2013 级多元微积分期中考题（A）答案

一．填空题

1. $\dfrac{ydx+xdy}{1+e^z}$

2. 3

3. $f_1'+f_3'\dfrac{x}{x^2+y^2}$

4. $\dfrac{32}{9}$

5. $\int_0^2 dy\int_{\frac{y}{3}}^{\frac{y}{2}} f(x,y)dx+\int_2^3 dy\int_{\frac{y}{3}}^{1} f(x,y)dx$

6. $\dfrac{\sin x^2}{x}+\int_0^x \cos xydy$

7. 2
8. 1
9. $x+2y=4$
10. $\begin{cases} x+z=2 \\ y+2=0 \end{cases}$

11. $1+x+\frac{1}{2}(x^2-y^2)+o(x^2+y^2)$

12. $\frac{\partial f}{\partial u}=0$

13. $(-1,1)$

14. $\dfrac{2}{5}$

15. 是

二．计算题

1. 解：$\dfrac{\partial z}{\partial x}=\dfrac{f'}{1-f'}$

$$\frac{\partial^2 z}{\partial x^2}=\frac{f''}{(1-f')^3}$$

2. **解**：问题转化为求 $f(x,y)=x^2+y^2$ 在条件 $5x^2+4xy+2y^2=1$ 下的极值。令

$$\varphi(x,y)=x^2+y^2+\lambda(5x^2+4xy+2y^2-1)$$

则
$$\begin{cases}\varphi_x'(x,y)=2x+10\lambda x+4\lambda y=0, & (1)\\ \varphi_y'(x,y)=2y+4\lambda x+4\lambda y=0, & (2)\\ \varphi_\lambda'(x,y)=5x^2+4xy+2y^2-1=0, & (3)\end{cases}$$

由问题的实际意义知条件极值存在，方程组必有非零解。视 $(1),(2)$ 为关于 x,y 的方程组，化简为

$$\begin{cases}(1+5\lambda)x+2\lambda y=0,\\ 2\lambda x+(1+2\lambda)y=0,\end{cases}$$

该方程组也有非零解，故系数行列式为 0，即

$$\det\begin{pmatrix}1+5\lambda & 2\lambda\\ 2\lambda & 1+2\lambda\end{pmatrix}=1+7\lambda+6\lambda^2=0,\lambda_1=-1,\lambda_2=-\frac{1}{6}.$$

$(1)\cdot x_i+(2)\cdot y_i$，得

$$x_i^2+y_i^2+\lambda(5x_i^2+4x_iy_i+2y_i^2)=0.$$

利用 (3)，得 $x_i^2+y_i^2=-\lambda_i$. 故椭圆的长半轴为 1，

短半轴为 $\frac{1}{\sqrt{6}}$.

3. 解：$\iint\limits_D ydxdy=\int_0^2 dy\int_{-2}^{-\sqrt{2y-y^2}} ydx$，

$$=4-\int_0^2 y\sqrt{2y-y^2}dy$$

令 $y-1=\sin t$，$\iint\limits_D ydxdy=4-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+\sin t)\cos^2 tdt=4-\frac{\pi}{2}$。

4. （1）$p>0$ 时，$f(x,y)$ 在原点连续；

（2）$p>\frac{1}{2}$ 时，$f_x'(x,y)$ 和 $f_y'(x,y)$ 都存在；

（3） $p>\frac{1}{2}$ 时，$f(x,y)$ 在原点可微。

三。证明题

1.证明：$\frac{\partial w}{\partial u}=\frac{\partial h}{\partial x}\frac{\partial f}{\partial u}+\frac{\partial h}{\partial y}\frac{\partial g}{\partial u}$

$$\frac{\partial^2 w}{\partial u^2}=\left(\frac{\partial^2 h}{\partial x^2}\frac{\partial f}{\partial u}+\frac{\partial^2 h}{\partial x\partial y}\frac{\partial g}{\partial u}\right)\frac{\partial f}{\partial u}+\left(\frac{\partial^2 h}{\partial x\partial y}\frac{\partial f}{\partial u}+\frac{\partial^2 h}{\partial y^2}\frac{\partial g}{\partial u}\right)\frac{\partial g}{\partial u}+\frac{\partial h}{\partial x}\frac{\partial^2 f}{\partial u^2}+\frac{\partial h}{\partial y}\frac{\partial^2 g}{\partial u^2}$$

$$\frac{\partial^2 w}{\partial v^2}=\left(\frac{\partial^2 h}{\partial x^2}\frac{\partial f}{\partial v}+\frac{\partial^2 h}{\partial x\partial y}\frac{\partial g}{\partial v}\right)\frac{\partial f}{\partial v}+\left(\frac{\partial^2 h}{\partial x\partial y}\frac{\partial f}{\partial v}+\frac{\partial^2 h}{\partial y^2}\frac{\partial g}{\partial v}\right)\frac{\partial g}{\partial v}+\frac{\partial h}{\partial x}\frac{\partial^2 f}{\partial v^2}+\frac{\partial h}{\partial y}\frac{\partial^2 g}{\partial v^2}$$

因为$\frac{\partial f}{\partial u}=\frac{\partial g}{\partial v}$，$\frac{\partial f}{\partial v}=-\frac{\partial g}{\partial u}$，$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}=0$，则

$$\frac{\partial^2 w}{\partial u^2}+\frac{\partial^2 w}{\partial v^2}=\frac{\partial^2 h}{\partial x^2}\left(\left(\frac{\partial f}{\partial u}\right)^2+\left(\frac{\partial f}{\partial v}\right)^2\right)+2\frac{\partial^2 h}{\partial x\partial y}\left(\frac{\partial f}{\partial u}\frac{\partial g}{\partial u}+\frac{\partial f}{\partial v}\frac{\partial g}{\partial v}\right)+\frac{\partial^2 h}{\partial y^2}\left(\left(\frac{\partial g}{\partial u}\right)^2+\left(\frac{\partial g}{\partial v}\right)^2\right)$$

$$+\frac{\partial h}{\partial x}\left(\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2}\right)+\frac{\partial h}{\partial y}\left(\frac{\partial^2 g}{\partial u^2}+\frac{\partial^2 g}{\partial v^2}\right)=0.$$

2．证明：构造函数 $g(x,y)=f(x,y)+2(x^2+y^2)$

则在单位圆周上，有 $g(x,y)\geq 1$, 而在原点， $g(0,0)=f(0,0)\leq 1$.

这样$g(x,y)$在单位圆内取到最小值，同时为极值点.

设(x_0,y_0)为单位圆内$g(x,y)$的一个极值点，则 $\frac{\partial g(x_0,y_0)}{\partial x}=\frac{\partial g(x_0,y_0)}{\partial y}=0$

即 $\left|\frac{\partial f(x_0,y_0)}{\partial x}\right|=|4x_0|$, $\left|\frac{\partial f(x_0,y_0)}{\partial y}\right|=|4y_0|$

从而$\left[\frac{\partial f(x_0,y_0)}{\partial x}\right]^2+\left[\frac{\partial f(x_0,y_0)}{\partial y}\right]^2=16(x_0{}^2+y_0{}^2)\leq 16$。