Review

Thm.(Fermat) x_0是f的极值点, $f'(x_0)$存在, 则$f'(x_0)=0$.

Thm.(Darboux) f在$[a,b]$上可导, $f_+'(a)\neq f_-'(b)$, 则对介于$f_+'(a)$与$f_-'(b)$之间的任意实数λ, $\exists\xi\in(a,b)$, s.t. $f'(\xi)=\lambda$.

Thm.(Rolle) $f\in C[a,b]$, f在(a,b)可导.若$f(a)=f(b)$, 则存在$\xi\in(a,b)$, s.t. $f'(\xi)=0$.

Thm.(Cauchy) $f,g\in C[a,b]$, f,g在(a,b)可导, 且$\forall t\in(a,b)$, 有$g'(t)\neq 0$. 则存在$\xi\in(a,b)$, s.t.

$$\frac{f'(\xi)}{g'(\xi)}=\frac{f(b)-f(a)}{g(b)-g(a)}.$$

Thm.(Lagrange) $f\in C[a,b]$, f在(a,b)可导,则$\exists\xi\in(a,b)$, $s.t.$

$$f'(\xi)=\frac{f(b)-f(a)}{b-a}.$$

Remark. $f\in C[a,b]$, f在(a,b)可导,则

(1)$\exists\xi\in(a,b)$, $s.t.$ $\quad f(b)-f(a)=f'(\xi)(b-a).$

(2)$\forall x,x_0\in[a,b]$,$\exists$介于x与x_0之间的ξ, $s.t.$

$$f(x)-f(x_0)=f'(\xi)(x-x_0).$$

(3)$\forall x_0,x_0+\Delta x\in[a,b]$,$\exists\theta\in(0,1)$, $s.t.$

$$f(x_0+\Delta x)-f(x_0)=f'(x_0+\theta\Delta x)\cdot\Delta x.$$

§ 2. L'Hospital法则

Thm. f, g在$(x_0, x_0+\rho)$中可导, $g'(x)\neq 0$, $\lim\limits_{x\to x_0+}\dfrac{f'(x)}{g'(x)}=A$,

(1)(0 / 0型)若 $\lim\limits_{x\to x_0+} f(x)=\lim\limits_{x\to x_0+} g(x)=0$, 则 $\lim\limits_{x\to x_0+}\dfrac{f(x)}{g(x)}=A$.

(2)(∞ / ∞型)若 $\lim\limits_{x\to x_0+} g(x)=\infty$, 则 $\lim\limits_{x\to x_0+}\dfrac{f(x)}{g(x)}=A$.

Remark.(1)极限过程 $\lim\limits_{x\to x_0+}$ 替换成 $\lim\limits_{x\to x_0-}$ 或 $\lim\limits_{x\to x_0}$, 定理仍成立.

(2)A替换成$+\infty, -\infty$或∞, 定理仍然成立.

Proof.(1) $\lim\limits_{x\to x_0+} f(x)=\lim\limits_{x\to x_0+} g(x)=0$, 不妨设

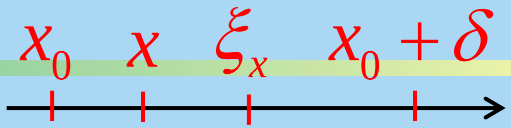
$$f(x_0)=g(x_0)=0.$$

$g'(x)\neq 0$,由Cauchy中值定理, $\exists\xi_x\in(x_0,x), s.t.$

$$\frac{f(x)}{g(x)}=\frac{f(x)-f(x_0)}{g(x)-g(x_0)}=\frac{f'(\xi_x)}{g'(\xi_x)},$$

且 $x\to x_0^+$时,$\xi_x\to x_0^+$. 于是

$$\lim_{x\to x_0+}\frac{f(x)}{g(x)}=\lim_{x\to x_0+}\frac{f'(\xi_x)}{g'(\xi_x)}=\lim_{x\to x_0+}\frac{f'(x)}{g'(x)}=A.$$



(2) $\lim\limits_{x\to x_0+}\dfrac{f'(x)}{g'(x)}=\mathrm{A}$, 则$\forall\varepsilon\in(0,1),\exists\delta>0, s.t.$

$$\left|\frac{f'(x)}{g'(x)}-\mathrm{A}\right|<\varepsilon,\quad \forall x\in(x_0,x_0+\delta).$$

$g'(x)\neq 0$,由Cauchy中值定理, $\forall x\in(x_0,x_0+\delta),\exists\xi_x\in(x,x_0+\delta)$,

$s.t.$ $$\frac{f(x)-f(x_0+\delta)}{g(x)-g(x_0+\delta)}=\frac{f'(\xi_x)}{g'(\xi_x)},\quad \frac{\dfrac{f(x)}{g(x)}-\dfrac{f(x_0+\delta)}{g(x)}}{1-\dfrac{g(x_0+\delta)}{g(x)}}=\frac{f'(\xi_x)}{g'(\xi_x)},$$

$$\frac{f(x)}{g(x)}-\mathrm{A}=\frac{f(x_0+\delta)}{g(x)}+\frac{f'(\xi_x)}{g'(\xi_x)}-\mathrm{A}-\frac{f'(\xi_x)}{g'(\xi_x)}\frac{g(x_0+\delta)}{g(x)},$$

$$\left|\frac{f(x)}{g(x)}-\mathrm{A}\right| \le \left|\frac{f(x_0+\delta)}{g(x)}\right| + \left|\frac{f'(\xi_x)}{g'(\xi_x)}-\mathrm{A}\right| + \left|\frac{f'(\xi_x)}{g'(\xi_x)}\right| \cdot \left|\frac{g(x_0+\delta)}{g(x)}\right|$$

$\lim\limits_{x\to x_0+} g(x)=\infty$, 则$\exists 0<\delta_1<\delta, s.t.$

$$\left|\frac{f(x_0+\delta)}{g(x)}\right|<\varepsilon,\ \left|\frac{g(x_0+\delta)}{g(x)}\right|<\varepsilon,\quad \forall x\in(x_0,x_0+\delta_1).$$

因此,

$$\left|\frac{f(x)}{g(x)}-\mathrm{A}\right| \le \varepsilon+\varepsilon+\left(|\mathrm{A}|+\varepsilon\right)\varepsilon<\left(|\mathrm{A}|+3\right)\varepsilon,$$

$$\forall x\in(x_0,x_0+\delta_1).$$

故 $\lim\limits_{x\to x_0+}\dfrac{f(x)}{g(x)}=\mathrm{A}.\square$

Thm. f, g在$(a,+\infty)$中可导, $g'(x)\neq 0$, $\lim\limits_{x\to+\infty}\dfrac{f'(x)}{g'(x)}=A$,

(1)(0/0型)若 $\lim\limits_{x\to+\infty}f(x)=\lim\limits_{x\to+\infty}g(x)=0$, 则 $\lim\limits_{x\to+\infty}\dfrac{f(x)}{g(x)}=A$.

(2)(∞/∞型)若 $\lim\limits_{x\to+\infty}g(x)=\infty$, 则 $\lim\limits_{x\to+\infty}\dfrac{f(x)}{g(x)}=A$.

Remark.(1)极限过程 $\lim\limits_{x\to+\infty}$ 替换成 $\lim\limits_{x\to-\infty}$ 或 $\lim\limits_{x\to\infty}$, 定理仍成立.

(2)A替换成$+\infty, -\infty$或∞, 定理仍然成立.

Proof. 不妨设$a>0$.令$\varphi(t)=f(\frac{1}{t}),\psi(t)=g(\frac{1}{t}),t\in(0,\frac{1}{a})$.

$$\lim_{t\to 0+}\varphi(t)=\lim_{x\to+\infty}f(x),\quad \lim_{t\to 0+}\psi(t)=\lim_{x\to+\infty}g(x).$$

$$\psi'(t)=g'(1/t)\cdot\frac{-1}{t^2}\neq 0,\forall t>0,$$

$\lim\limits_{x\to+\infty}f(x)=\lim\limits_{x\to+\infty}g(x)=0$ 或 $\lim\limits_{x\to+\infty}g(x)=\infty$时, 由上一定理,

$$\lim_{x\to+\infty}\frac{f(x)}{g(x)}=\lim_{t\to 0+}\frac{\varphi(t)}{\psi(t)}=\lim_{t\to 0+}\frac{\varphi'(t)}{\psi'(t)}=\lim_{t\to 0+}\frac{f'(1/t)\cdot\frac{-1}{t^2}}{g'(1/t)\cdot\frac{-1}{t^2}}$$

$$=\lim_{t\to 0+}\frac{f'(1/t)}{g'(1/t)}=\lim_{x\to+\infty}\frac{f'(x)}{g'(x)}=\text{A}.\square$$

Ex. $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=\lim\limits_{x\to 0}\dfrac{\cos x}{1}=1.$

Ex. $a>1, b>0,$ 则 $\lim\limits_{x\to+\infty}\dfrac{x^b}{a^x}=\lim\limits_{x\to+\infty}\dfrac{bx^{b-1}}{a^x\ln a}=\cdots=0.$

Ex. $\lambda>0, P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0,$ 则

$$\lim_{x\to+\infty}\frac{P(x)}{e^{\lambda x}}=\lim_{x\to+\infty}\frac{P'(x)}{\lambda e^{\lambda x}}=\cdots=\lim_{x\to+\infty}\frac{n!a_n}{\lambda^n e^{\lambda x}}=0.$$

Ex. $\alpha>0,$ 则 $\lim\limits_{x\to+\infty}\dfrac{\ln x}{x^\alpha}=\lim\limits_{x\to+\infty}\dfrac{1/x}{\alpha x^{\alpha-1}}=\lim\limits_{x\to+\infty}\dfrac{1}{\alpha x^\alpha}=0.$

$$\text{Ex. } \lim_{x\to\infty}(1+1/x)^{x^2}e^{-x} \boxed{= \lim_{x\to\infty}\left((1+1/x)^x\right)^x e^{-x} \overset{?}{=} \lim_{x\to\infty} e^x e^{-x} = 1}$$

$$= \lim_{x\to\infty}\exp\left\{x^2\ln(1+1/x)-x\right\}$$

$$= \exp\left\{\lim_{x\to\infty}\left(x^2\ln(1+1/x)-x\right)\right\}$$

$$= \exp\left\{\lim_{t\to 0}\frac{\ln(1+t)-t}{t^2}\right\} = \exp\left\{\lim_{t\to 0}\frac{1/(1+t)-1}{2t}\right\}$$

$$= \exp\left\{\lim_{t\to 0}\frac{-1/(1+t)^2}{2}\right\} = e^{-1/2}.$$

$$\text{Ex.}\lim_{x\to0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\lim_{x\to0}\frac{x-\sin x}{x\sin x}=\lim_{x\to0}\frac{1-\cos x}{\sin x+x\cos x}$$

$$=\lim_{x\to0}\frac{\sin x}{2\cos x-x\sin x}=\frac{\lim\limits_{x\to0}\sin x}{\lim\limits_{x\to0}(2\cos x-x\sin x)}=\frac{0}{2}=0.$$

$$\text{法二:}\lim_{x\to0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\lim_{x\to0}\frac{x-\sin x}{x\sin x}=\lim_{x\to0}\frac{x-\sin x}{x^2}$$

$$=\lim_{x\to0}\frac{1-\cos x}{2x}=\lim_{x\to0}\frac{\sin x}{2}=0.$$

Ex. $\lim\limits_{x\to 0}\left(\dfrac{1}{\sin^2 x}-\dfrac{1}{x^2}\right)$

$$=\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^2\sin^2 x}=\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^4}$$

$$=\lim_{x\to 0}\frac{x+\sin x}{x}\cdot\frac{x-\sin x}{x^3}=\lim_{x\to 0}\frac{x+\sin x}{x}\cdot\lim_{x\to 0}\frac{x-\sin x}{x^3}$$

$$=2\lim_{x\to 0}\frac{1-\cos x}{3x^2}=\lim_{x\to 0}\frac{x^2}{3x^2}=\frac{1}{3}.$$

Remark. 适时分离！ 等价因子替换！

Question. $\lim\limits_{x\to x_0+}\dfrac{f(x)}{g(x)}=\mathrm{A}$,能否推出 $\lim\limits_{x\to x_0+}\dfrac{f'(x)}{g'(x)}=\mathrm{A}$?否!

反例:

$$\lim_{x\to 0}\frac{x^2\sin(1/x)}{x}=0,$$

$$\lim_{x\to 0}\frac{\left(x^2\sin(1/x)\right)'}{(x)'}=\lim_{x\to 0}\frac{2x\sin\frac{1}{x}-\cos\frac{1}{x}}{1}\text{不存在.}$$

Remark. L'Hospital法则并非万能！

Ex. 判断正误.

$$\lim_{x\to 0}\frac{\ln(1+e^x)}{x}=\lim_{x\to 0}\frac{\left(\ln(1+e^x)\right)'}{x'}=\lim_{x\to 0}\frac{\frac{e^x}{1+e^x}}{1}=\frac{1}{2} \qquad \times$$

Ex. $\lim\limits_{x\to 0+} x^{x^x-1}=\lim\limits_{x\to 0+} e^{(x^x-1)\ln x}=\lim\limits_{x\to 0+} e^{(e^{x\ln x}-1)\ln x}$ $\qquad 0^0$型

$$=\exp\left\{\lim_{x\to 0+}(e^{x\ln x}-1)\ln x\right\}=\exp\left\{\lim_{x\to 0+}\frac{e^{x\ln x}-1}{x\ln x}\cdot x\ln^2 x\right\}$$

$$=\exp\left\{\lim_{x\to 0+}\frac{e^{x\ln x}-1}{x\ln x}\cdot\lim_{x\to 0+}x\ln^2 x\right\}=e^{1\cdot 0}=1.$$

L'Hospital?

Ex. $\lim\limits_{x\to+\infty}\dfrac{x^{\ln x}}{(\ln x)^x} = \lim\limits_{x\to+\infty} e^{(\ln x)^2 - x\ln\ln x}$

$\dfrac{\infty}{\infty}$型

$$= \exp\left\{\lim_{x\to+\infty}\left((\ln x)^2 - x\ln\ln x\right)\right\}$$

$$= \exp\left\{\lim_{x\to+\infty} x\left(\frac{(\ln x)^2}{x} - \ln\ln x\right)\right\}$$

$$= \exp\left\{\lim_{x\to+\infty} x\cdot\lim_{x\to+\infty}\left(\frac{(\ln x)^2}{x} - \ln\ln x\right)\right\}$$

$$= \exp\left\{+\infty\cdot(0-\infty)\right\} = e^{-\infty} = 0.$$

L′Hospital?

Question. $f(x)=2x+\sin 2x, g(x)=e^{\sin x} f(x),$

$$f'(x)=2+2\cos 2x=4\cos^2 x,$$

$$g'(x)=e^{\sin x}(f'(x)+f(x)\cos x)$$

$$=e^{\sin x}(4\cos x+2x+\sin 2x)\cos x,$$

$$\left|\frac{f'(x)}{g'(x)}\right|=\left|\frac{4\cos x}{e^{\sin x}(4\cos x+2x+\sin 2x)}\right|\le\frac{4}{e^{-1}(2x-5)}, x>>1\text{时},$$

$$\lim_{x\to+\infty} g(x)=+\infty,\ \lim_{x\to+\infty}\frac{f'(x)}{g'(x)}=0,\ \lim_{x\to+\infty}\frac{f(x)}{g(x)}=\lim_{x\to+\infty}\frac{1}{e^{\sin x}}\text{不存在.}$$

这是否是 L'Hosptial 法则的一个反例？否!不满足$g'(x)\neq 0$.

Ex. f在$(0,+\infty)$上可导, $a>0$.

(1) $\lim\limits_{x\to+\infty}(af(x)+f'(x))=l$, 则 $\lim\limits_{x\to+\infty} f(x)=l/a$.

(2) $\lim\limits_{x\to+\infty}(af(x)-f'(x))=l$, $|f(x)|\le M$, 则 $\lim\limits_{x\to+\infty} f(x)=l/a$.

(3) $\lim\limits_{x\to+\infty}(af(x)+2\sqrt{x}f'(x))=l$, 则 $\lim\limits_{x\to+\infty} f(x)=l/a$.

Proof.(1)

$$\lim_{x\to+\infty} f(x)=\lim_{x\to+\infty}\frac{e^{ax}f(x)}{e^{ax}}=\lim_{x\to+\infty}\frac{e^{ax}(af(x)+f'(x))}{ae^{ax}}$$

$$=\lim_{x\to+\infty}\frac{af(x)+f'(x)}{a}=l/a.$$

(2) $|f(x)| \le M$，则

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{e^{-ax} f(x)}{e^{-ax}} = \lim_{x\to+\infty} \frac{e^{-ax}(-af(x) + f'(x))}{-ae^{-ax}}$$

$$= \lim_{x\to+\infty} \frac{af(x) - f'(x)}{a} = \frac{l}{a}.$$

(3)

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{e^{a\sqrt{x}} f(x)}{e^{a\sqrt{x}}} = \lim_{x\to+\infty} \frac{e^{a\sqrt{x}}\left(\frac{a}{2\sqrt{x}} f(x) + f'(x)\right)}{\frac{a}{2\sqrt{x}} e^{a\sqrt{x}}}$$

$$= \lim_{x\to+\infty} \frac{af(x) + 2\sqrt{x} f'(x)}{a} = \frac{l}{a}. \square$$

Ex. $\lim\limits_{x\to+\infty}\left(\tan\dfrac{\pi x}{2x+1}\right)^{1/x}$ $\quad \infty^0$型

$$=\exp\left\{\lim_{x\to+\infty}\frac{1}{x}\ln\tan\frac{\pi x}{2x+1}\right\}\overset{\mathrm{L}}{=}\exp\left\{\lim_{x\to+\infty}\frac{\dfrac{\pi}{(2x+1)^2}}{\tan\dfrac{\pi x}{2x+1}\cdot\cos^2\dfrac{\pi x}{2x+1}}\right\}$$

$$=\exp\left\{\lim_{x\to+\infty}\frac{2\pi}{(2x+1)^2\sin\dfrac{2\pi x}{2x+1}}\right\}=\exp\left\{\lim_{x\to+\infty}\frac{2\pi}{(2x+1)^2\sin\dfrac{\pi}{2x+1}}\right\}$$

$$=\exp\left\{2\lim_{x\to+\infty}\frac{\dfrac{\pi}{2x+1}}{\sin\dfrac{\pi}{2x+1}}\cdot\lim_{x\to+\infty}\frac{1}{2x+1}\right\}=e^0=1.\square$$

Ex. $\lim\limits_{x\to+\infty}\left(\dfrac{a^x-1}{(a-1)x}\right)^{1/x}\quad (a>0, a\neq 1)$ ∞^0型, 0^0型

解: 原式 $=\lim\limits_{x\to+\infty}\exp\left\{\dfrac{1}{x}\ln\dfrac{a^x-1}{(a-1)x}\right\}$

$$=\exp\left\{\lim_{x\to+\infty}\frac{\ln\left|a^x-1\right|-\ln|x|-\ln|a-1|}{x}\right\}$$

$$\overset{\text{L}}{=\!=}\exp\left\{\lim_{x\to+\infty}\left(\frac{a^x\ln a}{a^x-1}-\frac{1}{x}\right)\right\}=\begin{cases}\exp\{\ln a-0\}=a, & (a>1)\\ \exp\{0-0\}=1, & (0<a<1)\end{cases}.$$

$$\text{Ex.}\lim_{x\to 0}\frac{e^{(1+x)^{1/x}}-\left((1+x)^{1/x}\right)^{e}}{x^2}$$

$\frac{0}{0}$型

解: $\left(e^{(1+x)^{1/x}}\right)'=e^{(1+x)^{1/x}}\cdot\left((1+x)^{1/x}\right)'$

$$=e^{(1+x)^{1/x}}\cdot\left(e^{\frac{1}{x}\ln(1+x)}\right)'=e^{(1+x)^{1/x}}\cdot(1+x)^{1/x}\cdot\left(\frac{1}{x}\ln(1+x)\right)'$$

$$\left(\left((1+x)^{1/x}\right)^{e}\right)'=\left((1+x)^{e/x}\right)'$$

$$=\left(e^{\frac{e}{x}\ln(1+x)}\right)'=e(1+x)^{e/x}\cdot\left(\frac{1}{x}\ln(1+x)\right)'$$

$$\lim_{x\to 0}\left(\frac{1}{x}\ln(1+x)\right)' = \lim_{x\to 0}\frac{\frac{x}{1+x}-\ln(1+x)}{x^2}$$

$$=\lim_{x\to 0}\frac{1-\frac{1}{1+x}-\ln(1+x)}{x^2} \overset{\text{L}}{=\!=} \lim_{x\to 0}\frac{\frac{1}{(1+x)^2}-\frac{1}{1+x}}{2x}$$

$$=\lim_{x\to 0}\frac{\frac{-x}{(1+x)^2}}{2x}=\lim_{x\to 0}\frac{\frac{-1}{(1+x)^2}}{2}=-\frac{1}{2}$$

清華大學

$$\lim_{x\to 0}\frac{e^{(1+x)^{1/x}}-(1+x)^{e/x}}{x^2}$$

$$=\lim_{x\to 0}\frac{\left(e^{(1+x)^{1/x}}(1+x)^{1/x}-e(1+x)^{e/x}\right)\cdot\left(\frac{1}{x}\ln(1+x)\right)'}{2x}$$

$$=\lim_{x\to 0}\frac{\left(e^{(1+x)^{1/x}}(1+x)^{1/x}-e(1+x)^{e/x}\right)}{2x}\cdot\lim_{x\to 0}\left(\frac{1}{x}\ln(1+x)\right)'$$

$$= -\frac{1}{4}\lim_{x\to 0}\frac{\left(e^{(1+x)^{1/x}}(1+x)^{1/x} - e(1+x)^{e/x}\right)}{x}$$

$$= -\frac{1}{4}\lim_{x\to 0}\left(e^{(1+x)^{1/x}}(1+x)^{2/x} + e^{(1+x)^{1/x}}(1+x)^{1/x} - e^{2}(1+x)^{e/x}\right)$$

$$\cdot\lim_{x\to 0}\left(\frac{1}{x}\ln(1+x)\right)'$$

$$= \frac{1}{8}\left(e^{e}e^{2} + e^{e}e - e^{2}e^{e}\right) = \frac{1}{8}e^{e+1}.\square$$

作业：习题4.2

No. 2(2,7,8,18,19,20),3,4