Review

- $\int_a^b f(x)\mathrm{d}x = \mathrm{I}: \forall \varepsilon > 0, \exists \delta > 0$, 当$|T| < \delta$时, 无论$\xi_i \in [x_{i-1}, x_i]$ 如何取, 都有

$$\left| \sum_{i=1}^{n} f(\xi_i)\Delta x_i - \mathrm{I} \right| < \varepsilon.$$

- Darboux上和$U(f,T) = \sum_{i=1}^{n} M_i \Delta x_i$, $\quad M_i \triangleq \sup_{x\in[x_{i-1},x_i]} f(x)$,

Darboux下和$L(f,T) = \sum_{i=1}^{n} m_i \Delta x_i$, $\quad m_i \triangleq \inf_{x\in[x_{i-1},x_i]} f(x)$.

Riemann和$\sigma(f,T,\{\xi_i\}) = \sum_{i=1}^{n} f(\xi_i)\Delta x_i$.

$$m(b-a) \le L(f,T) \le \sigma(f,T,\{\xi_i\}) \le U(f,T) \le M(b-a).$$

- 在T中加入k个新分点得到T_k, 则

$$0 \le U(f,T) - U(f,T_k) \le k|T|(M-m);$$

$$0 \le L(f,T_k) - L(f,T) \le k|T|(M-m).$$

- $L(f,T_1) \le U(f,T_2)$.

- Darboux上积分 : $\overline{\int_a^b} f(x)\mathrm{d}x = \inf\{U(f,T) : T$为$[a,b]$的分割$\}$,

Darboux下积分 : $\underline{\int_a^b} f(x)\mathrm{d}x = \sup\{L(f,T) : T$为$[a,b]$的分割$\}$.

- $L(f,T) \le \underline{\int_a^b} f(x)\mathrm{d}x \le \overline{\int_a^b} f(x)\mathrm{d}x \le U(f,T)$.

- f在$[a,b]$有界,则

$$f \in R[a,b];$$

$$\Leftrightarrow \forall \varepsilon > 0, \exists [a,b]\text{的分割}T, s.t.\ U(f,T) - L(f,T) < \varepsilon;$$

$$\Leftrightarrow \underline{\int_a^b} f(x)\mathrm{d}x = \overline{\int_a^b} f(x)\mathrm{d}x.$$

- $[a,b]$上的可积函数类.

§ 2.Riemann积分的性质

Prop1. (线性性质)

$$f,g\in R[a,b],\alpha,\beta\in\mathbb{R}\Rightarrow\alpha f+\beta g\in R[a,b],\text{且}$$

$$\int_a^b\big(\alpha f(x)+\beta g(x)\big)\mathrm{d}x=\alpha\int_a^b f(x)\mathrm{d}x+\beta\int_a^b g(x)\mathrm{d}x.$$

Proof. $\lim\limits_{|T|\to0}\sigma(\alpha f+\beta g,T,\{\xi_i\})$

$$=\lim_{|T|\to0}\alpha\sigma(f,T,\{\xi_i\})+\lim_{|T|\to0}\beta\sigma(g,T,\{\xi_i\}).\square$$

Prop2. (积分区间的可加性) $a<b<c$,则

$$f\in R[a,c]\Leftrightarrow f\in R[a,b]\,\&\, f\in R[b,c].$$

此时$\int_a^c f(x)\mathrm{d}x=\int_a^b f(x)\mathrm{d}x+\int_b^c f(x)\mathrm{d}x.$

Proof. $\Leftarrow$:设$f\in R[a,b],\ f\in R[b,c].\forall\varepsilon>0,\exists[a,b]$的分割$T_1$, $[b,c]$的分割T_2, $s.t.$

$$U(f,T_1)-L(f,T_1)<\varepsilon,\quad U(f,T_2)-L(f,T_2)<\varepsilon.$$

合并T_1,T_2的分点得到$[a,c]$的分割T,则

$$U(f,T)-L(f,T)$$

$$=U(f,T_1)-L(f,T_1)+U(f,T_2)-L(f,T_2)<2\varepsilon.$$

$\Rightarrow$:设$f \in R[a,c]$.则$\forall \varepsilon > 0, \exists [a,c]$的分割$T_0, s.t.$

$$U(f,T_0) - L(f,T_0) < \varepsilon.$$

在T_0中添加分点b得到$[a,c]$的分割T,则

$$U(f,T) - L(f,T) \le U(f,T_0) - L(f,T_0) < \varepsilon.$$

T在$[a,b],[b,c]$上的限制分别记为T_1, T_2,则

$$U(f,T_i) - L(f,T_i) \le U(f,T) - L(f,T) < \varepsilon, \quad i = 1, 2.$$

故$f \in R[a,b], f \in R[b,c]$.

至此,我们证明了$f\in R[a,c]\Leftrightarrow f\in R[a,b]\ \&\ f\in R[b,c]$.

下证 $\int_a^c f(x)\mathrm{d}x=\int_a^b f(x)\mathrm{d}x+\int_b^c f(x)\mathrm{d}x$.

设T_1, T_2分别为$[a,b],[b,c]$上的分割,合并T_1, T_2的分点得到$[a,c]$上的分割T. 则 $|T|\to 0\Leftrightarrow |T_i|\to 0, i=1,2$, 且

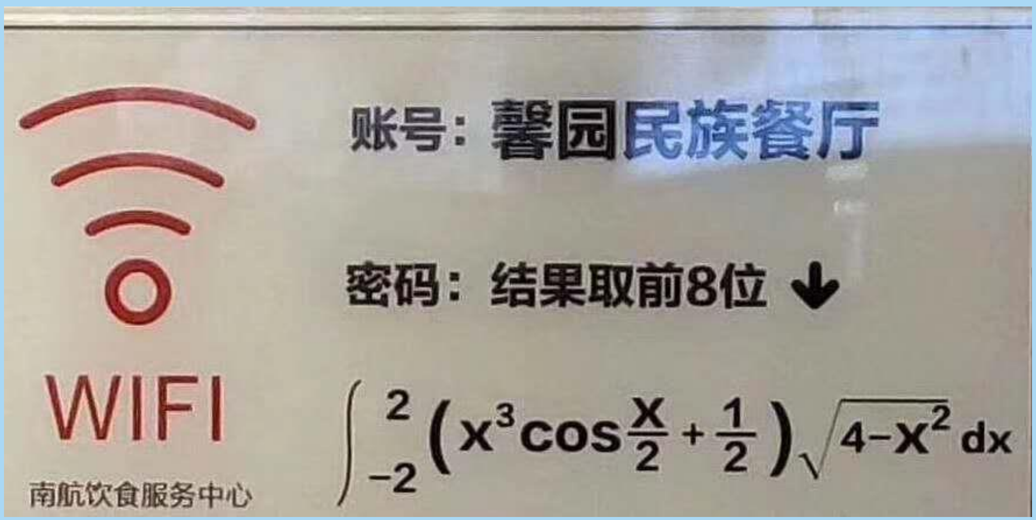
$$\int_a^b f(x)\mathrm{d}x+\int_b^c f(x)\mathrm{d}x$$

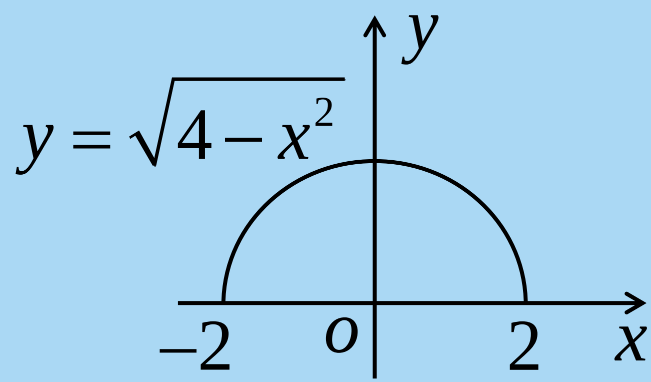
$$=\lim_{|T_1|\to 0}\sigma(f,T_1,\{\xi_i\})+\lim_{|T_2|\to 0}\sigma(f,T_2,\{\eta_i\})$$

$$=\lim_{|T|\to 0}\sigma(f,T,\{\xi_i\}\cup\{\eta_i\})=\int_a^c f(x)\mathrm{d}x.\ \square$$

Prop3. f 为 $[-a,a]$ 上的奇函数,则 $\int_{-a}^{a} f(x)\mathrm{d}x = 0$;

f 为 $[-a,a]$ 上的偶函数,则 $\int_{-a}^{a} f(x)\mathrm{d}x = 2\int_{0}^{a} f(x)\mathrm{d}x$.





$$=0+\frac{1}{2}\int_{-2}^{2}\sqrt{4-x^2}\,\mathrm{d}x \quad =\pi = 3.1415926\cdots$$

Prop4. (单调性) $f,g\in R[a,b]$, 且$f(x)\le g(x)$,则

$$\int_a^b f(x)\mathrm{d}x\le\int_a^b g(x)\mathrm{d}x.$$

特别地, 若$m\le f(x)\le M\,(a\le x\le b)$, 则

$$m(b-a)\le\int_a^b f(x)\mathrm{d}x\le M(b-a).$$

Proof. $g(x)-f(x)\ge 0(a\le x\le b)$, 则

$$\int_a^b g(x)\mathrm{d}x-\int_a^b f(x)\mathrm{d}x=\int_a^b (g(x)-f(x))\mathrm{d}x$$

$$=\lim_{|T|\to 0}\sigma(g-f,T,\{\xi_i\})\ \ge 0.\square$$

Prop5.(积分估值) $f \in R[a,b] \Rightarrow |f| \in R[a,b]$, 且

$$\left|\int_a^b f(x)\mathrm{d}x\right| \le \int_a^b |f(x)|\mathrm{d}x.$$

Proof. $U(|f|,T) - L(|f|,T) \le U(f,T) - L(f,T)$.□

Question. $|f| \in R[a,b] \overset{?}{\Rightarrow} f \in R[a,b]$

$$f = \begin{cases} 1, & x \in \mathbb{Q} \cap [0,1], \\ -1, & x \in [0,1] \setminus \mathbb{Q}. \end{cases} \quad f \notin R[0,1].$$

$$|f| = 1, \forall x \in [0,1], \quad |f| \in R[0,1].$$

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Prop6. $f,g\in R[a,b]\Rightarrow fg\in R[a,b]$.

Proof. 1)设 $f\geq 0, g\geq 0$. 记$M=\sup\limits_{a\leq x\leq b} f(x), N=\sup\limits_{a\leq x\leq b} g(x)$.

任给$T: a=x_0<x_1<\cdots<x_n=b$, 记

$$M_i=\sup_{x_{i-1}\leq x\leq x_i} f(x),\quad m_i=\inf_{x_{i-1}\leq x\leq x_i} f(x),$$

$$N_i=\sup_{x_{i-1}\leq x\leq x_i} g(x),\quad n_i=\inf_{x_{i-1}\leq x\leq x_i} g(x).$$

$$U(fg,T)-L(fg,T)\leq\sum_{i=1}^{n}(M_iN_i-m_in_i)\Delta x_i$$

$$=\sum_{i=1}^{n}M_i(N_i-n_i)\Delta x_i+\sum_{i=1}^{n}n_i(M_i-m_i)\Delta x_i$$

$$\leq M\big[U(g,T)-L(g,T)\big]+N\big[U(f,T)-L(f,T)\big].$$

2)对任意 f, 记$f^{\pm}(x)=\frac{1}{2}\left[\left|f(x)\right|\pm f(x)\right]$,则

$$f^{\pm}\geq 0,\quad f^{\pm}\in R[a,b],\quad f=f^{+}-f^{-},$$

于是 $$fg=(f^{+}-f^{-})(g^{+}-g^{-})$$

$$=f^{+}g^{+}-f^{+}g^{-}-f^{-}g^{+}+f^{-}g^{-}\in R[a,b].\square$$

Prop7. $f\in R[a,b], f\geq 0\Rightarrow\sqrt{f}\in R[a,b].$

Proof. $\omega_i(\sqrt{f})=\sqrt{M_i}-\sqrt{m_i}\leq\dfrac{\omega_i(f)}{\sqrt{M}}.\square$

Prop8. $f\in R[a,b],\left|f\right|\geq\lambda>0\Rightarrow\dfrac{1}{f}\in R[a,b].$

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Thm. 设$f\in C[a,b]$, $f(x)\geq 0$, $\int_a^b f(x)\mathrm{d}x=0$. 求证 : $f(x)\equiv 0$.

Proof. 反证. 设$f(x)$在不恒为0, 则$\exists x_0\in[a,b], s.t. f(x_0)>0$. 不妨设$x_0\in(a,b)$. $f\in C[a,b]$, 则$\exists\delta>0, s.t.\ N_\delta(x_0)\subset[a,b]$, 且

$$f(x)>f(x_0)/2>0,\quad \forall x\in[x_0-\delta,x_0+\delta].$$

而$f(x)\geq 0$, 于是

$$0=\int_a^b f(x)\mathrm{d}x=\int_a^{x_0-\delta} f(x)\mathrm{d}x+\int_{x_0-\delta}^{x_0+\delta} f(x)\mathrm{d}x+\int_{x_0+\delta}^{b} f(x)\mathrm{d}x$$

$$\geq 0+\int_{x_0-\delta}^{x_0+\delta}\frac{f(x_0)}{2}dx+0\ \geq f(x_0)\delta>0,\text{矛盾}.\square$$

Corollary. 设$f, g \in C[a,b]$, $f(x) \geq g(x)$, $f \neq g$, 则

$$\int_a^b f(x)\mathrm{d}x > \int_a^b g(x)\mathrm{d}x.$$

Thm.(Cauchy不等式) $f, g \in R[a,b]$, 则

$$\left(\int_a^b f(x)g(x)\mathrm{d}x\right)^2 \leq \int_a^b f^2(x)\mathrm{d}x \cdot \int_a^b g^2(x)\mathrm{d}x.$$

Proof. 令$A = \int_a^b f^2(x)\mathrm{d}x$, $B = \int_a^b f(x)g(x)\mathrm{d}x$, $C = \int_a^b g^2(x)\mathrm{d}x$.

则 $0 \leq \int_a^b [tf(x) + g(x)]^2 dx = At^2 + 2Bt + C, \quad \forall t \in \mathbb{R}.$

故 $(2B)^2 - 4AC \leq 0.$ □

证法二: $\left(\sigma(fg, T, \{\xi_i\})\right)^2 \leq \sigma(f^2, T, \{\xi_i\}) \cdot \sigma(g^2, T, \{\xi_i\}).$

Thm.(积分第一中值定理) $f\in C[a,b], g\in R[a,b]$, g不变号,

则$\exists\xi\in[a,b], s.t.$ $\int_a^b f(x)g(x)\mathrm{d}x = f(\xi)\int_a^b g(x)\mathrm{d}x.$ (*)

特别地, $g(x)\equiv 1$时, $\int_a^b f(x)dx = f(\xi)(b-a).$

Proof.不妨设$g\geq 0$.记f在$[a,b]$上的最大值与最小值为M, m,则

$$m\int_a^b g(x)\mathrm{d}x \leq \int_a^b f(x)g(x)\mathrm{d}x \leq M\int_a^b g(x)\mathrm{d}x.$$

若$\int_a^b g(x)\mathrm{d}x = 0$,则$\int_a^b f(x)g(x)dx = 0, \forall\xi\in[a,b]$, (*)成立.

若$\int_a^b g(x)\mathrm{d}x > 0$,由连续函数的介值定理, $\exists\xi\in[a,b], s.t.$

$$f(\xi) = \frac{\int_a^b f(x)g(x)\mathrm{d}x}{\int_a^b g(x)\mathrm{d}x} \in [m, M].\square$$

Question. g变号, 反例?

$$\text{提示: 构造} \int_a^b g(x)\mathrm{d}x = 0, \int_a^b f(x)g(x)\mathrm{d}x > 0.$$

Ex. $f \in R[a,b],\ f \geq 0, \int_a^b xf(x)\mathrm{d}x = 0$, 则

$$\int_a^b x^2 f(x)\mathrm{d}x \leq -ab\int_a^b f(x)\mathrm{d}x.$$

Proof. $(x-a)(x-b)f(x) \leq 0, \forall x \in [a,b].$

$$0 \geq \int_a^b (x-a)(x-b)f(x)\mathrm{d}x$$

$$= \int_a^b x^2 f(x)\mathrm{d}x - (a+b)\int_a^b xf(x)\mathrm{d}x + ab\int_a^b f(x)\mathrm{d}x$$

$$= \int_a^b x^2 f(x)\mathrm{d}x + ab\int_a^b f(x)\mathrm{d}x. \square$$

Ex. $x\to 0$时, $f(x)=\int_{\sin x}^{x}\ln(1+e^t)\mathrm{d}t$ 与x^p是同阶无穷小, 则

$p=$ ________, 此时 $\lim\limits_{x\to 0} f(x)/x^p=$ ________.

解: 由积分中值定理, 存在介于x与$\sin x$之间的ξ, s.t.

$$f(x)=(x-\sin x)\ln(1+e^{\xi})=\left(\frac{x^3}{3!}+o(x^3)\right)\ln(1+e^{\xi}),\ x\to 0.$$

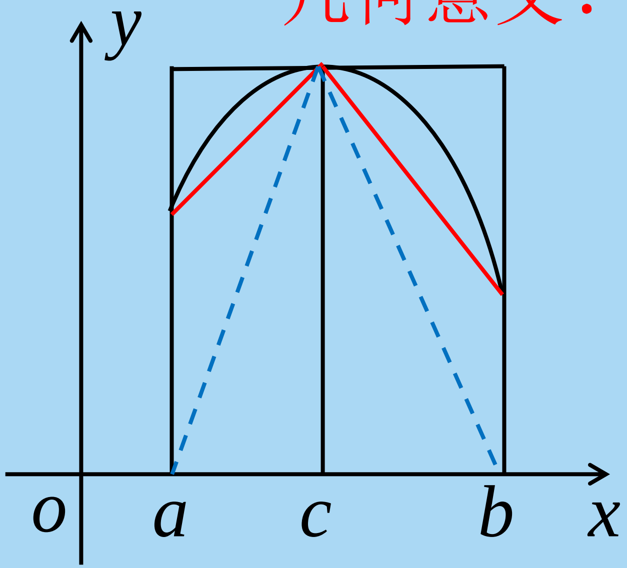
$f(x)$与x^p是同阶无穷小, 则

$$\lim_{x\to 0}\frac{f(x)}{x^p}=\ln 2\cdot\lim_{x\to 0}\frac{\frac{x^3}{3!}+o(x^3)}{x^p}\text{存在且非0,}$$

因而$p=3$, $\lim\limits_{x\to 0}\dfrac{f(x)}{x^p}=\dfrac{\ln 2}{6}$. □

Ex. $f \ge 0, f'' \le 0 \Rightarrow \max\limits_{a \le x \le b} f(x) \le \dfrac{2}{b-a}\int_a^b f(x)\mathrm{d}x.$

几何意义?



Proof. 不妨设 $f(c) = \max\limits_{a \le x \le b} f(x), c \in (a,b).$

$f'' \le 0$, 则 f 上凸, 因此, $\forall x \in [a,c]$, 有

$$f(x) \ge f(a) + \frac{f(c) - f(a)}{c-a}(x-a).$$

两边在 $[a,c]$ 上积分, 再利用 $f \ge 0$, 得

$$\int_a^c f(x)\mathrm{d}x \ge \frac{1}{2}(c-a)(f(a)+f(c)) \ge \frac{1}{2}(c-a)f(c).$$

同理, $\int_c^b f(x)\mathrm{d}x \ge \frac{1}{2}(b-c)f(c)$, 从而

$$\int_a^b f(x)\mathrm{d}x \ge \frac{1}{2}(b-a)f(c) = \frac{1}{2}(b-a)\max_{a \le x \le b} f(x). \square$$

Ex. f在$[0,1]$可导, $f(1)=4\int_0^{1/4}e^{1-x^3}f(x)\mathrm{d}x$.则$\exists\xi\in(0,1), s.t.$

$f'(\xi)=3\xi^2 f(\xi)$.

Proof. 由积分第一中值定理, $\exists\eta\in[0,1/4]\subset[0,1), s.t.$

$$f(1)=4\int_0^{1/4}e^{1-x^3}f(x)\mathrm{d}x=e^{1-\eta^3}f(\eta).$$

令$g(x)=e^{1-x^3}f(x)$, 则 $g(\eta)=g(1)=f(1)$,

$$g'(x)=e^{1-x^3}(f'(x)-3x^2 f(x)).$$

由Rolle定理, $\exists\xi\in(\eta,1)\subset(0,1), s.t. g'(\xi)=0$, 从而

$$f'(\xi)=3\xi^2 f(\xi).\square$$

Ex. $f\in C[a,b]$, 则

$$\left(\int_a^b f(x)\cos x\mathrm{d}x\right)^2+\left(\int_a^b f(x)\sin x\mathrm{d}x\right)^2\le\left(\int_a^b |f(x)|\mathrm{d}x\right)^2.$$

Proof. 由积分估值及Cauchy不等式,

$$\left(\int_a^b f(x)\cos x\mathrm{d}x\right)^2\le\left(\int_a^b \sqrt{|f(x)|}\cdot\sqrt{|f(x)|}\,|\cos x|\mathrm{d}x\right)^2$$

$$\le\int_a^b |f(x)|\mathrm{d}x\cdot\int_a^b |f(x)|\cos^2 x\mathrm{d}x.$$

$$\left(\int_a^b f(x)\sin x\mathrm{d}x\right)^2\le\int_a^b |f(x)|\mathrm{d}x\cdot\int_a^b |f(x)|\sin^2 x\mathrm{d}x.$$

两式相加即证所需结论. □

Ex. $\lim\limits_{n\to+\infty}\int_0^1 x^n\mathrm{d}x = 0.$

Proof. 令$a_n=\int_0^1 x^n\mathrm{d}x$, 则$a_n$单调下降有下界0, $\lim\limits_{n\to+\infty} a_n$存在.

$\forall\varepsilon\in(0,1)$,

$$0\le\int_0^1 x^n\mathrm{d}x=\int_0^{1-\varepsilon} x^n\mathrm{d}x+\int_{1-\varepsilon}^1 x^n\mathrm{d}x \le(1-\varepsilon)^{n+1}+\varepsilon.$$

因此, $$0\le\lim_{n\to+\infty}\int_0^1 x^n\mathrm{d}x\le\varepsilon.$$

由ε的任意性, $\lim\limits_{n\to+\infty}\int_0^1 x^n\mathrm{d}x = 0.$ □

作业：习题5.2

No.6(1),7(2),9,10