习题课 2021.11.24.

两个常识:

$$\begin{vmatrix} & & & a_n \\ & & \cdots & \\ & a_2 & & \\ a_1 & & & \end{vmatrix} = (-1)^{\frac{n(n-1)}{2}} a_1 a_2 \cdots a_n$$

$$\begin{vmatrix} a_1 & & & & \\ \vdots & \ddots & & & \\ b_1 & \cdots & a_i & \cdots & b_n \\ & & & \ddots & \vdots \\ & & & & a_n \end{vmatrix} = a_1 a_2 \cdots a_n$$

一、拆分法

例
$$\begin{vmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & & \vdots \\ b & b & b & \cdots & a \end{vmatrix} = [a+(n-1)b](a-b)^{n-1}$$

原式 $= \begin{vmatrix} b+(a-b) & b & b & \cdots & b \\ b & b+(a-b) & b & \cdots & b \\ & & \cdots & & \\ b & b & b & \cdots & b+(a-b) \end{vmatrix} = \begin{vmatrix} b & b & b & \cdots & b \\ b & b+(a-b) & b & \cdots & b \\ & & \cdots & & \\ b & b & b & \cdots & b+(a-b) \end{vmatrix} + \begin{vmatrix} a-b & 0 & 0 & \cdots & 0 \\ b & b+(a-b) & b & \cdots & b \\ & & \cdots & & \\ b & b & b & \cdots & b+(a-b) \end{vmatrix}$

$= \cancel{\begin{vmatrix} b & b & b & \cdots & b \\ b & b+(a-b) & b & \cdots & b \\ & & \cdots & & \\ b & b & b & \cdots & b+(a-b) \end{vmatrix}} = \begin{vmatrix} b & b & b & \cdots & b \\ b & b & b & \cdots & b \\ & & \cdots & & \\ b & b & b & \cdots & b+(a-b) \end{vmatrix} + \begin{vmatrix} b & b & b & \cdots & b \\ 0 & a-b & 0 & \cdots & 0 \\ & & \cdots & & \\ b & b & b & \cdots & b+(a-b) \end{vmatrix}$

$+ \begin{vmatrix} a-b & 0 & 0 & \cdots & 0 \\ b & b & b & \cdots & b \\ & & \cdots & & \\ b & b & b & \cdots & b+(a-b) \end{vmatrix} + \begin{vmatrix} a-b & 0 & 0 & \cdots & 0 \\ 0 & a-b & 0 & \cdots & 0 \\ & & \cdots & & \\ b & b & b & \cdots & b+(a-b) \end{vmatrix}$

$= \cdots$

注. 最后拆出 2^n 个行列式，但只有两种情况行列式不为0. 一种是 $[b, \cdots b]$ 只有一行，~~样~~ $\begin{vmatrix} a-b & & & \\ \vdots & \ddots & & \\ b & \cdots b & \cdots & b \\ & & \ddots & a-b \end{vmatrix}$，另一种是 $\begin{vmatrix} a-b & & \\ & a-b & \\ & & \ddots \\ & & & a-b \end{vmatrix}$.

所以原行列式 $= (a-b)^{n-1} b \cdot n + (a-b)^n$

例 $\begin{vmatrix} x_1 & a_2 & a_3 & \cdots & a_n \\ a_1 & x_2 & a_3 & \cdots & a_n \\ a_1 & a_2 & x_3 & \cdots & a_n \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 & a_2 & a_3 & \cdots & x_n \end{vmatrix} = \begin{vmatrix} (x_1-a_1)+a_1 & a_2 & a_3 & \cdots & a_n \\ a_1 & (x_2-a_2)+a_2 & a_3 & \cdots & a_n \\ a_1 & a_2 & (x_3-a_3)+a_3 & \cdots & a_n \\ \cdots & & & & \\ a_1 & a_2 & a_3 & \cdots & (x_n-a_n)+a_n \end{vmatrix}$

同样拆开. 只有 $\begin{vmatrix} x_1-a_1 & & & \\ & x_2-a_2 & & \\ & & \ddots & \\ & & & x_n-a_n \end{vmatrix}$ 和 $\begin{vmatrix} x_1-a_1 & & & & \\ \vdots & \ddots & & & \\ a_1 & a_2 & \cdots a_i & \cdots & a_n \\ & & & \ddots & \\ & & & & x_n-a_n \end{vmatrix}$ 不为0.

得原式 $= \prod_{i=1}^{n}(x_i-a_i) + \sum_{i=1}^{n}\left(\frac{a_i}{x_i-a_i}\prod_{j=1}^{n}(x_i-a_i)\right)$

$= \left[1+\sum_{i=1}^{n}\frac{a_i}{x_i-a_i}\right]\prod_{i=1}^{n}(x_i-a_i)$

二、递推法.(常用于三对角形)

补充: 对数列 $a_{n+1}+pa_n+qa_{n-1}=0$. 可构造特征方程 $x^2+px+q=0$

$\Delta>0$. 特征方程 2个~~解~~实根 x_1, x_2. 则 $a_n = C_1x_1^n + C_2x_2^n$

$\Delta=0$. ······1个~~解~~重根 x_0 则 ~~$a_n=(nC_1+C_2)x_0^n$~~ $a_n = C_1x_0^n + C_2nx_0^n$

$\Delta<0$ ······$x_{1,2}=r(\cos\theta \pm i\sin\theta)$ 则 $a_n = C_1r^n\cos n\theta + C_2r^n\sin n\theta$

例: $D_n = \begin{vmatrix} \alpha+\beta & \alpha\beta & & & & \\ 1 & \alpha+\beta & \alpha\beta & & & \\ & 1 & \alpha+\beta & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \alpha+\beta & \alpha\beta \\ & & & & 1 & \alpha+\beta \end{vmatrix}$

$D_n = (\alpha+\beta)\begin{vmatrix} \alpha+\beta & \alpha\beta & & \\ 1 & \alpha+\beta & \ddots & \\ & \ddots & \alpha+\beta & \alpha\beta \\ & & 1 & \alpha+\beta \end{vmatrix} - \alpha\beta\begin{vmatrix} 1 & \alpha\beta & & & \\ & \alpha+\beta & \alpha\beta & & \\ & 1 & \alpha+\beta & \alpha\beta & \\ & & \ddots & \ddots & \alpha\beta \\ & & & 1 & \alpha+\beta \end{vmatrix}$

$= (\alpha+\beta)D_{n-1} - \alpha\beta D_{n-2}$.

特征方程有两个根 α, β.

于是 $D_n = c_1\alpha^n + c_2\beta^n$ $(\alpha \neq \beta)$

又 $D_1 = \alpha+\beta$、 $D_2 = \alpha^2+\alpha\beta+\beta^2$.

有 $c_1 = \frac{\alpha}{\alpha-\beta}$ $c_2 = \frac{-\beta}{\alpha-\beta}$

于是 $D_n = \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}$. ~~$(\alpha\neq\beta)$~~.

$\alpha=\beta$ 时. ~~按~~ $D_n = c_1 n\alpha^n + c_2\alpha^n$

$D_1 = 2\alpha$, $D_2 = 3\alpha^2$. 有 $c_1 = c_2 = 1$

于是 $D_n = (n+1)\alpha^n$.

~~同时~~ (另外. $\alpha\neq\beta$ 时. 令 $\beta\to\alpha$ 也有 $\lim\limits_{\beta\to\alpha}\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta} = \frac{d}{dx}x^{n+1}\Big|_{x=\alpha} = (n+1)\alpha^n$)

例:

$$D_n = \begin{vmatrix} \cos\alpha & 1 & 0 & \cdots & 0 & 0 \\ 1 & 2\cos\alpha & 1 & \cdots & 0 & 0 \\ 0 & 1 & 2\cos\alpha & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2\cos\alpha & 1 \\ 0 & 0 & 0 & & 1 & 2\cos\alpha \end{vmatrix}$$

~~$D_n = 2\cos\alpha$~~ 按最后一行展开(注意,绝对不能是第1行)

$D_n = 2\cos\alpha - D_{n-2}$. 解特征方程 $x^2 - 2\cos\alpha + 1 = 0$.

有 $x_1 = \cos\alpha + i\sin\alpha$. $x_2 = \cos\alpha - i\sin\alpha$.

于是 $D_n = c_1\cos n\alpha + c_2\sin n\alpha$.

又 $D_1 = \cos\alpha$. $D_2 = 2\cos^2\alpha - 1 = \cos 2\alpha$.

知 $c_1 = 1$, $c_2 = 0$. 所以 $D_n = \cos n\alpha$.

三. 分块矩阵.

~~A+B∈R~~ $A\in\mathbb{R}^{n\times m}$ $B\in\mathbb{R}^{m\times n}$

则 (1) $\det(I_n\pm AB)=\det(I_m\pm BA)$

(2). $\det(\lambda I_n\pm AB)=\det(\lambda I_m\pm BA)\lambda^{n-m}$

例 1: $D_n=\begin{vmatrix} 1+a_1+x_1 & a_1+x_2 & \cdots & a_1+x_n \\ a_2+x_1 & 1+a_2+x_2 & \cdots & a_2+x_n \\ \vdots & \vdots & & \vdots \\ a_n+x_1 & a_n+x_2 & \cdots & 1+a_n+x_n \end{vmatrix}$

$D_n=\left|I_n+\begin{bmatrix} a_1 & 1 \\ \vdots & \vdots \\ a_n & 1 \end{bmatrix}\begin{bmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \end{bmatrix}\right|=\left|I_2+\begin{bmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \end{bmatrix}\begin{bmatrix} a_1 & 1 \\ \vdots & \vdots \\ a_n & 1 \end{bmatrix}\right|$

$=\begin{vmatrix} 1+\sum_{i=1}^n a_i & n \\ \sum_{i=1}^n a_i x_i & 1+\sum_{i=1}^n x_i \end{vmatrix}=(1+\sum_{i=1}^n a_i)(1+\sum_{i=1}^n x_i)-n\sum_{i=1}^n a_i x_i$

例 2:

(4) $A=\begin{vmatrix} \sin(\alpha_1+\beta_1) & \sin(\alpha_1+\beta_2) & \cdots & \sin(\alpha_1+\beta_n) \\ \sin(\alpha_2+\beta_1) & \sin(\alpha_2+\beta_2) & \cdots & \sin(\alpha_2+\beta_n) \\ \vdots & & & \\ \sin(\alpha_n+\beta_1) & \sin(\alpha_n+\beta_2) & \cdots & \sin(\alpha_n+\beta_n) \end{vmatrix}$.

① $n=1$ $|A|=\sin(\alpha_1+\beta_1)$

② $n=2$ $|A|=\sin(\alpha_1+\beta_1)\sin(\alpha_2+\beta_2)-\sin(\alpha_1+\beta_2)\sin(\alpha_2+\beta_1)$

③ $n>2$ 时.

$A=\begin{bmatrix} \sin\alpha_1 & \cos\alpha_1 \\ \sin\alpha_2 & \cos\alpha_2 \\ \cdots & \\ \sin\alpha_n & \cos\alpha_n \end{bmatrix}\begin{bmatrix} \cos\beta_1 & \cos\beta_2 & \cdots & \cos\beta_n \\ \sin\beta_1 & \sin\beta_2 & \cdots & \sin\beta_n \end{bmatrix}$

$\operatorname{rank}(A)\leqslant 2$.

所以 $|A|=0$.

(2) ~~求值 $\det(I_n+$~~ $\det(aI_n+A)$

利用 $\det(aI_n+BC)=\det(aI_2+CB)a^{n-2}$ 即可.

箭形行列式

$$D_n=\begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ 1 & b_2 & & \\ \vdots & & \ddots & \\ 1 & & & b_n \end{vmatrix}$$

~~$(b_i\neq 0)$~~ 第1列中的1换成一般数可用同样的办法做.

第i列 $-\frac{1}{b_i}$ 倍加至第一列.

$$D_n=\begin{vmatrix} a_1-\sum_{i=2}^{n}\frac{a_i}{b_i} & a_2 & \cdots & a_n \\ 0 & b_2 & & \\ \vdots & & \ddots & \\ 0 & & & b_n \end{vmatrix}=\left(a_1-\sum_{i=2}^{n}\frac{a_i}{b_i}\right)\prod_{i=2}^{n}b_i \quad (b_i\neq 0)$$

$$=a_1\prod_{i=2}^{n}b_i-\sum_{i=2}^{n}\left(a_i\prod_{\substack{j=2\\ j\neq i}}^{n}b_j\right) \quad (*)$$

~~b_i不能有2个为0.~~ ~~b_i有2个以上为0，则 $D_n=0$~~ b_i有2个以上为0，则 $D_n=0$，符合~~等式~~ $(*)$式

只有~~$b_i=0$，则按第i行展开~~ 仅有b_i为0.

则
$$D_n=\begin{vmatrix} a_1 & a_2 & \cdots & & a_n \\ 1 & b_2 & & & \\ \vdots & & \ddots & & \\ 1 & & \cdots & b_i=0 & \\ 1 & & & & b_n \end{vmatrix}=-\begin{vmatrix} 1 & & & & \\ 1 & b_2 & & & \\ \vdots & & \ddots & & \\ a_1 & a_2 & \cdots a_i & \cdots & a_n \\ \vdots & & & \ddots & \\ 1 & & \cdots & & b_n \end{vmatrix}$$

$$=-\begin{vmatrix} b_2 & & & \\ & \ddots & & \\ a_1 & \cdots a_i & \cdots & a_n \\ & & \ddots & b_n \end{vmatrix}=a_i\prod_{\substack{j=2\\ j\neq i}}^{n}b_j$$

也符合 $(*)$式

加边法

例：
$$D_n=\begin{vmatrix} c_1 & a_2 & a_3 & \cdots & a_n \\ a_1 & c_2 & a_3 & \cdots & a_n \\ a_1 & a_2 & c_3 & \cdots & a_n \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 & a_2 & a_3 & \cdots & c_n \end{vmatrix}$$

$$D_n=\begin{vmatrix} 1 & -a_1 & -a_2 & -a_3 & \cdots & -a_n \\ 0 & c_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & c_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & c_3 & \cdots & a_n \\ \vdots & & & & \ddots & \\ 0 & a_1 & a_2 & a_3 & \cdots & c_n \end{vmatrix}$$

$$=\begin{vmatrix} 1 & -a_1 & -a_2 & -a_3 & \cdots & -a_n \\ 1 & c_1-a_1 & 0 & 0 & \cdots & 0 \\ 1 & & c_2-a_2 & & & \\ \vdots & & & \ddots & & \\ 1 & & & & & c_n-a_n \end{vmatrix}$$

套入箭形行列式结论即可.