Review

- f为I上的下凸函数(定义与几何意义)

$$f(\lambda x_1+(1-\lambda)x_2)\le \lambda f(x_1)+(1-\lambda)f(x_2),$$
$$\forall x_1,x_2\in I,\forall \lambda\in[0,1].$$

$$f(\lambda_1 x_1+\cdots+\lambda_n x_n)\le \lambda_1 f(x_1)+\cdots+\lambda_n f(x_n),$$
$$\forall x_1,\cdots,x_n\in I,\lambda_1+\cdots+\lambda_n=1,\lambda_1,\cdots,\lambda_n\ge 0.$$

$$\frac{f(x)-f(x_1)}{x-x_1}\le \frac{f(x_2)-f(x_1)}{x_2-x_1}\le \frac{f(x_2)-f(x)}{x_2-x},$$
$$\forall x_1,x,x_2\in I,x_1<x<x_2.$$

- $f\in C[a,b]$, f在(a,b)上可导,则

  f在$[a,b]$下凸$\Leftrightarrow f'$在(a,b)单调递增.

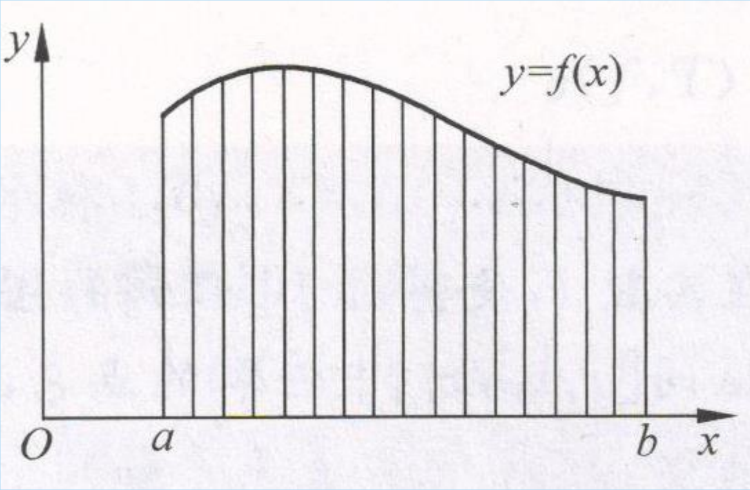
- $f\in C[a,b]$, f在(a,b)上二阶可导,则

  f在$[a,b]$下凸$\Leftrightarrow$在(a,b)中$f''(x)\geq 0$.

- 拐点的定义
- $(x_0, f(x_0))$为$y=f(x)$的拐点, $f''(x_0)$存在,则$f''(x_0)=0$.
- 函数作图

§1.Riemann积分的几何意义与概念

1. 曲边梯形的有向面积



Step1.分割

$$T: a = x_0 < x_1 < \cdots < x_n = b.$$

$$\Delta x_i \triangleq x_i - x_{i-1},\ |T| = \max_{1\le i\le n}\{\Delta x_i\}.$$

Step2.取标志点 $\xi_i \in [x_{i-1}, x_i]$.

以直代曲的思想!

Step3.近似求和. $S \approx \sum_{i=1}^{n} f(\xi_i)\Delta x_i,$

Step4.取极限. $\lim_{|T|\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i = S.$

2. 变力做功

直线运动, 位移x, 力$F(x)$.

Step1.分割

$$T: a = x_0 < x_1 < \cdots < x_n = b.$$

Step2.取标志点$\xi_i \in [x_{i-1}, x_i]$.

Step3.近似求和. $\mathrm{W} \approx \sum_{i=1}^{n} F(\xi_i)\Delta x_i$,

Step4.取极限. $\lim_{|T|\to 0} \sum_{i=1}^{n} F(\xi_i)\Delta x_i = \mathrm{W}$.

Def. 设f为闭区间$[a,b]$上的有界函数, 若存在实数I,s.t.对$[a,b]$的任何一个分割$T: a=x_0<x_1<\cdots<x_n=b$, 对任意$\{\xi_i\}, \xi_i\in[x_{i-1},x_i], 1\le i\le n$, 只要$|T|=\max\limits_{1\le i\le n}\{\Delta x_i\}\to 0$, 就有

$$\lim_{|T|\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i = \mathrm{I},$$

即$\forall\varepsilon>0, \exists\delta>0$, 当$|T|<\delta$时, 无论$\xi_i\in[x_{i-1},x_i]$如何取, 都有

$$\left|\sum_{i=1}^{n} f(\xi_i)\Delta x_i - \mathrm{I}\right| < \varepsilon,$$

则称f在$[a,b]$上Riemann可积,称I为f在$[a,b]$上的Riemann积分, 记为

$$\int_a^b f(x)\mathrm{d}x = \mathrm{I}.$$

b, a, f, x分别称为积分上、下限, 被积函数和积分变量.

Def. $[a,b]$上全体Riemann可积函数记为$R[a,b]$.

Remark. 曲边梯形的有向面积$S=\int_a^b f(x)\mathrm{d}x$.

曲边梯形的面积为 $\int_a^b |f(x)|\mathrm{d}x$.

变力做功$W=\int_a^b F(x)\mathrm{d}x$.

Remark. $\int_a^b f(x)\mathrm{d}x=\int_a^b f(t)\mathrm{d}t=\int_a^b f$.

$$\int_b^a f(x)\mathrm{d}x \triangleq -\int_a^b f(x)\mathrm{d}x$$

$$\int_a^a f(x)\mathrm{d}x=0.$$

3. 积分存在的条件

$$T: a = x_0 < x_1 < \cdots < x_n = b. \quad \Delta x_i \triangleq x_i - x_{i-1}, \quad \xi_i \in [x_{i-1}, x_i].$$

$$M \triangleq \sup_{x\in[a,b]} f(x), \quad m \triangleq \inf_{x\in[a,b]} f(x). \quad \omega(f) \triangleq M - m.$$

$$M_i \triangleq \sup_{x\in[x_{i-1},x_i]} f(x), \quad m_i \triangleq \inf_{x\in[x_{i-1},x_i]} f(x). \quad \omega_i(f) \triangleq M_i - m_i.$$

$$L(f,T) = \sum_{i=1}^{n} m_i \Delta x_i, \quad U(f,T) = \sum_{i=1}^{n} M_i \Delta x_i,$$

$$\sigma(f,T,\{\xi_i\}) = \sum_{i=1}^{n} f(\xi_i)\Delta x_i.$$

$$m(b-a) \le L(f,T) \le \sigma(f,T,\{\xi_i\}) \le U(f,T) \le M(b-a).$$

Def. $U(f,T), L(f,T), \sigma(f,T,\{\xi_i\})$分别称为$f$在$[a,b]$关于$T$的Darboux上和、Darboux下和与Riemann和.

Lemma1. f在$[a,b]$有界, M, m为上、下确界, T为$[a,b]$的任一分割, T_k是在T中加入k个新分点得到的分割, 则有

$$0 \le U(f,T) - U(f,T_k) \le k|T|(M-m);$$

$$0 \le L(f,T_k) - L(f,T) \le k|T|(M-m).$$

Proof. 只证上和(下和同理).设$T: a = x_0 < x_1 < \cdots < x_n = b$. T依次添加一个分点后得到的分割记为$T_1, T_2, \cdots, T_k$.

$$T_1: a = x_0 < x_1 < \cdots < x_{i-1} < t < x_i < \cdots < x_n = b.$$

$$0 \le U(f,T) - U(f,T_1)$$

$$= M_i(x_i - x_{i-1}) - \sup_{x_{i-1}\le x\le t} f(x)(t - x_{i-1}) - \sup_{t\le x\le x_i} f(x)(x_i - t)$$

$$\le (M_i - m_i)(x_i - x_{i-1}) \le |T|(M - m).$$

$$0 \le U(f,T) - U(f,T_k)$$

$$= [U(f,T) - U(f,T_1)] + [U(f,T_1) - U(f,T_2)]$$

$$+ \cdots + [U(f,T_{k-1}) - U(f,T_k)]$$

$$\le (M - m)|T| + (M - m)|T_1| + \cdots + (M - m)|T_{k-1}|$$

$$\le k(M - m)|T|. \square$$

Lemma2. f在$[a,b]$有界,T_1,T_2为$[a,b]$的任意两个分割,则

$$L(f,T_1)\le U(f,T_2).$$

Proof. 合并T_1,T_2的分点得到$[a,b]$的分割T,则

$$L(f,T_1)\le L(f,T)\le U(f,T)\le U(f,T_2).\square$$

Def. f在$[a,b]$有界,分别称

$$\overline{\int_a^b} f(x)\mathrm{d}x=\inf\{U(f,T):T\text{为}[a,b]\text{的分割}\},$$

$$\underline{\int_a^b} f(x)\mathrm{d}x=\sup\{L(f,T):T\text{为}[a,b]\text{的分割}\}$$

为f在$[a,b]$上的Darboux上积分与Darboux下积分.

Lemma3. $L(f,T)\le\underline{\int_a^b} f(x)\mathrm{d}x\le\overline{\int_a^b} f(x)\mathrm{d}x\le U(f,T).$

Thm. f在$[a,b]$有界,则以下命题等价:

$(1)\ f\in R[a,b]$;

$(2)\ \forall\varepsilon>0,\exists[a,b]$的分割$T$, s.t. $U(f,T)-L(f,T)<\varepsilon$;

$(3)\ \underline{\int_a^b} f(x)\mathrm{d}x=\overline{\int_a^b} f(x)\mathrm{d}x.$

Proof. $(1)\Rightarrow(2)$: $f\in R[a,b]$, 记$\mathrm{I}=\int_a^b f(x)\mathrm{d}x$, 则

$\forall\varepsilon>0,\exists T,\forall\xi_i\in[x_{i-1},x_i]$, 有 $\left|\sum_{i=1}^n f(\xi_i)\Delta x_i-\mathrm{I}\right|<\varepsilon/3.$

上式两端对$\xi_i\in[x_{i-1},x_i]$取上、下确界,有

$$|U(f,T)-\mathrm{I}|\le\varepsilon/3,\qquad |L(f,T)-\mathrm{I}|\le\varepsilon/3.$$

$(2) \Rightarrow (3)$: 由(2)及Lemma3, $\forall \varepsilon > 0, \exists T, s.t.$

$$0 \le \overline{\int_a^b} f(x)\mathrm{d}x - \underline{\int_a^b} f(x)\mathrm{d}x \le U(f,T) - L(f,T) < \varepsilon.$$

$(3) \Rightarrow (1)$: 记$\mathrm{I} = \overline{\int_a^b} f(x)\mathrm{d}x = \underline{\int_a^b} f(x)\mathrm{d}x$. 由上积分定义,

$\forall \varepsilon > 0, \exists T_0 : a = t_0 < t_1 < \cdots < t_k = b, s.t.$

$$\mathrm{I} \le U(f,T_0) < \mathrm{I} + \varepsilon/2.$$

令$\delta_1 = \dfrac{\varepsilon}{2k(M-m)}$, 任给$[a,b]$的分割$T$, 当$|T| < \delta_1$时,

合并T与T_0的分点得T',

$$0 \le U(f,T) - \mathrm{I} \le U(f,T') + k|T|(M-m) - \mathrm{I}$$

$$\le U(f,T_0) - \mathrm{I} + k|T|(M-m) < \varepsilon.$$

同理, $\exists\delta_2>0$, 当$|T|<\delta_2$时,

$$0\le \mathrm{I}-L(f,T)<\varepsilon.$$

令$\delta=\min\{\delta_1,\delta_2\}$, 对$[a,b]$的任意分割$T$, 当$|T|<\delta$时, 不论如何选取标志点$\{\xi_i\}$, 都有

$$\mathrm{I}-\varepsilon< L(f,T)\le\sigma(f,T,\{\xi_i\})\le U(f,T)<\mathrm{I}+\varepsilon.\square$$

Ex. Dirichlet函数$D(x)$在$[0,1]$上不可积.

Proof. 对$[0,1]$的任意分割T,

$$U(D,T)-L(D,T)=1-0=1.\square$$

4. 可积函数类

Thm. $f\in C[a,b]\Rightarrow f\in R[a,b]$.

Proof. $f\in C[a,b]$,则f在$[a,b]$上一致连续. $\forall\varepsilon>0,\exists\delta>0$, 只要$x,y\in[a,b],|x-y|<\delta$,就有

$$|f(x)-f(y)|<\varepsilon.$$

令T为$[a,b]$的n等分分割,使得$|T|=\dfrac{b-a}{n}<\delta$,则

$$0\le M_i-m_i\le\varepsilon\quad(i=1,2,\cdots,n).$$

$$U(f,T)-L(f,T)=\sum_{i=1}^{n}(M_i-m_i)\Delta x_i\le\varepsilon(b-a).$$

故$f\in R[a,b]$.□

Thm. f在$[a,b]$上有界,且只有有限个间断点,则$f\in R[a,b]$.

Proof. 不妨设f有唯一间断点b.其他情形可类似证明.

f在$[a,b]$上有界,$\exists M>0, s.t. |f(x)|\le M\ \ (a\le x\le b).\forall\varepsilon>0,$

取$c\in(a,b), s.t.\ b-c<\varepsilon/(4M).\ f\in C[a,c]$,则$f\in R[a,c]$,

$\exists T_1: a=x_0<x_1<\cdots<x_n=c, s.t.\ U(f,T_1)-L(f,T_1)<\varepsilon/2.$

对$T_2: a=x_0<x_1<\cdots<x_{n-1}<c<b$,有

$$U(f,T_2)-L(f,T_2)$$

$$=U(f,T_1)+\sup_{c\le x\le b} f(x)(b-c)-L(f,T_1)-\inf_{c\le x\le b} f(x)(b-c)$$

$$\le U(f,T_1)-L(f,T_1)+2M(b-c)\ <\varepsilon.\square$$

Def. 数集$E \subset \mathbb{R}$,若$\forall \delta > 0$,$\exists$一列开区间(α_k, β_k), s.t.

$$E \subset \bigcup_{k=1}^{+\infty} (\alpha_k, \beta_k), \quad \text{且} \sum_{k=1}^{n} (\beta_k - \alpha_k) < \delta \ (\forall n),$$

则称E为零测集.

Ex. $[0,1] \cap \mathbb{Q}$ 为零测集.

Ex. $\mathbb{Q}$ 为零测集.

Thm. 有界函数 $f \in R[a,b]$的充要条件是: f在$[a,b]$上的间断点集E为零测集.

Thm. f在$[a,b]$上单调$\Rightarrow f\in R[a,b]$.

Proof. 不妨设f在$[a,b]$上单增. 对$[a,b]$的任一分割T, 有

$$0\le U(f,T)-L(f,T)=\sum_{i=1}^{n}\big(f(x_i)-f(x_{i-1})\big)\Delta x_i$$

$$\le |T|\sum_{i=1}^{n}\big(f(x_i)-f(x_{i-1})\big)=|T|(f(b)-f(a)).\square$$

作业：习题5.1 No.1(4)