Review

- $\inf A,\ \sup A,\ \min A,\ \max A$
- $\lim\limits_{n\to\infty} a_n = A$的$\varepsilon$-N语言描述

$\Leftrightarrow \forall\varepsilon>0, \exists N\in\mathbb{N}$,s.t. 当$n>N$时,有$a_n\in(A-\varepsilon, A+\varepsilon)$

$\Leftrightarrow \forall\varepsilon>0, \exists N\in\mathbb{N}$,s.t. 当$n\geq N$时,有$A-\varepsilon\leq a_n\leq A+\varepsilon$

$\Leftrightarrow \forall\varepsilon>0, \exists N\in\mathbb{N}$,s.t. 当$n>N$时,有$|a_n-A|<\varepsilon/2$

$\Leftrightarrow \forall\varepsilon\in(0,1), \exists N\in\mathbb{N}$,s.t. 当$n\geq N$时,有$|a_n-A|\leq 2\varepsilon$.

$\Leftrightarrow \forall k\in\mathbb{N}, \exists N=N(k)\in\mathbb{N}$,s.t. 当$n\geq N$时,有$|a_n-A|\leq \dfrac{1}{2^k}$.

- ε-N语言中N的选取.

(1) 放缩法求解不等式$|a_n - \mathrm{A}| < \varepsilon$

$$|a_n - \mathrm{A}| < \cdots < \boxed{n\text{的简单表达式}} < \varepsilon$$

(2) 分段法确定N

$$\mathrm{N} = \max\{N_1, N_2, \cdots, N_k\}$$

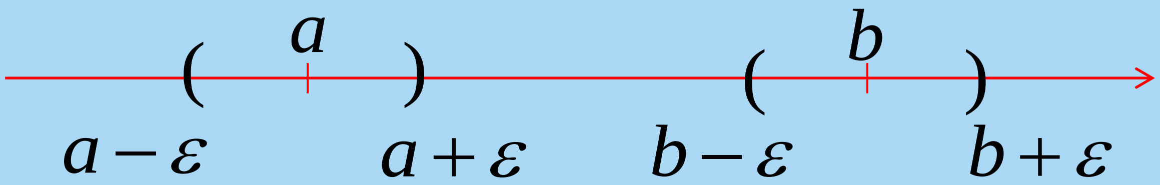
- 记住一些基本结论.

$$\lim_{n\to\infty} \sqrt[n]{n} = 1, \qquad \lim_{n\to\infty} a_n\text{存在} \Rightarrow \lim_{n\to\infty} e^{a_n} = e^{\lim\limits_{n\to\infty} a_n},$$

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0, \qquad \left.\begin{aligned} &a_n > 0, \\ &\lim_{n\to\infty} a_n\text{存在且非零} \end{aligned}\right\} \Rightarrow \lim_{n\to\infty} \ln a_n = \ln\left(\lim_{n\to\infty} a_n\right)$$

§ 3. 收敛列的性质

Prop1. 收敛列的极限唯一.



Proof. 假设a,b均为$\{a_n\}$的极限，且$a\neq b$. $\forall 0<\varepsilon<\dfrac{|a-b|}{2}$,

由$\lim\limits_{n\to\infty} a_n = a$, $\exists N_1$, 当$n>N_1$时, $|a_n-a|<\varepsilon$,

由$\lim\limits_{n\to\infty} a_n = b$, $\exists N_2$, 当$n>N_2$时, $|a_n-b|<\varepsilon$.

令$N=\max\{N_1,N_2\}+1$, 则$N>N_i, i=1,2$,

$$|a-b| = |(a-a_N)-(b-a_N)| \le |a-a_N|+|b-a_N|$$

$$<2\varepsilon<|a-b|,$$ 矛盾.□

$$\lim_{n\to\infty} a_n = \mathrm{A} \Leftrightarrow \forall \varepsilon>0, \exists \mathrm{N} \in \mathbb{N}, \text{s.t. } \forall n > \mathrm{N}, 有 |a_n - \mathrm{A}| < \varepsilon.$$

$$\lim_{n\to\infty} a_n \neq \mathrm{A} \Leftrightarrow \exists \varepsilon>0, \forall \mathrm{N} \in \mathbb{N}, \exists n > \mathrm{N}, \text{s.t.} |a_n - \mathrm{A}| > \varepsilon.$$

Prop2. 在数列中添加、删除有限项, 或者改变有限项的值, 不改变数列的敛散性与极限值.

Def. $0 < n_1 < n_2 < \cdots < n_k < \cdots$ 为一列自然数, 称 $\{a_{n_k}\}$ 为 $\{a_n\}$ 的一个子列.

Prop3. (收敛列的任意子列具有相同的极限)

$$\lim_{n\to\infty} a_n = a \Rightarrow \lim_{k\to\infty} a_{n_k} = a. \qquad (n_k \geq k)$$

Corollary. $\lim_{k\to\infty} a_{n_k} = a \neq b = \lim_{k\to\infty} a_{m_k} \Rightarrow \{a_n\}$ 发散.

Ex. $\{(-1)^n\}$ 发散.

Question. $\lim\limits_{n\to\infty} a_{2n} = \lim\limits_{n\to\infty} a_{2n+1} = \text{A} \overset{?}{\Rightarrow} \lim\limits_{n\to\infty} a_n = \text{A}.$ *Yes*!

Prop4. 收敛列一定有界.

Proof. 设 $\lim\limits_{n\to\infty} a_n = a$. 对 $\varepsilon = 1, \exists N$, 当 $n > N$ 时, $|a_n - a| < 1$.

因此, $|a_n| = |(a_n - a) + a| \le |a_n - a| + |a| < |a| + 1, \forall n > N$.

令 $M = \max\{|a_1|, |a_2|, \cdots, |a_N|, |a| + 1\}$, 则 $|a_n| \le M, \forall n \in \mathbb{N}$. $\square$

Question. 有界列是否必为收敛列? *No*!

Def. 若 $\lim\limits_{n\to\infty} a_n = 0$, 则称 $\{a_n\}$ 为无穷小数列.

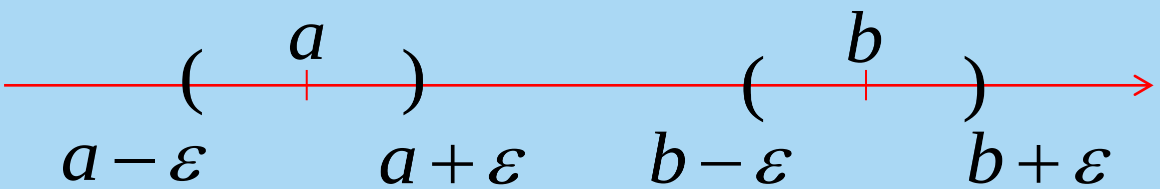
Prop5. $\{a_n\}$为无穷小列,$\{b_n\}$为有界列,则$\{a_nb_n\}$为无穷小列.

Prop6. (极限的保序性) $\lim\limits_{n\to\infty} a_n = a, \lim\limits_{n\to\infty} b_n = b.$

(1)若$a<b$,则$\exists N$,当$n>N$时有$a_n<b_n$.

(2)若$\exists N$,当$n>N$时有$a_n\le b_n$,则$a\le b$.

Proof. (1)



(2)反设$a>b$.由(1)中结论,$\exists N_1$,当$n>N_1$时有$a_n>b_n$.矛盾.□

Question. $\left.\begin{array}{l}\lim\limits_{n\to\infty} a_n = a, \lim\limits_{n\to\infty} b_n = b \\ a_n < b_n, \quad \forall n\end{array}\right\} \overset{?}{\Rightarrow} a<b$ $(\times)$

Prop7. (极限的四则运算) 若$\{a_n\}$与$\{b_n\}$都收敛,则

(1)$\forall c\in\mathbb{R},\ \lim\limits_{n\to\infty}(ca_n)=c\lim\limits_{n\to\infty}a_n$;

(2)$\lim\limits_{n\to\infty}(a_n\pm b_n)=\lim\limits_{n\to\infty}a_n\pm\lim\limits_{n\to\infty}b_n$;

(3)$\lim\limits_{n\to\infty}(a_n\cdot b_n)=\lim\limits_{n\to\infty}a_n\cdot\lim\limits_{n\to\infty}b_n$;

(4) $\lim\limits_{n\to\infty}b_n\neq 0$时, $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n}=\dfrac{\lim\limits_{n\to\infty}a_n}{\lim\limits_{n\to\infty}b_n}$.

Remark. 可以推广到$\lim\limits_{n\to\infty}a_n,\lim\limits_{n\to\infty}b_n$为$\infty$或$\pm\infty$的情形, 但要右端运算有意义。

Remark. (3)$a>0,\ a\cdot\pm\infty=\pm\infty$; $0\cdot\infty, 0\cdot\pm\infty$没有意义.

(4)$a\neq\infty,\dfrac{a}{\infty}=0$; $a\neq 0,\dfrac{a}{0}=\infty$; $\dfrac{0}{0},\dfrac{(\pm)\infty}{(\pm)\infty}$没有意义.

Proof. (3) $\{b_n\}$收敛,则有界, $\exists M>0, s.t.$

$$|b_n|<M, \quad \forall n.$$

因 $\lim\limits_{n\to\infty} a_n = a, \lim\limits_{n\to\infty} b_n = b,\ \forall \varepsilon>0, \exists N_1, N_2, s.t.$

$$|a_n-a|<\varepsilon, \quad \forall n>N_1,$$

$$|b_n-b|<\varepsilon, \quad \forall n>N_2.$$

当$n>N=\max\{N_1,N_2\}$时,

$$\begin{aligned}|a_nb_n-ab| &= |a_nb_n-ab_n+ab_n-ab| \\ &\le |a_n-a||b_n|+|a||b_n-b| \\ &\le (M+|a|)\varepsilon.\end{aligned}$$

(4) $\lim\limits_{n\to\infty} b_n = b \neq 0$, 对$\varepsilon_0 = |b|/2, \exists N_1$, 当$n > N_1$时, $|b_n - b| < \varepsilon_0$.

因此, $|b_n| = |b_n - b + b| \geq |b| - |b_n - b|$

$$> |b| - \varepsilon_0 = |b|/2, \quad \forall n > N_1.$$

因$\lim\limits_{n\to\infty} a_n = a, \lim\limits_{n\to\infty} b_n = b, \forall \varepsilon > 0, \exists N_2, N_3, s.t.$

$$|a_n - a| < \varepsilon, \ \forall n > N_2; \quad |b_n - b| < \varepsilon, \ \forall n > N_3.$$

当$n > N = \max\{N_1, N_2, N_3\}$时,

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| = \frac{|ba_n - ab_n|}{|bb_n|} \leq \frac{2}{b^2}|ba_n - ab + ab - ab_n|$$

$$\leq \frac{2}{b^2}\left(|b||a_n - a| + |a||b - b_n|\right) \leq \frac{2}{b^2}\left(|b| + |a|\right)\varepsilon. \square$$

Prop8. (夹挤原理) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = a$, 且$\exists n_0, s.t. \forall n > n_0$,

有$a_n \le x_n \le b_n$, 则$\lim\limits_{n\to\infty} x_n = a$.

Proof. 因$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = a, \forall \varepsilon > 0, \exists N_1, N_2, s.t.$

$$-\varepsilon < a_n - a < \varepsilon, \quad \forall n > N_1,$$

$$-\varepsilon < b_n - a < \varepsilon, \quad \forall n > N_2.$$

又$n > n_0$时, $a_n \le x_n \le b_n$, 令$N = \max\{n_0, N_1, N_2\}$, 则

$$-\varepsilon < a_n - a \le x_n - a \le b_n - a < \varepsilon, \quad \forall n > N.$$

因此$\lim\limits_{n\to\infty} x_n = a$.□

Question. 能否用极限的保序性直接得到夹挤原理？否！

Ex. 设$b>0, a>1$.证明：$\lim\limits_{n\to\infty}\dfrac{n^b}{a^n}=0$.

Proof. 法一：$\ln\dfrac{n^b}{a^n}=\ln\dfrac{\left(\sqrt[n]{n}\right)^{nb}}{\left(\sqrt[b]{a}\right)^{nb}}=nb\left(\ln\sqrt[n]{n}-\ln\sqrt[b]{a}\right)$

$$\begin{aligned}\lim_{n\to\infty}\ln\frac{n^b}{a^n} &= \lim_{n\to\infty}nb\cdot\lim_{n\to\infty}\left(\ln\sqrt[n]{n}-\ln\sqrt[b]{a}\right)\\ &= +\infty\cdot(\ln 1-\ln\sqrt[b]{a})\\ &= -\infty\end{aligned}$$

$$\lim_{n\to\infty}\frac{n^b}{a^n}=\lim_{n\to\infty}\exp\left(\ln\frac{n^b}{a^n}\right)=\exp\left(\lim_{n\to\infty}\ln\frac{n^b}{a^n}\right)=\exp(-\infty)=0.$$

法二：(1)$b=k$为整数时. 令$d=a-1$，则$d>0$，当$n\geq 2k$时

$$0<\frac{n^k}{a^n}=\frac{n^k}{(1+d)^n}<\frac{n^k}{C_n^{k+1}d^{k+1}}=\frac{n^k(k+1)!}{n(n-1)\cdots(n-k)d^{k+1}}$$

$$=\frac{(k+1)!}{n(1-\frac{1}{n})\cdots(1-\frac{k}{n})d^{k+1}}\leq\frac{2^k(k+1)!}{nd^{k+1}}$$

$$\lim_{n\to\infty}\frac{2^k(k+1)!}{nd^{k+1}}=0,$$ 由夹挤原理，$$\lim_{n\to\infty}\frac{n^k}{a^n}=0.$$

(2) $b>0$ 不为整数时. $0<\dfrac{n^b}{a^n}<\dfrac{n^{\lfloor b\rfloor+1}}{a^n}$.

由(1)中结论, $\lim\limits_{n\to\infty}\dfrac{n^{\lfloor b\rfloor+1}}{a^n}=0$. 由夹挤原理, $\lim\limits_{n\to\infty}\dfrac{n^b}{a^n}=0$.□

Ex. $a>0$, 证明: $\lim\limits_{n\to\infty}\sqrt[n]{a}=1$.

Proof. 当 $a\geq 1$ 时, $\forall n>\lceil a\rceil$, 有 $1\leq\sqrt[n]{a}<\sqrt[n]{n}$, 而 $\lim\limits_{n\to\infty}\sqrt[n]{n}=1$,

由夹挤原理, $\lim\limits_{n\to\infty}\sqrt[n]{a}=1$.

当 $0<a<1$ 时, $\lim\limits_{n\to\infty}\sqrt[n]{a}=\lim\limits_{n\to\infty}\dfrac{1}{\sqrt[n]{1/a}}=\dfrac{1}{\lim\limits_{n\to\infty}\sqrt[n]{1/a}}=1$.□

Question. 错在哪里？

$$1=\lim_{n\to\infty}\underbrace{\left(\frac{1}{n}+\frac{1}{n}+\cdots\frac{1}{n}\right)}_{n\text{个}}$$

$$=\underbrace{\lim_{n\to\infty}\frac{1}{n}+\lim_{n\to\infty}\frac{1}{n}+\cdots+\lim_{n\to\infty}\frac{1}{n}}_{n\text{个}}=0+0+\cdots 0=0$$

无穷个0相加

Question. 谁对谁错？错在哪里？

$$(1)\lim_{n\to\infty}\frac{1}{\sqrt[n]{n!}}=\lim_{n\to\infty}\underbrace{\left(\sqrt[n]{1}\cdot\sqrt[n]{\frac{1}{2}}\cdot\sqrt[n]{\frac{1}{3}}\cdot\cdots\cdot\sqrt[n]{\frac{1}{n}}\right)}_{n个}$$

$$=\underbrace{\lim_{n\to\infty}\sqrt[n]{1}\cdot\lim_{n\to\infty}\sqrt[n]{\frac{1}{2}}\cdot\cdots\cdot\lim_{n\to\infty}\sqrt[n]{\frac{1}{n}}}_{n个}=1\cdot1\cdots\cdot1=1$$

$$（2）\ n!>\left(\frac{n}{2}\right)^{n/2},0<\frac{1}{\sqrt[n]{n!}}<\frac{1}{\sqrt{n/2}},\lim_{n\to\infty}\frac{1}{\sqrt[n]{n!}}=0.$$

作业：习题1.3 No. 4(单),6,8

No.8 $$\frac{1}{2\sqrt{n}} \le \frac{1}{2}\cdot\frac{3}{4}\cdot\cdots\cdots\frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$$