



Review

积分的应用--微元法

- $\rho = \rho(\theta), \theta \in [\alpha, \beta]$, 向径扫过的面积 $S = \int_{\alpha}^{\beta} \frac{1}{2} \rho^2(\theta) d\theta$.
- 曲线的弧长

$$L: x = x(t), y = y(t), z = z(t), \quad t \in [\alpha, \beta].$$

$$l = \int_{\alpha}^{\beta} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt.$$



- 旋转体的体积:

$y = f(x), a \leq x \leq b$, 绕 x 轴旋转

$$V(\Omega) = \pi \int_a^b f^2(x) dx.$$

- 旋转面的面积

$y = f(x), a \leq x \leq b$, 绕 x 轴旋转

$$S = 2\pi \int_a^b |y(x)| \sqrt{1 + (f'(x))^2} dx.$$

$x = x(t), y = y(t), a \leq t \leq b$, 绕 x 轴旋转

$$S = 2\pi \int_a^b |y(t)| \sqrt{(x'(t))^2 + (y'(t))^2} dt.$$

- 积分在物理中的应用 (功, 质量, 质心, 引力)



§ 1. 广义Riemann积分的概念

• 无穷限积分

Def. 若 $\lim_{A \rightarrow +\infty} \int_a^A f(x)dx = I$, 则称 f 在 $[a, +\infty)$ 上的广义积分收敛, 称 I 为 f 在 $[a, +\infty)$ 上的广义积分(值), 记作

$$\int_a^{+\infty} f(x)dx = \lim_{A \rightarrow +\infty} \int_a^A f(x)dx.$$

若 $\lim_{A \rightarrow +\infty} \int_a^A f(x)dx$ 不存在, 则称广义积分 $\int_a^{+\infty} f(x)dx$ 发散.

Remark. $\int_{-\infty}^a f(x)dx \triangleq \lim_{A \rightarrow -\infty} \int_A^a f(x)dx.$



Ex. $\int_0^{+\infty} \frac{1}{1+x^2} dx = \lim_{A \rightarrow +\infty} \int_0^A \frac{1}{1+x^2} dx = \lim_{A \rightarrow +\infty} \arctan x \Big|_0^A = \frac{\pi}{2}.$

$$\left(\int_0^{+\infty} \frac{1}{1+x^2} dx = \arctan x \Big|_0^{+\infty} = \frac{\pi}{2}. \right)$$

Ex. $\int_1^{+\infty} \frac{\ln x}{x^2} dx = \lim_{A \rightarrow +\infty} \int_1^A \frac{\ln x}{x^2} dx = - \lim_{A \rightarrow +\infty} \int_1^A \ln x d \frac{1}{x}$

$$= - \lim_{A \rightarrow +\infty} \left(\frac{\ln x}{x} \Big|_1^A - \int_1^A \frac{1}{x^2} dx \right) = - \lim_{A \rightarrow +\infty} \left(\frac{\ln x}{x} + \frac{1}{x} \right) \Big|_1^A = 1.$$

$$\left(\int_1^{+\infty} \frac{\ln x}{x^2} dx = - \int_1^{+\infty} \ln x d \frac{1}{x} = - \frac{\ln x}{x} \Big|_1^{+\infty} + \int_1^{+\infty} \frac{1}{x^2} dx = 0 - \frac{1}{x} \Big|_1^{+\infty} = 1. \right)$$



Ex. 讨论广义积分 $\int_1^{+\infty} \frac{1}{x^p} dx$ 的敛散性.

解: $p = 1$ 时, $\int_1^{+\infty} \frac{1}{x^p} dx = \int_1^{+\infty} \frac{1}{x} dx = \ln x \Big|_1^{+\infty} = +\infty$, 发散.

$$p \neq 1 \text{ 时, } \int_1^{+\infty} \frac{1}{x^p} dx = \frac{1}{1-p} x^{1-p} \Big|_1^{+\infty} = \begin{cases} \frac{1}{p-1}, & p > 1, \\ +\infty, & p < 1. \end{cases}$$

综上, $p > 1$ 时, $\int_1^{+\infty} \frac{1}{x^p} dx = \frac{1}{p-1}$; $p \leq 1$ 时, $\int_1^{+\infty} \frac{1}{x^p} dx$ 发散. \square



Ex. 讨论广义积分 $\int_e^{+\infty} \frac{1}{x(\ln x)^p} dx$ 的敛散性.

解: $p = 1$ 时, $\int_e^{+\infty} \frac{1}{x(\ln x)^p} dx = \int_e^{+\infty} \frac{1}{\ln x} d \ln x = \ln \ln x \Big|_e^{+\infty} = +\infty.$

$$\begin{aligned} p \neq 1 \text{ 时, } \int_e^{+\infty} \frac{1}{x(\ln x)^p} dx &= \int_e^{+\infty} \frac{1}{(\ln x)^p} d \ln x \\ &= \frac{1}{1-p} (\ln x)^{1-p} \Big|_e^{+\infty} = \begin{cases} \frac{1}{p-1}, & p > 1, \\ +\infty, & p < 1. \end{cases} \end{aligned}$$

综上, $\int_1^{+\infty} \frac{1}{x(\ln x)^p} dx$ 当 $p > 1$ 时收敛, 当 $p \leq 1$ 时发散. \square



Remark. 若 $\int_a^{+\infty} f(x)dx$ 与 $\int_{-\infty}^a f(x)dx$ 均收敛, 则 $\forall b \in \mathbb{R}$,

$$\int_b^{+\infty} f(x)dx = \int_b^a f(x)dx + \int_a^{+\infty} f(x)dx \text{ 收敛,}$$

$$\int_{-\infty}^b f(x)dx = \int_{-\infty}^a f(x)dx - \int_b^a f(x)dx \text{ 收敛.}$$

Def. 若 $\exists a \in \mathbb{R}$, s.t. $\int_a^{+\infty} f(x)dx$ 与 $\int_{-\infty}^a f(x)dx$ 均收敛, 则称广义

积分 $\int_{-\infty}^{+\infty} f(x)dx$ 收敛, 且

$$\int_{-\infty}^{+\infty} f(x)dx \triangleq \int_{-\infty}^a f(x)dx + \int_a^{+\infty} f(x)dx = \lim_{\substack{A \rightarrow -\infty \\ B \rightarrow +\infty}} \int_A^B f(x)dx.$$



Ex.

$$\int_{-\infty}^{+\infty} \frac{dx}{e^x + e^{2-x}} = \int_{-\infty}^{+\infty} \frac{dx}{e^{2-x}(e^{2x-2} + 1)}$$

$$= \frac{1}{e} \int_{-\infty}^{+\infty} \frac{e^{x-1} dx}{(e^{x-1})^2 + 1} = \frac{1}{e} \arctan e^{x-1} \Big|_{-\infty}^{+\infty} = \frac{\pi}{2e}.$$

Question. 变上(下)限的广义积分如何求导?

$$\left(\int_{-\infty}^x f(t) dt \right)' = \left(\int_{-\infty}^a f(t) dt + \int_a^x f(t) dt \right)' = f(x)$$

$$\left(\int_{\alpha(x)}^{+\infty} f(t) dt \right)' = \left(\int_0^{+\infty} f(t) dt - \int_0^{\alpha(x)} f(t) dt \right)' = -f(\alpha(x)) \cdot \alpha'(x)$$



Ex. $F(x) = e^{x^2/2} \int_x^{+\infty} e^{-t^2/2} dt, x \in [0, +\infty).$

证明: (1) $\lim_{x \rightarrow +\infty} F(x) = 0$, (2) $F(x)$ 在 $[0, +\infty)$ 上单减.

Proof. (1) $x > 1$ 时, $0 < \int_x^{+\infty} e^{-t^2/2} dt \leq \int_x^{+\infty} t e^{-t^2/2} dt$
 $= -e^{-t^2/2} \Big|_x^{+\infty} = e^{-x^2/2} \quad (\rightarrow 0, x \rightarrow +\infty \text{ 时. })$

$$\lim_{x \rightarrow +\infty} F(x) = \lim_{x \rightarrow +\infty} \frac{\int_x^{+\infty} e^{-t^2/2} dt}{e^{-x^2/2}} = \lim_{x \rightarrow +\infty} \frac{-e^{-x^2/2}}{-x e^{-x^2/2}} = \lim_{x \rightarrow +\infty} \frac{1}{x} = 0.$$

$$(2) F'(x) = x e^{x^2/2} \int_x^{+\infty} e^{-t^2/2} dt - 1 \leq e^{x^2/2} \int_x^{+\infty} t e^{-t^2/2} dt - 1 = 0.$$

故 $F(x)$ 在 $[0, +\infty)$ 上单减. \square



•瑕积分(无界函数积分)

Def. f 在 $[a, b)$ 上定义, 在 b 点附近无界(此时称 $x = b$ 为 f 的一个瑕点), 若 $\forall \delta \in (0, b - a), f \in R[a, b - \delta]$, 且

$$\lim_{\delta \rightarrow 0^+} \int_a^{b-\delta} f(x) dx = I,$$

则称 f 在 $[a, b)$ 上的瑕积分收敛, 称 I 为 f 在 $[a, b)$ 上的瑕积分(值), 记作

$$\int_a^b f(x) dx = \lim_{\delta \rightarrow 0^+} \int_a^{b-\delta} f(x) dx.$$

若 $\lim_{\delta \rightarrow 0^+} \int_a^{b-\delta} f(x) dx$ 不存在, 则称瑕积分 $\int_a^b f(x) dx$ 发散.



Def. f 在 (a, b) 上定义, a, b 为瑕点, 若 $\exists c \in (a, b)$, s.t. 瑕积分

$\int_a^c f(x)dx$ 与 $\int_c^b f(x)dx$ 均收敛, 则

$$\int_a^b f(x)dx \triangleq \int_a^c f(x)dx + \int_c^b f(x)dx.$$

此时, $\int_a^b f(x)dx = \int_a^d f(x)dx + \int_d^b f(x)dx, \forall d \in (a, b);$

$$\begin{aligned} \int_a^b f(x)dx &= \lim_{\alpha \rightarrow a^+} \int_{\alpha}^c f(x)dx + \lim_{\beta \rightarrow b^-} \int_c^{\beta} f(x)dx \\ &= \lim_{\substack{\alpha \rightarrow a^+, \\ \beta \rightarrow b^-}} \int_{\alpha}^{\beta} f(x)dx. \end{aligned}$$



$$\text{Ex. } \int_0^1 \ln x dx = \lim_{\delta \rightarrow 0^+} \int_{\delta}^1 \ln x dx = \lim_{\delta \rightarrow 0^+} (x \ln x - x) \Big|_{\delta}^1 = -1.$$

$$\left(\int_0^1 \ln x dx = x \ln x \Big|_0^1 - \int_0^1 x \cdot \frac{1}{x} dx = -1. \right)$$

$$\text{Ex. } \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{\substack{\alpha \rightarrow (-1)^+ \\ \beta \rightarrow 1^-}} \int_{\alpha}^{\beta} \frac{dx}{\sqrt{1-x^2}} = \lim_{\substack{\alpha \rightarrow (-1)^+ \\ \beta \rightarrow 1^-}} \arcsin x \Big|_{\alpha}^{\beta} = \pi.$$

$$\left(\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = \arcsin x \Big|_{-1}^1 = \pi. \right)$$



Ex. 讨论广义积分 $\int_0^1 \frac{1}{x^p} dx$ 的敛散性.

解: $p = 1$ 时, $\int_0^1 \frac{1}{x^p} dx = \int_0^1 \frac{1}{x} dx = \ln x \Big|_0^1 = +\infty.$

$$p \neq 1 \text{ 时, } \int_0^1 \frac{1}{x^p} dx = \frac{1}{1-p} x^{1-p} \Big|_0^1 = \begin{cases} \frac{1}{1-p}, & p < 1, \\ +\infty, & p > 1. \end{cases}$$

综上, $p < 1$ 时, $\int_0^1 \frac{1}{x^p} dx = \frac{1}{p-1}$; $p \geq 1$ 时, $\int_0^1 \frac{1}{x^p} dx$ 发散. \square

Remark. $f \in R[a, b]$, 则 $\int_a^b f(x) dx = \lim_{c \rightarrow b^-} \int_a^c f(x) dx = \lim_{c \rightarrow a^+} \int_c^b f(x) dx.$

所以, $p \leq 0$ 时, 尽管 $\int_0^1 \frac{1}{x^p} dx$ 不是瑕积分, 我们也称其收敛.



Ex.

$$\int_0^{+\infty} \frac{1}{\sqrt{x}(1+x)} dx$$

$$= \int_0^1 \frac{1}{\sqrt{x}(1+x)} dx + \int_1^{+\infty} \frac{1}{\sqrt{x}(1+x)} dx \quad \text{瑕积分} + \text{无穷限积分}$$

$$= \int_0^1 \frac{2}{1+(\sqrt{x})^2} d\sqrt{x} + \int_1^{+\infty} \frac{2}{1+(\sqrt{x})^2} d\sqrt{x}$$

$$= 2 \arctan \sqrt{x} \Big|_0^1 + 2 \arctan \sqrt{x} \Big|_1^{+\infty} = \pi.$$

$$\left(\int_0^{+\infty} \frac{1}{\sqrt{x}(1+x)} dx = \int_0^{+\infty} \frac{2}{1+(\sqrt{x})^2} d\sqrt{x} = 2 \arctan \sqrt{x} \Big|_0^{+\infty} = \pi. \right)$$



Ex. 计算 $\int_0^1 x^n \ln^n x dx$, n 为正整数.

解: $\int_0^1 x^n \ln^n x dx = \frac{1}{n+1} \int_0^1 \ln^n x dx^{n+1}$

$$= \frac{1}{n+1} x^{n+1} \ln^n x \Big|_0^1 - \frac{n}{n+1} \int_0^1 x^n \ln^{n-1} x dx = -\frac{n}{n+1} \int_0^1 x^n \ln^{n-1} x dx$$

$$= -\frac{n}{(n+1)^2} \int_0^1 \ln^{n-1} x dx^{n+1} = \frac{n(n-1)}{(n+1)^2} \int_0^1 x^n \ln^{n-2} x dx$$

$$= \cdots = (-1)^n \frac{n!}{(n+1)^n} \int_0^1 x^n dx = (-1)^n \frac{n!}{(n+1)^{n+1}}. \square$$



Ex. $I = \int_0^{\pi/2} \ln(\cos x) dx.$

解: $I = \int_0^{\pi/2} \ln(\sin x) dx$

$$= \int_0^{\pi/2} \ln 2 dx + \int_0^{\pi/2} \ln \sin \frac{x}{2} dx + \int_0^{\pi/2} \ln \cos \frac{x}{2} dx$$

$$= \frac{\pi}{2} \ln 2 + 2 \int_0^{\pi/4} \ln \sin t dt + 2 \int_0^{\pi/4} \ln \cos t dt$$

$$= \frac{\pi}{2} \ln 2 + 2 \int_0^{\pi/4} \ln \sin t dt + 2 \int_{\pi/4}^{\pi/2} \ln \sin t dt$$

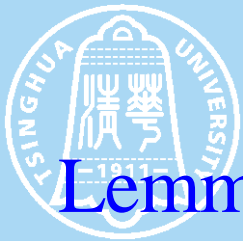
$$= \frac{\pi}{2} \ln 2 + 2I, \quad I = -\frac{\pi}{2} \ln 2. \square$$



Ex. $I = \int_0^{\pi/2} \sin x \cdot \ln \sin x dx$

分析: $I = -\int_0^{\pi/2} \ln \sin x d \cos x$
 $= -\cos x \ln \sin x \Big|_0^{\pi/2} + \int_0^{\pi/2} \frac{\cos^2 x}{\sin x} dx$, 无法计算

解: $I = \int_0^{\pi/2} \ln \sin x d(1 - \cos x)$
 $= (1 - \cos x) \ln \sin x \Big|_0^{\pi/2} - \int_0^{\pi/2} \frac{1 - \cos x}{\sin x} \cos x dx$
 $= 0 - \int_0^{\pi/2} \frac{\sin^2 x}{\sin x(1 + \cos x)} \cos x dx = \int_0^{\pi/2} \frac{\cos x}{1 + \cos x} d \cos x$
 $= \int_1^0 \frac{t}{1+t} dt = \int_0^1 \left(\frac{1}{1+t} - 1 \right) dt = -1 + \ln(1+t) \Big|_0^1 = -1 + \ln 2. \square$



Lemma (Riemann-Lebesgue). f 在 $[a, b]$ 上可积或广义绝对可积(即 f 与 $|f|$ 均在 $[a, b]$ 上广义可积), 则

$$\lim_{\lambda \rightarrow \infty} \int_a^b f(x) \cos \lambda x dx = 0, \quad \lim_{\lambda \rightarrow \infty} \int_a^b f(x) \sin \lambda x dx = 0.$$

Proof. 只证第一式, 第二式同理.

Case 1. 设 f 在 $[a, b]$ 上可积, 则 f 在 $[a, b]$ 上有界, 即

$$\exists M > 0, \text{ s.t. } |f(x)| \leq M, \forall x \in [a, b].$$

任意给定 $\lambda > 1$, 令 $n = \lfloor \sqrt{\lambda} \rfloor$. n 等分 $[a, b]$:

$$x_i = a + (b - a)i/n, \quad i = 0, 1, 2, \dots, n.$$

$$\omega_i(f) = \sup \{ f(\xi) - f(\eta) : \xi, \eta \in [x_{i-1}, x_i] \}, \quad i = 1, 2, \dots, n.$$



f 在 $[a, b]$ 上可积, 则 $\lim_{n \rightarrow \infty} \sum_{i=1}^n \omega_i(f) \Delta x_i = 0$. 于是

$$\begin{aligned} \left| \int_a^b f(x) \cos \lambda x dx \right| &= \left| \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f(x) \cos \lambda x dx \right| \\ &\leq \left| \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (f(x) - f(x_i)) \cos \lambda x dx \right| + \left| \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f(x_i) \cos \lambda x dx \right| \\ &\leq \sum_{i=1}^n \omega_i(f) \Delta x_i + \sum_{i=1}^n |f(x_i)| \left| \int_{x_{i-1}}^{x_i} \cos \lambda x dx \right| \\ &\leq \sum_{i=1}^n \omega_i(f) \Delta x_i + \frac{2Mn}{\lambda} = \sum_{i=1}^{\lfloor \sqrt{\lambda} \rfloor} \omega_i(f) \Delta x_i + \frac{2M \lfloor \sqrt{\lambda} \rfloor}{\lambda} \\ &\rightarrow 0, \text{ 当 } \lambda \rightarrow +\infty \text{ 时.} \end{aligned}$$



Case 2. f 在 $[a, b]$ 上广义绝对可积, 不妨设 a 为唯一的瑕点.

则 $\forall \varepsilon > 0, \exists \delta > 0$, s.t., f 在 $[a + \delta, b]$ 上可积, 且

$$\int_a^{a+\delta} |f(x)| dx < \varepsilon/2,$$

从而
$$\left| \int_a^{a+\delta} f(x) \cos \lambda x dx \right| \leq \int_a^{a+\delta} |f(x)| dx < \varepsilon/2,$$

$$\lim_{\lambda \rightarrow +\infty} \int_{a+\delta}^b f(x) \cos \lambda x dx = 0.$$

于是 $\exists \Lambda > 0$, 当 $\lambda > \Lambda$ 时, $\left| \int_{a+\delta}^b f(x) \cos \lambda x dx \right| < \varepsilon/2$, 进而有

$$\begin{aligned} \left| \int_a^b f(x) \cos \lambda x dx \right| &\leq \left| \int_a^{a+\delta} f(x) \cos \lambda x dx \right| + \left| \int_{a+\delta}^b f(x) \cos \lambda x dx \right| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon, \quad \forall \lambda > \Lambda. \square \end{aligned}$$



作业：习题6.1

No.2(2,8),3(3,5),4(4,6)