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$\mathbb{R}^m$中向量内积 $\vec{a}^T\vec{b}$

- 向量长度：$\|\vec{a}\| = \sqrt{\vec{a}^T\vec{a}}$；单位化：$\frac{\vec{a}}{\|\vec{a}\|}$.
- 正交：$\vec{a}^T\vec{b}=0$
  - 正交(单位)向量 → 标准正交基
  - 列正交矩阵：$Q^TQ = I_m$ or I_r；一系列等价结论
  - 子空间M与其正交补$M^\perp$.
- QR分解
  - $A = QR = [Q_1\ Q_2]\begin{bmatrix} R_1 \\ O_{(m-n)\times n}\end{bmatrix} = Q_1R_1 \in \mathbb{R}^{m\times n}$：$rank(A)=n$, or $<n$.
  - Gram-Schmidt 正交化：$\vec{a}_j = \sum_{i=1}^{j} r_{ij}\vec{q}_i \Rightarrow \begin{cases} r_{ij} = \vec{q}_i^{\,T}\vec{a}_j & \text{for } 1\le i\le j-1 \\ r_{jj}\vec{q}_j = \vec{a}_j - \sum_{i=1}^{j-1} r_{ij}\vec{q}_i \end{cases} \Rightarrow r_{jj}, \vec{q}_j$.
  - $span\{a_j\}_{j=1}^{k} \subseteq span\{\vec{q}_j\}_{j=1}^{k}$ for $1\le k\le n$.
  - (标准正交)基的扩充：使用Q_1^T的阶梯形及其自由列
- 正交投影
  - $\begin{cases} M = span\{\vec{q}_j\}_{j=1}^{r} \\ M^\perp = span\{\vec{q}_j\}_{j=r+1}^{m} \end{cases} \rightarrow$
    - $\forall \vec{a} = \vec{a}_1 + \vec{a}_2 = \sum_{j=1}^{r}(\vec{q}_j^{\,T}\vec{a})\vec{a} + \sum_{j=r+1}^{m}(\vec{q}_j^{\,T}\vec{a})\vec{a} = \cancel{P_M} Q_1Q_1^T\vec{a} + Q_2Q_2^T\vec{a} = P_M\vec{a} + P_{M^\perp}\vec{a}$.
    - $\|\vec{a} - P_M\vec{a}\| = \min_{\vec{y}\in M}\|\vec{a} - \vec{y}\|$
    - P是个正交投影变换/矩阵 $\Leftrightarrow \forall \vec{a}\in\mathbb{R}^n$，有$\begin{cases} P\vec{a} \in \mathcal{R}(P) \\ I - P\vec{a} \in \mathcal{R}(P)^\perp \end{cases} \Leftrightarrow P^2 = P^T = P$.
    - $\forall B$ with $\mathcal{R}(B) = \mathcal{R}(P)$.
  - $\mathcal{N}(A) = \mathcal{R}(A^T)^\perp$，$\mathcal{N}(A^TA) = \mathcal{N}(A)$，其它都是推论.
- 最小二乘法
  - 问题转化：$\vec{x}^* = \arg\min_{\vec{x}\in\mathbb{R}^n}\|\vec{b} - A\vec{x}\| \Leftrightarrow \begin{cases} \tilde{b}^* = \arg\min_{\tilde{b}\in\mathcal{R}(A)}\|\vec{b} - \tilde{b}\| \\ A\vec{x}^* = \tilde{b}^* \in \mathcal{R}(A) \end{cases} \Leftrightarrow \begin{cases} \tilde{b}^* = P_A\vec{b} & \text{唯一} \\ A\vec{x}^* = \tilde{b}^* & \text{3但可能多个解 } \vec{x}^* \end{cases}$
  - 正则化方程：$A^TA\vec{x}^* = A^T\vec{b} \xrightarrow{rank(A)=n} \vec{x}^* = (A^TA)^{-1}A^T\vec{b} \rightarrow \tilde{b}^* = A\vec{x}^* = \underline{A(A^TA)^{-1}A^T}\vec{b} = P_A\vec{b}$
    - A的广义逆A^+；非列满秩A的广义逆=?
  - QR分解 $(rank(A)=n)$ $\begin{cases} A = Q_1R_1，有\ \mathcal{R}(A) = \mathcal{R}(Q_1) \\ \tilde{b}^* = P_A\vec{b} = Q_1Q_1^T\vec{b}. \\ A\vec{x}^* = \tilde{b}^* \rightarrow Q_1R_1\vec{x}^* = Q_1Q_1^T\vec{b} \xrightarrow{左乘Q_1^T} R_1\vec{x}^* = Q_1^T\vec{b}. \end{cases}$
  - $rank(A) < n$：多个解$\vec{x}^*$，rank deficiency.