



Review

- 有理函数 $\frac{p(x)}{q(x)}$ (p, q 为多项式) 的积分
- 三角有理式 $R(\sin x, \cos x)$ 万能变换 $t = \tan \frac{x}{2}$
- 可化为有理式的简单无理式

$$1) \int R(x, \sqrt[n]{\frac{ax+b}{cx+d}}) dx \quad (ad-bc \neq 0) \quad \text{令 } t = \sqrt[n]{\frac{ax+b}{cx+d}}$$

$$2) \int R(x, \sqrt{ax^2+bx+c}) dx, (a \neq 0)$$

三角变换开根号、Euler变换



§ 6. 定积分的计算

Newton-Leibniz公式 $\int_a^b F'(x)dx = F(x)\Big|_{x=a}^b$

Thm.(换元法) $f \in C[a, b]$, $\varphi \in C^1[\alpha, \beta]$ (或 $\varphi \in C^1[\beta, \alpha]$), $\varphi(\alpha) = a$, $\varphi(\beta) = b$, $a \leq \varphi(t) \leq b$, 则 $\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt$.

Proof. $f \in C[a, b]$, 则 f 在 $[a, b]$ 上有原函数 $F(x)$,

$$\frac{d}{dt} F(\varphi(t)) = f(\varphi(t))\varphi'(t), \quad \forall t \in [\alpha, \beta].$$

$$\int_\alpha^\beta f(\varphi(t))\varphi'(t)dt = F(\varphi(t))\Big|_\alpha^\beta = F(b) - F(a) = \int_a^b f(x)dx. \square$$

Remark. $\varphi \in C^1[\alpha, \beta]$, 不要求 $x = \varphi(t)$ 可逆.



Question. " $\varphi(t) \in [a, b]$ " 可去吗? 可以, f 在 $\varphi(t)$ 的值域连续.

Question. 不定积分的第一、二换元法以及定积分换元法中, 对 $x = \varphi(t)$ 分别有什么要求?

Ex. 判断正误

$$\int_{-1}^1 \frac{1}{1+x^2} dx = \arctan x \Big|_{-1}^1 = \frac{\pi}{2} \quad (\checkmark)$$

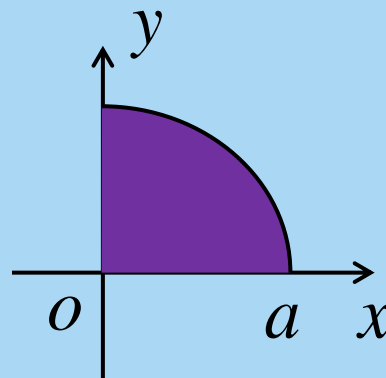
$$\int_{-1}^1 \frac{1}{1+x^2} dx = - \int_{-1}^1 \frac{1}{1+\frac{1}{x^2}} d\frac{1}{x} = - \arctan \frac{1}{x} \Big|_{-1}^1 = -\frac{\pi}{2}. \quad (\times)$$

$\left(-\arctan \frac{1}{x} \right)$ 在 $x=0$ 处不连续; $x = \frac{1}{t}$ 在 $t=0$ 处不连续



Ex. $\int_0^a \sqrt{a^2 - x^2} dx$ ($a > 0$) 几何意义?

解: 令 $x = a \sin t, t \in [0, \pi/2]$, $t \in [0, 5\pi/2]$?



$$\begin{aligned} \int_0^a \sqrt{a^2 - x^2} dx &= \int_0^{\pi/2} a^2 \cos^2 t dt \\ &= \frac{1}{2} a^2 \int_0^{\pi/2} (1 + \cos 2t) dt \\ &= \frac{1}{2} a^2 \left(t + \frac{1}{2} \sin 2t \right) \Big|_0^{\pi/2} = \frac{\pi}{4} a^2. \quad \square \end{aligned}$$



Ex. $f \in C[-a, a]$ 为偶函数, 则 $\int_{-a}^a \frac{f(x)}{1+e^x} dx = \int_0^a f(x) dx$.

Proof.

$$\begin{aligned} \int_{-a}^a \frac{f(x)}{1+e^x} dx &= \int_0^a \frac{f(x)}{1+e^x} dx + \int_{-a}^0 \frac{f(x)}{1+e^x} dx \\ &= \int_0^a \frac{f(x)}{1+e^x} dx - \int_a^0 \frac{1}{1+e^{-t}} f(-t) dt \quad (x = -t) \\ &= \int_0^a \frac{f(x)}{1+e^x} dx + \int_0^a \frac{e^t}{1+e^t} f(t) dt \quad (f \text{ 偶}) \\ &= \int_0^a \frac{f(x)}{1+e^x} dx + \int_0^a \frac{e^x}{1+e^x} f(x) dx = \int_0^a f(x) dx. \square \end{aligned}$$



Ex. $f \in C[0, a]$, $f(x) + f(a-x) \neq 0$, 则 $\int_0^a \frac{f(x)}{f(x) + f(a-x)} dx = \frac{a}{2}$.

Proof. $I = \int_0^a \frac{f(x)}{f(x) + f(a-x)} dx$

(令 $x = a - t$)
$$= - \int_a^0 \frac{f(a-t)}{f(a-t) + f(t)} dt = \int_0^a \frac{f(a-t)}{f(a-t) + f(t)} dt$$

(令 $t = x$)
$$= \int_0^a \frac{f(a-x)}{f(a-x) + f(x)} dx$$

$$2I = \int_0^a \left(\frac{f(x)}{f(x) + f(a-x)} + \frac{f(a-x)}{f(a-x) + f(x)} \right) dx = \int_0^a dx = a. \square$$



Ex. $f \in C[1, +\infty)$, $a > 1$, 则 $\int_1^a f(x^2 + \frac{a^2}{x^2}) \frac{dx}{x} = \int_1^a f(x + \frac{a^2}{x}) \frac{dx}{x}$.

Proof. $\int_1^a f(x^2 + \frac{a^2}{x^2}) \frac{dx}{x} = \frac{1}{2} \int_1^{a^2} f(t + \frac{a^2}{t}) \frac{dt}{t} \quad (t = x^2)$

$$= \frac{1}{2} \int_1^a f(t + \frac{a^2}{t}) \frac{dt}{t} + \frac{1}{2} \int_a^{a^2} f(t + \frac{a^2}{t}) \frac{dt}{t} \triangleq \frac{1}{2} (I_1 + I_2).$$

令 $s = \frac{a^2}{t}$, 则

$$I_2 = - \int_a^1 f(s + \frac{a^2}{s}) \frac{ds}{s} = \int_1^a f(s + \frac{a^2}{s}) \frac{ds}{s} = I_1.$$

故 $\int_1^a f(x^2 + \frac{a^2}{x^2}) \frac{dx}{x} = \int_1^a f(x + \frac{a^2}{x}) \frac{dx}{x}.$ \square



Ex. (1) $f \in C[a, b]$, 则 $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$;

$$(2) I = \int_{\pi/6}^{\pi/3} \frac{\cos^2 x}{x(\pi-2x)} dx = \frac{1}{\pi} \ln 2.$$

Proof. (1) $\int_a^b f(a+b-x) dx \xrightarrow{t=a+b-x} -\int_b^a f(t) dt = \int_a^b f(t) dt$

$$(2) \text{由(1), } I = \int_{\pi/6}^{\pi/3} \frac{\sin^2 x}{x(\pi-2x)} dx = \int_a^b f(x) dx.$$

$$= \int_{\pi/6}^{\pi/3} \frac{\cos^2 x}{x(\pi-2x)} dx = \frac{1}{2} \int_{\pi/6}^{\pi/3} \frac{\cos^2 x + \sin^2 x}{x(\pi-2x)} dx$$

$$= \frac{1}{\pi} \int_{\pi/6}^{\pi/3} \left(\frac{1}{2x} + \frac{1}{\pi-2x} \right) dx = \frac{1}{2\pi} \ln \frac{2x}{\pi-2x} \Big|_{\pi/6}^{\pi/3} = \frac{1}{\pi} \ln 2. \square$$



Ex. $I = \int_0^1 \frac{\ln(1+x)}{1+x^2} dx.$

解: $I = \int_0^{\pi/4} \ln(1+\tan t) dt \quad (t = \arctan x)$

$$= \int_0^{\pi/4} \ln(\sin t + \cos t) dt - \int_0^{\pi/4} \ln(\cos t) dt = I_1 - I_2.$$

$$I_1 = \int_0^{\pi/4} \left(\ln \sqrt{2} + \ln \sin\left(t + \frac{\pi}{4}\right) \right) dt = \frac{\pi}{8} \ln 2 + \int_0^{\pi/4} \ln \cos\left(\frac{\pi}{4} - t\right) dt$$

$$= \frac{\pi}{8} \ln 2 - \int_{\pi/4}^0 \ln \cos u du = \frac{\pi}{8} \ln 2 + I_2.$$

$$I = \frac{\pi}{8} \ln 2. \square$$



Thm.(定积分的分部积分法) $u, v \in C^1[a, b]$, 则

$$\int_a^b u(x)v'(x)dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)dx.$$

Proof. $(u(x)v(x))' = u'(x)v(x) + u(x)v'(x)$

$$\Rightarrow \int_a^b u'(x)v(x)dx + \int_a^b u(x)v'(x)dx = u(x)v(x)\Big|_a^b$$

$$\Rightarrow \int_a^b u(x)v'(x)dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)dx. \square$$



Ex. 证明 $I_n \triangleq \int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \cos^n x dx$, 并求 I_n .

Proof. 令 $t = \frac{\pi}{2} - x$, 则

$$\int_0^{\pi/2} \sin^n x dx = -\int_{\pi/2}^0 \sin^n \left(\frac{\pi}{2} - t\right) dt = \int_0^{\pi/2} \cos^n t dt.$$

$$\begin{aligned} I_n &= -\int_0^{\pi/2} \sin^{n-1} x d\cos x \\ &= -\sin^{n-1} x \cos x \Big|_0^{\pi/2} + (n-1) \int_0^{\pi/2} \cos^2 x \sin^{n-2} x dx \\ &= (n-1) \int_0^{\pi/2} (1 - \sin^2 x) \sin^{n-2} x dx \\ &= (n-1) I_{n-2} - (n-1) I_n. \end{aligned}$$



$$I_n = \frac{n-1}{n} I_{n-2}.$$

$$I_0 = \int_0^{\pi/2} dx = \frac{\pi}{2},$$

$$I_1 = \int_0^{\pi/2} \sin x dx = -\cos x \Big|_0^{\pi/2} = 1,$$

$$I_{2n} = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2} = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2},$$

$$I_{2n-1} = \frac{2n-2}{2n-1} \cdot \frac{2n-4}{2n-3} \cdots \frac{2}{3} \cdot 1 = \frac{(2n-2)!!}{(2n-1)!!}. \square$$



Ex. $f, g \in C[a, b], \int_a^x f(t)dt \geq \int_a^x g(t)dt \quad (a \leq x \leq b),$

$\int_a^b f(x)dx = \int_a^b g(x)dx, \text{ 则 } \int_a^b xf(x)dx \leq \int_a^b xg(x)dx.$

Proof. 令 $F(x) = \int_a^x f(x)dx, G(x) = \int_a^x g(x)dx,$ 则

$F(a) = G(a) = 0, F(b) = G(b), F(x) \geq G(x) (a \leq x \leq b).$

$$\begin{aligned} \int_a^b x(f(x) - g(x))dx &= \int_a^b xd(F(x) - G(x)) \\ &= x(F(x) - G(x))\Big|_a^b - \int_a^b (F(x) - G(x))dx \\ &= -\int_a^b (F(x) - G(x))dx \leq 0. \square \end{aligned}$$



Ex. $f \in C^1[a, b]$, $f(a) = 0$, 则

$$\int_a^b f^2(x) dx \leq \frac{(b-a)^2}{2} \int_a^b (f'(x))^2 dx - \frac{1}{2} \int_a^b (x-a)^2 (f'(x))^2 dx.$$

Proof. $f(a) = 0$, 则 $f^2(x) = \left(\int_a^x 1 \cdot f'(t) dt \right)^2 \leq (x-a) \int_a^x (f'(t))^2 dt$.

$$\begin{aligned} \int_a^b f^2(x) dx &\leq \int_a^b \left(\int_a^x (f'(t))^2 dt \right) d \frac{(x-a)^2}{2} \\ &= \frac{(x-a)^2}{2} \int_a^x (f'(t))^2 dt \Big|_{x=a}^b - \int_a^b \frac{(x-a)^2}{2} (f'(x))^2 dx \\ &= \frac{(b-a)^2}{2} \int_a^b (f'(x))^2 dx - \frac{1}{2} \int_a^b (x-a)^2 (f'(x))^2 dx. \square \end{aligned}$$



Ex. $f \in C^1[a, b]$, $f(a) = f(b) = 0$, $\int_a^b f^2(x) dx = 1$, 则

$$\int_a^b (f'(x))^2 dx \cdot \int_a^b x^2 f^2(x) dx > \frac{1}{4}.$$

Proof. 若 $f'(x) + \lambda x f(x) \equiv 0$, 则

$$\left(f(x) e^{\frac{1}{2} \lambda x^2} \right)' = (f'(x) + \lambda x f(x)) e^{\frac{1}{2} \lambda x^2} \equiv 0,$$

$$f(x) e^{\frac{1}{2} \lambda x^2} \equiv C, \quad f(x) = C e^{\frac{-1}{2} \lambda x^2},$$

$f(a) = f(b) = 0$, 则 $C = 0$, $f(x) \equiv 0$, 与 $\int_a^b f^2(x) dx = 1$ 矛盾.

故 $\forall \lambda \in \mathbb{R}$, $\int_a^b (f'(x) + \lambda x f(x))^2 dx > 0$, 即

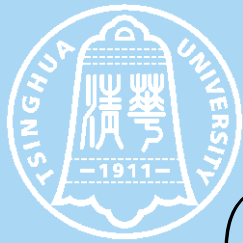


$$\int_a^b (f'(x))^2 dx + 2\lambda \int_a^b xf(x)f'(x)dx + \lambda^2 \int_a^b x^2 f^2(x)dx > 0.$$

$$\text{于是} \int_a^b (f'(x))^2 dx \cdot \int_a^b x^2 f^2(x)dx > \left(\int_a^b xf(x)f'(x)dx \right)^2.$$

$$\begin{aligned} \text{而} \int_a^b xf(x)f'(x)dx &= \frac{1}{2} \int_a^b x df^2(x) \\ &= \frac{1}{2} xf^2(x) \Big|_a^b - \frac{1}{2} \int_a^b f^2(x)dx = -\frac{1}{2} \end{aligned}$$

$$\text{故} \int_a^b (f'(x))^2 dx \cdot \int_a^b x^2 f^2(x)dx > \frac{1}{4}. \square$$



Ex. $\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1 \cdot 3 \cdot 5 \cdots (2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}.$

Proof. $S_n \triangleq 1 \cdot 3 \cdot 5 \cdots (2n-1),$

$$2 \ln S_n = 2 \sum_{k=2}^n \ln(2k-1) < \sum_{k=2}^n \int_{2k-1}^{2k+1} \ln x dx$$

$$= \int_3^{2n+1} \ln x dx = x \ln x \Big|_3^{2n+1} - \int_3^{2n+1} x \cdot \frac{1}{x} dx$$

$$= (x \ln x - x) \Big|_3^{2n+1} = x \ln \frac{x}{e} \Big|_3^{2n+1} < (2n+1) \ln \frac{2n+1}{e}.$$

同理, $2 \ln S_n > \sum_{k=2}^n \int_{2k-3}^{2k-1} \ln x dx = \int_1^{2n-1} \ln x dx = (2n-1) \ln \frac{2n-1}{e} + 1. \square$



Thm.(带积分余项的Taylor公式) $f \in C^{n+1}[a, b], x_0 \in [a, b]$,

则 $\forall x \in [a, b]$, 有

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{1}{n!} \int_{x_0}^x (x - t)^n f^{(n+1)}(t) dt.$$

Proof. $n = 0$ 时, 即Newton-Leibniz公式.

假设 $n = m - 1$ 时, 定理成立, 即

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{1}{(m-1)!} \int_{x_0}^x (x - t)^{m-1} f^{(m)}(t) dt.$$

对余项分部积分, 得



$$\begin{aligned}\frac{1}{(m-1)!} \int_{x_0}^x (x-t)^{m-1} f^{(m)}(t) dt &= \frac{-1}{m!} \int_{x_0}^x f^{(m)}(t) d(x-t)^m \\&= -\frac{1}{m!} f^{(m)}(t)(x-t)^m \Big|_{t=x_0}^x + \frac{1}{m!} \int_{x_0}^x (x-t)^m f^{(m+1)}(t) dt \\&= \frac{1}{m!} f^{(m)}(x_0)(x-x_0)^m + \frac{1}{m!} \int_{x_0}^x (x-t)^m f^{(m+1)}(t) dt.\end{aligned}$$

$$\text{故 } f(x) = \sum_{k=0}^m \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k + \frac{1}{m!} \int_{x_0}^x (x-t)^m f^{(m+1)}(t) dt,$$

即 $n = m$ 时, 定理成立. \square



作业：习题5.6

No.1(2,11,14),2(2,4,8)

3(1,4,8),5,**8**,11,13.