Review

- 隐函数求导 $F:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^m,(x,y)\mapsto F(x,y)$, 若 $\dfrac{\partial F}{\partial y}$ 可逆, 则 $F(x,y)=0$ 确定隐"函数" $y=y(x)$, 求 $\dfrac{\partial y}{\partial x}$ 时有两种方法:

(1) 套用定理: $\dfrac{\partial y}{\partial x}=-\left(\dfrac{\partial F}{\partial y}\right)^{-1}\dfrac{\partial F}{\partial x}$.

求Jaccobi矩阵 $\dfrac{\partial F}{\partial x},\dfrac{\partial F}{\partial y}$ 时 x,y 相互独立!

(2) 将 $F(x,y)=0$ 中 y 视为 $y=y(x)$, 利用复合映射的链式法则, 方程组 $F(x,y(x))=0$ 两边对 x 求Jaccobi矩阵.

Remark: 对具体的例子, 不必死记硬背隐函数定理中的公式, 只要将某些变量视为其它变量的隐函数, 再利用复合函数的求导法则即可.

Remark: m个方程确定m个隐函数, 将某m个变量看成函数, 其它变量相互独立.

- 逆映射的Jacobi矩阵 $$\frac{\partial(x,y)}{\partial(u,v)}=\left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1}$$

§4. 空间曲面和曲线

曲线$y=f(x)$在x_0可导, 即

$$y-y_0=f'(x_0)(x-x_0)+o(x-x_0),$$

$$x\to x_0\text{时}.$$

则曲线$y=f(x)$在x_0的切线方程为

$$y-y_0=f'(x_0)(x-x_0).$$

以直代曲:

以全微分代替函数值的改变量.

类比曲线的情形,曲面$z=g(x,y)$在点(x_0,y_0)可微, 即

$$z-z_0=g'_x(x_0,y_0)(x-x_0)+g'_y(x_0,y_0)(y-y_0)$$

$$+o(\sqrt{(x-x_0)^2+(y-y_0)^2}),$$

当$(x,y)\to(x_0,y_0)$时.

则曲面$z=g(x,y)$在点(x_0,y_0)的切平面方程为:

$$z-z_0=g'_x(x_0,y_0)(x-x_0)+g'_y(x_0,y_0)(y-y_0).$$

1. 参数方程下空间曲线的切线

空间C^1曲线$L: \mathrm{r} = \mathrm{r}(t) = \big(x(t), y(t), z(t)\big)$

记 $\Delta x = x(t+\Delta t) - x(t), \quad \Delta y = y(t+\Delta t) - y(t),$

$\Delta z = z(t+\Delta t) - z(t), \quad \Delta \mathrm{r} = \mathrm{r}(t+\Delta t) - \mathrm{r}(t).$

Def. $\mathrm{r}'(t) = \lim\limits_{\Delta t\to 0} \dfrac{\Delta \mathrm{r}}{\Delta t} = \lim\limits_{\Delta t\to 0} \left(\dfrac{\Delta x}{\Delta t}, \dfrac{\Delta y}{\Delta t}, \dfrac{\Delta z}{\Delta t} \right)$

$= \big(x'(t), y'(t), z'(t)\big).$

Def. 若$\mathrm{r}'(t_0) \neq 0$,则称$\mathrm{r}(t_0)$为曲线$L: \mathrm{r} = \mathrm{r}(t)$的正则点.

Question.正则点的意义(几何意义、逆映射定理).

Remark1: (几何意义) $T=\dfrac{r'(t)}{\|r'(t)\|}$为曲线L在点$r(t)$处的单位切向量.

Remark2: (物理意义)设质点的位移为$r(t)$,则速度为$r'(t)$,加速度为$r''(t)$.

Remark3: $r'(t)$既反映了$r(t)$在长度上的变化,又反映了$r(t)$在方向上的变化.

Remark4. L在$r(t_0)=(x_0, y_0, z_0)$处的切线方程为

$$\frac{x-x_0}{x'(t_0)}=\frac{y-y_0}{y'(t_0)}=\frac{z-z_0}{z'(t_0)},$$

法平面方程为

$$x'(t_0)(x-x_0)+y'(t_0)(y-y_0)+z'(t_0)(z-z_0)=0.$$

2. 参数方程下曲面的切平面

设曲面S的参数方程为$\mathbf{r}=\mathbf{r}(u,v)$,即

$$S:\begin{cases} x=x(u,v)\\ y=y(u,v)\\ z=z(u,v)\end{cases}$$

Def. $\mathbf{r}(u,v)$连续可微,称$\mathbf{r}_0=\mathbf{r}(u_0,v_0)$为曲面$S$的正则点,若$\operatorname{rank}\left.\dfrac{\partial(x,y,z)}{\partial(u,v)}\right|_{(u_0,v_0)}=2.$

Question. 正则点的意义(几何意义、逆映射定理).

Question. 求曲面S在正则点$\mathbf{r}_0$处的切平面Π.

考虑S上两条特殊的光滑曲线:

$$\ell_1: \mathrm{r}=\mathrm{r}(u,v_0),\ \ell_2: \mathrm{r}=\mathrm{r}(u_0,v).$$

ℓ_1在r_0的切向量为$\mathrm{r}'_u(u_0,v_0)=\left(x'_u, y'_u, z'_u\right)\Big|_{(u_0,v_0)}$,

ℓ_2在r_0的切向量为$\mathrm{r}'_v(u_0,v_0)=\left(x'_v, y'_v, z'_v\right)\Big|_{(u_0,v_0)}$.

$$\operatorname{rank}\left.\frac{\partial(x,y,z)}{\partial(u,v)}\right|_{(u_0,v_0)}=2,$$ $\mathrm{r}'_u(u_0,v_0)$与$\mathrm{r}'_v(u_0,v_0)$不平行,

则Π过点r_0,由$\mathrm{r}'_u(u_0,v_0)$与$\mathrm{r}'_v(u_0,v_0)$张成. 故

- S的切平面Π: $\quad \mathrm{r}-\mathrm{r}_0=s\mathrm{r}'_u+t\mathrm{r}'_v$,
- S在r_0的法向量: $\quad \vec{n}=\left(\mathrm{r}'_u\times\mathrm{r}'_v\right)\Big|_{(u_0,v_0)}$.

Remark. $S: z = f(x, y)$可以看成以x, y为参数的曲面

$$x = x, y = y, z = f(x, y).$$

于是在点(x_0, y_0, z_0)处

$$\mathrm{r}'_x = (1, 0, f'_x(x_0, y_0))^{\mathrm{T}}, \mathrm{r}'_y = (0, 1, f'_y(x_0, y_0))^{\mathrm{T}}.$$

•切平面为 $\Pi: \begin{cases} x = x_0 + t \\ y = y_0 + s \\ z = z_0 + t f'_x(x_0, y_0) + s f'_y(x_0, y_0) \end{cases},$

即$z = z_0 + f'_x(x_0, y_0)(x - x_0) + f'_y(x_0, y_0)(y - y_0).$

•法向量 $\vec{n} = (-f'_x(x_0, y_0), -f'_y(x_0, y_0), 1)^{\mathrm{T}}$

例:求球面$\begin{cases} x = a\sin\varphi\cos\theta \\ y = a\sin\varphi\sin\theta \\ z = a\cos\varphi \end{cases}$ $\begin{pmatrix} 0 \le \varphi \le \pi \\ 0 \le \theta < 2\pi \end{pmatrix}$在$\varphi=\pi/6$,

$\theta = \pi/3$的切平面和法向量.

解: $\mathrm{r}'_\theta = (-a\sin\varphi\sin\theta, a\sin\varphi\cos\theta, 0)$

$\mathrm{r}'_\varphi = (a\cos\varphi\cos\theta, a\cos\varphi\sin\theta, -a\sin\varphi)$.

当$\varphi=\pi/6, \theta = \pi/3$时,

$$(x, y, z) = (a/4, \sqrt{3}a/4, \sqrt{3}a/2),$$

$$\mathrm{r}'_\varphi = (\sqrt{3}a/4, 3a/4, -a/2),$$

$$\mathrm{r}'_\theta = (-\sqrt{3}a/4, a/4, 0).$$

$$\vec{n} // \left(\mathrm{r}'_\varphi \times \mathrm{r}'_\theta\right) = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \sqrt{3}a/4 & 3a/4 & -a/2 \\ -\sqrt{3}a/4 & a/4 & 0 \end{pmatrix}$$

$$\vec{n} // \left(1/8, \sqrt{3}/8, \sqrt{3}/4\right).$$

切平面方程为

$$(x - a/4, y - \sqrt{3}a/4, z - \sqrt{3}a/2) \cdot \vec{n} = 0,$$

即

$$x + \sqrt{3}y + 2\sqrt{3}z - 4a = 0 \ . \square$$

3. 一般方程下曲面的切平面

Def. $F(x,y,z)$连续可微,若$\mathrm{grad}F(x_0,y_0,z_0)\neq 0$,
则称$\mathrm{r}_0=(x_0,y_0,z_0)$为曲面

$$S:F(x,y,z)=0,\ F(x_0,y_0,z_0)=0$$

的正则点.

Question.如何求曲面

$$S:F(x,y,z)=0,F(x_0,y_0,z_0)=0$$

在正则点$\mathrm{r}_0=(x_0,y_0,z_0)$处的法线和切平面?

$\mathrm{grad}F(x_0,y_0,z_0)\neq 0$,不妨设$F_z'(x_0,y_0,z_0)\neq 0$,则 $S: F(x,y,z)=0, F(x_0,y_0,z_0)=0$局部确定的隐函数$z=f(x,y)$,且

$$f_x'(x_0,y_0)=-\left.\frac{F_x'}{F_z'}\right|_{\mathrm{r}_0},\ f_y'(x_0,y_0)=-\left.\frac{F_y'}{F_z'}\right|_{\mathrm{r}_0}.$$

- S在r_0的法向量为

$$\vec{n}=(-f_x'(x_0,y_0),-f_y'(x_0,y_0),1)^{\mathrm{T}}=\left.\left(\frac{F_x'}{F_z'},\frac{F_y'}{F_z'},1\right)^{\mathrm{T}}\right|_{\mathrm{r}_0}$$

即$\vec{n}\,/\!/\,\mathrm{grad}F(x_0,y_0,z_0)$.

求$\mathrm{grad}F$时, F是x,y,z的三元函数!

• S在r_0的切平面方程为

$$(\mathrm{r}-\mathrm{r}_0)\cdot \mathrm{grad}F(\mathrm{r}_0)=0$$

即

$$(x-x_0)F_x'(\mathrm{r}_0)+(y-y_0)F_y'(\mathrm{r}_0)+(z-z_0)F_z'(\mathrm{r}_0)=0.$$

• S在r_0的法线方程为 $\mathrm{r}=\mathrm{r}_0+t\cdot \mathrm{grad}F(\mathrm{r}_0)$

$$即\begin{cases} x=x_0+F_x'(\mathrm{r}_0)t \\ y=y_0+F_y'(\mathrm{r}_0)t \\ x=z_0+F_z'(\mathrm{r}_0)t. \end{cases}$$

例:球面S_1:$x^2+y^2+z^2=R^2$与锥面S_2:$x^2+y^2=a^2z^2$正交(即交点处的法向量相互垂直).

证明:记$F(x,y,z)=x^2+y^2+z^2-R^2$,

$$G(x,y,z)=x^2+y^2-a^2z^2.$$

交点(x,y,z)处S_1与S_2的法向量分别为

$$\mathrm{grad}F(x,y,z)=(2x,2y,2z)$$

$$\mathrm{grad}G(x,y,z)=(2x,2y,-2a^2z).$$

而 $\mathrm{grad}F(x,y,z)\cdot\mathrm{grad}G(x,y,z)=4(x^2+y^2-a^2z^2)$ $=0$,故S_1与S_2正交. □

例:设f可微.求证曲面$S: f\left(\frac{x-a}{z-c},\frac{y-b}{z-c}\right)=0$上任意一点处的切平面通过一定点.

证明:记$F(x,y,z)\triangleq f\left(\frac{x-a}{z-c},\frac{y-b}{z-c}\right)$.则曲面$S$在点$\left(x_0,y_0,z_0\right)$的法向量为

$$\vec{n}=\mathrm{grad}F\left(x_0,y_0,z_0\right)$$

$$=\left(\frac{f_1'}{z_0-c},\frac{f_2'}{z_0-c},\frac{a-x_0}{\left(z_0-c\right)^2}f_1'+\frac{b-y_0}{\left(z_0-c\right)^2}f_2'\right)^{\mathrm{T}}$$

S在点(x_0, y_0, z_0)的切平面方程为

$$(x-x_0)\frac{f_1'}{z_0-c}+(y-y_0)\frac{f_2'}{z_0-c}$$

$$+(z-z_0)\frac{a-x_0}{(z_0-c)^2}f_1'+(z-z_0)\frac{b-y_0}{(z_0-c)^2}f_2'=0.$$

可见所有的切平面都过定点(a,b,c).□

例: 求$\lambda > 0$,使以下两曲面相切:

$$S_1 : xyz = \lambda, \qquad S_2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

解: 设S_1与S_2在点(x, y, z)相切, 则两曲面在(x, y, z)的切平面的法向量平行,即

$$\left(yz, xz, xy\right) // \left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}\right).$$

于是存在$\mu \in \mathbb{R}$, s.t.

$$yz = \mu\frac{x}{a^2}, xz = \mu\frac{y}{b^2}, xy = \mu\frac{z}{c^2}.$$

用x, y, z分别乘各等式, 得

$$xyz = \mu \frac{x^2}{a^2} = \mu \frac{y^2}{b^2} = \mu \frac{z^2}{c^2} \qquad (*)$$

于是 $3xyz = \mu \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right).$

点(x, y, z)在两曲面上,因此$3\lambda = \mu.$

注意到$xyz = \lambda$,由(*)式得

$$x^2 = a^2 xyz/\mu = a^2\lambda/\mu = a^2/3,$$

$$y^2 = b^2 xyz/\mu = b^2\lambda/\mu = b^2/3,$$

$$z^2 = c^2 xyz/\mu = c^2\lambda/\mu = c^2/3.$$

故$\lambda = \sqrt{x^2 y^2 z^2} = \sqrt{3}abc/9.$ □

4. 一般方程表示的空间曲线的切线

$$\operatorname{rank}\frac{\partial(F,G)}{\partial(x,y,z)}=2,\mathrm{L}:\begin{cases}(S_1:)F(x,y,z)=0,\\(S_2:)G(x,y,z)=0.\end{cases}$$

求L在点$\mathrm{r}_0=(x_0,y_0,z_0)$处的切线.

L在点r_0处的切线必落在S_1,S_2在点r_0的切平面上.

因而L在r_0的切向量T与S_1,S_2在点r_0的法向量垂直.

于是, $\mathrm{T}=\mathrm{grad}F(\mathrm{r}_0)\times\mathrm{grad}G(\mathrm{r}_0)$,

L在点$\mathrm{r}_0=(x_0,y_0,z_0)$处的切线方程为

$$\mathrm{r}-\mathrm{r}_0=t\left(\mathrm{grad}F(\mathrm{r}_0)\times\mathrm{grad}G(\mathrm{r}_0)\right).$$

例:求曲线 $\begin{cases} x^2+y^2+z^2-6=0, \\ z-x^2-y^2=0 \end{cases}$ 在点$M_0(1,1,2)$处的切线方程.

解:令$F(x,y,z)=x^2+y^2+z^2-6,$

$$G(x,y,z)=z-x^2-y^2.$$

则$\mathrm{grad}F(1,1,2)=(2,2,4)^{\mathrm{T}},\mathrm{grad}G(1,1,2)=(-2,-2,1)^{\mathrm{T}}.$

曲线在点$M_0(1,1,2)$的切向量为

$$\vec{v}=\mathrm{grad}F(M_0)\times\mathrm{grad}G(M_0)=(10,-10,0)^{\mathrm{T}}.$$

曲线在点M_0的切线方程为$\begin{cases} x=1+t, \\ y=1-t, \\ z=2. \end{cases}$ □

4. 总结

曲面的切平面与法线:

曲面方程	点	法向量
$\mathrm{r}=\mathrm{r}(u,v)$	$\mathrm{r}_0=\mathrm{r}(u_0,v_0)$	$\left.(\mathrm{r}_u'\times\mathrm{r}_v')\right\|_{(u_0,v_0)}$
$z=f(x,y)$	(x_0,y_0,z_0) $z_0=f(x_0,y_0)$	$\left.(-f_x',-f_y',1)^{\mathrm{T}}\right\|_{(x_0,y_0)}$
$F(x,y,z)=0$	$\mathrm{r}_0=(x_0,y_0,z_0)$	$\operatorname{grad}F(\mathrm{r}_0)$

曲线的切向量:

曲线方程	点	切向量
$\mathrm{r}=\mathrm{r}(t)$	$\mathrm{r}_0=\mathrm{r}(t_0)$	$\mathrm{r}'(t_0)=\left(x'(t_0),y'(t_0),z'(t_0)\right)$
$\begin{cases}F(x,y,z)=0\\G(x,y,z)=0\end{cases}$	$\mathrm{r}_0=(x_0,y_0,z_0)$	$\mathrm{grad}F(\mathrm{r}_0)\times\mathrm{grad}G(\mathrm{r}_0)$

作业:习题1.7

No.1(6),2,3,5,6