



Review

- 单调收敛原理：单调有界列必收敛.
- $a_n \uparrow \Rightarrow \lim_{n \rightarrow \infty} a_n = \sup\{a_n\} \in (-\infty, +\infty]$
- $a_{2n} \uparrow A, a_{2n+1} \downarrow A \Rightarrow \lim_{n \rightarrow \infty} a_n = A.$
- 重要极限 $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e; \quad \lim_{n \rightarrow \infty} n \ln \left(1 + \frac{1}{n}\right) = 1;$
 $\lim_{n \rightarrow \infty} \frac{\ln \left(1 + \frac{1}{n}\right)}{\frac{1}{n}} = 1; \quad \lim_{n \rightarrow \infty} \frac{\ln \left(1 - \frac{1}{n}\right)}{-\frac{1}{n}} = 1.$



● Stolz定理

$$(1) \quad \left. \begin{array}{l} \{b_n\} \text{ 严格 } \uparrow \\ \lim_{n \rightarrow \infty} b_n = +\infty \\ \lim_{n \rightarrow \infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = A \end{array} \right\} \Rightarrow \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = A;$$

$$(2) \quad \left. \begin{array}{l} \{b_n\} \text{ 严格 } \downarrow \\ \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = 0 \\ \lim_{n \rightarrow \infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = A \end{array} \right\} \Rightarrow \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = A.$$



§ 5. 实数系的几个基本定理

Thm.(确界原理) 非空有上界的集合必有上确界.

Thm.(单调收敛原理) 单调有界列必收敛.

Thm.(闭区间套定理) 若闭区间列 $[a_n, b_n]$ 满足条件:

$$(1) [a_{n+1}, b_{n+1}] \subset [a_n, b_n] (n = 1, 2, \cdots),$$

$$(2) \lim_{n \rightarrow \infty} (b_n - a_n) = 0,$$

则 $\exists! \xi \in \mathbb{R}, s.t. \xi \in \bigcap_{n \geq 1} [a_n, b_n]; \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = \xi$.

Thm.(Bolzano-Weirstrass定理) 有界列必有收敛子列.

Thm.(Cauchy收敛原理) 收敛列 \Leftrightarrow Cauchy列.

以上五个定理相互等价



●确界原理 \Rightarrow 单调收敛原理 (前一节已证).

●单调收敛原理 \Rightarrow 闭区间套定理:

存在性. 由 $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$, $a_n \uparrow, b_n \downarrow$, 且

$$a_1 \leq a_n \leq b_n \leq b_1, \quad \forall n.$$

由单调收敛原理, $\lim_{n \rightarrow \infty} a_n$ 与 $\lim_{n \rightarrow \infty} b_n$ 存在. 又 $\lim_{n \rightarrow \infty} (b_n - a_n) = 0$, 故

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n - \lim_{n \rightarrow \infty} (b_n - a_n) = \lim_{n \rightarrow \infty} b_n \triangleq \xi.$$

若 $\exists a_k > \xi$. 由 $\{a_n\}$ 单增, 有 $a_n \geq a_k, \forall n > k$. 令 $n \rightarrow +\infty$, 有 $\xi = \lim_{n \rightarrow \infty} a_n \geq a_k > \xi$, 矛盾. 所以 $a_n \leq \xi, \forall n$. 同理, $\xi \leq b_n, \forall n$.

故

$$a_n \leq \xi \leq b_n, \forall n.$$



唯一性. 若 η 满足 $a_n \leq \eta \leq b_n, \forall n$. 由极限的保序性, 有

$$\lim_{n \rightarrow \infty} a_n \leq \eta \leq \lim_{n \rightarrow \infty} b_n. \text{ 而 } \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = \xi, \text{ 故 } \eta = \xi. \square$$

• 闭区间套定理 \Rightarrow Bolzano-Weirstrass 定理 (列紧性定理):

设 $\{x_n\}$ 为有界列. 则 $\exists a_1 < b_1, s.t. x_n \in [a_1, b_1], \forall n$. 用中点 $\frac{a_1 + b_1}{2}$ 将 $[a_1, b_1]$ 分为两个区间, 其中至少有一个含有 $\{x_n\}$ 中无穷多项, 记之为 $[a_2, b_2]$. 用中点 $\frac{a_2 + b_2}{2}$ 将 $[a_2, b_2]$ 分为两个区间, 其中至少有一个含有 $\{x_n\}$ 中无穷多项, 记之为 $[a_3, b_3]$. 如此继续, 得到一系列区间 $[a_n, b_n], n = 1, 2, \dots$, 满足



$$[a_{n+1}, b_{n+1}] \subset [a_n, b_n] (n = 1, 2, \dots),$$

$$\lim_{n \rightarrow \infty} (b_n - a_n) = \lim_{n \rightarrow \infty} \frac{b_1 - a_1}{2^{n-1}} = 0.$$

由闭区间套定理, $\exists! \xi \in \bigcap_{n \geq 1} [a_n, b_n]$, 且 $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = \xi$.

$[a_1, b_1]$ 中包含 $\{x_n\}$ 的无穷多项, 因此 $\exists x_{n_1} \in [a_1, b_1]$. $[a_2, b_2]$ 中包含 $\{x_n\}$ 的无穷多项, 因此 $\exists x_{n_2} \in [a_2, b_2]$, 且 $n_2 > n_1$. 依此推, $\exists x_{n_{k+1}} \in [a_{k+1}, b_{k+1}]$, 且 $n_{k+1} > n_k$. 由此得到 $\{x_n\}$ 的子列 $\{x_{n_k}\}$, 满足 $a_k \leq x_{n_k} \leq b_k$, $\forall k$. 令 $k \rightarrow \infty$, 由夹挤原理得

$$\lim_{k \rightarrow \infty} x_{n_k} = \lim_{k \rightarrow \infty} a_k = \lim_{k \rightarrow \infty} b_k = \xi. \square$$



Def.(Cauchy列)

$\{x_n\}$ 为Cauchy列

$$\Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, s.t. \forall n, m > N, \text{有 } |x_n - x_m| < \varepsilon.$$

$$\Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, s.t. \forall n > N, \forall p \in \mathbb{N}, \text{有 } |x_n - x_{n+p}| < \varepsilon.$$

$$\Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, s.t. \forall n \geq N, \forall p \in \mathbb{N}, \text{有 } |x_n - x_{n+p}| \leq \varepsilon.$$

$$\Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, s.t. \forall n, m > N, \text{有 } |x_n - x_m| < 2\varepsilon$$

Remark. $\{x_n\}$ 不是Cauchy列

$$\Leftrightarrow \exists \varepsilon > 0, \forall N \in \mathbb{N}, \exists n, m > N, s.t. |x_n - x_m| > \varepsilon.$$

$$\Leftrightarrow \exists \varepsilon > 0, \forall N \in \mathbb{N}, \exists n > N, \exists p > 0, s.t. |x_{n+p} - x_n| > \varepsilon.$$



Lemma. Cauchy列必为有界列.

Proof. 设 $\{x_n\}$ 为Cauchy列. 则对 $\varepsilon = 1, \exists N \in \mathbb{N}, s.t. \forall n > N$, 有 $|x_n - x_{N+1}| < 1$. 于是

$$|x_n| \leq \max\{|x_1|, |x_2|, \dots, |x_N|, |x_{N+1}| + 1\}, \quad \forall n. \square$$



●Bolzano-Weirstrass定理 \Rightarrow Cauchy收敛原理:

先证Cauchy列必为收敛列. 设 $\{x_n\}$ 为Cauchy列. $\forall \varepsilon > 0$,
 $\exists N, s.t.$

$$|x_n - x_m| < \varepsilon, \quad \forall n, m > N.$$

Cauchy列 $\{x_n\}$ 为有界列, 由Bolzano-Weirstrass定理, \exists 收敛子列 $\{x_{n_k}\}$, 设 $\lim_{k \rightarrow \infty} x_{n_k} = a$. 对前面的 $\varepsilon > 0$, $\exists K > N, s.t.$

$$|x_{n_k} - a| < \varepsilon, \quad \forall k \geq K.$$

于是

$$|x_n - a| \leq |x_n - x_{n_K}| + |x_{n_K} - a| < 2\varepsilon, \quad \forall n > N.$$

故 $\lim_{n \rightarrow \infty} x_n = a$.



再证收敛列必为Cauchy列.

设 $\lim_{n \rightarrow \infty} x_n = a$. 则 $\forall \varepsilon > 0, \exists N, s.t.$

$$|x_n - a| < \frac{\varepsilon}{2}, \quad \forall n > N.$$

于是

$$|x_n - x_m| \leq |x_n - a| + |x_m - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall n, m > N.$$

故 $\{x_n\}$ 为Cauchy列. \square

• Cauchy收敛原理 \Rightarrow 确界原理

Hint: 二分区间法构造闭区间套, 进而取Cauchy列.



Ex. $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, $\{x_n\}$ 发散.

Proof. $|x_{2n} - x_n| = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > \frac{1}{2n} \cdot n = \frac{1}{2},$

$\{x_n\}$ 不为Cauchy列, 故 $\{x_n\}$ 发散. \square

Ex. $x_n = \sum_{k=1}^n \frac{(-1)^k}{k^2}$, $\{x_n\}$ 收敛.

Proof. $|x_n - x_{n+p}| \leq \sum_{k=n+1}^{n+p} \frac{1}{k^2} \leq \sum_{k=n+1}^{n+p} \frac{1}{k(k-1)} = \frac{1}{n} - \frac{1}{n+p} < \frac{1}{n}.$

$\forall \varepsilon > 0, \exists N = \lceil \varepsilon^{-1} \rceil, s.t., \forall n > N, \forall p \in \mathbb{N}, \text{有 } |x_n - x_{n+p}| < \varepsilon.$

$\{x_n\}$ 为Cauchy列, 故 $\{x_n\}$ 收敛. \square



Question. 相较于定义, 利用Cauchy收敛原理判别数列敛散性的优势?

Question. $\lim_{n \rightarrow \infty} |x_n - x_{n+p}| = 0, \forall p \in \mathbb{N} \stackrel{?}{\Rightarrow} \{x_n\}$ 为Cauchy列

No! 反例: $\{\sqrt{n}\}, \{\ln n\}, \left\{ \sum_{k=1}^n \frac{1}{k} \right\}$.

Question. $|x_n - x_{n+p}| \leq \frac{p}{n}, \forall p, n \in \mathbb{N} \stackrel{?}{\Rightarrow} \{x_n\}$ 为Cauchy列.

No! 反例: $\left\{ \sum_{k=1}^n \frac{1}{k} \right\}$.



Question. $|x_n - x_{n+p}| \leq \frac{p}{n^2}, \forall p, n \in \mathbb{N} \stackrel{?}{\Rightarrow} \{x_n\}$ 为 Cauchy 列. Yes!

Proof. $|x_n - x_{n+p}| \leq \frac{p}{n^2}, \forall p, n \in \mathbb{N}$, 则 $|x_n - x_{n+1}| \leq \frac{1}{n^2}, \forall n$.

于是

$$\begin{aligned} |x_n - x_{n+p}| &\leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{n+p-1} - x_{n+p}| \\ &\leq \frac{1}{n^2} + \cdots + \frac{1}{(n+p-1)^2} \\ &\leq \frac{1}{n(n-1)} + \cdots + \frac{1}{(n+p-1)(n+p-2)} \\ &= \frac{1}{n-1} - \frac{1}{n+p-1} < \frac{1}{n-1}, \quad \forall p, \forall n > 1. \square \end{aligned}$$



Ex. $\exists M > 0, s.t. \sum_{k=1}^n |x_{k+1} - x_k| \leq M, \forall n \Rightarrow \{x_n\}$ 为Cauchy列.

Proof. 令 $y_n = \sum_{k=1}^n |x_{k+1} - x_k|, n \in \mathbb{N}$. $\{y_n\}$ 单增有上界, $\{y_n\}$ 收敛,

$\{y_n\}$ 为Cauchy列. $\forall \varepsilon > 0, \exists N, s.t.$

$$0 \leq y_{n+p} - y_n \leq \varepsilon, \quad \forall n > N, \forall p.$$

从而有

$$\begin{aligned} |x_{n+p} - x_n| &\leq |x_{n+p} - x_{n+p-1}| + \cdots + |x_{n+1} - x_n| \\ &= y_{n+p-1} - y_n < \varepsilon, \quad \forall n > N, \forall p. \square \end{aligned}$$



Ex. $0 \leq x_{n+m} \leq x_n + x_m$, 则 $\inf_{n \geq 1} \{\frac{x_n}{n}\}$ 存在, 且 $\lim_{n \rightarrow \infty} \frac{x_n}{n} = \inf_{n \geq 1} \{\frac{x_n}{n}\}$.

Proof. $0 \leq \frac{x_n}{n} \leq x_1$, 则 $\inf_{n \geq 1} \{\frac{x_n}{n}\} = A$ 存在. $\forall \varepsilon > 0, \exists m, s.t.$

$$A \leq \frac{x_m}{m} < A + \varepsilon.$$

$\forall n > m$, 有 $n = km + r, k, r \in \mathbb{Z}, 0 \leq r < m$. 记 $x_0 = 0$, 由已知条件得

$$A \leq \frac{x_n}{n} \leq \frac{kx_m + x_r}{n} = \frac{kx_m}{km + r} + \frac{x_r}{n} \leq \frac{x_m}{m} + \frac{x_r}{n} \leq A + \varepsilon + \frac{x_r}{n}.$$

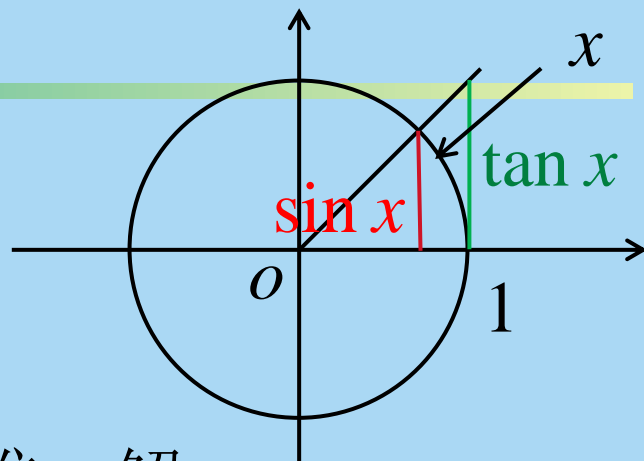
$\exists N > m, s.t. \max_{0 \leq r < m} \{\frac{x_r}{n}\} < \varepsilon, \forall n > N$. 于是,

$$A \leq \frac{x_n}{n} \leq A + 2\varepsilon, \forall n > N. \square$$



$$|\sin x| \leq |x|, \forall x \in \mathbb{R}.$$

$$|x| \leq |\tan x|, \forall x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$



Ex. $0 < a < 1, b \in \mathbb{R}$, 则 $x - a \sin x = b$ 有唯一解.

Proof. 存在性. 令 $x_0 = b, x_{n+1} = b + a \sin x_n$, 则

$$\begin{aligned} |x_{n+p} - x_n| &= a |\sin x_{n+p-1} - \sin x_{n-1}| \\ &= 2a \left| \sin \frac{x_{n+p-1} - x_{n-1}}{2} \cos \frac{x_{n+p-1} + x_{n-1}}{2} \right| \leq a |x_{n+p-1} - x_{n-1}| \\ &\leq \cdots \leq a^n |x_p - x_0| = a^{n+1} |\sin x_{p-1}| \leq a^{n+1}. \end{aligned}$$

$0 < a < 1$, 故 $\{x_n\}$ 为Cauchy列. 设 $\lim_{n \rightarrow +\infty} x_n = \xi$.



$$|\sin x_n - \sin \xi| \leq |x_n - \xi|,$$

由夹挤原理得

$$\lim_{n \rightarrow \infty} \sin x_n = \sin \xi.$$

在 $x_{n+1} = b + a \sin x_n$ 中令 $n \rightarrow \infty$, 得 $\xi = b + a \sin \xi$.

唯一性. 设 $\eta = b + a \sin \eta$, 则

$$|\xi - \eta| = a |\sin \xi - \sin \eta| \leq a |\xi - \eta|.$$

由 $0 < a < 1$ 得 $\xi = \eta$. \square



作业：习题1.5

No. 2(4)(5), 3(2)(3), 8