Review

- $f\uparrow \Leftrightarrow f'(x)\geq 0;$
- f严格$\uparrow \Leftrightarrow f'(x)\geq 0$，且在任意$(c,d)$上$f'(x)$不恒为0.
- $f'(x_0)=f''(x_0)=\cdots=f^{(2n-1)}(x_0)=0$，则

  $f^{(2n)}(x_0)>(<)0 \Rightarrow x_0$为$f$的极小(大)值点.
- $f'(x_0)=f''(x_0)=\cdots=f^{(2n)}(x_0)=0,\ f^{(2n+1)}(x_0)\neq 0$

  $\Rightarrow x_0$不是极值点.
- f在$[a,b]$上、$\mathbb{R}$上的极值和最值问题.

§5.函数的凸凹性

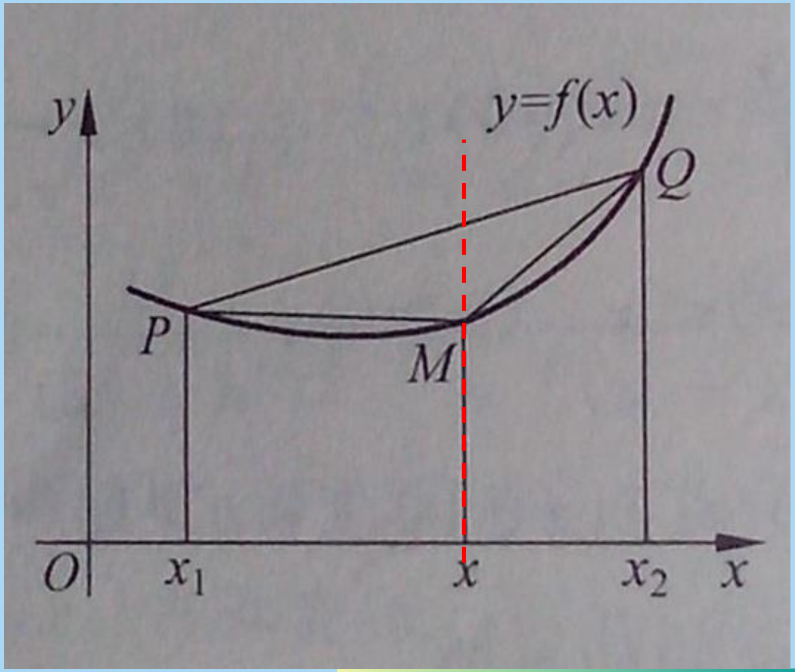
Def. f在区间I上定义,若$\forall x_1, x_2 \in I, \forall \lambda \in (0,1)$,都有

$$f(\lambda x_1 + (1-\lambda)x_2) \le (\ge) \lambda f(x_1) + (1-\lambda) f(x_2),$$

则称f为I上的下凸(上凸)函数.

Remark.下凸函数的几何意义?

Question. 如何定义严格上(下)凸函数?



Thm. f 为 I 上的下凸函数, 当且仅当 $\forall x_1,\cdots,x_n \in I$, 以及满足 $\lambda_1+\cdots+\lambda_n=1$ 的正数 $\lambda_1,\cdots,\lambda_n$, 都有

$$f(\lambda_1 x_1+\cdots+\lambda_n x_n)\le \lambda_1 f(x_1)+\cdots+\lambda_n f(x_n).$$

Proof. 充分性显然, 下证必要性.

$n=1,2$ 时, 显然.

设 $n=k$ 时结论成立, 当 $n=k+1$ 时,

$$\frac{\lambda_1}{1-\lambda_{k+1}}+\frac{\lambda_2}{1-\lambda_{k+1}}+\cdots+\frac{\lambda_k}{1-\lambda_{k+1}}=1,$$

$$f(\lambda_1 x_1+\cdots+\lambda_{k+1}x_{k+1}) = f\left(\lambda_{k+1}x_{k+1}+(1-\lambda_{k+1})\frac{\lambda_1 x_1+\cdots+\lambda_k x_k}{1-\lambda_{k+1}}\right)$$

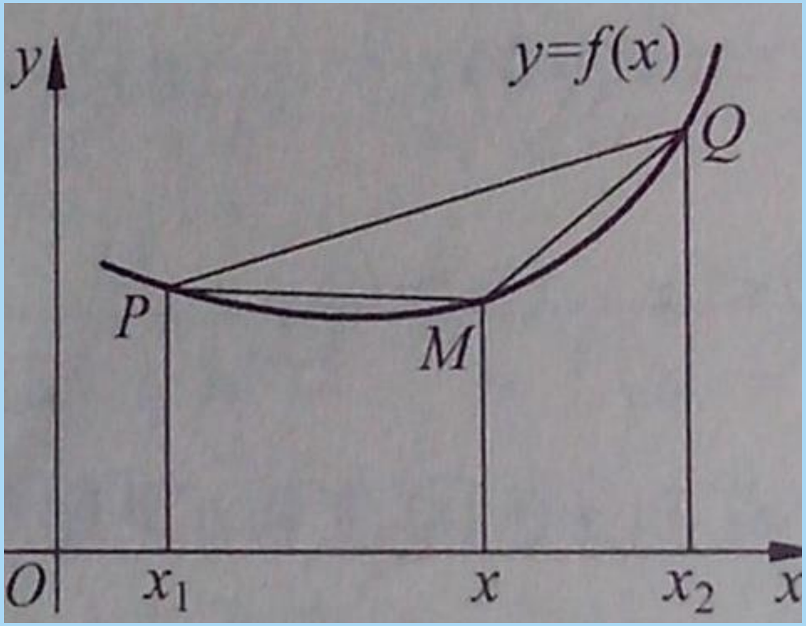
$$\le \lambda_{k+1} f(x_{k+1})+(1-\lambda_{k+1}) f\left(\frac{\lambda_1 x_1+\cdots+\lambda_k x_k}{1-\lambda_{k+1}}\right)$$

$$\le \lambda_{k+1} f(x_{k+1})+(1-\lambda_{k+1})\left(\frac{\lambda_1 f(x_1)}{1-\lambda_{k+1}}+\cdots+\frac{\lambda_k f(x_k)}{1-\lambda_{k+1}}\right)$$

$$=\lambda_1 f(x_1)+\cdots+\lambda_k f(x_k)+\lambda_{k+1} f(x_{k+1}).\square$$

Thm. f为I上的下凸函数,当且仅当$\forall x_1, x_2 \in I$及$x \in (x_1, x_2)$,

有 $$\frac{f(x)-f(x_1)}{x-x_1} \le \frac{f(x_2)-f(x_1)}{x_2-x_1} \le \frac{f(x_2)-f(x)}{x_2-x}.$$



Remark. 几何意义.

$$k_{PM} \le k_{PQ} \le k_{MQ}.$$

Remark. 我们证明以下更强的定理.

Thm. 以下各命题等价：(1) f 为 I 上的下凸函数;

(2) $\forall x_1, x_2 \in I$ 及 $x \in (x_1, x_2)$，有

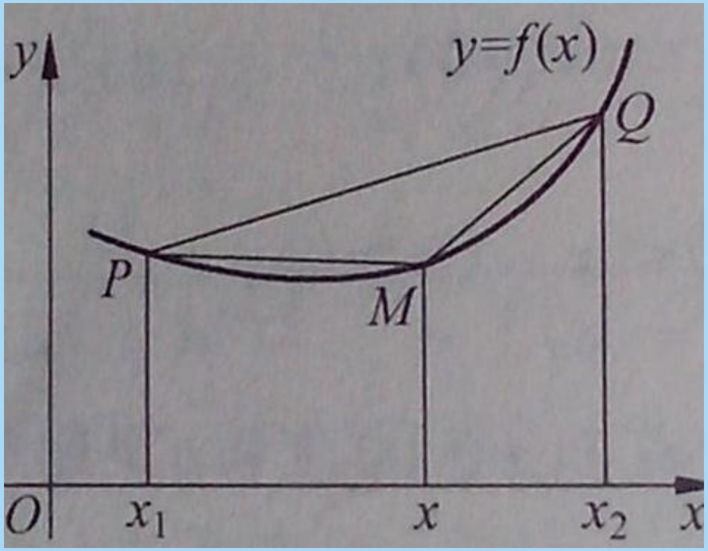
$$\frac{f(x)-f(x_1)}{x-x_1} \le \frac{f(x_2)-f(x_1)}{x_2-x_1};$$

(3) $\forall x_1, x_2 \in I$ 及 $x \in (x_1, x_2)$，有

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} \le \frac{f(x_2)-f(x)}{x_2-x};$$

(4) $\forall x_1, x_2 \in I$ 及 $x \in (x_1, x_2)$，有

$$\frac{f(x)-f(x_1)}{x-x_1} \le \frac{f(x_2)-f(x)}{x_2-x}.$$



Proof. $\forall x_1, x, x_2 \in I, x_1 < x < x_2$, 记$x = \lambda x_1 + (1-\lambda)x_2$, 则

$$\lambda = \frac{x_2 - x}{x_2 - x_1} \in (0,1),\ 1-\lambda = \frac{x - x_1}{x_2 - x_1}.$$

(1) $\Leftrightarrow$ (2):

$$f\text{下凸} \Leftrightarrow f(x) \le \lambda f(x_1) + (1-\lambda) f(x_2)$$

$$\Leftrightarrow f(x) - f(x_1) \le (1-\lambda)(f(x_2) - f(x_1))$$

$$= \frac{x - x_1}{x_2 - x_1}(f(x_2) - f(x_1)),$$

$$\Leftrightarrow \frac{f(x) - f(x_1)}{x - x_1} \le \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

$$\boxed{\begin{aligned}&x\in(x_1,x_2),\ x=\lambda x_1+(1-\lambda)x_2,\\ &\qquad\lambda=\frac{x_2-x}{x_2-x_1}\in(0,1),\quad 1-\lambda=\frac{x-x_1}{x_2-x_1}.\end{aligned}}$$

$(1)\Leftrightarrow(3):$

$$f\text{下凸}\Leftrightarrow f(x)\le\lambda f(x_1)+(1-\lambda)f(x_2)$$

$$\Leftrightarrow f(x)-f(x_2)\le\lambda(f(x_1)-f(x_2))$$

$$=\frac{x_2-x}{x_2-x_1}(f(x_1)-f(x_2))$$

$$\Leftrightarrow\frac{f(x_2)-f(x_1)}{x_2-x_1}\le\frac{f(x_2)-f(x)}{x_2-x}.$$

$$x \in (x_1, x_2), x = \lambda x_1 + (1-\lambda) x_2, \lambda = \frac{x_2 - x}{x_2 - x_1} \in (0,1), 1-\lambda = \frac{x - x_1}{x_2 - x_1}.$$

$(1) \Leftrightarrow (4):$

$$f\text{下凸} \Leftrightarrow f(x) \le \lambda f(x_1) + (1-\lambda) f(x_2)$$

$$\Leftrightarrow \lambda f(x) + (1-\lambda) f(x) \le \lambda f(x_1) + (1-\lambda) f(x_2)$$

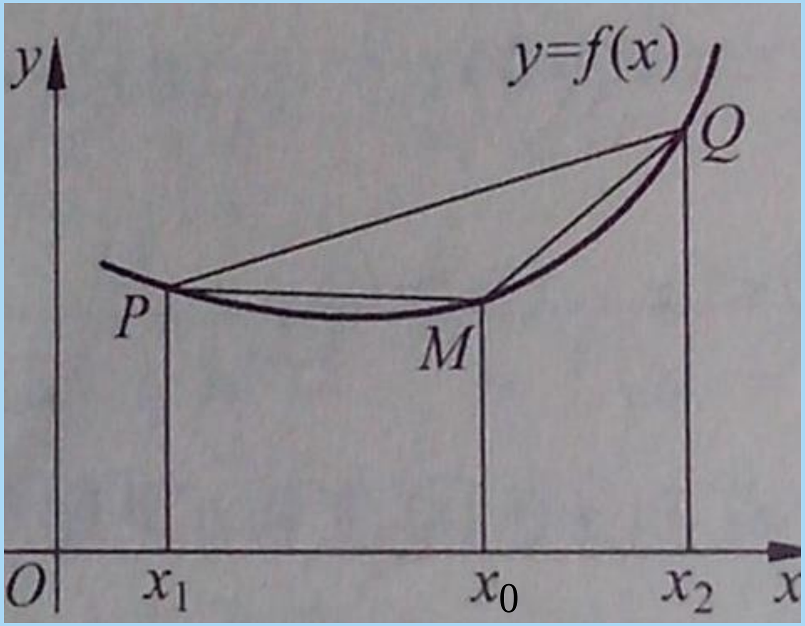
$$\Leftrightarrow \lambda(f(x) - f(x_1)) \le (1-\lambda)(f(x_2) - f(x))$$

$$\Leftrightarrow \frac{x_2 - x}{x_2 - x_1}(f(x) - f(x_1)) \le \frac{x - x_1}{x_2 - x_1}(f(x_2) - f(x))$$

$$\Leftrightarrow \frac{f(x) - f(x_1)}{x - x_1} \le \frac{f(x_2) - f(x)}{x_2 - x}.$$

Remark. f在I上下凸, $N_\delta(x_0)\subset I$,则

$\dfrac{f(x_1)-f(x_0)}{x_1-x_0}$关于$x_1(<x_0)$单增,有上界$\dfrac{f(x_0)-f(x_2)}{x_0-x_2}$,



故 $f'_-(x_0)=\lim\limits_{x_1\to x_0^-}\dfrac{f(x_1)-f(x_0)}{x_1-x_0}$

存在 $(\Rightarrow \lim\limits_{x\to x_0^-} f(x)=f(x_0))$,且

$$\frac{f(x)-f(x_0)}{x-x_0}\le f'_-(x_0),\quad \forall x<x_0, x\in I,$$

$$f(x)\ge f(x_0)+f'_-(x_0)(x-x_0),\quad \forall x<x_0, x\in I.$$

Corollary.开区间I上的(上、下)凸函数每一点处左右导数存在。闭区间?

Corollary.开区间I上的(上、下)凸函数连续。

闭区间?(左右端点处不一定连续)

Thm. f在(a,b)上可导,则f在$[a,b]$下凸的充要条件是:

$\forall x_0 \in (a,b), \forall x \in [a,b]$,有

$$f(x) \geq f(x_0) + f'(x_0)(x - x_0),$$

即曲线$y = f(x)$上的每一点的切线在曲线下方.

Proof. 略。

Thm. f在$[a,b]$上可导,则f在$[a,b]$下凸的充要条件是:

$\forall x_0, x \in [a,b]$,有

$$f(x) \geq f(x_0) + f'(x_0)(x - x_0),$$

即曲线$y = f(x)$上的每一点的切线在曲线下方.

Thm. $f\in C[a,b]$, f在(a,b)上可导, 则

f在$[a,b]$下(上)凸 $\Leftrightarrow$ f'在(a,b)单调递增(递减).

Proof. $\Rightarrow$: 任取$x_1,x_2\in(a,b), x_1<x_2$, f下凸, 则

$$\frac{f(x)-f(x_1)}{x-x_1}\le\frac{f(x_2)-f(x)}{x_2-x},\qquad \forall x\in(x_1,x_2).$$

分别令$x\to x_1^+, x\to x_2^-$, 得

$$f'(x_1)\le\frac{f(x_2)-f(x_1)}{x_2-x_1}\le f'(x_2).$$

故 $f'(x)$单调递增.

$\Leftarrow$: 任取$x_1, x, x_2 \in [a,b], x_1 < x < x_2$, 由Lagrange中值定理,

$\exists \xi_1 \in (x_1, x), \xi_2 \in (x, x_2), s.t.$

$$\frac{f(x)-f(x_1)}{x-x_1} = f'(\xi_1), \quad \frac{f(x_2)-f(x)}{x_2-x} = f'(\xi_2).$$

f'单调递增, $f'(\xi_1) \le f'(\xi_2)$, 则

$$\frac{f(x)-f(x_1)}{x-x_1} \le \frac{f(x_2)-f(x)}{x_2-x}.$$

故f下凸.□

Thm. $f\in C[a,b]$, f在(a,b)上二阶可导,则

f在$[a,b]$下(上)凸$\Leftrightarrow$在(a,b)中$f''(x)\geq(\leq)0$.

Proof. 此为上一定理之推论.□

Def. 若$y=f(x)$在$(x_0,f(x_0))$两侧有不同的凸凹性,则称$(x_0,f(x_0))$为曲线$y=f(x)$的拐点.

Thm. $(x_0,f(x_0))$为$y=f(x)$的拐点, $f''(x_0)$存在,则$f''(x_0)=0$.

Proof. $f''(x_0)$存在,则f'在x_0的邻域中有定义. $(x_0,f(x_0))$为拐点,则$f'(x)$在x_0两侧有不同的单调性, x_0为$f'(x)$的极值点,故$f''(x_0)=0$.□

Ex. $x_1, x_2, \cdots, x_n > 0$, 则 $\sqrt[n]{x_1 x_2 \cdots x_n} \le \dfrac{x_1 + x_2 + \cdots + x_n}{n}$.

Proof. 令 $f(x) = \ln x$, 则

$$f'(x) = \frac{1}{x}, \quad f''(x) = -\frac{1}{x^2} < 0,$$

故 f 上凸,

$$\frac{\ln x_1 + \ln x_2 + \cdots + \ln x_n}{n} \le \ln \frac{x_1 + x_2 + \cdots + x_n}{n}$$

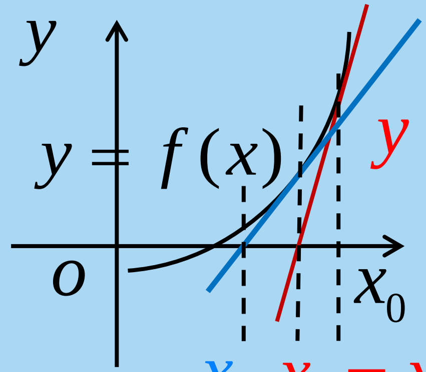
$$\Leftrightarrow \sqrt[n]{x_1 x_2 \cdots x_n} \le \frac{x_1 + x_2 + \cdots + x_n}{n}. \square$$

Ex. 曲线$x=t-\sin t, y=1-\cos t\ (0\le t\le 2\pi)$的凸凹性.

解: $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{y'(t)}{x'(t)}=\dfrac{\sin t}{1-\cos t}$,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}(\frac{\mathrm{d}y}{\mathrm{d}x})}{\mathrm{d}x}=\frac{\left(\dfrac{\sin t}{1-\cos t}\right)'}{x'(t)}=\frac{-1}{(1-\cos t)^2}<0.$$

故曲线在$t\in[0,2\pi]$上上凸.□



$$y = f(x_0) + f'(x_0)(x - x_0)$$

$$x_2 \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Thm.(Newton法)

设 $f \in C^2[a,b]$, 且

$$f(a)f(b) < 0, \quad f'(x)f''(x) \neq 0, \forall x \in [a,b],$$

若$x_0 \in [a,b]$,满足 $f(x_0)f''(x_0) > 0$, 则由迭代公式

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

得到的$\{x_n\}$收敛到 $f(x) = 0$ 在 $[a,b]$ 中的唯一解c.

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Proof.不妨设 $f'>0, f''>0$,即f 在$[a,b]$上严格单增、下凸.

又因$f(a)f(b)<0$,所以f在$[a,b]$上有唯一零点c.

$f(x_0)f''(x_0)>0$, 则 $f(x_0)>0$. 而 $f(c)=0$, f严格单增,故

$$x_0>c, \qquad x_1=x_0-\frac{f(x_0)}{f'(x_0)}<x_0.$$

令$\varphi(x)=x-\dfrac{f(x)}{f'(x)}$,则 $\varphi'(x)=\dfrac{f(x)f''(x)}{(f'(x))^2}$, $\exists\xi\in(c,x_0)$.s.t.

$$x_1-c=\varphi(x_0)-\varphi(c)=\varphi'(\xi)(x_0-c)>0.$$

至此,我们证明了:只要$x_0>c$,就有$c<x_1<x_0$. 归纳可证

$$c < x_n < x_{n-1}, \quad n \geq 1.$$

$\{x_n\}$单调递减有下界c,因而收敛, 则$\lim\limits_{n\to\infty} x_n = x_* \in [c,b]$. 在

$x_n = x_{n-1} - \dfrac{f(x_{n-1})}{f'(x_{n-1})}$ 中令$n \to \infty$, 由f, f'的连续性, 得

$$x_* = x_* - \frac{f(x_*)}{f'(x_*)}, \quad f(x_*) = 0.$$

而 f在$[c,b]$上严格单增, $f(c)=0$, 故$\lim\limits_{n\to\infty} x_n = x_* = c$.□

Remark.Newton法中条件$f(x_0)f''(x_0) > 0$可以去掉.

$$x_0 < c, f\text{下凸(切线在曲线下方)} \Rightarrow x_1 > c.$$

Thm.(Newton法2次收敛)设 f 在 $[a,b]$ 上二阶连续可微,

$c\in(a,b)$ 是 $f(x)=0$ 的根, 且 $f'(x)f''(x)\neq 0, \forall x\in[a,b]$,

则Newton法产生的迭代数列 $x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)}$ 满足

$$|x_{n+1}-c|\le q|x_n-c|^2,$$

其中 $q=\dfrac{M}{2m}, M=\sup\limits_{x\in[a,b]}|f''(x)|, m=\inf\limits_{x\in[a,b]}|f'(x)|$.

Proof. $0=f(c)=f(x_n)+f'(x_n)(c-x_n)+\dfrac{1}{2}f''(\xi)(c-x_n)^2$.

$$x_{n+1}-c=x_n-\frac{f(x_n)}{f'(x_n)}-c=\frac{f''(\xi)}{2f'(x_n)}(c-x_n)^2. \ \square$$

§ 6.函数作图

Def.(1)若 $\lim\limits_{x\to x_0^+} f(x)=\infty$或$\lim\limits_{x\to x_0^-} f(x)=\infty$,则称$x=x_0$为$y=$

$f(x)$的一条竖直渐近线;

(2)若 $\lim\limits_{x\to+\infty} f(x)=a$或$\lim\limits_{x\to-\infty} f(x)=a$,则称$y=a$为$y=f(x)$

的一条水平渐近线;

(3)若$\exists a\neq 0$及b,使得

$\lim\limits_{x\to+\infty}(f(x)-ax-b)=0$ 或 $\lim\limits_{x\to-\infty}(f(x)-ax-b)=0$,

则称$y=ax+b$为$y=f(x)$的一条斜渐近线.

Remark. $\lim\limits_{x\to+\infty}(f(x)-ax-b)=0$

$$\Leftrightarrow \lim_{x\to+\infty}\frac{f(x)}{x}=a \ \ \& \lim_{x\to+\infty}(f(x)-ax)=b.$$

函数作图关键因素:

(1)定义域;　　(2)奇偶性、周期性、对称性

(3)渐近线

(4)极值点与增减区间

(5)拐点与凸凹性

(6)特殊点, 如$f(x_0)=0$, 极值点, 拐点的函数值

Ex.作图$y=\dfrac{(x-1)^3}{2(x+1)^2}$.

解:定义域: $(-\infty,-1)\cup(-1,+\infty)$.

$$\lim_{x\to-1} y(x)=-\infty,\quad \lim_{x\to\pm\infty} y(x)=\pm\infty,$$

有竖直渐近线 $x=-1$, 无水平渐近线.

$$\lim_{x\to\infty}\frac{y(x)}{x}=\lim_{x\to\infty}\frac{(x-1)^3}{2x(x+1)^2}=\frac{1}{2},$$

$$\lim_{x\to\infty}\left(y(x)-\frac{1}{2}x\right)=\lim_{x\to\infty}\frac{-5x^2+2x-1}{2(x+1)^2}=-\frac{5}{2}.$$

有斜渐近线: $y=\dfrac{1}{2}x-\dfrac{5}{2}$.

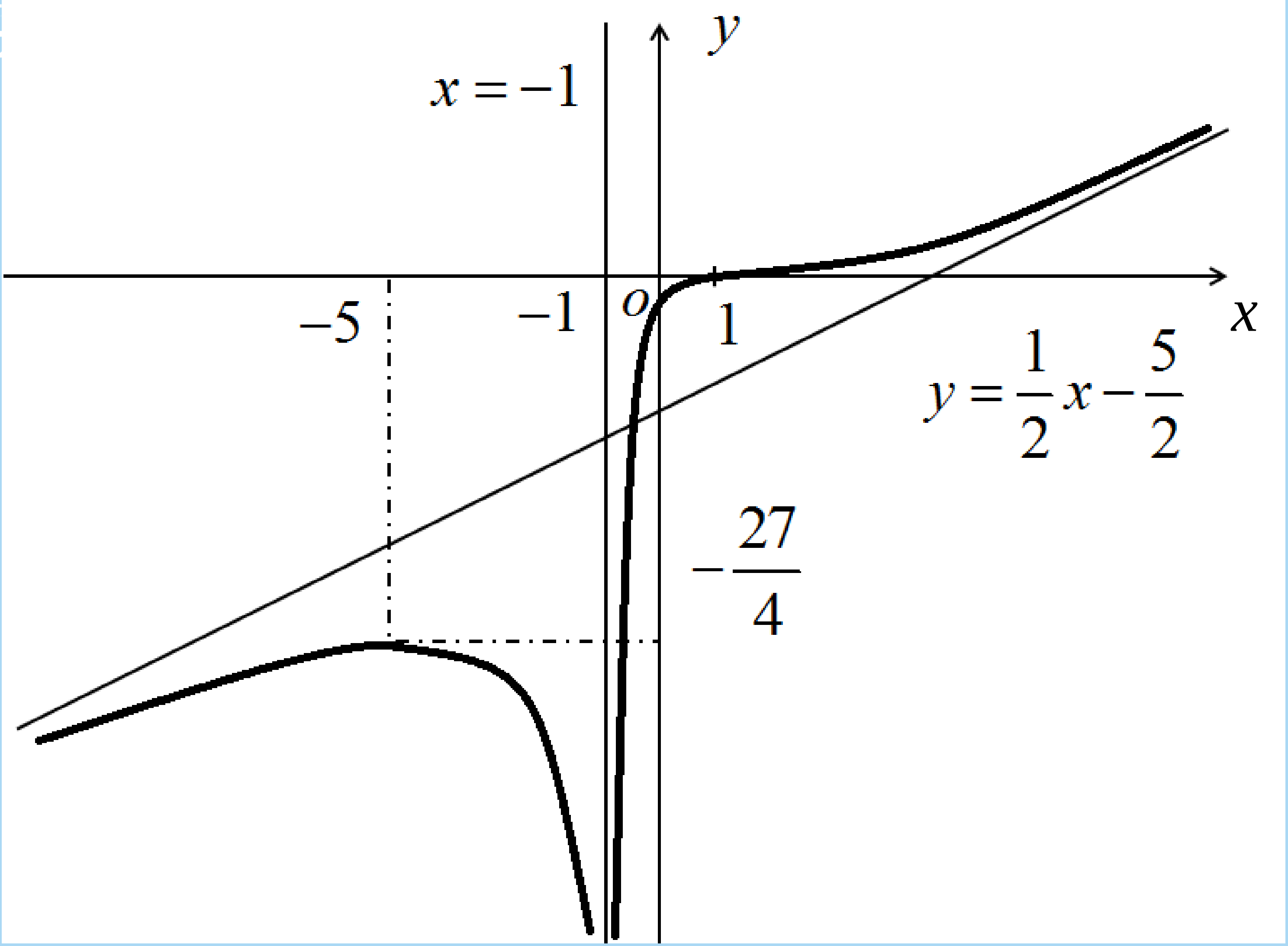
$$\ln|y| = 3\ln|x-1| - 2\ln|x+1| - \ln 2,\quad \frac{y'}{y} = \frac{3}{x-1} - \frac{2}{x+1} = \frac{x+5}{x^2-1},$$

$$y' = \frac{x+5}{x^2-1}y = \frac{(x+5)(x-1)^2}{2(x+1)^3},\ \cdots,\ y'' = \frac{12(x-1)}{(x+1)^4},$$

$$y'(-5) = y'(1) = 0.\quad y''(1) = 0.$$

x	$(-\infty,-5)$	-5	$(-5,-1)$	-1	$(-1,1)$	1	$(1,+\infty)$
$f'(x)$	$+$	0	$-$	无	$+$	0	$+$
$f''(x)$	$-$	$-$	$-$	定	$-$	0	$+$
$f(x)$	$\nearrow$	$\frac{-27}{4}$	$\searrow$	义	$\nearrow$	0	$\nearrow$

极大　　　　　　　　　　拐点



y
$x=-1$
-5
-1
O
1
x
$y=\frac{1}{2}x-\frac{5}{2}$
$-\frac{27}{4}$

Question.如何求隐函数、参数函数的函数渐近线？

Ex.曲线 $y^3 - x^3 + 2xy = 0$ 的渐近线.

解:令 $x = r(\theta)\cos\theta, y = r(\theta)\sin\theta, r \geq 0, \theta \in [0, 2\pi)$, 则

$$r(\theta) = \frac{2\sin\theta\cos\theta}{\cos^3\theta - \sin^3\theta}, \quad \theta \in [0, \frac{\pi}{4}) \cup [\frac{\pi}{2}, \pi] \cup (\frac{5\pi}{4}, \frac{3\pi}{2}].$$

$$\lim_{\theta \to (\frac{\pi}{4})^-} r(\theta) = \lim_{\theta \to (\frac{5\pi}{4})^+} r(\theta) = +\infty.$$

$$\lim_{x \to +\infty} \frac{y}{x} = \lim_{\theta \to (\frac{\pi}{4})^-} \frac{r(\theta)\sin\theta}{r(\theta)\cos\theta} = 1, \quad \lim_{x \to -\infty} \frac{y}{x} = \lim_{\theta \to (\frac{5\pi}{4})^+} \frac{r(\theta)\sin\theta}{r(\theta)\cos\theta} = 1,$$

$$\lim_{x\to\infty}(y-x)=\lim_{x\to\infty}\frac{-2xy}{x^2+xy+y^2}=\lim_{x\to\infty}\frac{-2\dfrac{y}{x}}{1+\dfrac{y}{x}+(\dfrac{y}{x})^2}=-\frac{2}{3}.$$

因此 $y^3-x^3+2xy=0$ 有斜渐近线 $y=x-\dfrac{2}{3}$.

若$y^3-x^3+2xy=0$ 有竖直渐近线,不妨设 $\lim\limits_{x\to x_0^+} y(x)=+\infty$,

则 $$\infty=\lim_{x\to x_0^+} y^2(x)=\lim_{x\to x_0^+}(\frac{x^3}{y(x)}-2x)=2x_0,$$

矛盾,因此无竖直渐近线. 同理,没有水平渐近线.□

question. $y^3 - x^3 + 2xy = 0$ 作图？

作业：习题4.5 No.3,4,5(1),8

习题4.6 No.1(3),2(3)