

Review

曲面的切平面与法线：

曲面方程	点	法向量
$\mathbf{r} = \mathbf{r}(u, v)$	$\mathbf{r}_0 = \mathbf{r}(u_0, v_0)$	$(\mathbf{r}'_u \times \mathbf{r}'_v) \big _{(u_0, v_0)}$
$z = f(x, y)$	(x_0, y_0, z_0) $z_0 = f(x_0, y_0)$	$(-f'_x, -f'_y, 1)^T \big _{(x_0, y_0)}$
$F(x, y, z) = 0$	$\mathbf{r}_0 = (x_0, y_0, z_0)$	$\text{grad}F(\mathbf{r}_0)$

曲线的切向量：

曲线方程	点	切向量
$\mathbf{r} = \mathbf{r}(t)$	$\mathbf{r}_0 = \mathbf{r}(t_0)$	$\mathbf{r}'(t_0) =$ $(x'(t_0), y'(t_0), z'(t_0))$
$\begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases}$	$\mathbf{r}_0 =$ (x_0, y_0, z_0)	$\text{grad}F(\mathbf{r}_0) \times \text{grad}G(\mathbf{r}_0)$

§ 8. 多元函数的Taylor公式

对充分光滑的一元函数 $f(x)$, 有

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n \\ + \frac{1}{(n+1)!} f^{(n+1)}(x_0 + \theta(x - x_0))(x - x_0)^{n+1},$$

$(0 < \theta < 1)$, 带*Lagrange*余项的*n*阶Taylor公式

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n \\ + o((x - x_0)^n), \text{ 带Peano余项的}n\text{阶Taylor公式}$$

*Taylor*公式的应用之一是近似计算.

对二元可微函数 $g(x, y)$, 利用全微分也可以进行近似计算.

$$\begin{aligned} g(x, y) &= g(x_0, y_0) + g'_x(x_0, y_0)(x - x_0) \\ &\quad + g'_y(x_0, y_0)(y - y_0) \\ &\quad + o\left(\sqrt{(x - x_0)^2 + (y - y_0)^2}\right) \end{aligned}$$

$$\begin{aligned} g(x, y) &\approx g(x_0, y_0) + g'_x(x_0, y_0)(x - x_0) \\ &\quad + g'_y(x_0, y_0)(y - y_0) \end{aligned}$$

Question: 如何进一步提高精度?

Lemma. 设 $A = (a_{ij})_{n \times n}$ 为 n 阶实对称矩阵, 则

$$|x^T A x| \leq \left(n \sum_{i=1}^n \sum_{j=1}^n |a_{ij}| \right) \cdot \|x\|^2, \quad \forall x \in \mathbb{R}^n.$$

Proof. 记 $A = (\alpha_1, \dots, \alpha_n)$, $\forall x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$, 有

$$\begin{aligned} \|Ax\| &= \|x_1 \alpha_1 + \dots + x_n \alpha_n\| \leq |x_1| \|\alpha_1\| + \dots + |x_n| \|\alpha_n\| \\ &\leq (|x_1| + \dots + |x_n|) (\|\alpha_1\| + \dots + \|\alpha_n\|) \leq n \|x\| \cdot \sum_{i=1}^n \sum_{j=1}^n |a_{ij}|, \end{aligned}$$

$$\text{于是 } |x^T A x| \leq \|x\| \cdot \|Ax\| \leq \left(n \sum_{i=1}^n \sum_{j=1}^n |a_{ij}| \right) \cdot \|x\|^2. \quad \square$$

Remark. $|x^T A x| \leq \max |\lambda(A)| \cdot \|x\|^2, \quad \forall x \in \mathbb{R}^n.$

Def. 设 n 元函数 f 在 $B(\mathbf{x}_0, \delta)$ 中二阶连续可微, 称实对称阵

$$H_f(\mathbf{x}_0) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}_{\mathbf{x}_0}$$

为 f 在点 \mathbf{x}_0 的Hessian矩阵.

Thm. 设 n 元函数 f 在 $B(x_0, \delta)$ 中二阶连续可微, 则

$\forall x_0 + \Delta x \in B(x_0, \delta), \exists \theta \in (0, 1), s.t.$

$$f(x_0 + \Delta x) = f(x_0) + J_f(x_0)\Delta x \\ + \frac{1}{2}(\Delta x)^T H_f(x_0 + \theta\Delta x)\Delta x$$

(称为带 $Lagrange$ 余项的一阶 $Taylor$ 公式), 且

$$f(x_0 + \Delta x) = f(x_0) + J_f(x_0)\Delta x \\ + \frac{1}{2}(\Delta x)^T H_f(x_0)\Delta x + o(\|\Delta x\|^2), \Delta x \rightarrow 0 \text{ 时}$$

(称为带 $Peano$ 余项的二阶 $Taylor$ 公式).

Proof. 构造一元函数

$$g(t) = f(x_0 + t\Delta x) = f(x_0^{(1)} + t\Delta x_1, \dots, x_0^{(n)} + t\Delta x_n),$$

$$f \in C^2, \text{ 则 } g'(t) = \sum_{i=1}^n f'_i(x_0 + t\Delta x) \Delta x_i = J_f(x_0 + t\Delta x) \Delta x$$

$$g''(t) = \sum_{i=1}^n \sum_{j=1}^n f''_{ij}(x_0 + t\Delta x) \Delta x_i \Delta x_j = (\Delta x)^T H_f(x_0 + t\Delta x) \Delta x$$

$$g(t) = g(0) + g'(0)t + \frac{1}{2} g''(\theta t)t^2, \theta \in (0, 1).$$

$$\begin{aligned} f(x_0 + \Delta x) &= g(1) = g(0) + g'(0) + \frac{1}{2} g''(\theta) \\ &= f(x_0) + J_f(x_0) \Delta x + \frac{1}{2} (\Delta x)^T H_f(x_0 + \theta \Delta x) \Delta x, \theta \in (0, 1). \end{aligned}$$

带Lagrange余项的一阶Taylor公式得证.

记 $P(\Delta x) = H_f(x_0 + \theta\Delta x) - H_f(x_0) = (h_{ij})_{n \times n}$,

$$\begin{aligned}\alpha(\Delta x) &= \frac{1}{2}(\Delta x)^T H_f(x_0 + \theta\Delta x)\Delta x - \frac{1}{2}(\Delta x)^T H_f(x_0)\Delta x \\ &= \frac{1}{2}(\Delta x)^T P(\Delta x)\Delta x.\end{aligned}$$

则
$$f(x_0 + \Delta x) = f(x_0) + J_f(x_0)\Delta x + \frac{1}{2}(\Delta x)^T H_f(x_0)\Delta x + \alpha(\Delta x).$$

$$|\alpha(\Delta x)| \leq \frac{1}{2} \left(n \sum_{i=1}^n \sum_{j=1}^n |h_{ij}| \right) \cdot \|\Delta x\|^2.$$

由 $f \in C^2$, 有 $\lim_{\Delta x \rightarrow 0} h_{ij} = 0$, $\alpha(\Delta x) = o(\|\Delta x\|^2)$, 当 $\Delta x \rightarrow 0$ 时. \square

Question. 能否用一元函数带Peano余项的Taylor公式直接证明多元函数带Peano余项的Taylor公式?

Thm. 设函数 $f(x, y)$ 在区域 D 中 $n+1$ 阶连续可微,
 $M_0(x_0, y_0) \in D, M(x, y) \in D$, 且线段 $\overline{M_0M}$ 完全包含在 D 中. 记 $h = x - x_0, k = y - y_0$, 记算子

$$\left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^m \triangleq \sum_{i=0}^m C_m^i h^i k^{m-i} \frac{\partial^m}{\partial x^i \partial y^{m-i}},$$

则 f 在点 (x_0, y_0) 有

(1)带 $Lagrange$ 余项的 n 阶 $Taylor$ 公式

$$\begin{aligned} f(x, y) = & f(x_0, y_0) + \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) f \Big|_{(x_0, y_0)} \\ & + \cdots + \frac{1}{n!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^n f \Big|_{(x_0, y_0)} \\ & + \frac{1}{(n+1)!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^{n+1} f \Big|_{(x_0 + \theta h, y_0 + \theta k)} \\ & \quad (0 < \theta < 1) \end{aligned}$$

(2)带 $Peano$ 余项的 $n+1$ 阶 $Taylor$ 公式

$$\begin{aligned} f(x, y) = & f(x_0, y_0) + \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) f \Big|_{(x_0, y_0)} \\ & + \cdots + \frac{1}{(n+1)!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^{n+1} f \Big|_{(x_0, y_0)} \\ & + o \left(\left(\sqrt{h^2 + k^2} \right)^{n+1} \right). \end{aligned}$$

Question. n 元函数的 m 阶 $Taylor$ 公式?

Thm. 设 m 元函数 $f(x)$ 在 $x_0 = (x_1^0, \dots, x_m^0)$ 的邻域 D 中
 $n+1$ 阶连续可微, $x \in D$. 记 $h_i = x_i - x_i^0, i = 1, \dots, m$, 则

$$f(x) = \sum_{k=0}^n \frac{1}{k!} \left(h_1 \frac{\partial}{\partial x_1} + \dots + h_m \frac{\partial}{\partial x_m} \right)^k f \Big|_{x_0} \\ + \frac{1}{(n+1)!} \left(h_1 \frac{\partial}{\partial x_1} + \dots + h_m \frac{\partial}{\partial x_m} \right)^{n+1} f \Big|_{x_0 + \theta h} \quad (0 < \theta < 1),$$

$$f(x) = \sum_{k=0}^{n+1} \frac{1}{k!} \left(h_1 \frac{\partial}{\partial x_1} + \dots + h_m \frac{\partial}{\partial x_m} \right)^k f \Big|_{x_0} \\ + o(\|x - x_0\|^{n+1}), \quad x \rightarrow x_0 \text{ 时.}$$

例：写出 $f(x, y) = x^y$ 在点 $(1, 1)$ 的邻域内带 $Peano$ 余项的3阶 $Taylor$ 公式, 并求 $(1.1)^{1.02}$.

解： $f(x, y) = x^y, f'_x = yx^{y-1}, f'_y = x^y \ln x,$

$$f''_{xx} = y(y-1)x^{y-2}, f''_{yy} = x^y \ln^2 x, f''_{xy} = x^{y-1} + yx^{y-1} \ln x,$$

$$f'''_{xxx} = y(y-1)(y-2)x^{y-3}, f'''_{yyy} = x^y \ln^3 x$$

$$f'''_{xxy} = (2y-1)x^{y-2} + y(y-1)x^{y-2} \ln x,$$

$$f'''_{xyy} = yx^{y-1} \ln^2 x + 2x^{y-1} \ln x.$$

$$\text{令 } x_0 = y_0 = 1, h = x - 1, k = y - 1.$$

$$f(1,1) = f'_x(1,1) = f''_{xy}(1,1) = f'''_{xxy}(1,1) = 1,$$

$$f'_y(1,1) = f''_{xx}(1,1) = f''_{yy}(1,1)$$

$$= f'''_{xxx}(1,1) = f'''_{yyy}(1,1) = f'''_{xyy}(1,1) = 0$$

$$f(x, y) = f(x_0, y_0) + \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right) f(x_0, y_0)$$

$$+ \frac{1}{2!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right)^2 f(x_0, y_0)$$

$$+ \frac{1}{3!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right)^3 f(x_0, y_0) + o((\sqrt{h^2 + k^2})^3)$$

$$\begin{aligned} f(x, y) &= 1 + (x-1) + (x-1)(y-1) \\ &\quad + \frac{1}{2}(x-1)^2(y-1) \\ &\quad + o\left(\left(\sqrt{(x-1)^2 + (y-1)^2}\right)^3\right) \end{aligned}$$

$$\begin{aligned} (1.1)^{1.02} &\approx 1.1 + 0.1 \times 0.02 + \frac{1}{2} \times 0.01 \times 0.02 \\ &= 1.1021. \square \end{aligned}$$

Question. n 元函数在一点的Taylor多项式是否唯一？ 如何证明？ **唯一！**

例. $\cos(x^2 + y^2)$ 在 $(0,0)$ 的8阶带Peano余项的Taylor展开式.

解: $\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots + (-1)^n \frac{t^{2n}}{(2n)!} + o(t^{2n}), t \rightarrow 0$ 时.

$$\cos(x^2 + y^2) = 1 - \frac{(x^2 + y^2)^2}{2!} + \cdots + (-1)^n \frac{(x^2 + y^2)^{2n}}{(2n)!} + o((x^2 + y^2)^{2n}), x^2 + y^2 \rightarrow 0 \text{时}.$$

令 $n = 2$ 得

$$\cos(x^2 + y^2) = 1 - \frac{(x^2 + y^2)^2}{2!} + \frac{(x^2 + y^2)^4}{4!} + o((x^2 + y^2)^4), x^2 + y^2 \rightarrow 0 \text{时}. \square$$

例. $\ln(2+x+y+xy)$ 在 $(0,0)$ 带Peano余项的2阶Taylor展开.

解: $x+y+xy \rightarrow 0$ 时,

$$\begin{aligned}\ln(2+x+y+xy) &= \ln 2 + \ln\left(1 + \frac{x+y+xy}{2}\right) \\ &= \ln 2 + \frac{x+y+xy}{2} - \frac{1}{2} \left(\frac{x+y+xy}{2} \right)^2 + o\left((x+y+xy)^2\right)\end{aligned}$$

$x^2+y^2 \rightarrow 0$ 时, $x^2y = o(x^2+y^2)$, $xy^2 = o(x^2+y^2)$,

$$\begin{aligned}\frac{(x+y+xy)^2}{x^2+y^2} &\leq 2 + \frac{2(x^2y+xy^2+x^2y^2)}{x^2+y^2} \rightarrow 2, \\ o\left((x+y+xy)^2\right) &= o(x^2+y^2).\end{aligned}$$

$$\ln(2+x+y+xy) = \ln 2 + \frac{x+y}{2} - \frac{x^2+y^2-2xy}{8} + o(x^2+y^2). \square$$



作业：习题1.8 No. 2(2) (3)