Review

- 第一换元法（凑微分法）

$$\int f'(\varphi(x))\varphi'(x)\mathrm{d}x = \int f'(\varphi(x))\mathrm{d}\varphi(x) = f(\varphi(x)) + C$$

- 第二换元法

$$\int f'(u)\mathrm{d}u \xequal{u=\varphi(x)} \int f'(\varphi(x))\varphi'(x)\mathrm{d}x$$

$$= g(x) + C = g(\varphi^{-1}(u)) + C$$

- 分部积分法 $\quad \int u(x)\mathrm{d}v(x) = u(x)v(x) - \int v(x)\mathrm{d}u(x)$

- 联合积分法

§5.有理函数与三角有理函数的不定积分

1.有理函数 $\dfrac{p(x)}{q(x)}(p,q$为多项式$)$

Thm.有理真分式 $\dfrac{p(x)}{q(x)}$ 可分解成最简分式之代数和:

(1)$q(x)$的一次k重因式$(x-a)^k$对应k项

$$\frac{A_1}{x-a},\frac{A_2}{(x-a)^2},\cdots,\frac{A_k}{(x-a)^k};$$

(2)$q(x)$的二次k重因式$((x+a)^2+b^2)^k$对应k项

$$\frac{B_1x+C_1}{(x+a)^2+b^2},\frac{B_2x+C_2}{((x+a)^2+b^2)^2},\cdots,\frac{B_kx+C_k}{((x+a)^2+b^2)^k}.$$

Remark. $\dfrac{Bx+C}{\left((x+a)^2+b^2\right)^k}$ 可以表示成形如 $\dfrac{x+a}{\left((x+a)^2+b^2\right)^k}$,

$\dfrac{1}{\left((x+a)^2+b^2\right)^k}$ 的简单分式的代数和.

• $\displaystyle\int \dfrac{\mathrm{d}x}{x-a} = \ln|x-a| + C,$

• $\displaystyle\int \dfrac{\mathrm{d}x}{(x-a)^k} = \dfrac{-1}{(k-1)(x-a)^{k-1}} + C, \quad (k>1)$

$$\bullet \int \frac{x+a}{(x+a)^2+b^2}\,\mathrm{d}x = \frac{1}{2}\ln((x+a)^2+b^2)+C,$$

$$\bullet \int \frac{x+a}{((x+a)^2+b^2)^k}\,\mathrm{d}x = \frac{-1}{2(k-1)((x+a)^2+b^2)^{k-1}}, \ (k>1)$$

$$\bullet J_k = \int \frac{1}{((x+a)^2+b^2)^k}\,\mathrm{d}x$$

$$J_1 = \frac{1}{b}\arctan\frac{x+a}{b}+C,$$

$$J_{k+1} = \frac{1}{2kb^2}\Big((x+a)((x+a)^2+b^2)^{-k}+(2k-1)J_k\Big).$$

Ex. $I = \int \dfrac{\mathrm{d}x}{1 + x^2 + x^4}.$

解：$1 + x^2 + x^4 = (x^2 + 1)^2 - x^2 = (x^2 + x + 1)(x^2 - x + 1).$

$$\frac{1}{1 + x^2 + x^4} = \frac{ax + b}{x^2 + x + 1} + \frac{cx + d}{x^2 - x + 1}$$

$$\Rightarrow a = b = d = \frac{1}{2}, c = -\frac{1}{2}.$$

$$\frac{1}{1 + x^2 + x^4} = \frac{1}{2}\frac{x + 1}{x^2 + x + 1} - \frac{1}{2}\frac{x - 1}{x^2 - x + 1}.$$

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$$\int \frac{dx}{1+x^2+x^4} = \frac{1}{2}\int \frac{x+1}{x^2+x+1}dx - \frac{1}{2}\int \frac{x-1}{x^2-x+1}dx$$

$$= \frac{1}{2}\int \frac{x+1/2}{x^2+x+1}dx + \frac{1}{4}\int \frac{dx}{x^2+x+1}$$

$$- \frac{1}{2}\int \frac{x-1/2}{x^2-x+1}dx + \frac{1}{4}\int \frac{dx}{x^2-x+1}$$

$$= \frac{1}{4}\ln(x^2+x+1) + \frac{1}{2\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}$$

$$- \frac{1}{4}\ln(x^2-x+1) + \frac{1}{2\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}} + C.$$

2.三角有理式 $R(\sin x, \cos x)$：$\sin x, \cos x$ 有限次四则运算

万能变换 $t = \tan\dfrac{x}{2}$, $x = 2\arctan t$, $dx = \dfrac{2}{1+t^2}dt$,

$$\sin x = \frac{2\sin\dfrac{x}{2}\cos\dfrac{x}{2}}{\sin^2\dfrac{x}{2}+\cos^2\dfrac{x}{2}} = \frac{2\tan\dfrac{x}{2}}{\tan^2\dfrac{x}{2}+1} = \frac{2t}{1+t^2}$$

$$\cos x = \frac{\cos^2\dfrac{x}{2}-\sin^2\dfrac{x}{2}}{\sin^2\dfrac{x}{2}+\cos^2\dfrac{x}{2}} = \frac{1-\tan^2\dfrac{x}{2}}{1+\tan^2\dfrac{x}{2}} = \frac{1-t^2}{1+t^2}.$$

$$\int R(\sin x, \cos x)dx = \int R(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2})\frac{2dt}{1+t^2}.$$

Ex. $I = \int \dfrac{1 + \sin x}{1 + \cos x} \, dx.$

解法一 : 令 $t = \tan \dfrac{x}{2}$, 则 $x = 2\arctan t$, $dx = \dfrac{2}{1 + t^2} \, dt$,

$$\int \frac{1 + \sin x}{1 + \cos x} \, dx = \int \frac{1 + \dfrac{2t}{1 + t^2}}{1 + \dfrac{1 - t^2}{1 + t^2}} \cdot \frac{2}{1 + t^2} \, dt = \int \frac{(1 + t)^2}{2} \cdot \frac{2}{1 + t^2} \, dt$$

$$= \int \left(1 + \frac{2t}{1 + t^2} \right) dt = t + \ln(1 + t^2) + C = \tan \frac{x}{2} - 2\ln \left| \cos \frac{x}{2} \right| + C.$$

解法二 : $I = \int \dfrac{1}{2\cos^2 \dfrac{x}{2}} \, dx + \int \dfrac{\sin x}{1 + \cos x} \, dx$

$$= \tan \frac{x}{2} - \ln(1 + \cos x) + C.$$

Remark.万能变换不一定简单.

Ex.$\displaystyle\int \frac{\tan x}{a^2 \cos^2 x + b^2 \sin^2 x}\,\mathrm{d}x$

$\displaystyle= \int \frac{\tan x}{a^2 + b^2 \tan^2 x}\cdot\frac{\mathrm{d}x}{\cos^2 x} = \int \frac{\tan x}{a^2 + b^2 \tan^2 x}\,\mathrm{d}\tan x$

$\displaystyle= \frac{1}{2}\int \frac{\mathrm{d}\tan^2 x}{a^2 + b^2 \tan^2 x} = \frac{1}{2b^2}\ln(a^2 + b^2 \tan^2 x) + C.$

3.可化为有理式的简单无理式

1) $\int R(x, \sqrt[n]{\dfrac{ax+b}{cx+d}})\mathrm{d}x \quad (ad-bc \neq 0)$

令 $t = \sqrt[n]{\dfrac{ax+b}{cx+d}}$, 则 $x = \dfrac{b-dt^n}{ct^n-a}$, $\mathrm{d}x = \dfrac{ad-bc}{(ct^n-a)^2}nt^{n-1}\mathrm{d}t$,

$$\int R(x, \sqrt[n]{\dfrac{ax+b}{cx+d}})\mathrm{d}x = \int R(\dfrac{b-dt^n}{ct^n-a}, t)\dfrac{ad-bc}{(ct^n-a)^2}nt^{n-1}\mathrm{d}t.$$

Ex. $I = \displaystyle\int \frac{\mathrm{d}x}{\sqrt[3]{(x-1)(x+1)^2}}$

解：令 $t = \sqrt[3]{\dfrac{x+1}{x-1}}$，则 $x = \dfrac{t^3+1}{t^3-1}$，$\mathrm{d}x = \dfrac{-6t^2}{(t^3-1)^2}\,dt$.

$I = \displaystyle\int \sqrt[3]{\frac{x+1}{x-1}}\,\frac{\mathrm{d}x}{x+1} = \int \frac{-3}{t^3-1}\,\mathrm{d}t = \int\left(\frac{-1}{t-1} + \frac{t+2}{t^2+t+1}\right)\mathrm{d}t$

$= -\ln|t-1| + \dfrac{1}{2}\ln(t^2+t+1) + \sqrt{3}\arctan\dfrac{2t+1}{\sqrt{3}} + C$

$= \dfrac{1}{2}\ln|x-1| - \dfrac{3}{2}\ln\left|\sqrt[3]{\dfrac{x+1}{x-1}} - 1\right| + \sqrt{3}\arctan\dfrac{2}{\sqrt{3}}\left(\sqrt[3]{\dfrac{x+1}{x-1}} - \dfrac{1}{2}\right) + C.$

2) $\displaystyle\int R(x, \sqrt{ax^2 + bx + c}\,)\mathrm{d}x, (a \neq 0)$

- $\displaystyle\int R(x, \sqrt{(x+p)^2 + q^2}\,)\mathrm{d}x$ 令 $x + p = q\tan t, |t| < \dfrac{\pi}{2}$

- $\displaystyle\int R(x, \sqrt{(x+p)^2 - q^2}\,)\mathrm{d}x$ 令 $x + p = q\csc t, t \in (0, \pi)$

- $\displaystyle\int R(x, \sqrt{q^2 - (x+p)^2}\,)\mathrm{d}x$ 令 $x + p = q\sin t, |t| < \dfrac{\pi}{2}$

Euler第一替换：若 $a > 0$, 令 $\sqrt{ax^2 + bx + c} = \pm\sqrt{a}\,x + t.$

此时, $x = \dfrac{t^2 - c}{b \mp 2\sqrt{a}\,t}$, $\sqrt{ax^2 + bx + c} = \dfrac{\sqrt{a}(t^2 - c)}{-2\sqrt{a}\,t \pm b} + t.$

Euler第二替换：若$c > 0$, 令$\sqrt{ax^2 + bx + c} = xt \pm \sqrt{c}$.

此时, $x = \dfrac{-b \pm 2\sqrt{c}t}{a - t^2}$, $\sqrt{ax^2 + bx + c} = \dfrac{-b \pm 2\sqrt{c}t}{a - t^2}t \pm \sqrt{c}$.

Euler第三替换：若$ax^2 + bx + c$有相异实根α, β, 令

$$\sqrt{ax^2 + bx + c} = (x - \alpha)t,$$

此时, $a(x - \alpha)(x - \beta) = (x - \alpha)^2 t^2$, $\quad x = \dfrac{a\beta - \alpha t^2}{a - t^2}$.

$$\text{Ex.} I = \int \frac{\mathrm{d}x}{\sqrt{x^2 \pm a^2}}$$

解: 令 $\sqrt{x^2 \pm a^2} = x + t$, 则

$$x = \frac{-t^2 \pm a^2}{2t}, \mathrm{d}x = \frac{t^2 \pm a^2}{-2t^2} \mathrm{d}t, \sqrt{x^2 \pm a^2} = \frac{t^2 \pm a^2}{2t},$$

$$\int \frac{\mathrm{d}x}{\sqrt{x^2 \pm a^2}} = \int \frac{\mathrm{d}t}{-t} = -\ln|t| + C$$

$$= -\ln\left|\sqrt{x^2 \pm a^2} - x\right| + C = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C. \ \square$$

$$\text{Ex.} I = \int \frac{\mathrm{d}x}{x^2 \sqrt{4x^2 - 3x - 1}}$$

解法一： $4x^2 - 3x - 1 = (4x+1)(x-1)$, 令

$$\sqrt{(4x+1)(x-1)} = (4x+1)t,$$

则 $x = \dfrac{t^2+1}{1-4t^2}$, $\mathrm{d}x = \dfrac{10t\mathrm{d}t}{(1-4t^2)^2}$, $\sqrt{4x^2-3x-1} = \dfrac{5t}{1-4t^2}$,

$$I = \int \frac{2(1-4t^2)\mathrm{d}t}{(t^2+1)^2} = -8\int \frac{\mathrm{d}t}{t^2+1} + 10\int \frac{\mathrm{d}t}{(t^2+1)^2} = -8I_1 + 10I_2,$$

$$\arctan t + C = I_1 = \frac{t}{t^2+1} + \int \frac{2t^2 \mathrm{d}t}{(t^2+1)^2} = \frac{t}{t^2+1} + 2\arctan t - 2I_2,$$

$$I_2 = \frac{1}{2}\left(\frac{t}{t^2+1} + \arctan t\right) + C,$$

$$I = -8I_1 + 10I_2 = -3\arctan t + \frac{5t}{t^2+1} + C$$

$$= -3\arctan \frac{\sqrt{4x^2-3x-1}}{4x+1} + \frac{5\sqrt{4x^2-3x-1}}{x} + C.$$

解法二: 令 $\sqrt{4x^2-3x-1}=-2x+t$, 则 $x=\dfrac{1+t^2}{4t-3}$,

$$\mathrm{d}x=\dfrac{2(2t^2-3t-2)}{(4t-3)^2}\mathrm{d}t, \ \sqrt{4x^2-3x-1}=\dfrac{2t^2-3t-2}{4t-3},$$

$$I=2\int\dfrac{(4t-3)\mathrm{d}t}{(1+t^2)^2}=4\int\dfrac{\mathrm{d}(t^2+1)}{(1+t^2)^2}-6\int\dfrac{\mathrm{d}t}{(1+t^2)^2}$$

$$=-\dfrac{4}{1+t^2}-\dfrac{3t}{t^2+1}-3\arctan t+C$$

$$=\dfrac{\sqrt{4x^2-3x-1}-2x}{x}-3\arctan(2x+\sqrt{4x^2-3x-1})+C.\ \square$$

Remark. 不同解法得出的结果可能形式不同.

Remark.初等函数的原函数不一定是初等函数,如

$$e^{x^2}, \sin x^2, \cos x^2, \frac{\sin x}{x}, \frac{\cos x}{x},$$

$$\frac{1}{\ln x}, \sqrt{1 - a^2 \sin^2 x} (0 < a < 1).$$

作业：习题5.5
　　No.1(2,6),2(4,7),
　　　3(3,6),4(6,8,9)