Review

- Thm.(微积分基本定理)

$f\in R[a,b], F(x)=\int_a^x f(t)\mathrm{d}t\ (a\le x\le b)$,则

(1)$F\in C[a,b]$;

(2)若f在$x_0\in[a,b]$连续,则F在x_0可导,且$F'(x_0)=f(x_0)$;

(3)若$f\in C[a,b]$,则F是f在$[a,b]$上的一个原函数. 若G为f的任一个原函数,则

$$\int_a^b f(t)\mathrm{d}t=G(b)-G(a)\triangleq G(x)\Big|_a^b.$$ (Newton-Leibniz)

- $F'(x)=f(x)\Leftrightarrow\int f(x)\mathrm{d}x=F(x)+C$

§ 4.不定积分换元法与分部积分

1.换元积分法 $$\frac{\mathrm{d}}{\mathrm{d}x} f(\varphi(x)) = f'(\varphi(x)) \cdot \varphi'(x)$$

$$\Rightarrow \int f'(\varphi(x))\varphi'(x)\mathrm{d}x = f(\varphi(x)) + C.$$

与 $\int f'(u)\mathrm{d}u = f(u) + C$ 比较, 得

$$\int f'(\varphi(x))\varphi'(x)\mathrm{d}x = \int f'(\varphi(x))\mathrm{d}\varphi(x) = f(\varphi(x)) + C$$

$$\overset{u=\varphi(x)}{=\!=\!=\!=} \int f'(u)\mathrm{d}u = f(u) + C = f(\varphi(x)) + C$$

——第一换元法(凑微分法)

$$\int f'(u)\mathrm{d}u \underset{\text{可逆}}{\overset{u=\varphi(x)}{=\!=\!=\!=}} \int f'(\varphi(x))\varphi'(x)\mathrm{d}x$$ ——第二换元法

$$= g(x) + C = g(\varphi^{-1}(u)) + C$$

Ex. $\int 2xe^{x^2}\mathrm{d}x = \int e^{x^2}\mathrm{d}x^2 = e^{x^2} + C$

Ex. $\int \cot x\mathrm{d}x = \int \frac{\cos x}{\sin x}\mathrm{d}x = \int \frac{1}{\sin x}\mathrm{d}\sin x = \ln|\sin x| + C.$

Ex. $\int \frac{\mathrm{d}x}{\sqrt{a^2 - x^2}} = \int \frac{\mathrm{d}(x/a)}{\sqrt{1-(x/a)^2}} = \arcsin\frac{x}{a} + C, \ (a > 0).$

Ex. $\int \frac{\mathrm{d}x}{x(1+x^5)} = \int \frac{x^4\mathrm{d}x}{x^5(1+x^5)} = \frac{1}{5}\int \frac{\mathrm{d}x^5}{x^5(1+x^5)}$

$$= \frac{1}{5}\int\left(\frac{1}{x^5} - \frac{1}{1+x^5}\right)\mathrm{d}x^5 = \ln|x| - \frac{1}{5}\ln\left|1+x^5\right| + C.$$

$$\text{Ex.}\int \sec x\mathrm{d}x = \int \frac{1}{\cos x}\mathrm{d}x = \int \frac{\cos x}{\cos^2 x}\mathrm{d}x = \int \frac{\mathrm{d}\sin x}{1-\sin^2 x}$$

$$= \frac{1}{2}\int\left(\frac{1}{1+\sin x}+\frac{1}{1-\sin x}\right)\mathrm{d}\sin x$$

$$= \frac{1}{2}\ln(1+\sin x)-\frac{1}{2}\ln(1-\sin x)+C \qquad \surd$$

$$= \frac{1}{2}\ln\frac{1+\sin x}{1-\sin x}+C \qquad \surd$$

$$= \ln\left|\sec x+\tan x\right|+C \qquad \surd$$

Ex. $\int\sqrt{a^2-x^2}\,\mathrm{d}x\,(a>0)$

解：令 $x=a\sin t,\ |t|\le\frac{\pi}{2}$，则 $\mathrm{d}x=a\cos t\,\mathrm{d}t,\ \sqrt{a^2-x^2}=a\cos t$,

$$\int\sqrt{a^2-x^2}\,\mathrm{d}x=\int a^2\cos^2 t\,\mathrm{d}t=\frac{1}{2}a^2\int(1+\cos 2t)\,\mathrm{d}t$$

$$=\frac{1}{2}a^2\left(t+\frac{1}{2}\sin 2t\right)+C$$

$$t=\arcsin\frac{x}{a},\qquad \frac{1}{2}\sin 2t=\sin t\sqrt{1-\sin^2 t}=\frac{x}{a^2}\sqrt{a^2-x^2}$$

$$\int\sqrt{a^2-x^2}\,\mathrm{d}x=\frac{a^2}{2}\arcsin\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+C.\square$$

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Ex. $\displaystyle\int \frac{\mathrm{d}x}{1+\sqrt[3]{1+x}}$

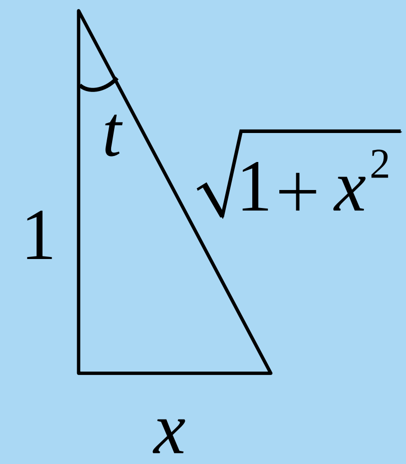
解: 令 $t=\sqrt[3]{1+x}$, 则 $x=t^3-1, \mathrm{d}x=3t^2\mathrm{d}t$,

$$\int \frac{\mathrm{d}x}{1+\sqrt[3]{1+x}}=3\int \frac{t^2\mathrm{d}t}{1+t}=3\int\left(t-1+\frac{1}{1+t}\right)\mathrm{d}t$$

$$=3\left(\frac{1}{2}t^2-t+\ln|1+t|\right)+C$$

$$=3\left(\frac{1}{2}\sqrt[3]{(1+x)^2}-\sqrt[3]{1+x}+\ln\left|1+\sqrt[3]{1+x}\right|\right)+C.$$

Ex. $\displaystyle\int \frac{\mathrm{d}x}{x^2\sqrt{1+x^2}}$



解: 令 $x=\tan t, |t|<\dfrac{\pi}{2}$,

$$\int \frac{\mathrm{d}x}{x^2\sqrt{1+x^2}}=\int \frac{\sec^2 t\mathrm{d}t}{\tan^2 t\sec t}$$

$$=\int \frac{\cos t\mathrm{d}t}{\sin^2 t}=\int \frac{\mathrm{d}\sin t}{\sin^2 t}=-\frac{1}{\sin t}+C=-\frac{\sqrt{1+x^2}}{x}+C.$$

Ex. $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x^2\pm a^2}}=\ln\left|x+\sqrt{x^2\pm a^2}\right|+C.$ 如何积分?

2.分部积分法

$$\left(u(x)v(x)\right)' = u'(x)v(x) + u(x)v'(x)$$

$$\Rightarrow \int u'(x)v(x)\mathrm{d}x + \int u(x)v'(x)\mathrm{d}x = \int \left(u(x)v(x)\right)' \mathrm{d}x$$

$$\Rightarrow \int u(x)v'(x)\mathrm{d}x = u(x)v(x) - \int u'(x)v(x)\mathrm{d}x$$

也记作 $\int u(x)\mathrm{d}v(x) = u(x)v(x) - \int v(x)\mathrm{d}u(x).$

——分部积分法

Ex. $\int x\cos x\mathrm{d}x = \int x\mathrm{d}\sin x$

$$= x\sin x - \int \sin x\mathrm{d}x = x\sin x + \cos x + C$$

Ex. $\int \ln x\mathrm{d}x = x\ln x - \int x\mathrm{d}\ln x$

$$= x\ln x - \int x\cdot\frac{1}{x}\mathrm{d}x = x\ln x - x + C$$

Ex. $\int \arcsin x\mathrm{d}x = x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}\mathrm{d}x$

$$= x\arcsin x - \frac{1}{2}\int \frac{\mathrm{d}x^2}{\sqrt{1-x^2}} = x\arcsin x + \sqrt{1-x^2} + C.$$

Ex. $\underline{\int \sqrt{x^2+a^2}\,\mathrm{d}x} = x\sqrt{x^2+a^2} - \int \dfrac{x\cdot 2x}{2\sqrt{x^2+a^2}}\,\mathrm{d}x$

$$= x\sqrt{x^2+a^2} - \underline{\int \sqrt{x^2+a^2}\,\mathrm{d}x} + \int \frac{a^2}{\sqrt{x^2+a^2}}\,\mathrm{d}x$$

$$\int \sqrt{x^2+a^2}\,\mathrm{d}x = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\int \frac{1}{\sqrt{x^2+a^2}}\,\mathrm{d}x$$

$$= \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right) + C.$$

Ex. $\int \sqrt{x^2-a^2}\,\mathrm{d}x = \dfrac{x}{2}\sqrt{x^2-a^2} - \dfrac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + C.$

Ex. $J_k=\int((x+a)^2+b^2)^{-k}\,\mathrm{d}x$

解: $J_1=\int\frac{1}{(x+a)^2+b^2}\mathrm{d}x=\frac{1}{b}\int\frac{1}{\left(\frac{x+a}{b}\right)^2+1}\mathrm{d}\frac{x+a}{b}$

$$=\frac{1}{b}\arctan\frac{x+a}{b}+C.$$

$$J_k=\int((x+a)^2+b^2)^{-k}\mathrm{d}(x+a)$$

$$=\frac{x+a}{((x+a)^2+b^2)^k}+2k\int\frac{(x+a)^2+b^2-b^2}{((x+a)^2+b^2)^{k+1}}\mathrm{d}(x+a)$$

$$=(x+a)((x+a)^2+b^2)^{-k}+2kJ_k-2kb^2J_{k+1}.$$

$$J_{k+1}=\frac{1}{2kb^2}\Big((x+a)((x+a)^2+b^2)^{-k}+(2k-1)J_k\Big).\square$$

Ex. $I=\int e^{ax}\cos bx\mathrm{d}x.$

解法一：$I=\frac{1}{b}\int e^{ax}\mathrm{d}\sin bx=\frac{1}{b}e^{ax}\sin bx-\frac{a}{b}\int e^{ax}\sin bx\mathrm{d}x$

$$=\frac{1}{b}e^{ax}\sin bx+\frac{a}{b^2}\int e^{ax}\mathrm{d}\cos bx$$

$$=\frac{1}{b}e^{ax}\sin bx+\frac{a}{b^2}\left(e^{ax}\cos bx-a\int e^{ax}\cos bx\mathrm{d}x\right)$$

$$=\frac{e^{ax}(b\sin bx+a\cos bx)}{b^2}-\frac{a^2}{b^2}I$$

$$I=\frac{e^{ax}(b\sin bx+a\cos bx)}{a^2+b^2}+C.$$

解法二:定义$e^{(a+\mathrm{i}b)}=e^{a}(\cos b+\mathrm{i}\sin b), \forall a,b\in\mathbb{R}$;

$$(x(t)+\mathrm{i}y(t))'=x'(t)+\mathrm{i}y'(t);$$

$$\int(x(t)+\mathrm{i}y(t))\mathrm{d}t=\int x(t)\mathrm{d}t+\mathrm{i}\int y(t)\mathrm{d}t.$$

可以验证$e^{\lambda_1+\lambda_2}=e^{\lambda_1}\cdot e^{\lambda_2}, \forall\lambda_1,\lambda_2\in\mathbb{C}$; $(e^{\lambda t})'=\lambda e^{\lambda t}, \forall\lambda\in\mathbb{C}$;

$$\int e^{(a+\mathrm{i}b)x}\mathrm{d}x=\frac{1}{a+\mathrm{i}b}e^{(a+\mathrm{i}b)x}+C=\frac{(a-\mathrm{i}b)}{a^2+b^2}e^{ax}(\cos bx+\mathrm{i}\sin bx)+C$$

比较实部虚部,得

$$\int e^{ax}\cos bx\mathrm{d}x=\frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)+C,$$

$$\int e^{ax}\sin bx\mathrm{d}x=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)+C.\square$$

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3.联合积分法 Ex. $I=\int e^{ax}\cos bx\mathrm{d}x,\ J=\int e^{ax}\sin bx\mathrm{d}x.$

解: $I=\frac{1}{b}\int e^{ax}\mathrm{d}\sin bx=\frac{1}{b}e^{ax}\sin bx-\frac{a}{b}\int e^{ax}\sin bx\mathrm{d}x$

$$J=-\frac{1}{b}\int e^{ax}\mathrm{d}\cos bx=-\frac{1}{b}e^{ax}\cos bx+\frac{a}{b}\int\cos bx e^{ax}\mathrm{d}x$$

即 $I=\frac{1}{b}e^{ax}\sin bx-\frac{a}{b}J,\quad J=-\frac{1}{b}e^{ax}\cos bx+\frac{a}{b}I.$

联立得 $$I=\frac{e^{ax}(b\sin bx+a\cos bx)}{a^2+b^2}+C,$$

$$J=\frac{e^{ax}(a\sin bx-b\cos bx)}{a^2+b^2}+C.$$ □

$$\text{Ex.} I=\int\frac{dx}{1+x^2+x^4}, J=\int\frac{x^2dx}{1+x^2+x^4}.$$

解:

$$I+J=\int\frac{(1+x^2)dx}{1+x^2+x^4}=\int\frac{(1+\frac{1}{x^2})dx}{x^2+\frac{1}{x^2}+1}=\int\frac{d\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+3}=A(x)+C,$$

其中, $A(x)=\begin{cases}\frac{1}{\sqrt{3}}\arctan\frac{x^2-1}{\sqrt{3}x}+\frac{\pi}{\sqrt{3}}, & x>0\\ \frac{\pi}{2\sqrt{3}}, & x=0\\ \frac{1}{\sqrt{3}}\arctan\frac{x^2-1}{\sqrt{3}x}, & x<0\end{cases}$

$$J-I=\int\frac{(x^2-1)\mathrm{d}x}{1+x^2+x^4}=\int\frac{(1-\frac{1}{x^2})\mathrm{d}x}{x^2+\frac{1}{x^2}+1}=\int\frac{\mathrm{d}\left(x+\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)^2-1}$$

$$=\frac{1}{2}\ln\left|\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right|+C=\frac{1}{2}\ln\frac{x^2-x+1}{x^2+x+1}+C$$

$$I=\frac{1}{2}A(x)-\frac{1}{4}\ln\frac{x^2-x+1}{x^2+x+1}+C,\ J=\frac{1}{2}A(x)+\frac{1}{4}\ln\frac{x^2-x+1}{x^2+x+1}+C.\square$$

Remark. 通常记为$I=\frac{1}{2\sqrt{3}}\arctan\frac{x^2-1}{\sqrt{3}x}-\frac{1}{4}\ln\frac{x^2-x+1}{x^2+x+1}+C.$

Ex. $I=\int\frac{\cos^3 x\mathrm{d}x}{\cos x+\sin x},\ J=\int\frac{\sin^3 x\mathrm{d}x}{\cos x+\sin x}.$

解: $I+J=\int\left(1-\frac{1}{2}\sin 2x\right)\mathrm{d}x=x+\frac{1}{4}\cos 2x+C$

$$I-J=\int\frac{(\cos x-\sin x)(1+\frac{1}{2}\sin 2x)\mathrm{d}x}{\cos x+\sin x}$$

$$=\int\frac{\cos 2x\cdot(1+\frac{1}{2}\sin 2x)\mathrm{d}x}{1+\sin 2x}=\frac{1}{2}\int\frac{(1+\frac{1}{2}\sin 2x)\mathrm{d}\sin 2x}{1+\sin 2x}$$

$$=\frac{1}{4}\int\left(1+\frac{1}{1+\sin 2x}\right)\mathrm{d}\sin 2x=\frac{1}{4}\sin 2x+\frac{1}{4}\ln(1+\sin 2x)+C.\square$$

作业：习题5.4

No.3(3,7),4(1,11),5(5,12),

6(1),7(3)(11)