



Review

- 偏导数的定义与几何意义
- 方向导数的定义与几何意义

$$g(t) = f\left(x_0 + \frac{\vec{v}}{\|\vec{v}\|}t\right) = f\left(x_0^{(1)} + \frac{v_1}{\|\vec{v}\|}t, \dots, x_0^{(n)} + \frac{v_n}{\|\vec{v}\|}t\right)$$

$$\frac{\partial f(x_0)}{\partial \vec{v}} = \lim_{t \rightarrow 0^+} \frac{f(x_0 + \vec{v}t) - f(x_0)}{\|\vec{v}\|t} = \lim_{t \rightarrow 0^+} \frac{g(t) - g(0)}{t} = g'_+(0)$$

$$\frac{\partial f(x_0, y_0)}{\partial \vec{v}} \quad \underline{\underline{f \text{ 在 } (x_0, y_0) \text{ 可微时}}} \quad \text{grad} f(x_0, y_0) \cdot \frac{\vec{v}}{\|\vec{v}\|}$$

- 梯度 $\text{grad} f(x_0)$ 的定义与意义



• n 元函数可微的定义与判别

f 在 \mathbf{x}_0 可微 $\Leftrightarrow \exists$ 常数 $a_1, a_2, \dots, a_n \in \mathbb{R}, s.t.$

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} \frac{f(\mathbf{x}) - f(\mathbf{x}_0) - a_1 \Delta x_1 - a_2 \Delta x_2 - \dots - a_n \Delta x_n}{\|\mathbf{x} - \mathbf{x}_0\|} = 0.$$

$$\Leftrightarrow f(\mathbf{x}_0 + \Delta \mathbf{x}) - f(\mathbf{x}_0)$$

$$= \frac{\partial f}{\partial x_1}(\mathbf{x}_0) \Delta x_1 + \dots + \frac{\partial f}{\partial x_n}(\mathbf{x}_0) \Delta x_n + o(\|\Delta \mathbf{x}\|), \Delta \mathbf{x} \rightarrow 0 \text{ 时.}$$

$$\Leftrightarrow f(\mathbf{x}_0 + \Delta \mathbf{x}) - f(\mathbf{x}_0) = \sum_{i=1}^n f'_{x_i}(\mathbf{x}_0) \Delta x_i + \sum_{i=1}^n \varepsilon_i \Delta x_i,$$

$$\lim_{\Delta \mathbf{x} \rightarrow 0} \varepsilon_i(\Delta \mathbf{x}) = 0, \quad i = 1, 2, \dots, n.$$



- 多元函数连续、可微、偏导数存在、偏导函数连续之间的关系

- f''_{xy}, f''_{yx} 都在 (x_0, y_0) 连续

$$\Rightarrow f''_{xy}(x_0, y_0) = f''_{yx}(x_0, y_0)$$

- f''_{xy}, f''_{yx} 也记为 f''_{12}, f''_{21}



§ 5. 向量值函数的微分

1. 线性映射

Def. 称向量值函数 $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ 为线性的, 若对任意 $\alpha, \beta \in \mathbb{R}, x, y \in \mathbb{R}^n$, 都有 $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$.

Thm. $\mathbb{R}^n \rightarrow \mathbb{R}^m$ 的线性映射与 M_{mn} 中矩阵一一对应.

Proof. 记 $\mathbb{R}^n \rightarrow \mathbb{R}^m$ 的线性映射构成的集合为 Y , 定义

$$\varphi: M_{mn} \rightarrow Y$$

$$A \mapsto \varphi(A)$$

使得 $\varphi(A)x = Ax, \forall x \in \mathbb{R}^n$. 下证 φ 为一一映射.



任给 $f \in Y$, 记 e_1, e_2, \dots, e_n 为 \mathbb{R}^n 中自然基, 则

$$A \triangleq (f(e_1), f(e_2), \dots, f(e_n)) \in M_{mn}.$$

对任意 $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$, 有

$$\begin{aligned} f(x) &= f(x_1 e_1 + x_2 e_2 + \dots + x_n e_n) \\ &= x_1 f(e_1) + x_2 f(e_2) + \dots + x_n f(e_n) = Ax. \end{aligned}$$

因此 φ 为满射.

任给 $f \in Y$, 若 $\exists A, B \in M_{m,n}$, s.t. $\varphi(A) = \varphi(B) = f$, 则

$f(x) = Ax = Bx, \forall x \in \mathbb{R}^n$, 从而 $A = B$. 因此 φ 为单射. \square



Thm. $A \in M_{mn}$, 则存在 $C \geq 0$, s.t.

$$\|Ax\|_m \leq C\|x\|_n, \forall x \in \mathbb{R}^n.$$

使得此不等式成立的**最小的C**记为 $\|A\|$.

Proof. $y = \|Ax\|_m$ 连续, $C = \max \{\|Ax\|_m : x \in \mathbb{R}^n, \|x\|_n = 1\}$ 存在,

$$\|Ax\|_m = \left\| A \left(\|x\|_n \frac{x}{\|x\|_n} \right) \right\|_m = \left\| A \frac{x}{\|x\|_n} \right\|_m \|x\|_n \leq C\|x\|_n, \forall x \in \mathbb{R}^n. \square$$



2. 向量值函数的微分

Def. 设 $f : \Omega (\subset \mathbb{R}^n) \rightarrow \mathbb{R}^m$, $x_0 \in \Omega$. 若存在 $m \times n$ 矩阵 A , s.t.

$$\Delta f(x_0) = f(x_0 + \Delta x) - f(x_0) = A\Delta x + \alpha,$$

其中, $\alpha = \alpha(\Delta x) = (\alpha_1(\Delta x), \dots, \alpha_m(\Delta x))^T$, 且当 $\Delta x \rightarrow 0$ 时,

$\alpha = o(\Delta x)$, 即

$$\lim_{\Delta x \rightarrow 0} \frac{\|\alpha\|_m}{\|\Delta x\|_n} = 0,$$

则称 f 在点 x_0 **可微**, 称 $A\Delta x$ 为 f 在点 x_0 的微分, 记为

$$df(x_0) = A\Delta x = Adx,$$

称 A 为 f 在 x_0 的 **Jacobi** 矩阵, 记作 $A = J(f)|_{x_0} = J_f(x_0)$.



Thm. $f = (f_1, f_2, \dots, f_m)^T : \Omega (\subset \mathbb{R}^n) \rightarrow \mathbb{R}^m$ 在 x_0 可微

$\Leftrightarrow n$ 元函数 $f_i : \Omega (\subset \mathbb{R}^n) \rightarrow \mathbb{R}$ 在 x_0 可微, $i = 1, 2, \dots, m$.

Proof.(必要性) 设 f 在点 x_0 可微, 则

$$\Delta f(x_0) = f(x_0 + \Delta x) - f(x_0) = A\Delta x + \alpha, \lim_{\Delta x \rightarrow 0} \frac{\|\alpha\|_m}{\|\Delta x\|_n} = 0,$$

比较第 i 个分量 ($1 \leq i \leq m$), 得

$$f_i(x_0 + \Delta x) - f_i(x_0) = \sum_{j=1}^n a_{ij} \Delta x_j + \alpha_i(\Delta x), \lim_{\Delta x \rightarrow 0} \frac{\alpha_i(\Delta x)}{\|\Delta x\|_n} = 0,$$

即 f_i 在 x_0 可微, $i = 1, 2, \dots, m$, 且 $a_{ij} = \frac{\partial f_i}{\partial x_j}(x_0)$.



(充分性) 设 f_i 在 x_0 可微, $i = 1, 2, \dots, m$, 则

$$\begin{pmatrix} \Delta f_1(x_0) \\ \Delta f_2(x_0) \\ \vdots \\ \Delta f_m(x_0) \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \dots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \bigg|_{x_0} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_n \end{pmatrix} + \begin{pmatrix} \alpha_1(\Delta x) \\ \alpha_2(\Delta x) \\ \vdots \\ \alpha_m(\Delta x) \end{pmatrix},$$

且 $\alpha_i(\Delta x) = o(\|\Delta x\|_n)$, $\Delta x \rightarrow 0$ 时, $1 \leq i \leq m$. 故 f 在 x_0 可微. \square



Remark: $f = (f_1, f_2, \dots, f_m)^T : \Omega (\subset \mathbb{R}^n) \rightarrow \mathbb{R}^m$ 在 x_0 可微,
则 f 在 x_0 的 *Jacobi* 矩阵为

$$J(f)|_{x_0} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \dots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \Big|_{x_0} \triangleq \frac{\partial (f_1, f_2, \dots, f_m)}{\partial (x_1, x_2, \dots, x_n)} \Big|_{x_0},$$

也记为 $\frac{\partial f}{\partial x}(x_0)$.



3. 复合映射的微分

Thm. $u = g(x) : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$, $y = f(u) : g(\Omega) \subset \mathbb{R}^m \rightarrow \mathbb{R}^k$,
 $x_0 \in \Omega$, $u_0 = g(x_0)$, 若 $g(x)$ 在 x_0 可微, $f(u)$ 在 u_0 可微, 则复合映射 $y = (f \circ g)(x) = f(g(x))$ 在 x_0 可微, 且

$$J(f \circ g)|_{x_0} = J(f)|_{u_0} \cdot J(g)|_{x_0},$$

$$\text{即 } \frac{\partial(y_1, y_2, \dots, y_k)}{\partial(x_1, x_2, \dots, x_n)} \Big|_{x_0} = \frac{\partial(y_1, y_2, \dots, y_k)}{\partial(u_1, u_2, \dots, u_m)} \Big|_{u_0} \cdot \frac{\partial(u_1, u_2, \dots, u_m)}{\partial(x_1, x_2, \dots, x_n)} \Big|_{x_0},$$

$$\text{简记为 } \frac{\partial y}{\partial x} \Big|_{x_0} = \frac{\partial y}{\partial u} \Big|_{u_0} \cdot \frac{\partial u}{\partial x} \Big|_{x_0}$$

Question. 可微能否

替换成偏导存在? 否!

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Proof. $u = g(x)$ 在 x_0 可微, $y = f(u)$ 在 u_0 可微, 记

$$A = \frac{\partial(y_1, y_2, \dots, y_k)}{\partial(u_1, u_2, \dots, u_m)} \bigg|_{u_0}, B = \frac{\partial(u_1, u_2, \dots, u_m)}{\partial(x_1, x_2, \dots, x_n)} \bigg|_{x_0},$$

则

$$\Delta u = g(x_0 + \Delta x) - g(x_0) = B\Delta x + o(\Delta x),$$

$$\Delta y = f(u_0 + \Delta u) - f(u_0) = A\Delta u + o(\Delta u),$$

于是

$$\Delta(f \circ g)(x_0) = f(g(x_0 + \Delta x)) - f(g(x_0))$$

$$= f(u_0 + \Delta u) - f(u_0) = A\Delta u + o(\Delta u)$$

$$= A(B\Delta x + o(\Delta x)) + o(\Delta u) = AB\Delta x + Ao(\Delta x) + o(\Delta u),$$

往证: $Ao(\Delta x) + o(\Delta u) = o(\Delta x), \Delta x \rightarrow 0.$



$$\frac{\|Ao(\Delta x)\|}{\|\Delta x\|} \leq \frac{\|A\| \|o(\Delta x)\|}{\|\Delta x\|} \rightarrow 0, \text{当} \Delta x \rightarrow 0 \text{时.}$$

$$\begin{aligned} \frac{\|\Delta u\|}{\|\Delta x\|} &= \frac{\|B\Delta x + o(\Delta x)\|}{\|\Delta x\|} \leq \frac{\|B\Delta x\|}{\|\Delta x\|} + \frac{\|o(\Delta x)\|}{\|\Delta x\|} \\ &\leq \|B\| + \frac{\|o(\Delta x)\|}{\|\Delta x\|} \rightarrow \|B\|, \text{当} \Delta x \rightarrow 0 \text{时.} \end{aligned}$$

$$\frac{\|o(\Delta u)\|}{\|\Delta x\|} = \frac{\|o(\Delta u)\|}{\|\Delta u\|} \cdot \frac{\|\Delta u\|}{\|\Delta x\|} \rightarrow 0, \text{当} \Delta x \rightarrow 0 \text{时.}$$

$$\frac{\|Ao(\Delta x) + o(\Delta u)\|}{\|\Delta x\|} \leq \frac{\|Ao(\Delta x)\|}{\|\Delta x\|} + \frac{\|o(\Delta u)\|}{\|\Delta x\|} \rightarrow 0, \text{当} \Delta x \rightarrow 0 \text{时.} \square$$



Remark. $k = 1$ 时, $y = f(u_1, u_2, \dots, u_m), u_i = g_i(x_1, x_2, \dots, x_n),$

$1 \leq i \leq m$, 均可微, 则 $\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} \frac{\partial u}{\partial x},$

$$\left(\frac{\partial y}{\partial x_1}, \dots, \frac{\partial y}{\partial x_n} \right) = \left(\frac{\partial y}{\partial u_1}, \dots, \frac{\partial y}{\partial u_m} \right) \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \dots & \frac{\partial u_1}{\partial x_n} \\ \vdots & \dots & \vdots \\ \frac{\partial u_m}{\partial x_1} & \dots & \frac{\partial u_m}{\partial x_n} \end{pmatrix},$$

因此 $y = y(x_1, x_2, \dots, x_n)$ 的偏导数满足 **链式法则**

$$\frac{\partial y}{\partial x_i} = \frac{\partial y}{\partial u_1} \frac{\partial u_1}{\partial x_i} + \frac{\partial y}{\partial u_2} \frac{\partial u_2}{\partial x_i} + \dots + \frac{\partial y}{\partial u_m} \frac{\partial u_m}{\partial x_i}, i = 1, 2, \dots, n.$$



例. 球坐标变换

$$\begin{cases} x = r \sin \varphi \cos \theta, \\ y = r \sin \varphi \sin \theta, \\ z = r \cos \varphi, \end{cases}$$

是 $\Omega = \{(r, \varphi, \theta) \mid 0 \leq r < \infty, 0 \leq \varphi \leq \pi, 0 \leq \theta \leq 2\pi\}$ 到 \mathbb{R}^3 的连续可微映射.

$$\frac{\partial(x, y, z)}{\partial(r, \varphi, \theta)} = \begin{pmatrix} \sin \varphi \cos \theta & r \cos \varphi \cos \theta & -r \sin \varphi \sin \theta \\ \sin \varphi \sin \theta & r \cos \varphi \sin \theta & r \sin \varphi \cos \theta \\ \cos \varphi & -r \sin \varphi & 0 \end{pmatrix}$$

$$\frac{D(x, y, z)}{D(r, \varphi, \theta)} \triangleq \det \frac{\partial(x, y, z)}{\partial(r, \varphi, \theta)} = r^2 \sin \varphi. \square$$



例:
$$\begin{cases} y_1 = u_1 u_2 - u_1 u_3, \\ y_2 = u_1 u_3 - u_2^2, \end{cases}$$

$$\begin{cases} u_1 = x_1 \cos x_2 + (x_1 + x_2)^2 \\ u_2 = x_1 \sin x_2 + x_1 x_2 \\ u_3 = x_1^2 - x_1 x_2 + x_2^2 \end{cases}$$

求 $\left. \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} \right|_{(1,0)}$.

解: $(x_1, x_2) = (1, 0)$ 时, $(u_1, u_2, u_3) = (2, 0, 1)$,

$$\left. \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} \right|_{(1,0)} = \left. \frac{\partial(y_1, y_2)}{\partial(u_1, u_2, u_3)} \right|_{(2,0,1)} \cdot \left. \frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2)} \right|_{(1,0)}$$



$$= \begin{pmatrix} u_2 - u_3 & u_1 & -u_1 \\ u_3 & -2u_2 & u_1 \end{pmatrix} \Big|_{(2,0,1)}$$

$$\cdot \begin{pmatrix} \cos x_2 + 2(x_1 + x_2) & -x_1 \sin x_2 + 2(x_1 + x_2) \\ \sin x_2 + x_2 & x_1 \cos x_2 + x_1 \\ 2x_1 - x_2 & -x_1 + 2x_2 \end{pmatrix} \Big|_{(1,0)}$$

$$= \begin{pmatrix} -1 & 2 & -2 \\ 1 & 0 & 2 \end{pmatrix} \times \begin{pmatrix} 3 & 2 \\ 0 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -7 & 4 \\ 7 & 0 \end{pmatrix}. \square$$



例. $z = 2yf(x^2/y, 3y)$, f 二阶连续可微 (即所有二阶偏导函数连续). 求 z''_{xy} .

解: f 二阶连续可微, 则 $z''_{xy} = z''_{yx}$. 后者求解相对简单.

设 $u = x^2/y$, $v = 3y$, $z = 2yf(u, v)$. 则

$$z'_x = 2y \left[f'_u \cdot (2x/y) + f'_v \cdot 0 \right] = 4xf'_u(u, v)$$

$$\begin{aligned} z''_{xy} = z''_{yx} &= \frac{\partial}{\partial y} (4xf'_u) = 4x \left[f''_{uu} \cdot \frac{-x^2}{y^2} + f''_{vu} \cdot 3 \right] \\ &= -\frac{4x^3}{y^2} f''_{uu} + 12xf''_{vu}. \quad \square \end{aligned}$$



例. $u(x, y)$ 二阶连续可微, 且 $u''_{xx} - u''_{yy} = 0$. 令 $\xi = x - y$,

$\eta = x + y$. 试证: $u''_{\xi\eta} = 0$.

解: $x = (\xi + \eta)/2, y = (\eta - \xi)/2$,

$$u'_\eta = u'_x x'_\eta + u'_y y'_\eta = (u'_x + u'_y)/2$$

$$u''_{\xi\eta} = \frac{1}{2} \frac{\partial}{\partial \xi} (u'_x + u'_y)$$

Remark. 此处不能用 u'_1, u''_{12} 这样的记号.

$$= \frac{1}{2} (u''_{xx} x'_\xi + u''_{yx} y'_\xi + u''_{xy} x'_\xi + u''_{yy} y'_\xi)$$

$$= \frac{1}{4} (u''_{xx} - u''_{yx} + u''_{xy} - u''_{yy}) = \frac{1}{4} (u''_{xx} - u''_{yy}) = 0. \square$$



例. $f(x, y)$ 可微, $f(x, x^2) = 1$, $f'_x(x, x^2) = x$, 求 $f'_y(x, x^2)$, ($x \neq 0$).

分析: $f'_x(x, x^2) = \frac{\partial f}{\partial x} \Big|_{(x, y)=(x, x^2)}$.

解: 将 $f(x, x^2) = 1$ 两边同时对 x 求导, 有

$$f'_x(x, x^2) \cdot 1 + f'_y(x, x^2) \cdot 2x = 0,$$

$$x + 2xf'_y(x, x^2) = 0,$$

$$\text{故 } f'_y(x, x^2) = -\frac{1}{2}, \forall x \neq 0. \square$$

Question. $f'_x(x, x^2)$ 与 $\frac{d}{dx} f(x, x^2)$ 有何区别?



例. $y = (1/x)^{-(1/x)}$, 求 $y'(x)$.

解: 令 $u = 1/x$, $v = -1/x$, $y = u(x)^{v(x)}$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{\partial y}{\partial u} \cdot \frac{du}{dx} + \frac{\partial y}{\partial v} \cdot \frac{dv}{dx} \\ &= vu^{v-1}(-1/x^2) + u^v(\ln u)(1/x^2) \\ &= u^{v+2}(1 + \ln u) \\ &= (1/x)^{2-\frac{1}{x}}(1 - \ln x). \quad \square\end{aligned}$$



作业：习题1.5 No. 4,5,8.