Review

- 收敛列的性质

1. 收敛列的极限唯一.
2. 改变有限项, 不改变数列的敛散性与极限值.
3. 收敛列的任意子列收敛到原极限
4. 收敛列必为有界列.
5. $\lim\limits_{n\to\infty} a_n = 0, \{b_n\}$为有界列, 则$\lim\limits_{n\to\infty} a_n b_n = 0.$
6. 极限的保序性
7. 极限的四则运算
8. 夹挤原理

- 重要极限

$$\lim_{n\to+\infty}\frac{n^b}{a^n}=0\;(a>1, b\in\mathbb{R}),$$

$$\lim_{n\to+\infty}\sqrt[n]{a}=1\;(a>0)$$

§ 4. 单调数列

Def. 称$\{a_n\}$单调递增, 若$\forall n$, 有$a_{n+1} \geq a_n$;

称$\{a_n\}$严格单调递增, 若$\forall n$, 有$a_{n+1} > a_n$;

称$\{a_n\}$单调递减, 若$\forall n$, 有$a_{n+1} \leq a_n$;

称$\{a_n\}$严格单调递减, 若$\forall n$, 有$a_{n+1} < a_n$.

Thm.(单调收敛原理) 单调有界列必收敛.

Proof. 我们来证明:

(1)若$\{a_n\}$单调递增且有上界, 则$\lim\limits_{n\to\infty} a_n = \sup\{a_n\}$;

(2)若$\{a_n\}$单调递减且有下界, 则$\lim\limits_{n\to\infty} a_n = \inf\{a_n\}$.

(1)设$\{a_n\}\uparrow$, 有上界, 由确界原理, $\xi=\sup\{a_n\}\in\mathbb{R}$.

下证$\lim\limits_{n\to\infty}a_n=\xi$. 由上确界定义, 有

$$a_n\le\xi,\qquad\forall n;$$

$$\forall\varepsilon>0,\exists a_k, s.t.\ \ \xi-\varepsilon<a_k.$$

而$\{a_n\}\uparrow$,因此

$$\xi-\varepsilon<a_k\le a_n\le\xi,\qquad\forall n>k.$$

故$\lim\limits_{n\to\infty}a_n=\xi=\sup\{a_n\}$.

(2) 同理可证,$\{a_n\}\downarrow$有下界$\Rightarrow\lim\limits_{n\to\infty}a_n=\inf\{a_n\}$.□

Remark. $\{a_n\}$有上界,从某一项后单增$\Rightarrow\{a_n\}$收敛;

$\{a_n\}$有下界,从某一项后单减$\Rightarrow\{a_n\}$收敛.

Remark. $\{a_n\}\uparrow$,无上界$\Rightarrow \lim\limits_{n\to\infty} a_n = +\infty$;

$\{a_n\}\downarrow$,无下界$\Rightarrow \lim\limits_{n\to\infty} a_n = -\infty$.

Lemma. (Bernoulli不等式) 设$x\geq -1$, n为正整数,则

$$(1+x)^n \geq 1+nx.$$

Proof. 数学归纳法,略.□

Ex. $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ 存在.

Proof. 由单调收敛原理, 只要证$a_n=\left(1+\frac{1}{n}\right)^n$ 单增有上界.

$$\frac{a_n}{a_{n-1}}=\left(\frac{n+1}{n}\right)^n\cdot\left(\frac{n-1}{n}\right)^{n-1}=\left(\frac{n^2-1}{n^2}\right)^{n-1}\cdot\frac{n+1}{n}$$

$$=\left(1-\frac{1}{n^2}\right)^{n-1}\cdot\frac{n+1}{n}\geq\left(1-\frac{n-1}{n^2}\right)\cdot\frac{n+1}{n}$$

$$=\frac{n^2-n+1}{n^2}\cdot\frac{n+1}{n}=\frac{n^3+1}{n^3}>1,\quad 故a_n\uparrow.$$

$$a_n = \left(1+\frac{1}{n}\right)^n = 1+\sum_{k=1}^{n} C_n^k \frac{1}{n^k}$$

$$= 1+\sum_{k=1}^{n} \frac{n(n-1)\cdots(n-k+1)}{k!}\frac{1}{n^k} \le 1+\sum_{k=1}^{n}\frac{1}{k!}$$

$$\le 1+1+\frac{1}{2}+\frac{1}{6}+\sum_{k=4}^{n}\frac{1}{2^{k-1}} = 2+\frac{1}{2}+\frac{1}{6}+\frac{\frac{1}{2^3}(1-\frac{1}{2^{n-3}})}{1-\frac{1}{2}}$$

$$< 2+\frac{1}{2}+\frac{1}{6}+\frac{1}{4} = 2+\frac{11}{12} < 3,$$

a_n有上界.□

Remark. (1) $\lim\limits_{n\to+\infty}\left(1+\dfrac{1}{n}\right)^n = e \in (2, 2\dfrac{11}{12}] \subset (2,3)$;

(2) $\lim\limits_{n\to+\infty} n\ln\left(1+\dfrac{1}{n}\right) = 1$;

(3) $\lim\limits_{n\to+\infty}\dfrac{\ln\left(1+\dfrac{1}{n}\right)}{\dfrac{1}{n}} = 1$;   (4) $\lim\limits_{n\to+\infty}\dfrac{\ln\left(1-\dfrac{1}{n}\right)}{-\dfrac{1}{n}} = 1.$

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Proof of (4):

$$\lim_{n\to+\infty}\left(1-\frac{1}{n}\right)^{-n}=\lim_{n\to+\infty}\left(\frac{n}{n-1}\right)^{n}=\lim_{n\to+\infty}\left(1+\frac{1}{n-1}\right)^{n}$$

$$=\lim_{n\to+\infty}\left(1+\frac{1}{n-1}\right)^{n-1}\left(\frac{n}{n-1}\right)=\lim_{n\to+\infty}\left(1+\frac{1}{n-1}\right)^{n-1}\cdot\lim_{n\to+\infty}\left(\frac{n}{n-1}\right)$$

$$=e\cdot 1\quad =e.$$

因此 $\lim\limits_{n\to+\infty}\dfrac{\ln\left(1-\frac{1}{n}\right)}{-\frac{1}{n}}=1.\square$

Remark. $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\dfrac{1}{k!}=e.$ 如何证明?

Hint. 记$a_n=\left(1+\frac{1}{n}\right)^n, b_n=\sum_{k=0}^{n}\frac{1}{k!}$. $b_n\uparrow, a_n\le b_n<3$(上例已证),

故b_n有极限, 设为b, 则$a_n\le b_n\le b$. 令$n\to+\infty$, 得 $e\le b$. 又

任意固定$2<m\in\mathbb{N}, \forall n>m$,

$$a_n=1+\sum_{k=1}^{n}C_n^k\frac{1}{n^k}=2+\sum_{k=2}^{n}\frac{n(n-1)\cdots(n-k+1)}{k!}\frac{1}{n^k}$$

$$>2+\frac{1}{2!}(1-\frac{1}{n})+\cdots+\frac{1}{m!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{m-1}{n})\triangleq c_n.$$

令$n\to+\infty$, 得 $e=\lim\limits_{n\to\infty}a_n\ge\lim\limits_{n\to\infty}c_n=2+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{m!}=b_m$.

再令$m\to+\infty$, 得 $e\ge b$.□

Ex. 设$b\in\mathbb{R}, a>1$.证明:$\lim\limits_{n\to\infty}\frac{n^b}{a^n}=0$.

Proof. 令$x_n=\frac{n^b}{a^n}$,则

$$\lim_{n\to\infty}\frac{x_{n+1}}{x_n}=\frac{1}{a}\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^b=\frac{1}{a}\lim_{n\to\infty}\exp\left\{b\ln\frac{n+1}{n}\right\}=\frac{1}{a}<1.$$

由极限的保序性,$\exists N$, s.t. $\frac{x_{n+1}}{x_n}<1, \forall n>N$. $\{x_n\}$有下界0, 从第N项后单减,故$\{x_n\}$收敛,设$\lim\limits_{n\to\infty}x_n=x$. 又 $x_{n+1}=\frac{1}{a}\left(\frac{n+1}{n}\right)^b x_n$,

两边取极限得 $x=\frac{x}{a}$. 由$a>1$得$x=0$.□

Question. 能否去掉极限存在性的证明？否!考虑$\{(-1)^n\}$.

Remark. $a_{2n}\uparrow A,\ a_{2n+1}\downarrow A\Rightarrow\lim\limits_{n\to\infty}a_n=A.$（自证）

Ex. $a_1=1,\ a_{n+1}=1+\dfrac{1}{a_n}$, 证明$\lim\limits_{n\to\infty}a_n=\dfrac{1+\sqrt{5}}{2}.$

Proof. $a_{n+1}=1+\dfrac{1}{a_n},\ a_1=1,\ a_2=2, a_3=\dfrac{3}{2},\ a_4=\dfrac{5}{3}.$

归纳可证$a_{2n}\downarrow, a_{2n+1}\uparrow$. 又$1\le a_n\le 2$,由单调收敛原理可设

$$\lim_{n\to\infty}a_{2n}=a,\ \lim_{n\to\infty}a_{2n+1}=b.$$

由极限的保序性,$1\le a\le 2, 1\le b\le 2.$

$$a_{n+2}=1+\frac{1}{a_{n+1}}=1+\frac{a_n}{1+a_n},$$

令$n=2m\to\infty$,得

$$a=1+\frac{a}{1+a},\qquad a=\frac{1+\sqrt{5}}{2},a=\frac{1-\sqrt{5}}{2}(舍).$$

同理, 令$n=2m+1\to\infty$,得 $b=a=\dfrac{1+\sqrt{5}}{2}$,即

$$\lim_{n\to\infty}a_{2n}=\lim_{n\to\infty}a_{2n+1}=\frac{1+\sqrt{5}}{2}.$$

故$\displaystyle\lim_{n\to\infty}a_n=\frac{1+\sqrt{5}}{2}.\square$

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Thm.(Stolz定理)

(1) (∞/∞型)

$$\left.\begin{array}{r} \{b_n\}\text{严格}\uparrow \\ \lim\limits_{n\to\infty} b_n = +\infty \\ \lim\limits_{n\to\infty}\dfrac{a_n - a_{n-1}}{b_n - b_{n-1}} = A \end{array}\right\} \Rightarrow \lim_{n\to\infty}\frac{a_n}{b_n} = A;$$

(2) (0/0型)

$$\left.\begin{array}{r} \{b_n\}\text{严格}\downarrow \\ \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = 0 \\ \lim\limits_{n\to\infty}\dfrac{a_n - a_{n-1}}{b_n - b_{n-1}} = A \end{array}\right\} \Rightarrow \lim_{n\to\infty}\frac{a_n}{b_n} = A.$$

Proof.(1) $\lim\limits_{n\to\infty}\dfrac{a_n-a_{n-1}}{b_n-b_{n-1}}=\mathrm{A}$,则 $\lambda_n \triangleq \dfrac{a_n-a_{n-1}}{b_n-b_{n-1}}-\mathrm{A}\to 0.$

$\forall\varepsilon>0,\exists N\in\mathbb{N},\forall n>N$,有$|\lambda_n|<\varepsilon$. 于是,当$n>N$时,

$$a_n-\mathrm{A}b_n=a_{n-1}-\mathrm{A}b_{n-1}+\lambda_n(b_n-b_{n-1})$$

$$a_{n-1}-\mathrm{A}b_{n-1}=a_{n-2}-\mathrm{A}b_{n-2}+\lambda_{n-1}(b_{n-1}-b_{n-2})$$

$$\vdots$$

$$a_{N+1}-\mathrm{A}b_{N+1}=a_N-\mathrm{A}b_N+\lambda_{N+1}(b_{N+1}-b_N)$$

各式相加,得

$$a_n-\mathrm{A}b_n=a_N-\mathrm{A}b_N+\lambda_n(b_n-b_{n-1})+\cdots+\lambda_{N+1}(b_{N+1}-b_N),$$

$b_n\uparrow$,则

$$|a_n-\mathrm{A}b_n|\le|a_N-\mathrm{A}b_N|+\varepsilon(b_n-b_N),\forall n>N.$$

$$\left|\frac{a_n}{b_n}-\mathrm{A}\right|\le\frac{\left|a_N-\mathrm{A}b_N\right|}{\left|b_n\right|}+\varepsilon\frac{\left|b_n-b_N\right|}{\left|b_n\right|},\forall n>N.$$

$b_n\uparrow+\infty$,则 $\exists N_1>N, s.t.$

$$\frac{\left|a_N-\mathrm{A}b_N\right|}{\left|b_n\right|}<\varepsilon,\quad \frac{\left|b_n-b_N\right|}{\left|b_n\right|}\le 1+\frac{\left|b_N\right|}{\left|b_n\right|}<2,\quad \forall n>N_1.$$

于是,
$$\left|\frac{a_n}{b_n}-\mathrm{A}\right|\le 3\varepsilon,\forall n>N_1.$$

由极限的定义知 $\lim\limits_{n\to\infty}\frac{a_n}{b_n}=\mathrm{A}.$

$$(2)\lim_{n\to\infty}\frac{a_n-a_{n-1}}{b_n-b_{n-1}}=\mathrm{A},\text{则}\forall\varepsilon>0,\exists N,s.t.$$

$$\mathrm{A}-\varepsilon<\frac{a_n-a_{n-1}}{b_n-b_{n-1}}<\mathrm{A}+\varepsilon,\quad\forall n>N.$$

$\{b_n\}$严格$\downarrow$,则

$$(\mathrm{A}-\varepsilon)(b_{n-1}-b_n)<a_{n-1}-a_n<(\mathrm{A}+\varepsilon)(b_{n-1}-b_n),\quad\forall n>N.$$

于是

$$(\mathrm{A}-\varepsilon)(b_{n+m-1}-b_{n+m})<a_{n+m-1}-a_{n+m}<(\mathrm{A}+\varepsilon)(b_{n+m-1}-b_{n+m}),$$
$$\forall n>N,\forall m>0.$$

上式对m从1到k求和, 得

$$(\mathrm{A}-\varepsilon)(b_n-b_{n+k})<a_n-a_{n+k}<(\mathrm{A}+\varepsilon)(b_n-b_{n+k}),$$
$$\forall n>N,\forall k>0.$$

$$(\mathrm{A}-\varepsilon)(b_n-b_{n+k})<a_n-a_{n+k}<(\mathrm{A}+\varepsilon)(b_n-b_{n+k}),$$

$$\forall n>N, \forall k>0.$$

任意固定$n>N$, 令$k\to+\infty$, 由$\lim\limits_{k\to\infty}a_k=\lim\limits_{k\to\infty}b_k=0$, 得

$$(\mathrm{A}-\varepsilon)b_n\le a_n\le(\mathrm{A}+\varepsilon)b_n,\qquad \forall n>N.$$

$b_n\downarrow 0$, 故$b_n>0$,

$$\mathrm{A}-\varepsilon\le\frac{a_n}{b_n}\le\mathrm{A}+\varepsilon,\qquad \forall n>N.$$

从而$\lim\limits_{n\to\infty}\frac{a_n}{b_n}=\mathrm{A}$.□

Remark. Stolz定理与L′Hospital法则.

Remark. Stolz定理中其它条件不变,

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\mathrm{A}\quad \overset{?}{\Rightarrow}\quad \lim_{n\to\infty}\frac{a_n-a_{n-1}}{b_n-b_{n-1}}=\mathrm{A}.$$

Hint. 考虑$\lim\limits_{n\to\infty}\dfrac{\sin n}{n}$.

Ex. $\lim_{n\to\infty} a_n = A$.证明: $\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = A$.

Proof. 令$x_n = a_1 + a_2 + \cdots + a_n, y_n = n$,则

$$y_n\text{严格}\uparrow +\infty, \quad \lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} = \lim_{n\to\infty} a_n = A.$$

由Stolz定理,

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = \lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} = A. \square$$

Ex. $x_n = \dfrac{1}{\ln n}\left(1+\dfrac{1}{2}+\cdots+\dfrac{1}{n}\right)$, 求 $\lim\limits_{n\to\infty} x_n$.

解：令$a_n = 1+\dfrac{1}{2}+\cdots+\dfrac{1}{n}, b_n = \ln n$, 则 b_n严格$\uparrow +\infty$,

$$\lim_{n\to\infty}\frac{a_n - a_{n-1}}{b_n - b_{n-1}} = \lim_{n\to\infty}\frac{1/n}{\ln n - \ln(n-1)} = \lim_{n\to\infty}\frac{-1/n}{\ln(1-1/n)} = 1.$$

由Stolz定理,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{a_n - a_{n-1}}{b_n - b_{n-1}} = 1. \square$$

Remark. $\left\{1+\dfrac{1}{2}+\cdots+\dfrac{1}{n}-\ln n\right\}$收敛, 其极限称为Euler常数.

Ex. k为正整数, $x_n = \dfrac{1^k + 2^k + \cdots n^k}{n^{k+1}}$, 求 $\lim\limits_{n\to\infty} x_n$.

解: 令 $a_n = 1^k + 2^k + \cdots n^k, b_n = n^{k+1}$, 则 $b_n \uparrow +\infty$.

由Stolz定理,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = \lim_{n\to\infty} \frac{n^k}{n^{k+1} - (n-1)^{k+1}}$$

$$= \lim_{n\to\infty} \frac{n^k}{n^k + n^{k-1}(n-1) + n^{k-2}(n-1)^2 + \cdots + (n-1)^k}$$

$$= \lim_{n\to\infty} \frac{1}{1 + \left(1 - \frac{1}{n}\right) + \left(1 - \frac{1}{n}\right)^2 + \cdots + \left(1 - \frac{1}{n}\right)^k} = \frac{1}{k+1}. \square$$

作业：习题1.4

No. 3,4(2),5(2),12(1)(4),16