Review

- 函数项级数的逐点收敛与一致收敛

$$\sum_{n=1}^{+\infty} f_n(x)\text{在}x\in I\text{上一致收敛}$$

$$\Leftrightarrow \exists S(x), \forall \varepsilon > 0, \exists N(\varepsilon), s.t.,$$

$$\left|\sum_{k=1}^{n} f_k(x) - S(x)\right| < \varepsilon, \quad \forall n > N, \forall x \in I.$$

$\Leftrightarrow$ (Cauchy准则) $\forall \varepsilon > 0, \exists N(\varepsilon), s.t.,$

$$\left|\sum_{k=n+1}^{n+p} f_k(x)\right| < \varepsilon, \quad \forall n > N, \forall p \geq 1, \forall x \in I.$$

- 函数项级数一致收敛的判别法

Weierstrass

$$\left.\begin{array}{l} \displaystyle\sum_{n=1}^{+\infty} M_n \text{收敛}, \\ \left| f_n(x) \right| \le M_n, \forall n \in \mathbb{N}, \forall x \in I, \end{array}\right\}$$

$$\Rightarrow \sum_{n=1}^{+\infty} f_n(x) \text{在} I \text{上一致收敛}.$$

Dirichlet

$$\left.\begin{aligned} &\{a_n(x)\}\text{关于}n\text{单调},\text{在}I\text{上一致收敛到}0;\\ &\sum_{n=1}^{+\infty} b_n(x)\ \text{的部分和函数列在}I\text{上一致有界};\end{aligned}\right\}$$

$$\Rightarrow \sum_{n=1}^{+\infty} a_n(x)b_n(x)\text{在}I\text{上一致收敛}.$$

Abel

$$\left.\begin{aligned} &\{a_n(x)\}\text{关于}n\text{单调},\text{在}I\text{上一致有界};\\ &\sum_{n=1}^{+\infty} b_n(x)\ \text{在}I\text{上一致收敛};\end{aligned}\right\}$$

$$\Rightarrow \sum_{n=1}^{+\infty} a_n(x)b_n(x)\text{在}I\text{上一致收敛}.$$

§ 2. 一致收敛函数项级数和函数的性质

令$g(t,x)=f_n(x), t\in[n-1,n)$,则

$$\sum_{n=1}^{+\infty} f_n(x) = \sum_{n=1}^{+\infty}\int_{n-1}^{n} g(t,x)\mathrm{d}t = \int_0^{+\infty} g(t,x)\mathrm{d}t.$$

含参积分的一致收敛性

$\leftrightarrow$ 函数项级数的一致收敛性

1. 和函数的性质

目标：什么条件下, 以下极限过程可交换？

$$\lim_{x\to x_0}\sum_{n=1}^{+\infty} f_n(x)=\sum_{n=1}^{+\infty}\lim_{x\to x_0} f_n(x);\quad \text{(逐项求极限)}$$

$$\int_a^b \sum_{n=1}^{+\infty} f_n(x)\,\mathrm{d}x=\sum_{n=1}^{+\infty}\int_a^b f_n(x)\,\mathrm{d}x;\quad \text{(逐项积分)}$$

$$\left(\sum_{n=1}^{+\infty} f_n(x)\right)'=\sum_{n=1}^{+\infty} f_n'(x).\qquad \text{(逐项求导)}$$

Thm. $$\left.\begin{array}{l}\displaystyle\sum_{n=1}^{+\infty} f_n(x)\text{在区间}I\text{上一致收敛到}S(x)\\ f_n(x)\in C(I),\forall n\end{array}\right\}\Rightarrow S(x)\in C(I).$$

Proof.

$$|S(x)-S(x_0)|=\left|\sum_{n=1}^{+\infty} f_n(x)-\sum_{n=1}^{+\infty} f_n(x_0)\right|$$

$$=\left|\sum_{k=1}^{n} f_k(x)+\sum_{k=n+1}^{+\infty} f_k(x)-\sum_{k=1}^{n} f_k(x_0)-\sum_{k=n+1}^{+\infty} f_k(x_0)\right|$$

$$\le\sum_{k=1}^{n}|f_k(x)-f_k(x_0)|+\left|\sum_{k=n+1}^{+\infty} f_k(x)\right|+\left|\sum_{k=n+1}^{+\infty} f_k(x_0)\right| \qquad (*)$$

$\forall \varepsilon > 0,$ 由 $\sum_{n=1}^{+\infty} f_n(x)$ 在 I 上一致收敛到 $S(x), \exists N(\varepsilon), s.t.$

$$\left|\sum_{k=n+1}^{+\infty} f_k(x)\right| < \frac{\varepsilon}{3}, \forall n \geq N, \forall x \in I.$$

又 $f_k(x) \in C(I), k = 1, 2, \cdots N,$ 则 $\exists \delta(x_0) > 0, s.t.,$

$$|f_k(x) - f_k(x_0)| < \frac{\varepsilon}{3N}, \forall |x - x_0| < \delta, k = 1, 2, \cdots, N.$$

在(*)中取 $n = N,$ 则

$$|S(x) - S(x_0)| < \frac{\varepsilon}{3N} \cdot N + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \ \forall |x - x_0| < \delta.$$

□

Remark.

$$\lim_{x\to x_0} S(x) = S(x_0)$$

$$\Leftrightarrow \lim_{x\to x_0}\sum_{n=1}^{+\infty} f_n(x) = \sum_{n=1}^{+\infty} f_n(x_0)$$

$$\Leftrightarrow \lim_{x\to x_0}\sum_{n=1}^{+\infty} f_n(x) = \sum_{n=1}^{+\infty} \lim_{x\to x_0} f_n(x)$$ 逐项求极限！

例.判断$\sum_{n=0}^{+\infty}(1-x)x^n$在其收敛域上是否一致收敛.

解: $\lim_{n\to\infty}(1-x)x^n=0 \Rightarrow x\in(-1,1]$.

$x=1$时, $(1-x)x^n=0$, $\sum(1-x)x^n=0$.

$|x|<1$时, $\sum_{n=0}^{+\infty}(1-x)x^n=(1-x)\sum_{n=0}^{+\infty}x^n=(1-x)\cdot\frac{1}{1-x}=1$.

$\sum_{n=0}^{+\infty}(1-x)x^n$的和函数 $S(x)=\begin{cases}1, & -1<x<1\\ 0, & x=1\end{cases}$ 在其收敛域

$(-1,1]$上不连续, 故$\sum_{n=0}^{+\infty}(1-x)x^n$在$(-1,1]$上不一致收敛.□

Thm

$$\left.\begin{aligned}\sum_{n=1}^{+\infty} f_n(x)\text{在}[a,b]\text{上一致收敛到}S(x)\\ f_n(x)\in C[a,b],\forall n\end{aligned}\right\}\Rightarrow \int_a^b S(x)\mathrm{d}x=\sum_{n=1}^{+\infty}\int_a^b f_n(x)\,\mathrm{d}x.$$

Proof. $\sum\limits_{n=1}^{+\infty} f_n(x)$在$[a,b]$上一致收敛到$S(x)$, 则$\forall\varepsilon>0,\exists N, s.t.$

$\forall n>N, \forall x\in[a,b]$, 有$\left|S(x)-\sum\limits_{k=1}^{n} f_k(x)\right|<\dfrac{\varepsilon}{b-a}$. 于是,

$$\left|\int_a^b S(x)\mathrm{d}x-\sum_{k=1}^{n}\int_a^b f_k(x)\mathrm{d}x\right|=\left|\int_a^b\left(S(x)-\sum_{k=1}^{n} f_k(x)\right)\mathrm{d}x\right|$$

$$\le\int_a^b\left|S(x)-\sum_{k=1}^{n} f_k(x)\right|\mathrm{d}x<\frac{\varepsilon}{b-a}\cdot(b-a)=\varepsilon.\square$$

Remark. $\int_a^b S(x)\mathrm{d}x = \sum_{k=1}^{\infty}\int_a^b f_k(x)\mathrm{d}x$

$$\Leftrightarrow \int_a^b \sum_{k=1}^{\infty} f_k(x)\mathrm{d}x = \sum_{k=1}^{\infty}\int_a^b f_k(x)\mathrm{d}x$$ 逐项积分！

$$\Leftrightarrow \int_a^b S(x)\mathrm{d}x = \lim_{n\to\infty}\sum_{k=1}^{n}\int_a^b f_k(x)\mathrm{d}x = \lim_{n\to\infty}\int_a^b S_n(x)\mathrm{d}x$$

Corollary. $\left.\begin{array}{l} g_n(x)\in C[a,b] \\ g_n(x)\text{在}[a,b]\text{上一致收敛} \end{array}\right\}$

$$\Rightarrow \int_a^b \lim_{n\to\infty} g_n(x)\mathrm{d}x = \lim_{n\to\infty}\int_a^b g_n(x)\mathrm{d}x.$$

Corollary. $\left.\begin{array}{l}\sum\limits_{n=1}^{+\infty} f_n(x)\text{在}[a,b]\text{上一致收敛到}S(x) \\ f_n(x)\in C[a,b],\forall n\end{array}\right\}$

$$\Rightarrow \sum_{n=1}^{+\infty}\int_a^x f_n(t)\mathrm{d}t\text{在}x\in[a,b]\text{上一致收敛到}\int_a^x S(t)\mathrm{d}t.$$

Proof. 证明方法同定理,略.□

Remark. 以上逐项可积的定理和推论中, $f_n(x)\in C[a,b]$ 可以减弱为 $f_n(x)\in R[a,b]$.

例.求$\displaystyle\int_{\pi/2}^{3\pi/2}\left(\sum_{n=1}^{+\infty}\frac{\sin nx}{n}\right)\mathrm{d}x.$

解：$\displaystyle\sum_{n=1}^{+\infty}\frac{\sin nx}{n}$在$[\pi/2,3\pi/2]$上一致收敛(Dirichlet), 故

$$\int_{\pi/2}^{3\pi/2}\left(\sum_{n=1}^{+\infty}\frac{\sin nx}{n}\right)\mathrm{d}x=\sum_{n=1}^{+\infty}\int_{\pi/2}^{3\pi/2}\frac{\sin nx}{n}\mathrm{d}x$$

$$=\sum_{n=1}^{+\infty}\frac{-\cos nx}{n^2}\Bigg|_{\pi/2}^{3\pi/2}=0.\square$$

Thm

$$\left.\begin{array}{l} f_n(x)\in C^1[a,b],\forall n \\ \sum\limits_{n=1}^{+\infty} f_n'(x)\text{在}[a,b]\text{上一致收敛到}T(x) \\ \exists x_0\in[a,b],s.t.\sum\limits_{n=1}^{+\infty} f_n(x_0)\text{收敛} \end{array}\right\}$$

$$\Rightarrow\begin{cases} (1)\ \sum\limits_{n=1}^{+\infty} f_n(x)\text{在}[a,b]\text{上一致收敛, 设其和为}S(x); \\ (2)\ S'(x)=T(x),\text{即}\left(\sum\limits_{n=1}^{+\infty} f_n(x)\right)'=\sum\limits_{n=1}^{+\infty} f_n'(x). \end{cases}$$

Proof.(1) $S_n(x) \triangleq \sum_{k=1}^{n} f_k(x) = S_n(x_0) + \int_{x_0}^{x} S_n'(t)\mathrm{d}t$

已知$S_n(x_0)$收敛, $S_n'(x) = \sum_{k=1}^{n} f_k'(x)$在$x \in [a,b]$上一致收敛,

则$\forall \varepsilon > 0, \exists N \in \mathbb{N}, s.t., \forall n > N, \forall p \geq 1, \forall x \in [a,b],$

$$\left|S_{n+p}(x_0) - S_n(x_0)\right| < \varepsilon, \left|S_{n+p}'(x) - S_n'(x)\right| < \varepsilon/(b-a),$$

故 $\left|S_{n+p}(x) - S_n(x)\right|$

$\leq \left|S_{n+p}(x_0) - S_n(x_0)\right| + \left|\int_{x_0}^{x} \left|S_{n+p}'(t) - S_n'(t)\right| \mathrm{d}t\right|$

$< 2\varepsilon, \quad \forall x \in [a,b], \forall n > N.$

因此$\sum_{n=1}^{+\infty} f_n(x)$在$[a,b]$上一致收敛.

(2) $S_n'(x)=\sum_{k=1}^{n} f_k'(x)$在$[a,b]$上一致收敛, $f_k\in C^1[a,b],\forall k,$

则$\forall x,x_0\in[a,b],\sum_{k=1}^{+\infty} f_k'(x)$在区间$[x_0,x]$(或$[x,x_0]$)上可

逐项积分, 即

$$\lim_{n\to+\infty}\int_{x_0}^{x} S_n'(t)\mathrm{d}t=\lim_{n\to+\infty}\sum_{k=1}^{n}\int_{x_0}^{x} f_k'(t)\mathrm{d}t=\int_{x_0}^{x}\lim_{n\to+\infty} S_n'(t)\mathrm{d}t.$$

由$S_n(x)=S_n(x_0)+\int_{x_0}^{x} S_n'(t)\mathrm{d}t$, 令$n\to+\infty$,得

$$S(x)=S(x_0)+\int_{x_0}^{x}\lim_{n\to+\infty} S_n'(t)\mathrm{d}t=S(x_0)+\int_{x_0}^{x} T(t)\mathrm{d}t,$$

故$S'(x)=T(x),\forall x\in[a,b]$.□

例.Riemannζ函数$\zeta(x)=\sum\limits_{n=1}^{+\infty}\dfrac{1}{n^x}$在$(1,\infty)$上连续可微.

Proof. $\left(\dfrac{1}{n^x}\right)'=\dfrac{-\ln n}{n^x}\in C(1,\infty)$. 任给$b>a>1$, 有

$$0\le\frac{1}{n^x}\le\frac{1}{n^a},\quad 0\le\frac{\ln n}{n^x}\le\frac{\ln n}{n^a},\quad \forall x\in[a,b].$$

于是$\sum\limits_{n=1}^{+\infty}\dfrac{1}{n^x},\sum\left(\dfrac{1}{n^x}\right)'$均在$[a,b]$上一致收敛(Weierstrass),

故$\zeta'(x)=\left(\sum\limits_{n=1}^{+\infty}\dfrac{1}{n^x}\right)'=\sum\limits_{n=1}^{+\infty}\left(\dfrac{1}{n^x}\right)'\in C[a,b]$.

由a,b的任意性, $\zeta(x)\in C^1(1,\infty)$.□

例. $f_n(x)=\dfrac{(-1)^n\arctan\left(\dfrac{x}{\sqrt{n}}\right)}{\sqrt{n}}$, $\sum\limits_{n=1}^{+\infty} f_n(x)$ 在$\mathbb{R}$上连续可微?

解: $\sum\limits_{n=1}^{+\infty} f_n(0)$ 收敛.

$f_n'(x)=\dfrac{(-1)^n}{n+x^2}$, 由Dirichlet判别法, $\sum\limits_{n=1}^{+\infty} f_n'(x)$ 一致收敛.

综上, $\sum\limits_{n=1}^{+\infty} f_n(x)$ 在$\mathbb{R}$上连续可微.□

2. 函数项级数的应用

---一阶ODE初值问题解的存在唯一性定理

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases} \qquad (1)$$

$$\Updownarrow$$

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\mathrm{d}t \qquad (2)$$

Qeustion. 初值问题(1)容易求解还是积分方程(2)容易求解？

Qeustion. 如何用迭代法求解(2)？即构造$y_n(x) \to y(x)$.

构造Picard序列：$y_0(x) \equiv y_0$,

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))\mathrm{d}t, \quad n = 1, 2, 3, \cdots \qquad (3)$$

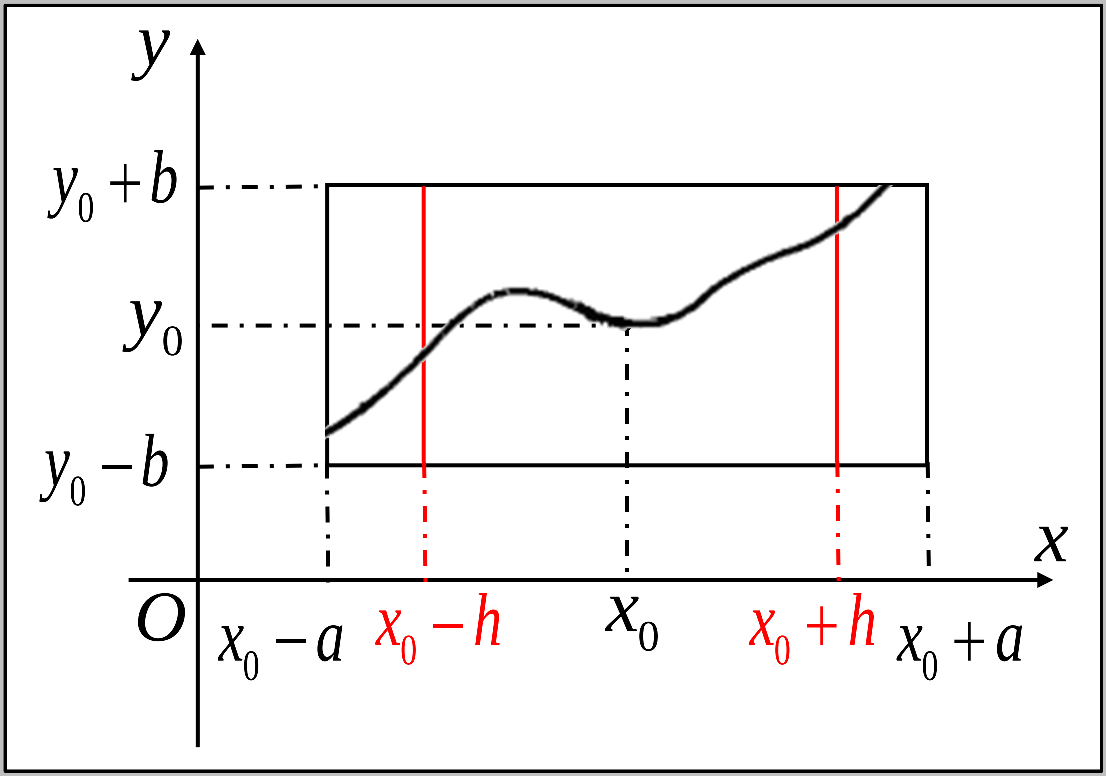
Qeustion. 对$f(x,y)$加什么样的条件,以确保ODE初值问题(1)或积分方程(2)的解存在、唯一？或者确保按(3)构造的Picard序列$y_n(x)$收敛到(1),(2)的解$y(x)$?

Def. 称$f(x,y)$在$D=\{(x,y):|x-x_0|\le a, |y-y_0|\le b\}$中关于$y$满足*Lipschitz*条件,若存在$L>0$, s.t.,

$$|f(x,y_1)-f(x,y_2)|\le L|y_1-y_2|,$$

$$\forall (x,y_1),(x,y_2)\in D.$$

Thm. $f(x,y)$在$D=\{(x,y):|x-x_0|\le a,|y-y_0|\le b\}$中连续, 关于$y$满足*Lipschitz*条件, 则ODE初值问题(1)在区间$[x_0-h,x_0+h]$上存在唯一解, 其中



$$h=\min\left\{a,\frac{b}{M}\right\},$$

$$M=\max_{(x,y)\in D}|f(x,y)|.$$

Proof. 先证存在性, 再证唯一性.

Step1.(1) $\Leftrightarrow$ (2)

Step2.往证Picard序列

$$y_0(x)=y_0,\ y_n(x)=y_0+\int_{x_0}^{x} f(t,y_{n-1}(t))\mathrm{d}t,\quad n=1,2,3,\cdots$$

在$I=[x_0-h,x_0+h]$上连续, 且

$$\left|y_n(x)-y_0\right|\le M\left|x-x_0\right|,\ n=1,2,3,\cdots.$$

事实上, $y_n(x)$的连续性由$f\in C(D)$可得, 而

$$\left|y_1(x)-y_0\right|=\left|\int_{x_0}^{x} f(t,y_0)dt\right|\le M\left|x-x_0\right|\le Mh\le b,$$

归纳地, 若 $|y_n(x) - y_0| \le b, \forall x \in I$, 则

$$|y_{n+1}(x) - y_0| = \left|\int_{x_0}^{x} f(t, y_n(t))\mathrm{d}t\right| \le \left|\int_{x_0}^{x} |f(t, y_n(t))|\mathrm{d}t\right|$$

$$\le M|x - x_0| \le Mh \le b, \qquad \forall x \in I.$$

Step3. 往证Picard序列$y_n(x)$在I上一致收敛.

序列$y_n(x)$的收敛性等价于级数

$$y_0 + \sum_{n=0}^{+\infty}[y_{n+1}(x) - y_n(x)]$$

的收敛性. 往证后者在I上一致收敛.

$$\left|y_1(x)-y_0\right| = \left|\int_{x_0}^{x} f(t,y_0)\mathrm{d}t\right| \le M\left|x-x_0\right|$$

$$\left|y_2(x)-y_1(x)\right| \le \left|\int_{x_0}^{x} \left|f(t,y_1(t))-f(t,y_0)\right|\mathrm{d}t\right|$$

$$\le L\left|\int_{x_0}^{x} \left|y_1(t)-y_0\right|\mathrm{d}t\right| \quad \text{(Lipschitz条件)}$$

$$\le LM\left|\int_{x_0}^{x} \left|t-x_0\right|\mathrm{d}t\right| = \frac{LM}{2}\left|x-x_0\right|^2$$

假设当$n=k$时，$\left|y_k(x)-y_{k-1}(x)\right| = \dfrac{ML^{k-1}}{k!}\left|x-x_0\right|^k$，则

$$\left|y_{k+1}(x)-y_k(x)\right|\le\left|\int_{x_0}^{x}\left|f(t,y_k(t))-f(t,y_{k-1}(t))\right|\mathrm{d}t\right|$$

$$\le L\left|\int_{x_0}^{x}\left|y_k(t)-y_{k-1}(t)\right|\mathrm{d}t\right| \le L\cdot\frac{ML^{k-1}}{k!}\left|\int_{x_0}^{x}\left|x-x_0\right|^k\mathrm{d}t\right|$$

$$\le\frac{ML^k}{(k+1)!}\left|x-x_0\right|^{k+1},\quad \forall x\in I=[x_0-h,x_0+h]$$

由数学归纳法,

$$\left|y_n(x)-y_{n-1}(x)\right|\le\frac{ML^{n-1}}{(n+1)!}\left|x-x_0\right|^n\le\frac{ML^{n-1}}{(n+1)!}h^n,\forall n\ge 1,\forall x\in I.$$

$\sum\frac{ML^{n-1}}{(n+1)!}h^n$收敛,由Weierstrass判别法,级数

$$y_0+\sum_{n=0}^{+\infty}[y_{n+1}(x)-y_n(x)]$$

在I上一致收敛, 从而$y_n(x)$在I上一致收敛.

Step4. 设$\varphi(x)=\lim\limits_{n\to\infty} y_n(x), x\in I$. 往证$\varphi(x)$是(1),(2)的解.

$$\left|f(t,y_n(t))-f(t,\varphi(t))\right|\le L\left|y_n(t)-\varphi(t)\right|,$$

$y_n(x)$在I上一致收敛到$\varphi(x)$, 则$f(t,y_n(t))$在I上一致收敛到$f(t,\varphi(t))$. 在下式中令$n\to+\infty$,

$$y_{n+1}(x)=y_0+\int_{x_0}^{x} f(t,y_n(t))\mathrm{d}t,$$

则有 $$\varphi(x)=y_0+\int_{x_0}^{x} f(t,\varphi(t))\mathrm{d}t,\quad \forall x\in I,$$

即$\varphi(x)$是(2)的解. 从而$\varphi(x)$是(1)的解.

至此, 我们已经证明了初值问题(1)的解的存在性.

Step5. 解的唯一性.

设积分方程(2)有解$u(x)$和$v(x)$, $x \in I = [x_0 - h, x_0 + h]$.则

$$u(x) - v(x) = \int_{x_0}^{x} [f(t, u(t)) - f(t, v(t))]\mathrm{d}t, \forall x \in J.$$

由f的Lipschitz条件得

$$\left|u(x) - v(x)\right| \le L\left|\int_{x_0}^{x} \left|u(t) - v(t)\right|\mathrm{d}t\right|. \qquad (5)$$

设连续函数$\left|u(x)-v(x)\right|$在区间J上的上界为K,则

$$\left|u(x)-v(x)\right|\le LK\left|x-x_0\right|,$$

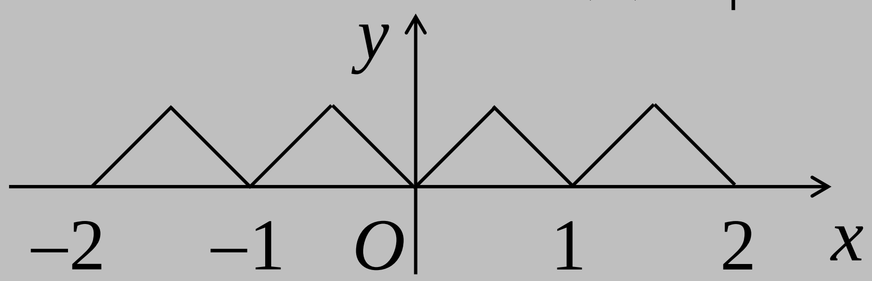
代入(5)式右端, 归纳可得, $\forall n\in\mathbb{N}$,

$$\left|u(x)-v(x)\right|\le K\frac{(L\left|x-x_0\right|)^n}{n!},\ x\in I.$$

令$n\to+\infty$得$u(x)\equiv v(x), x\in I$. 解的唯一性得证.□

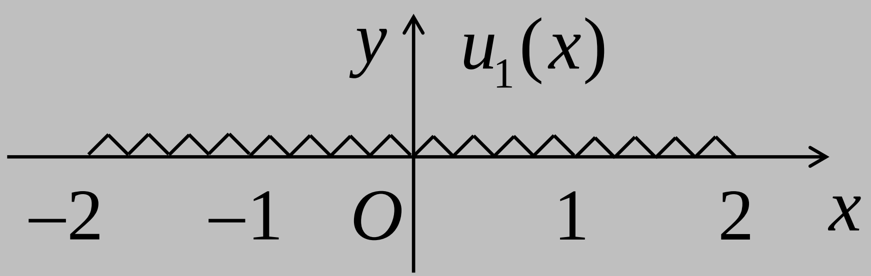
3. 函数项级数的应用 ---处处连续处处不可微的函数

$$u(x)=|x-m|,\ \ x\in[m-\frac{1}{2},m+\frac{1}{2}],m\in\mathbb{Z}.$$



$$u(x):\text{周期为}1$$

$$u'(x)=(-1)^m,x\in(\frac{m}{2},\frac{m+1}{2})$$



$$u_k(x)=\frac{u(4^k x)}{4^k},k=0,1,2,\cdots$$

$$u_k(x):\text{周期为}\frac{1}{4^k}$$

$$u_k'(x)=u'(4^k x)=(-1)^m,x\in(\frac{m}{2\cdot 4^k},\frac{m+1}{2\cdot 4^k})$$

$$0 \le u_k(x) \le 1/(2\cdot 4^k), \quad k = 0,1,2,\cdots.$$

$S(x) = \sum_{k=0}^{+\infty} u_k(x)$ 一致收敛(Weirstrass), 故 $S(x)$处处连续.

下证 $S(x)$ 处处不可微. 任意取定$c \in \mathbb{R}$, $\forall n \in \mathbb{N}$, $\exists m_n \in \mathbb{Z}$, $s.t.$

$$c \in \left[\frac{m_n}{2\cdot 4^n}, \frac{m_n+1}{2\cdot 4^n}\right) \triangleq I_n \supset I_{n+1}. \ \exists x_n \in I_n, s.t. |x_n - c| = \frac{1}{4^{n+1}}.$$

于是
$$\frac{u_k(x_n) - u_k(c)}{x_n - c} = \begin{cases} 0, & k \ge n+1, \quad (u_k\text{的周期性}) \\ (-1)^{m_k}, & 0 \le k \le n. \quad (x_n, c \in I_n \subset I_k) \end{cases}$$

$$\lim_{n\to+\infty} \frac{S(x_n) - S(c)}{x_n - c} = \lim_{n\to+\infty} \sum_{k=0}^{+\infty} \frac{u_k(x_n) - u_k(c)}{x_n - c} = \lim_{n\to+\infty} \sum_{k=0}^{n} (-1)^{m_k} \text{ 发散.}$$

4. 函数项级数的应用 ---填满正方形的连续曲线

目标: $x=\varphi(t), y=\psi(t), t\in[0,1],\ \varphi,\psi\in C[0,1],$

$$\forall a,b\in[0,1], \exists t\in[0,1], s.t.\ \ \varphi(t)=a, \psi(t)=b.$$

实数的p进制表示:

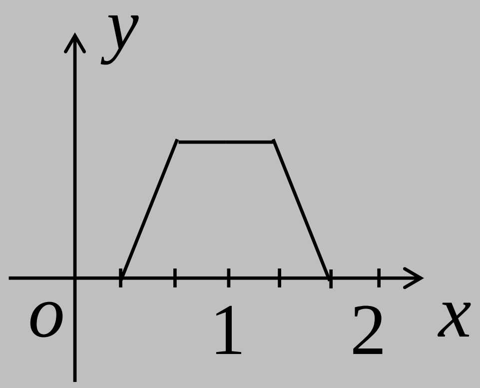
$$a=\sum_{n=1}^{+\infty}\frac{a_n}{2^n},\quad b=\sum_{n=1}^{+\infty}\frac{b_n}{2^n},\quad a_n, b_n\in\{0,1\}.$$

令$c_{2n-1}=a_n, c_{2n}=b_n$,则$c=2\sum_{n=1}^{+\infty}\frac{c_n}{3^n}\in[0,1]$.

如何构造连续函数φ,ψ,将 a,b 从c中"滤"出来?

构造2-周期连续函数$\omega(t), s.t.$

$$\omega(t)=\begin{cases} 0, & t\in[0,1/3], \\ 3t-1, & t\in[1/3,2/3], \\ 1, & t\in[2/3,4/3], \\ -3t+5, & t\in[4/3,5/3], \\ 0 & t\in[5/3,2]. \end{cases}$$



$c_{k+1}=1$时, $2\sum_{n=k+1}^{+\infty}\frac{c_n}{3^{n-k}}\in[2/3,1]$; $c_{k+1}=0$时, $2\sum_{n=k+1}^{+\infty}\frac{c_n}{3^{n-k}}\in[0,1/3]$;

$$\omega(3^k c)=\omega\left(2\sum_{n=1}^{k}3^{k-n}c_n+\sum_{n=k+1}^{+\infty}\frac{2c_n}{3^{n-k}}\right)=\omega\left(2\sum_{n=k+1}^{+\infty}\frac{c_n}{3^{n-k}}\right)=c_{k+1}.$$

令 $\varphi(t)=\sum_{n=1}^{+\infty}\frac{\omega(3^{2n-2}t)}{2^n},\ \psi(t)=\sum_{n=1}^{+\infty}\frac{\omega(3^{2n-1}t)}{2^n}, t\in[0,1].$

$\sum_{n=1}^{+\infty}\frac{1}{2^n}$ 为以上两函数项级数的优级数, 因此函数项级数

一致收敛, 和函数 $\varphi,\psi\in C[0,1]$. 而

$$\varphi(c)=\sum_{n=1}^{+\infty}\frac{c_{2n-1}}{2^n}=\sum_{n=1}^{+\infty}\frac{a_n}{2^n}=a,\ \psi(c)=\sum_{n=1}^{+\infty}\frac{c_{2n}}{2^n}=\sum_{n=1}^{+\infty}\frac{b_n}{2^n}=b,$$

故连续曲线 $\begin{cases}x=\varphi(t),\\ y=\psi(t),\end{cases} t\in[0,1]$ 填满正方形 $[0,1]\times[0,1]$.

例.证明$I=\lim_{n\to\infty}\int_0^1\frac{\mathrm{d}x}{1+(1+x/n)^n}=\ln\frac{2e}{e+1}$.

Proof. $\int_0^1\lim_{n\to\infty}\frac{\mathrm{d}x}{1+(1+x/n)^n}=\int_0^1\frac{\mathrm{d}x}{1+e^x}=\int_0^1\frac{e^{-x}\mathrm{d}x}{e^{-x}+1}$

$$=-\ln(e^{-x}+1)\Big|_{x=0}^{1}=\ln\frac{2e}{e+1}.$$

因此,只要证$\frac{1}{1+(1+x/n)^n}$在[0,1]上一致收敛到$\frac{1}{1+e^x}$.

而后一结论可以从下述不等式得出:

$$\left|\frac{1}{1+(1+x/n)^n}-\frac{1}{1+e^x}\right|=\left|\frac{e^x-(1+x/n)^n}{\left(1+(1+x/n)^n\right)\left(1+e^x\right)}\right|$$

$$\le\left|e^x-(1+x/n)^n\right| \ =\left|e^x-e^{n\ln(1+x/n)}\right|$$

$$=e^x-e^{n\left(\frac{x}{n}-\frac{x^2}{2(1+\xi)^2n^2}\right)} \qquad \xi\in(0,\frac{x}{n}),\ x\in[0,1]$$

$$=e^x\left(1-e^{\frac{-x^2}{2(1+\xi)^2n}}\right)\le e\left(1-e^{-\frac{1}{2n}}\right)\to 0, n\to+\infty\text{时}.$$

故 $\dfrac{1}{1+(1+x/n)^n}$ 在 $x\in[0,1]$ 上一致收敛到 $\dfrac{1}{1+e^x}$. □

例. $\alpha\in(0,1)$, 则 $\displaystyle\int_0^{+\infty}\frac{x^{\alpha-1}}{1+x}\,\mathrm{d}x=\frac{1}{\alpha}+\sum_{k=1}^{+\infty}(-1)^k\left(\frac{1}{\alpha+k}+\frac{1}{\alpha-k}\right)$.

Proof. $\displaystyle\int_0^{+\infty}\frac{x^{\alpha-1}}{1+x}\,\mathrm{d}x=\int_0^1\frac{x^{\alpha-1}}{1+x}\,\mathrm{d}x+\int_1^{+\infty}\frac{x^{\alpha-1}}{1+x}\,\mathrm{d}x\triangleq I_1+I_2.$

- $x\in(0,1)$时, $\displaystyle\frac{x^{\alpha-1}}{1+x}=\sum_{k=0}^{+\infty}(-1)^k x^{\alpha+k-1}$, （非一致收敛！）

$$\left|\sum_{k=n+1}^{+\infty}(-1)^k x^{\alpha+k-1}\right|=\frac{x^{\alpha+n}}{1+x}<x^{\alpha+n}.$$

$$\left|I_1-\int_0^1\sum_{k=0}^{n}(-1)^k x^{\alpha+k-1}\,\mathrm{d}x\right|=\left|\int_0^1\sum_{k=n+1}^{+\infty}(-1)^k x^{\alpha+k-1}\,\mathrm{d}x\right|$$

$$\le\int_0^1 x^{\alpha+n}\,\mathrm{d}x=\frac{1}{\alpha+n+1}\to 0,\text{当}n\to+\infty\text{时}.$$

故 $I_1=\sum_{k=0}^{+\infty}\int_0^1(-1)^k x^{\alpha+k-1}\,\mathrm{d}x=\sum_{k=0}^{+\infty}\frac{(-1)^k}{\alpha+k}.$ （非一致收敛但逐项可积！）

- $x\in(1,\infty)$时，令$t=1/x$，则

$$I_2=\int_1^{+\infty}\frac{x^{\alpha-1}}{1+x}\,\mathrm{d}x=\int_0^1\frac{t^{-\alpha}}{1+t}\,\mathrm{d}t=\int_0^1\frac{t^{(1-\alpha)-1}}{1+t}\,\mathrm{d}t$$

$1-\alpha\in(0,1)$,由前面的结论,

$$I_2=\sum_{k=0}^{+\infty}\frac{(-1)^k}{(1-\alpha)+k}=\sum_{k=1}^{+\infty}\frac{(-1)^{k+1}}{k-\alpha}=\sum_{k=1}^{+\infty}\frac{(-1)^k}{\alpha-k}.$$

综上, $\int_0^{+\infty}\frac{x^{\alpha-1}}{1+x}\,\mathrm{d}x=\frac{1}{\alpha}+\sum_{k=1}^{+\infty}(-1)^k\left(\frac{1}{\alpha+k}+\frac{1}{\alpha-k}\right).$ □

作业:习题6.2 No.2,5,7.