Review

- 导数

$$f'(x_0) \triangleq \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

$$f'_{\pm}(x_0) \triangleq \lim_{\Delta x \to 0^{\pm}} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

- $f'(x_0)$存在 $\Leftrightarrow$ $f'_-(x_0)$, $f'_+(x_0)$均存在且相等.
- 导数的几何、物理意义
- 可微 $\Leftrightarrow$ 可导 $\Rightarrow$ 连续
- $y = f(x)$, $f'(x)$也记为$\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

- f在x_0可微, 则 $f(x_0+\Delta x)\approx f(x_0)+f'(x_0)\Delta x.$

- f,g在x_0可导, $c\in\mathbb{R}$, 则

  (1) $(f+g)'(x_0)=f'(x_0)+g'(x_0)$;

  (2) $(cf)'(x_0)=cf'(x_0)$;

  (3) $(fg)'(x_0)=f'(x_0)g(x_0)+f(x_0)g'(x_0)$;

  (4) $\left(\dfrac{f}{g}\right)'(x_0)=\dfrac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{g^2(x_0)}.$

- 多个因子连乘的函数求导时先取对数再两端求导.

•(链式法则)$\varphi(x)$在x_0可导, $f(u)$在 $u_0=\varphi(x_0)$可导, 则

$h(x)=f(\varphi(x))$在x_0可导, 且

$$h'(x_0)=f'(\varphi(x_0))\cdot\varphi'(x_0),\text{即 } \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x}.$$

•(一阶微分形式的不变性)$u=\varphi(x)$在x_0可微, $y=f(u)$

在$u_0=\varphi(x_0)$可微, 则$y=f(\varphi(x))$在x_0可微, 且

$$\mathrm{d}y=f'(\varphi(x_0))\varphi'(x_0)\mathrm{d}x=f'(u_0)\mathrm{d}u.$$

无论将u视为中间变量还是自变量, 都有$\mathrm{d}y=f'(u)\mathrm{d}u$.

•(反函数求导)$\dfrac{\mathrm{d}x}{\mathrm{d}y}=\dfrac{1}{\dfrac{\mathrm{d}y}{\mathrm{d}x}}$.

$$c' = 0, \qquad (x^{\alpha})' = \alpha x^{\alpha-1},$$

$$(\sin x)' = \cos x, \qquad (\cos x)' = -\sin x,$$

$$(\tan x)' = \sec^2 x, \qquad (\cot x)' = -\csc^2 x,$$

$$(\sec x)' = \sec x \tan x, \qquad (\csc x)' = -\csc x \cot x$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}, \qquad \arctan x = \frac{1}{1+x^2}$$

$$(\arccos x)' = \frac{-1}{\sqrt{1-x^2}}, \qquad \operatorname{arc}\cot x = \frac{-1}{1+x^2}$$

$$(a^x)' = a^x \ln a, \qquad (e^x)' = e^x$$

$$(\log_a x)' = \frac{1}{x \ln a}, \qquad (\ln x)' = \frac{1}{x}$$

$$\left(\ln\left|x + \sqrt{x^2 \pm a^2}\right|\right)' = \frac{1}{\sqrt{x^2 \pm a^2}}$$

- 隐函数求导
- 参数函数求导

§ 3. 高阶导数

Def. $y = f(x)$.

二阶导(函)数: $y''(x) = \dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = f''(x) \triangleq (f'(x))'$,

三阶导(函)数: $y'''(x) = \dfrac{\mathrm{d}^3 y}{\mathrm{d}x^3} = f'''(x) \triangleq (f''(x))'$,

$\vdots$

$n+1$阶导(函)数: $y^{(n+1)}(x) = \dfrac{\mathrm{d}^{n+1} y}{\mathrm{d}x^{n+1}} = f^{(n+1)}(x) \triangleq (f^{(n)}(x))'$.

Def. $f \in C^n(a,b)$: f在(a,b)上n阶可导, 且$f^{(n)} \in C(a,b)$.

Question. $f \in C^n[a,b]$如何定义?

Ex. 求$\sin^{(n)} x, \cos^{(n)} x$.

解:$\sin' x = \cos x = \sin(x+\frac{\pi}{2})$,

$$\sin''(x) = -\sin x = \sin(x+2\cdot\frac{\pi}{2}),$$

$$\sin''' x = -\cos x = \sin(x+\frac{3\pi}{2}),$$

$$\sin^{(4)} x = \sin x = \sin(x+\frac{4\pi}{2}),$$

$$\vdots$$

$\sin^{(n)} x = \sin(x+\frac{n\pi}{2})$. 同理, $\cos^{(n)} x = \cos(x+\frac{n\pi}{2})$.

Ex. $y=\ln(1+x)$, 求$y^{(n)}$.

解： $y'=\dfrac{1}{1+x}=(1+x)^{-1}$,

$$y''=-(1+x)^{-2},$$

$$y'''=2!(1+x)^{-3},$$

$$\vdots$$

$$y^{(n)}=(-1)^{n-1}(n-1)!(1+x)^{-n}.$$

Thm. 设$f(x)$与$g(x)$在点x处有n阶导数,$c\in\mathbb{R}$,则

$$(1)(f+g)^{(n)}(x)=f^{(n)}(x)+g^{(n)}(x);$$

$$(2)(cf)^{(n)}(x)=c\cdot f^{(n)}(x);$$

$$(3)(f\cdot g)^{(n)}(x)=\sum_{k=0}^{n}C_n^k f^{(k)}(x)g^{(n-k)}(x).$$ (Leibniz公式)

Proof of (3). $n=1$时,$(fg)'=f'g+fg'$,结论成立.

设$n=m$时结论成立,即

$$(f\cdot g)^{(m)}(x)=\sum_{k=0}^{m}C_m^k f^{(k)}(x)g^{(m-k)}(x),$$

则$n=m+1$时,

$$(f\cdot g)^{(m+1)}(x)=\left(\sum_{k=0}^{m}C_m^k f^{(k)}(x)g^{(m-k)}(x)\right)'$$

$$=\sum_{k=0}^{m}C_m^k f^{(k+1)}(x)g^{(m-k)}(x)+\sum_{k=0}^{m}C_m^k f^{(k)}(x)g^{(m+1-k)}(x)$$

$$=\sum_{k=1}^{m}C_m^{k-1} f^{(k)}(x)g^{(m+1-k)}(x)+f^{(m+1)}(x)g(x)$$

$$+\sum_{k=1}^{m}C_m^k f^{(k)}(x)g^{(m+1-k)}(x)+f(x)g^{(m+1)}(x)$$

$$=\sum_{k=1}^{m}(C_m^{k-1}+C_m^k)f^{(k)}(x)g^{(m+1-k)}(x)+f^{(m+1)}(x)g(x)+f(x)g^{(m+1)}(x)$$

$$=\sum_{k=0}^{m+1}C_{m+1}^k f^{(k)}(x)g^{(m+1-k)}(x).\square$$

Ex. $y=\dfrac{1}{x^2-x-2}$, 求$y^{(n)}$.

解: $y=\dfrac{1}{(x+1)(x-2)}=\dfrac{1}{3}\left(\dfrac{1}{x-2}-\dfrac{1}{x+1}\right).$

$$y^{(n)}=\frac{1}{3}\left(\frac{1}{x-2}\right)^{(n)}-\frac{1}{3}\left(\frac{1}{x+1}\right)^{(n)}$$

$$=\frac{1}{3}(-1)^n n!(x-2)^{-(n+1)}-\frac{1}{3}(-1)^n n!(x+1)^{-(n+1)}.\square$$

Ex. $y=\dfrac{1+x}{\sqrt{1-x}}$, 求$y^{(n)}$.

解法一: $y=\dfrac{2-(1-x)}{\sqrt{1-x}}=2(1-x)^{-\frac{1}{2}}-(1-x)^{\frac{1}{2}}$,

$$y^{(n)}=2\left((1-x)^{-\frac{1}{2}}\right)^{(n)}-\left((1-x)^{\frac{1}{2}}\right)^{(n)}$$

$$=2\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\cdots\cdot\frac{2n-1}{2}(1-x)^{-\frac{1}{2}-n}+\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\cdots\cdot\frac{2n-3}{2}(1-x)^{\frac{1}{2}-n}$$

$$=\frac{(2n-1)!!}{2^{n-1}}(1-x)^{-\frac{2n+1}{2}}+\frac{(2n-3)!!}{2^{n}}(1-x)^{-\frac{2n-1}{2}}.$$

解法二： $y=(1+x)(1-x)^{-1/2}$,

$$y^{(n)}=(1+x)\left((1-x)^{-1/2}\right)^{(n)}+n\left((1-x)^{-1/2}\right)^{(n-1)}$$

$$=(1+x)\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\cdots\cdot\frac{2n-1}{2}(1-x)^{-\frac{1}{2}-n}$$

$$+n\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\cdots\cdot\frac{2n-3}{2}(1-x)^{\frac{1}{2}-n}$$

$$=\frac{(2n-1)!!}{2^n}(1+x)(1-x)^{-\frac{2n+1}{2}}+\frac{(2n-3)!!}{2^{n-1}}n(1-x)^{-\frac{2n-1}{2}}.\square$$

Ex. $y=(\arcsin x)^2$, 求 $y^{(n)}(0)$.

解： $y' = 2(\arcsin x)\big/\sqrt{1-x^2}$, $\sqrt{1-x^2}\,y' = 2\arcsin x$,

两边对 x 求导, 得 $-xy'\big/\sqrt{1-x^2} + \sqrt{1-x^2}\,y'' = 2\big/\sqrt{1-x^2}$,

$$xy' + (x^2-1)y'' = -2,$$

两边对 x 求 n 阶导, 得 $\left(y' + xy'' + 2xy'' + (x^2-1)y''' = 0, n=1\right)$

$$xy^{(n+1)} + ny^{(n)} + (x^2-1)y^{(n+2)} + 2nxy^{(n+1)} + n(n-1)y^{(n)} = 0,\ n \ge 1.$$

令 $x=0$, 得 $y^{(n+2)}(0) = n^2 y^{(n)}(0)$, $y'(0)=0$, $y''(0)=2$,

故 $$y^{(n)}(0) = \begin{cases} 0, & n = 2k-1, \\ 2^{2k-1}\left((k-1)!\right)^2, & n = 2k. \end{cases}$$ □

Ex. $x^2+xy+y^2=1$确定了隐函数$y=y(x)$, 求$y''(x)$.

解：视$x^2+xy+y^2=1$中$y=y(x)$, 两边对x求导, 得

$$2x+y+xy'+2yy'=0,\quad y'=-\frac{2x+y}{x+2y}.$$

于是

$$y''=-\frac{(2x+y)'(x+2y)-(2x+y)(x+2y)'}{(x+2y)^2}$$

$$=-\frac{(2+y')(x+2y)-(2x+y)(1+2y')}{(x+2y)^2}$$

$$=\frac{3(xy'-y)}{(x+2y)^2}=\frac{-6(x^2+xy+y^2)}{(x+2y)^3}=\frac{-6}{(x+2y)^3}.\square$$

Ex. $y = 2x + \sin x$, 求$x''(y)$.

解法一：视$y = 2x + \sin x$中$x = x(y)$, 两边对y求导, 得

$$1 = 2x'(y) + \cos x \cdot x'(y).$$

再对y求导, 得 $$0 = 2x'' - \sin x \cdot (x')^2 + \cos x \cdot x''.$$

解得 $$x'(y) = \frac{1}{2 + \cos x}, \quad x'' = \frac{\sin x \cdot (x')^2}{2 + \cos x} = \frac{\sin x}{(2 + \cos x)^3}.$$

解法二：$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}} = \frac{1}{2 + \cos x},$$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}y^2} = \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right) \cdot \frac{\mathrm{d}x}{\mathrm{d}y} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2 + \cos x}\right) \cdot \frac{1}{2 + \cos x}$$

$$= -\frac{-\sin x}{(2 + \cos x)^2} \cdot \frac{1}{2 + \cos x} = \frac{\sin x}{(2 + \cos x)^3}. \square$$

Ex. $\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}, t \in (0, 2\pi)$, 求$y'(x), y''(x)$.

解：$y'(x) = \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}t} \cdot \dfrac{\mathrm{d}t}{\mathrm{d}x} = \dfrac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \dfrac{a \sin t}{a(1 - \cos t)} = \dfrac{\sin t}{1 - \cos t}$.

$$y''(x) = \frac{\mathrm{d}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\mathrm{d}x} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\sin t}{1 - \cos t}\right)}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{\frac{\cos t(1 - \cos t) - \sin^2 t}{(1 - \cos t)^2}}{a(1 - \cos t)}$$

$$= \frac{-1}{a(1 - \cos t)^2}. \square$$

Ex. 证明$f(x)=\begin{cases} e^{-\frac{1}{x^2}}, & x\neq 0 \\ 0, & x=0 \end{cases}$ 任意阶可导,并求$f^{(n)}(x)$.

Proof. $f'(0)=\lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0}\dfrac{e^{-1/x^2}}{x}=\lim\limits_{t\to\infty}\dfrac{t}{e^{t^2}}=0.$

$$f'(x)=\begin{cases} 0, & x=0, \\ 2x^{-3}e^{-x^{-2}}, & x\neq 0. \end{cases}$$

$$f''(x)=\begin{cases} 0, & x=0, \\ (-6x^{-4}+4x^{-6})e^{-x^{-2}}, & x\neq 0. \end{cases}$$

归纳可证

$$f^{(n)}(x)=\begin{cases}0, & x=0,\\ P_{3n}(\frac{1}{x})e^{-x^{-2}}, & x\neq 0.\end{cases}$$

$P_{3n}(\cdot)$为$3n$次多项式.□

作业：习题3.3

No. 3(2,3,6,10),4(3),5(3),6