = \lim_{x \rightarrow 0} \frac{x \cdot \frac{1}{2} x^2}{x^2 \cos x \cdot x} = \frac{1}{2}. \square$$



Question. 何时不需强调因子亦可进行等价无穷小替换?

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^2 \ln(1+x)} &= \lim_{x \rightarrow 0} \frac{x + o(x) - (x + o(x))}{x^2 \cdot (x + o(x))} \\ &= \lim_{x \rightarrow 0} \frac{o(x)}{x^3 + o(x^3)} \quad \text{无法继续计算!}\end{aligned}$$

用 x 近似 $\sin x, \tan x$, 过于粗糙, 因此

$$\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^2 \ln(1+x)} \not\equiv \lim_{x \rightarrow 0} \frac{x - x}{x^2 \cdot x}$$



$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{\ln(1+x)} &= \lim_{x \rightarrow 0} \frac{x + o(x) - (x + o(x))}{x + o(x)} = \lim_{x \rightarrow 0} \frac{o(x)}{x + o(x)} \\ &= \lim_{x \rightarrow 0} \frac{\frac{o(x)}{x}}{1 + \frac{o(x)}{x}} = \frac{\lim_{x \rightarrow 0} \frac{o(x)}{x}}{1 + \lim_{x \rightarrow 0} \frac{o(x)}{x}} = \frac{0}{1} = 0.\end{aligned}$$

用 x 近似 $\sin x, \tan x$, 精度足够, 因此

$$\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{\ln(1+x)} \neq \lim_{x \rightarrow 0} \frac{x - x}{x}.$$



$$\lim_{t \rightarrow 0} \frac{(1+t)^{1/m} - 1}{(1+t)^{1/n} - 1} = \lim_{t \rightarrow 0} \frac{t/m + o(t)}{t/n + o(t)}$$

$$= \lim_{t \rightarrow 0} \frac{\frac{1}{m} + \frac{o(t)}{t}}{\frac{1}{n} + \frac{o(t)}{t}} = \frac{\frac{1}{m} + \lim_{t \rightarrow 0} \frac{o(t)}{t}}{\frac{1}{n} + \lim_{t \rightarrow 0} \frac{o(t)}{t}} = \frac{n}{m}$$

用 $t/m, t/n$ 近似 $(1+t)^{1/m} - 1, (1+t)^{1/n} - 1$, 精度足够, 因此

$$\lim_{t \rightarrow 0} \frac{(1+t)^{1/m} - 1}{(1+t)^{1/n} - 1} \neq \lim_{t \rightarrow 0} \frac{t/m}{t/n}.$$

Remark. 带小 $o(\cdot)$ 运算可以避免不恰当的无穷小替换!



Ex. $\lim_{x \rightarrow 0} \frac{1 - \sqrt{\cos x}}{\cos \sqrt{x} - 1 + x}$ $\left(\frac{0}{0} \text{型} \right)$

解: $1 - \sqrt{\cos x} = 1 - e^{\frac{1}{2} \ln \cos x} \sim -\frac{1}{2} \ln \cos x = -\frac{1}{2} \ln(1 - 2 \sin^2 \frac{x}{2})$
 $\sim \sin^2 \frac{x}{2} \sim \frac{x^2}{4} \quad (x \rightarrow 0).$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1 - \sqrt{\cos x}}{\cos \sqrt{x} - 1 + x} &= \frac{1}{4} \lim_{x \rightarrow 0} \frac{x^2}{\cos \sqrt{x} - 1 + x} \\ &= \frac{1}{4} \lim_{x \rightarrow 0} \frac{x}{\frac{\cos \sqrt{x} - 1}{x} + 1} = \frac{1}{4} \cdot \frac{0}{-\frac{1}{2} + 1} = 0. \square \end{aligned}$$



$$\text{Ex. } \lim_{x \rightarrow +\infty} \frac{\sqrt{x + \sqrt{x + \sqrt{x}}}}{\sqrt{x+1}} = \lim_{x \rightarrow +\infty} \frac{\sqrt{1 + \sqrt{\frac{1}{x}} + \sqrt{\frac{1}{x^3}}}}{\sqrt{1 + \frac{1}{x}}} = 1.$$

$$\text{Ex. } \lim_{x \rightarrow +\infty} \frac{\ln(2 + \sqrt{x})}{\ln(6 + \sqrt[6]{x})} = \lim_{x \rightarrow +\infty} \frac{\ln \sqrt{x} + \ln(1 + 2/\sqrt{x})}{\ln \sqrt[6]{x} + \ln(1 + 6/\sqrt[6]{x})} \neq \lim_{x \rightarrow +\infty} \frac{\ln \sqrt{x}}{\ln \sqrt[6]{x}} = 3. \square$$

$$\ln \sqrt{x} + \ln(1 + 2/\sqrt{x}) \sim \ln \sqrt{x} \quad (x \rightarrow +\infty)$$

$$\ln \sqrt[6]{x} + \ln(1 + 6/\sqrt[6]{x}) \sim \ln \sqrt[6]{x} \quad (x \rightarrow +\infty)$$

Remark. 等价无穷大因子替换！



Ex. $\lim_{x \rightarrow +\infty} \left(\sqrt{x^2 + 2x} - \sqrt[3]{x^3 - x^2} \right)$

解法一: $\lim_{x \rightarrow +\infty} (\sqrt{x^2 + 2x} - x)$

$$= \lim_{x \rightarrow +\infty} \frac{2x}{\sqrt{x^2 + 2x} + x} = \lim_{x \rightarrow +\infty} \frac{2}{\sqrt{1 + 2/x} + 1} = 1,$$

$$\lim_{x \rightarrow +\infty} (\sqrt[3]{x^3 - x^2} - x)$$

$$= \lim_{x \rightarrow +\infty} \frac{-x^2}{\left(\sqrt[3]{x^3 - x^2} \right)^2 + x \cdot \sqrt[3]{x^3 - x^2} + x^2} = -\frac{1}{3}.$$

$$\text{原式} = \lim_{x \rightarrow +\infty} (\sqrt{x^2 + 2x} - x) - \lim_{x \rightarrow +\infty} (\sqrt[3]{x^3 - x^2} - x) = \frac{4}{3}.$$



解法二： 令 $y = 1/x$. 则 $x \rightarrow +\infty$ 时, $y \rightarrow 0$, 且

$$\begin{aligned}\lim_{x \rightarrow +\infty} \left(\sqrt{x^2 + 2x} - \sqrt[3]{x^3 - x^2} \right) &= \lim_{y \rightarrow 0} \frac{(1+2y)^{1/2} - (1-y)^{1/3}}{y} \\&= \lim_{y \rightarrow 0} \left(\frac{(1+2y)^{1/2} - 1}{y} - \frac{(1-y)^{1/3} - 1}{y} \right) \\&= \lim_{y \rightarrow 0} \frac{(1+2y)^{1/2} - 1}{y} - \lim_{y \rightarrow 0} \frac{(1-y)^{1/3} - 1}{y} \\&= 2 \cdot \frac{1}{2} - (-1) \cdot \frac{1}{3} = \frac{4}{3}. \quad \square\end{aligned}$$



Ex. $\lim_{x \rightarrow +\infty} \left(\sqrt[n]{(x^2 + 1)(x^2 + 2) \cdots (x^2 + n)} - x^2 \right)$

$$= \lim_{x \rightarrow +\infty} x^2 \left(\sqrt[n]{\left(1 + \frac{1}{x^2}\right)\left(1 + \frac{2}{x^2}\right) \cdots \left(1 + \frac{n}{x^2}\right)} - 1 \right)$$

$$= \lim_{x \rightarrow +\infty} x^2 \left(e^{\frac{1}{n} \sum_{k=1}^n \ln(1 + k/x^2)} - 1 \right)$$

$$= \lim_{x \rightarrow +\infty} \frac{x^2}{n} \sum_{k=1}^n \ln(1 + k/x^2) \quad \left(\lim_{x \rightarrow +\infty} \frac{1}{n} \sum_{k=1}^n \ln(1 + k/x^2) = 0 \right)$$

$$= \lim_{x \rightarrow +\infty} \frac{1}{n} \sum_{k=1}^n k \ln(1 + k/x^2)^{x^2/k} = \frac{1}{n} \sum_{k=1}^n k = \frac{n+1}{2}. \square$$



Ex. $\lim_{x \rightarrow 0^+} \left(2 \sin \sqrt{x} + \sqrt{x} \sin \frac{1}{x} \right)^x$

解: $\lim_{x \rightarrow 0^+} \frac{\sin \sqrt{x}}{\sqrt{x}} = 1$, 故 $\exists \delta > 0$, s.t. $\frac{3}{4} < \frac{\sin \sqrt{x}}{\sqrt{x}} \leq 1, \forall 0 < |x| < \delta$.

于是 $\frac{1}{2} \sqrt{x} \leq 2 \sin \sqrt{x} + \sqrt{x} \sin \frac{1}{x} \leq 3\sqrt{x}, \forall 0 < |x| < \delta$.

$$\lim_{x \rightarrow 0^+} \left(\frac{1}{2} \sqrt{x} \right)^x = \frac{\lim_{x \rightarrow 0^+} (\sqrt{x})^x}{\lim_{x \rightarrow 0^+} 2^x} = \lim_{x \rightarrow 0^+} (\sqrt{x})^x = \lim_{x \rightarrow 0^+} e^{\frac{1}{2} x \ln x} = e^0 = 1.$$

$$\lim_{x \rightarrow 0^+} (3\sqrt{x})^x = \lim_{x \rightarrow 0^+} 3^x \cdot \lim_{x \rightarrow 0^+} (\sqrt{x})^x = 1 \cdot 1 = 1.$$

由夹挤原理, $\lim_{x \rightarrow 0^+} \left(2 \sin \sqrt{x} + \sqrt{x} \sin \frac{1}{x} \right)^x = 1. \square$



Ex. f 在 $(0, +\infty)$ 上单调, $\lim_{x \rightarrow +\infty} \frac{f(2x)}{f(x)} = 1, a > 0$, 则 $\lim_{x \rightarrow +\infty} \frac{f(ax)}{f(x)} = 1$.

Proof. 对任意正整数 n , 有

$$\lim_{x \rightarrow +\infty} \frac{f(2^n x)}{f(x)} = \lim_{x \rightarrow +\infty} \frac{f(2^n x)}{f(2^{n-1} x)} \cdot \frac{f(2^{n-1} x)}{f(2^{n-2} x)} \cdots \frac{f(2x)}{f(x)} = 1.$$

若 $a \geq 1$, 则 $\exists n > 0, s.t. 2^0 \leq a < 2^n$, 从而

$$1 = \frac{f(2^0 x)}{f(x)} \leq (\geq) \frac{f(ax)}{f(x)} \leq (\geq) \frac{f(2^n x)}{f(x)}, \quad \forall x > 1.$$

由夹挤原理, $\lim_{x \rightarrow +\infty} \frac{f(ax)}{f(x)} = 1$.

若 $0 < a < 1$, 则 $\lim_{x \rightarrow +\infty} \frac{f(ax)}{f(x)} = \lim_{t \rightarrow +\infty} \frac{f(t)}{f(t/a)} = 1. \square$



Ex. $\lim_{x \rightarrow 0} f(x) = 0, \lim_{x \rightarrow 0} \frac{f(x) - f(x/2)}{x} = 0$, 则 $\lim_{x \rightarrow 0} \frac{f(x)}{x} = 0$.

Proof. $\forall \varepsilon > 0$, 由 $\lim_{x \rightarrow 0} \frac{f(x) - f(x/2)}{x} = 0$, $\exists \delta > 0, s.t.$

$$|f(x) - f(x/2)| < \varepsilon |x|, \quad \forall 0 < |x| < \delta.$$

$$\begin{aligned} |f(x)| &\leq \sum_{k=1}^n \left| f\left(\frac{x}{2^{k-1}}\right) - f\left(\frac{x}{2^k}\right) \right| + \left| f\left(\frac{x}{2^n}\right) \right| \\ &\leq \sum_{k=1}^n \frac{\varepsilon |x|}{2^{k-1}} + \left| f\left(\frac{x}{2^n}\right) \right| < 2\varepsilon |x| + \left| f\left(\frac{x}{2^n}\right) \right|, \quad \forall 0 < |x| < \delta. \end{aligned}$$

令 $n \rightarrow +\infty$, 由 $\lim_{x \rightarrow 0} f(x) = 0$ 得 $|f(x)| \leq 2\varepsilon |x|, \quad \forall 0 < |x| < \delta$. \square



Ex. $\lim_{x \rightarrow \infty} (\sin \sqrt{x^2 + 1} - \sin \sqrt{x^2 - 1})$

解:
$$\begin{aligned} & \left| \sin \sqrt{x^2 + 1} - \sin \sqrt{x^2 - 1} \right| \\ &= 2 \left| \cos \frac{\sqrt{x^2 + 1} + \sqrt{x^2 - 1}}{2} \sin \frac{\sqrt{x^2 + 1} - \sqrt{x^2 - 1}}{2} \right| \\ &\leq \left| \sqrt{x^2 + 1} - \sqrt{x^2 - 1} \right| = \left| \frac{1}{\sqrt{x^2 + 1} + \sqrt{x^2 - 1}} \right| \rightarrow 0 \quad (x \rightarrow \infty). \end{aligned}$$

故 $\lim_{x \rightarrow \infty} (\sin \sqrt{x^2 + 1} - \sin \sqrt{x^2 - 1}) = 0. \square$



Ex. $\lim_{x \rightarrow 0} \left(\frac{e^x + e^{2x} + \cdots + e^{nx}}{n} \right)^{1/x}$ 1^∞ 型极限

$$= \lim_{x \rightarrow 0} \left(1 + \left(\frac{e^x + e^{2x} + \cdots + e^{nx}}{n} - 1 \right) \right)^{\frac{1}{\frac{e^x + e^{2x} + \cdots + e^{nx}}{n} - 1} \cdot \frac{\frac{e^x + e^{2x} + \cdots + e^{nx}}{n} - 1}{x}}$$

$$= \exp \left\{ \lim_{x \rightarrow 0} \left(\frac{e^x + e^{2x} + \cdots + e^{nx}}{n} - 1 \right) / x \right\}$$

$$= \exp \left\{ \lim_{x \rightarrow 0} \frac{(e^x - 1)/x + (e^{2x} - 1)/x + \cdots + (e^{nx} - 1)/x}{n} \right\}$$

$$= e^{(1+2+\cdots+n)/n} = e^{(n+1)/2}. \square$$



解法二. $\lim_{x \rightarrow 0} \left(\frac{e^x + e^{2x} + \cdots + e^{nx}}{n} \right)^{1/x}$

$$= \lim_{x \rightarrow 0} \left(1 + \frac{(e^x - 1) + (e^{2x} - 1) + \cdots + (e^{nx} - 1)}{n} \right)^{1/x}$$

$$= \lim_{x \rightarrow 0} \left(1 + \frac{x + o(x) + 2x + o(x) + \cdots + nx + o(x)}{n} \right)^{1/x}$$

$$= \lim_{x \rightarrow 0} \left(1 + \frac{(n+1)x}{2} + o(x) \right)^{1/x} = e^{\lim_{x \rightarrow 0} \left(\frac{(n+1)x}{2} + o(x) \right) / x} = e^{(n+1)/2}. \square$$



Ex. $\lim_{n \rightarrow +\infty} n^2 \left(\sqrt[n]{x} - \sqrt[n+1]{x} \right) \quad (x > 0)$

$$= \lim_{n \rightarrow +\infty} n^2 x^{1/(n+1)} \left(x^{1/n(n+1)} - 1 \right)$$

$$= \lim_{n \rightarrow +\infty} \left(x^{1/(n+1)} \cdot \frac{x^{1/n(n+1)} - 1}{1/n(n+1)} \cdot \frac{n^2}{n(n+1)} \right)$$

$$= \lim_{n \rightarrow +\infty} x^{1/(n+1)} \cdot \lim_{n \rightarrow +\infty} \frac{x^{1/n(n+1)} - 1}{1/n(n+1)} \cdot \lim_{n \rightarrow +\infty} \frac{n^2}{n(n+1)} = \ln x. \square$$

Remark. $a^t - 1 \sim t \ln a \quad (t \rightarrow 0) \Rightarrow \lim_{t \rightarrow 0} \frac{a^t - 1}{t} = \ln a.$



Ex. $\lim_{x \rightarrow 0} \frac{\sqrt{1 + \tan x} - \sqrt{1 - \tan x}}{e^x - 1} = 1$

Proof. $\frac{\sqrt{1 + \tan x} - \sqrt{1 - \tan x}}{e^x - 1} = \frac{\sqrt{1 + \tan x} - 1}{e^x - 1} - \frac{\sqrt{1 - \tan x} - 1}{e^x - 1},$

$$\frac{\sqrt{1 + \tan x} - 1}{e^x - 1} = \frac{1}{2} \cdot \frac{(1 + \tan x)^{1/2} - 1}{\frac{1}{2} \tan x} \cdot \frac{\tan x}{x} \cdot \frac{x}{e^x - 1} \rightarrow \frac{1}{2} \quad (x \rightarrow 0)$$

同理, $\frac{\sqrt{1 - \tan x} - 1}{e^x - 1} \rightarrow -\frac{1}{2} \quad (x \rightarrow 0). \square$



Remark. 上例可用等价因子替换法:

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{\sqrt{1 + \tan x} - \sqrt{1 - \tan x}}{e^x - 1} \\ &= \lim_{x \rightarrow 0} \frac{\sqrt{1 + \tan x} - 1}{e^x - 1} - \lim_{x \rightarrow 0} \frac{\sqrt{1 - \tan x} - 1}{e^x - 1} \\ &= \lim_{x \rightarrow 0} \frac{\frac{1}{2} \tan x}{x} - \lim_{x \rightarrow 0} \frac{-\frac{1}{2} \tan x}{x} = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1. \end{aligned}$$



采用 $o(\cdot)$ 更为简洁:

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{\sqrt{1 + \tan x} - \sqrt{1 - \tan x}}{e^x - 1} \\ &= \lim_{x \rightarrow 0} \frac{1 + \frac{1}{2} \tan x + o(\tan x) - \left(1 - \frac{1}{2} \tan x + o(\tan x)\right)}{x + o(x)} \\ &= \lim_{x \rightarrow 0} \frac{\frac{1}{2} x + o(x) - \left(-\frac{1}{2} x + o(x)\right)}{x + o(x)} = 1. \end{aligned}$$



Ex.(1) $\lim_{x \rightarrow 0+} \frac{(a^x - b^x)^2}{a^{x^2} - b^{x^2}} (a, b > 0, a \neq b),$ (2) $\lim_{x \rightarrow a} \frac{a^{a^x} - a^{x^a}}{a^x - x^a} (a > 0)$

解.(1) $\lim_{x \rightarrow 0+} \frac{(a^x - b^x)^2}{a^{x^2} - b^{x^2}} = \lim_{x \rightarrow 0+} \frac{b^{2x} ((a/b)^x - 1)^2}{b^{x^2} ((a/b)^{x^2} - 1)}$

$$= \lim_{x \rightarrow 0+} \frac{((a/b)^x - 1)^2}{(a/b)^{x^2} - 1} = \lim_{x \rightarrow 0+} \frac{(x \ln(a/b))^2}{x^2 \ln(a/b)} = \ln \frac{a}{b}.$$

(2) $\lim_{x \rightarrow a} \frac{a^{a^x} - a^{x^a}}{a^x - x^a} = \lim_{x \rightarrow a} \frac{a^{x^a} (a^{a^x - x^a} - 1)}{a^x - x^a}$

$$= \lim_{x \rightarrow a} a^{x^a} \lim_{x \rightarrow a} \frac{(a^{a^x - x^a} - 1)}{a^x - x^a} = a^{a^a} \ln a. \square$$



Ex.(1) $\lim_{x \rightarrow 0+} x \ln x$, (2) $\lim_{x \rightarrow 0+} (x^x - 1) \ln x$, (3) $\lim_{x \rightarrow 0+} x^{x^x - 1}$

解:(1) $\lim_{x \rightarrow 0+} x \ln x = \lim_{x \rightarrow 0+} \frac{-\ln(1/x)}{1/x} = \lim_{y \rightarrow +\infty} \frac{-\ln y}{y} = 0.$

(2) $\lim_{x \rightarrow 0+} (x^x - 1) \ln x = \lim_{x \rightarrow 0+} (e^{x \ln x} - 1) \ln x$

$$= \lim_{x \rightarrow 0+} x(\ln x)^2 = \lim_{x \rightarrow 0+} (\sqrt{x} \ln x)^2 = 0.$$

(3) $\lim_{x \rightarrow 0+} x^{x^x - 1} = \lim_{x \rightarrow 0+} e^{(x^x - 1) \ln x} = e^{\lim_{x \rightarrow 0+} (x^x - 1) \ln x} = e^0 = 1. \square$



Question. 以下解答是否正确？

$$\lim_{x \rightarrow +\infty} \frac{e^x}{\left(1 + \frac{1}{x}\right)^{x^2}} = \lim_{x \rightarrow +\infty} \frac{e^x}{\left(\left(1 + \frac{1}{x}\right)^x\right)^x}$$

$$= \lim_{x \rightarrow +\infty} \frac{e^x}{\left(\lim_{x \rightarrow +\infty} \left(1 + \frac{1}{x}\right)^x\right)^x} = \lim_{x \rightarrow +\infty} \frac{e^x}{e^x} = 1. \quad (\times)$$

不能局部取极限！

正确结果？ 正确解答？ \sqrt{e} , L'Hospital 或 *Taylor* 展开.



作业：习题2.4

No. 8,9(1)(2)(6)(8)(11),12

思考题（不交）：

1. 无穷多个无穷小量的乘积是否一定是无穷小量？

证明或给反例。

2. $\alpha \geq 1, n \rightarrow +\infty$ 时, n 个 $o(\frac{1}{n^\alpha})$ 之和是否为 $o(\frac{1}{n^{\alpha-1}})$ ？

证明或给反例。