Review

- 极限与连续
- 判断函数在一点没有极限的方法
- (n重)极限与累次极限
- 连续函数在有界闭集上的性质
- $o(\|x-x_0\|^k)$与$O(\|x-x_0\|^k)$, $x\to x_0$时.

§4. 多元函数的偏导数与全微分

1. 偏导数

Def. $u=f(\mathrm{x})=f(x_1,x_2,\cdots,x_n)$在$\mathrm{x}_0=(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)})\in\mathbb{R}^n$的某个邻域中有定义, 若极限

$$\lim_{\Delta x_i\to 0}\frac{\Delta_{x_i}u}{\Delta x_i}=\lim_{\Delta x_i\to 0}\frac{f(x_0^{(1)},\cdots,x_0^{(i-1)},x_0^{(i)}+\Delta x_i,x_0^{(i+1)},\cdots,x_0^{(n)})-f(\mathrm{x}_0)}{\Delta x_i}$$

存在, 则称之为$f(\mathrm{x})$在x_0关于x_i的偏导数, 记作$\dfrac{\partial f}{\partial x_i}(\mathrm{x}_0)$,

$\dfrac{\partial u}{\partial x_i}(\mathrm{x}_0)$, $\left.\dfrac{\partial f}{\partial x_i}\right|_{\mathrm{x}_0}$, $\left.\dfrac{\partial u}{\partial x_i}\right|_{\mathrm{x}_0}$, $u'_{x_i}(\mathrm{x}_0)$或$f'_{x_i}(\mathrm{x}_0)$.

Remark: 二元函数$f(x,y)$偏导数的几何意义.

$$f_x'(x_0,y_0)=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}$$

即平面$y=y_0$截曲面$z=f(x,y)$所得曲线$z=f(x,y_0)$在点$x=x_0$处切线的斜率.

Remark: 1) 对某个变量求偏导数时, 视其余变量为常数, 按一元函数求导法则(或定义)去求.

2)求分段函数的偏导函数时, 用定义求分界点处的偏导数, 用1) 中方法求其它点处的偏导数. 一般地, 分段函数的偏导函数仍为分段函数.

Remark: 视x_0为变量, 得偏导函数$\dfrac{\partial f}{\partial x_i}(x), i=1,2,\cdots,n.$

例. 设$f(x,y)=\begin{cases}(x+y)^2\sin\dfrac{1}{x^2+y^2} & x^2+y^2\neq 0\\ 0 & x^2+y^2=0\end{cases}$, 求$f_x'(x,y)$.

解: $f_x'(0,0)=\lim\limits_{x\to 0}\dfrac{f(x,0)-f(0,0)}{x}=\lim\limits_{x\to 0}\dfrac{x^2\sin\frac{1}{x^2}}{x}=0.$

$x^2+y^2\neq 0$时,

$$f_x'(x,y)=2(x+y)\sin\frac{1}{x^2+y^2}-\frac{2x(x+y)^2}{(x^2+y^2)^2}\cos\frac{1}{x^2+y^2}.\square$$

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例. $f(x,y)=x^2e^y+(x-1)\arctan\dfrac{y}{x}$, 求$f'_x(1,0)$.

解法一: $f(x,0)=x^2$, 所以$f'_x(1,0)=2$.

解法二:

$$f'_x(x,y)=2xe^y+\arctan\frac{y}{x}+(x-1)\cdot\frac{\frac{-y}{x^2}}{1+(\frac{y}{x})^2}$$

$$=2xe^y+\arctan\frac{y}{x}+\frac{y(1-x)}{x^2+y^2}.$$

所以$f'_x(1,0)=2.\square$

Remark: 求具体点处的偏导数时, 第一种方法通常较简单.

Remark: 多元函数偏导数存在与函数连续性互不蕴含：

例：设$f(x,y)=\begin{cases}1 & y=x^2, x>0\\ 0 & \text{其它情形}\end{cases}$，则$f(x,0)\equiv f(0,y)\equiv 0$，

$f_x'(0,0)=f_y'(0,0)=0$，但f在$(0,0)$不连续.

例：$f(x,y)=\sqrt{x^2+y^2}$在原点连续，但$f_x'(0,0)$，$f_y'(0,0)$

都不存在. 事实上，$\lim\limits_{x\to 0}\dfrac{f(x,0)-f(0,0)}{x}=\lim\limits_{x\to 0}\dfrac{\sqrt{x^2}}{x}$与

$\lim\limits_{y\to 0}\dfrac{f(0,y)-f(0,0)}{y}=\lim\limits_{y\to 0}\dfrac{\sqrt{y^2}}{y}$都不存在.□

例. $z = f(x,y), \dfrac{\partial z}{\partial y} = x^2 + 2y, f(x,x^2) = 1$, 求 $f(x,y)$.

解: 由 $\dfrac{\partial z}{\partial y} = x^2 + 2y$, 将 x 看成常数, 两边对 y 积分, 得

$$z = f(x,y) = \int \left(x^2 + 2y\right) \mathrm{d}y \;= x^2 y + y^2 + g(x),$$

其中 $g(x)$ 为待定函数. 由 $f(x,x^2) = 1$, 有

$$g(x) = 1 - 2x^4,$$

$$f(x,y) = 1 + x^2 y + y^2 - 2x^4. \square$$

2. 全微分

1）一元函数的微分

以直代曲

近似计算

$$\Delta f(x_0) = f(x_0 + \Delta x) - f(x_0)$$
$$= f'(x_0)\Delta x + \alpha,$$

其中 $\lim\limits_{\Delta x \to 0} \dfrac{\alpha}{\Delta x} = 0$, 即 $\alpha = o(\Delta x)$, 当 $\Delta x \to 0$ 时. 记

$$\mathrm{d}f(x_0) = f'(x_0)\Delta x = f'(x_0)\mathrm{d}x.$$

2）二元函数的微分

推广一元微分的概念, 形式上应该有,

$$\Delta f(x_0, y_0) = f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0)$$

$$= (a,b)\begin{pmatrix}\Delta x \\ \Delta y\end{pmatrix} + \alpha,$$

几何直观?

其中 $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} \dfrac{\alpha}{\sqrt{(\Delta x)^2+(\Delta y)^2}} = 0$, 即

$$\alpha = o\left(\sqrt{(\Delta x)^2+(\Delta y)^2}\right), \quad \Delta x \to 0, \Delta y \to 0\text{时}.$$

这里的a,b应该与f在(x_0, y_0)的两个一阶偏导数有关.

3）n元函数的微分

Def. $u=f(\mathrm{x})$在$\mathrm{x}_0\in\mathbb{R}^n$的某邻域中定义, 若存在常数$a_1,a_2,\cdots,a_n$, s.t. 当$\Delta\mathrm{x}=(\Delta x_1,\Delta x_2,\cdots,\Delta x_n)\to 0$时,

$$\Delta u(\mathrm{x}_0)=\Delta f(\mathrm{x}_0)\triangleq f(\mathrm{x}_0+\Delta\mathrm{x})-f(\mathrm{x}_0)$$

$$=a_1\Delta x_1+a_2\Delta x_2+\cdots+a_n\Delta x_n+o(\|\Delta\mathrm{x}\|),$$

则称$u=f(\mathrm{x})$在点x_0可微, 称$a_1\Delta x_1+a_2\Delta x_2+\cdots+a_n\Delta x_n$为$f$在$\mathrm{x}_0$的(全)微分, 记作

$$\mathrm{d}u(\mathrm{x}_0)=\mathrm{d}f(\mathrm{x}_0)=a_1\Delta x_1+a_2\Delta x_2+\cdots+a_n\Delta x_n,$$

或 $$\mathrm{d}u(\mathrm{x}_0)=\mathrm{d}f(\mathrm{x}_0)=a_1\mathrm{d}x_1+a_2\mathrm{d}x_2+\cdots+a_n\mathrm{d}x_n.$$

Remark: f在x_0可微$\Leftrightarrow\exists$常数$a_1,a_2,\cdots,a_n\in\mathbb{R}, s.t.$

$$\lim_{x\to x_0}\frac{f(\mathrm{x})-f(\mathrm{x}_0)-a_1\Delta x_1-a_2\Delta x_2-\cdots-a_n\Delta x_n}{\|\mathrm{x}-\mathrm{x}_0\|}=0.$$

例. f为有界函数, 即$\exists M>0$, 使得$|f(x,y)|\le M, \forall(x,y)$,则

$g(x,y)=(x^2+y^2)^{3/2}f(x,y)$在$(0,0)$可微.

Proof. $$\frac{|g(x,y)-g(0,0)|}{\sqrt{x^2+y^2}}=\frac{\left|(x^2+y^2)^{3/2}f(x,y)\right|}{\sqrt{x^2+y^2}}$$

$$=(x^2+y^2)|f(x,y)|\le M(x^2+y^2)\to 0,\text{当}(x,y)\to(0,0)\text{时}.\square$$

Thm. $u=f(\mathrm{x})$在$\mathrm{x}_0\in\mathbb{R}^n$可微, 则

1) f在x_0连续,

2) $\dfrac{\partial f}{\partial x_i}(\mathrm{x}_0)$存在, $i=1,2,\cdots,n$, 且f在x_0的全微分为

$$\mathrm{d}u=\frac{\partial f}{\partial x_1}(\mathrm{x}_0)\mathrm{d}x_1+\frac{\partial f}{\partial x_2}(\mathrm{x}_0)\mathrm{d}x_2+\cdots+\frac{\partial f}{\partial x_n}(\mathrm{x}_0)\mathrm{d}x_n.$$

Proof: 记f在x_0的全微分为$\mathrm{d}u=a_1\mathrm{d}x_1+a_2\mathrm{d}x_2+\cdots+a_n\mathrm{d}x_n$.

1) $f(\mathrm{x}_0+\Delta\mathrm{x})-f(\mathrm{x}_0)=a_1\Delta x_1+a_2\Delta x_2+\cdots+a_n\Delta x_n+o(\|\Delta\mathrm{x}\|)$

$$\to 0, 当\Delta\mathrm{x}\to 0时.$$

故f在x_0连续.

2) 当$\mathrm{\Delta x}=(\Delta x_1,0,0,\cdots,0)$时，由可微的定义，

$$f(\mathrm{x}_0+\Delta\mathrm{x})-f(\mathrm{x}_0)=a_1\Delta x_1+a_2\Delta x_2+\cdots+a_n\Delta x_n+o(\|\Delta\mathrm{x}\|)$$

$$\Delta_{x_1}f(\mathrm{x}_0)=a_1\Delta x_1+o(|\Delta x_1|),$$

于是，$$f'_{x_1}(\mathrm{x}_0)=\lim_{\Delta x_1\to 0}\frac{\Delta_{x_1}f(\mathrm{x}_0)}{\Delta x_1}=a_1.$$

同理，$$f'_{x_i}(\mathrm{x}_0)=a_i,\ i=1,2,\cdots,n.$$

故f在x_0的全微分为

$$\mathrm{d}u=\frac{\partial f}{\partial x_1}(\mathrm{x}_0)\mathrm{d}x_1+\frac{\partial f}{\partial x_2}(\mathrm{x}_0)\mathrm{d}x_2+\cdots+\frac{\partial f}{\partial x_n}(\mathrm{x}_0)\mathrm{d}x_n.\square$$

Thm. n元函数$f(\mathrm{x})$在$\mathrm{x}_0\in\mathbb{R}$可微的充要条件是

$$\Delta f(\mathrm{x}_0)=f(\mathrm{x}_0+\Delta\mathrm{x})-f(\mathrm{x}_0)$$

$$=\sum_{i=1}^{n} f'_{x_i}(\mathrm{x}_0)\Delta x_i+o(\|\Delta\mathrm{x}\|),\ \Delta\mathrm{x}\to 0\text{时}.$$

Remark: 函数的连续性与偏导数的存在性不蕴含函数的可微性. 看下例.

例. 讨论$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$在原点的连续性、偏导数的存在性与连续性、可微性.

解：1) f在$(0,0)$连续.

$$\left|\frac{xy}{\sqrt{x^2+y^2}}-f(0,0)\right|\leq|x|\to 0,\ \ (x,y)\to(0,0)\text{时}.$$

2) $$\frac{\partial f(x,y)}{\partial x}=\begin{cases}\dfrac{y^3}{(x^2+y^2)^{3/2}}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0),\end{cases}$$

f_x'在(0, 0)存在但不连续. 同理,

$$\frac{\partial f(x,y)}{\partial y}=\begin{cases}\dfrac{x^3}{(x^2+y^2)^{3/2}}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0),\end{cases}$$

f_y'在(0, 0)存在但不连续.

3) f 在(0,0)不可微. 若可微, 则

$$\Delta f(0,0)=\frac{\Delta x\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$$

$$=0\cdot\Delta x+0\cdot\Delta y+o\left(\sqrt{(\Delta x)^2+(\Delta y)^2}\right),(\Delta x,\Delta y)\to(0,0)\text{时}.$$

故 $\lim\limits_{(\Delta x,\Delta y)\to(0,0)}\dfrac{\Delta x\Delta y}{(\Delta x)^2+(\Delta y)^2}=0.$

这与 $\lim\limits_{\Delta x\to0,\Delta y=\Delta x}\dfrac{\Delta x\Delta y}{(\Delta x)^2+(\Delta y)^2}=\dfrac{1}{2}$ 矛盾.□

例. $f(x,y)=\begin{cases}(x^2+y^2)\sin\dfrac{1}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$ 在原点的连续性、可微性、偏导数的存在性与偏导函数的连续性.

解: 1) f 在$(0,0)$连续.

2) $f(x,0)=\begin{cases}x^2\sin\dfrac{1}{x^2}, & x\neq 0\\ 0, & x=0\end{cases}$,

$$f_x'(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=\lim_{x\to 0}x\sin\frac{1}{x^2}=0;$$

当$(x,y)\neq(0,0)$时,

$$f_x'(x,y)=2x\sin\frac{1}{x^2+y^2}-\frac{2x}{x^2+y^2}\cos\frac{1}{x^2+y^2}.$$

$\lim\limits_{(x,y)\to(0,0)} f_x'(x,y)$不存在, 故$f_x'(x,y)$在$(0,0)$不连续.

同理$f_y'(0,0)=0$, 但f_y'在$(0,0)$不连续.

3) f在$(0,0)$可微. 事实上,

$$\begin{aligned}&\Delta f(0,0)-f_x'(0,0)\Delta x-f_y'(0,0)\Delta y\\&=((\Delta x)^2+(\Delta y)^2)\sin\frac{1}{(\Delta x)^2+(\Delta y)^2}\\&=o\left(\sqrt{(\Delta x)^2+(\Delta y)^2}\right),(\Delta x,\Delta y)\to(0,0)\text{ 时.}\ \square\end{aligned}$$

例: $f=\sqrt{\sin|xy|}$在原点的连续性、偏导数与可微性.

解: • f在原点连续.

•因$f(x,0)\equiv 0,\ f(0,y)\equiv 0$, 故$f_x'(0,0)=f_y'(0,0)=0$.

•若f在原点可微, 则 $\lim\limits_{(x,y)\to(0,0)}\dfrac{\sqrt{\sin|xy|}}{\sqrt{x^2+y^2}}=0$. 而$k\neq 0$时,

$$\lim_{\substack{y=kx\\x\to 0}}\frac{\sqrt{\sin|xy|}}{\sqrt{x^2+y^2}}=\lim_{\substack{y=kx\\x\to 0}}\frac{\sqrt{\sin|xy|}}{\sqrt{|xy|}}\lim_{\substack{y=kx\\x\to 0}}\frac{\sqrt{|xy|}}{\sqrt{x^2+y^2}}=\frac{\sqrt{|k|}}{\sqrt{1+k^2}}\neq 0,$$

矛盾, 故f在原点不可微.□

3. 函数在一点可微的充要条件

Thm. n元函数$f(\mathrm{x})$在$\mathrm{x}_0\in\mathbb{R}$可微的充要条件是

$$\Delta f(\mathrm{x}_0)=f(\mathrm{x}_0+\Delta\mathrm{x})-f(\mathrm{x}_0)=\sum_{i=1}^{n}f'_{x_i}(\mathrm{x}_0)\Delta x_i+\sum_{i=1}^{n}\varepsilon_i\Delta x_i,$$

其中ε_i为$\Delta\mathrm{x}=(\Delta x_1,\Delta x_2,\cdots,\Delta x_n)$的$n$元函数, $i=1,2,\cdots,n$, 且

$$\lim_{\Delta\mathrm{x}\to 0}\varepsilon_i=0,\quad i=1,2,\cdots,n.$$

Proof: (必要性）若$f(\mathrm{x})$在x_0可微, 则

$$f(\mathrm{x}_0+\Delta\mathrm{x})-f(\mathrm{x}_0)=\sum_{i=1}^{n}f'_{x_i}(\mathrm{x}_0)\Delta x_i+\alpha,$$

其中$\alpha=o\left(\|\Delta\mathrm{x}\|\right)$, 当$\Delta\mathrm{x}\to 0$时.

则

$$\alpha=\sum_{i=1}^{n}\frac{\alpha\cdot sgn(\Delta x_i)}{|\Delta x_1|+\cdots+|\Delta x_n|}\Delta x_i=\sum_{i=1}^{n}\varepsilon_i\Delta x_i,$$

其中$\varepsilon_i=\dfrac{\alpha\cdot sgn(\Delta x_i)}{|\Delta x_1|+\cdots+|\Delta x_n|},i=1,2,\cdots,n.$

$$|\varepsilon_i|=\frac{|\alpha|}{|\Delta x_1|+\cdots+|\Delta x_n|}$$

$$=\frac{|\alpha|}{\|\Delta \mathrm{x}\|}\cdot\frac{\|\Delta \mathrm{x}\|}{|\Delta x_1|+\cdots+|\Delta x_n|}\le\frac{|\alpha|}{\|\Delta \mathrm{x}\|}$$

故$\lim\limits_{\Delta \mathrm{x}\to 0}|\varepsilon_i|=0,\ \lim\limits_{\Delta \mathrm{x}\to 0}\varepsilon_i=0,i=1,2,\cdots,n.$

(充分性) 设

$$f(\mathrm{x}_0+\Delta\mathrm{x})-f(\mathrm{x}_0)=\sum_{i=1}^{n} f'_{x_i}(\mathrm{x}_0)\Delta x_i+\sum_{i=1}^{n}\varepsilon_i\Delta x_i,$$

其中ε_i为$\Delta\mathrm{x}=(\Delta x_1,\Delta x_2,\cdots,\Delta x_n)$的$n$元函数,$i=1,2,\cdots,n$, 且

$$\lim_{\Delta\mathrm{x}\to 0}\varepsilon_i=0,\quad i=1,2,\cdots,n.$$

则

$$\frac{\left|\sum_{i=1}^{n}\varepsilon_i\Delta x_i\right|}{\|\Delta\mathrm{x}\|}\le\frac{\left(|\varepsilon_1|+\cdots+|\varepsilon_n|\right)\left(|\Delta x_1|+\cdots+|\Delta x_n|\right)}{\|\Delta\mathrm{x}\|}$$

$$\le n\left(|\varepsilon_1|+\cdots+|\varepsilon_n|\right),\quad\to 0,\text{当}\Delta\mathrm{x}\to 0\text{时}.$$

故$f(\mathrm{x})$在x_0可微.□

Thm. f'_{x_i} 在$\mathrm{x}_0 \in \mathbb{R}^n$ 连续, $i=1,2,\cdots,n$, 则f 在x_0可微.

Proof: 记$e_i, i=1,\cdots,n$, 为$\mathbb{R}^n$的自然基, $\Delta\mathrm{x}=(\Delta x_1,\Delta x_2,\cdots,\Delta x_n)$,

则
$$
\begin{aligned}
& f(\mathrm{x}_0+\Delta\mathrm{x})-f(\mathrm{x}_0) \\
=\ & f(\mathrm{x}_0+\Delta x_1 e_1)-f(\mathrm{x}_0) \\
& + f(\mathrm{x}_0+\Delta x_1 e_1+\Delta x_2 e_2)-f(\mathrm{x}_0+\Delta x_1 e_1) \\
& +\cdots+ f(\mathrm{x}_0+\Delta\mathrm{x})-f(\mathrm{x}_0+\Delta x_1 e_1+\cdots+\Delta x_{n-1}e_{n-1})
\end{aligned}
$$

由一元函数的微分中值定理, $\exists\theta_i\in(0,1), i=1,2,\cdots,n, s.t.$

$$
\begin{aligned}
& f(\mathrm{x}_0+\Delta\mathrm{x})-f(\mathrm{x}_0) \\
=\ & f'_{x_1}(\mathrm{x}_0+\theta_1\Delta x_1 e_1)\Delta x_1 + f'_{x_2}(\mathrm{x}_0+\Delta x_1 e_1+\theta_2\Delta x_2 e_2)\Delta x_2 \\
& +\cdots+ f'_{x_n}(\mathrm{x}_0+\Delta x_1 e_1+\cdots+\Delta x_{n-1}e_{n-1}+\theta_n\Delta x_n e_n)\Delta x_n
\end{aligned}
$$

记$\varepsilon_1 = f'_{x_1}(\mathrm{x}_0 + \theta_1 \Delta x_1 e_1) - f'_{x_1}(\mathrm{x}_0)$,

$$\varepsilon_2 = f'_{x_2}(\mathrm{x}_0 + \Delta x_1 e_1 + \theta_2 \Delta x_2 e_2) - f'_{x_2}(\mathrm{x}_0),$$

$$\vdots$$

$$\varepsilon_n = f'_{x_n}(\mathrm{x}_0 + \Delta x_1 e_1 + \cdots + \Delta x_{n-1} e_{n-1} + \theta_n \Delta x_n e_n) - f'_{x_n}(\mathrm{x}_0),$$

则 $$f(\mathrm{x}_0 + \Delta \mathrm{x}) - f(\mathrm{x}_0) = \sum_{i=1}^{n} f'_{x_i}(\mathrm{x}_0)\Delta x_i + \sum_{i=1}^{n} \varepsilon_i \Delta x_i,$$

其中ε_i为$\Delta \mathrm{x} = (\Delta x_1, \Delta x_2, \cdots, \Delta x_n)$的$n$元函数, $i = 1, 2, \cdots, n$.

由f'_{x_i}在x_0的连续性,

$$\lim_{\Delta x \to 0} \varepsilon_i = 0, i = 1, 2, \cdots, n.$$

因此f在x_0可微.□

Remark: 函数的连续性、可微性、偏导数存在性与偏导数连续性之间的蕴含关系图.

$$
\begin{tikzcd}
\text{连续} & \text{可微} \arrow[l] \arrow[d] \\
\text{偏导函数连续} \arrow[u] \arrow[ur] \arrow[r] & \text{偏导数存在}
\end{tikzcd}
$$

4. 高阶偏导数

视$f_x'(x,y)$, $f_y'(x,y)$为x, y的二元函数, 有时也记为f_1', f_2', 考虑它们的偏导数, 即高阶偏导数. 例如,

$$f''_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad f''_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

分别为f关于x和关于y的二阶偏导数, 也记为f''_{11}, f''_{22}. 而

$$f''_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \qquad f''_{yx} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

为f关于x, y的二阶混合偏导数, 也记为f''_{12}, f''_{21}.

例. $z=\frac{1}{x}f(xy)+yf(x+y)$, 求 $\frac{\partial^2 z}{\partial x \partial y}$.

解: $\frac{\partial z}{\partial y}=f'(xy)+f(x+y)+yf'(x+y)$

$$\frac{\partial^2 z}{\partial x \partial y}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$=yf''(xy)+f'(x+y)+yf''(x+y).\square$$

例. $f(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases},$

求$f''_{xy}(0,0)$和$f''_{yx}(0,0)$.

解: $\dfrac{\partial f(x,y)}{\partial x}=\begin{cases} y\dfrac{x^4-y^4+4x^2y^2}{(x^2+y^2)^2}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$

$$f''_{yx}(0,0)=\lim_{y\to 0}\frac{f'_x(0,y)-f'_x(0,0)}{y}=\lim_{y\to 0}\frac{-y}{y}=-1.$$

$$\frac{\partial f(x,y)}{\partial y}=\begin{cases} x\dfrac{x^4-y^4-4x^2y^2}{(x^2+y^2)^2}, & x^2+y^2\neq 0, \\ 0, & x^2+y^2=0, \end{cases}$$

$$f''_{xy}(0,0)=\lim_{x\to 0}\frac{f'_y(x,0)-f'_y(0,0)}{x}=\lim_{x\to 0}\frac{x}{x}=1.\square$$

Remark: 混合偏导数一般情况下与求导顺序有关.

Question: 要求$f''_{xx}(0,0)$, 是否必须计算出$f'_x(x,y)$?

$$f''_{xx}(0,0)=\lim_{x\to 0}\frac{f'_x(x,0)-f'_x(0,0)}{x},\text{只需计算出}f'_x(x,0).$$

上例中 $f(x,0)=0$,因此 $f'_x(x,0)=0,\ f''_{xx}(0,0)=0.$

Thm. 若f''_{xy}, f''_{yx}都在(x_0, y_0)连续,则$f''_{xy}(x_0, y_0) = f''_{yx}(x_0, y_0)$.

Proof. 令$\Delta = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0 + \Delta x, y_0)$

$$- f(x_0, y_0 + \Delta y) + f(x_0, y_0),$$

$$\varphi(t) = f(x_0 + t\Delta x, y_0 + \Delta y) - f(x_0 + t\Delta x, y_0),$$

则 $\varphi'(t) = f'_x(x_0 + t\Delta x, y_0 + \Delta y)\Delta x - f'_x(x_0 + t\Delta x, y_0)\Delta x,$

$$\Delta = \varphi(1) - \varphi(0) = \varphi'(\theta_1)$$

$$= \left(f'_x(x_0 + \theta_1\Delta x, y_0 + \Delta y) - f'_x(x_0 + \theta_1\Delta x, y_0) \right)\Delta x$$

$$= f''_{yx}(x_0 + \theta_1\Delta x, y_0 + \theta_2\Delta y)\Delta x\Delta y,$$ 其中$\theta_1, \theta_2 \in (0,1)$.

同理,令$\psi(t)=f(x_0+\Delta x, y_0+t\Delta y)-f(x_0, y_0+t\Delta y)$,则

$$\psi'(t)=f'_y(x_0+\Delta x, y_0+t\Delta y)\Delta y-f'_y(x_0, y_0+t\Delta y)\Delta y,$$

$$\Delta=\psi(1)-\psi(0)=\psi'(\theta_3)$$

$$=\left(f'_y(x_0+\Delta x, y_0+\theta_3\Delta y)-f'_y(x_0, y_0+\theta_3\Delta y)\right)\Delta y$$

$$=f''_{xy}(x_0+\theta_4\Delta x, y_0+\theta_3\Delta y)\Delta x\Delta y, \qquad \theta_3, \theta_4\in(0,1).$$

于是 $f''_{yx}(x_0+\theta_1\Delta x, y_0+\theta_2\Delta y)=f''_{xy}(x_0+\theta_4\Delta x, y_0+\theta_3\Delta y)$,

由于f''_{xy}, f''_{yx}在(x_0, y_0)连续,上式中令$(\Delta x, \Delta y)\to(0,0)$,得

$$f''_{yx}(x_0, y_0)=f''_{xy}(x_0, y_0).\square$$

5. 方向导数、梯度

Question: 用过点$\left(x_0, y_0, f(x_0, y_0)\right)$且平行于OZ轴的平面去截曲面$z = f(x, y)$,所得的交线在点$\left(x_0, y_0, f(x_0, y_0)\right)$处的斜率如何刻画?

点(x_0, y_0)及单位向量$\vec{v} = (v_1, v_2)^{\mathrm{T}} \in \mathbb{R}^2$确定直线

$$\ell = \{(x, y) \mid x = x_0 + v_1 t, y = y_0 + v_2 t\}.$$

其中t表示点(x_0, y_0)沿方向$\vec{v}$到点(x, y)的有向距离. 把直线ℓ类比为x轴, 方向$\vec{v}$类比为x轴正向. 则可以得到方向导数的定义.

Def. f在$\mathrm{x}_0\in\mathbb{R}^n$的邻域中有定义, $\vec{v}=(v_1,v_2,\cdots,v_n)\in\mathbb{R}^n$为非零向量, l为过x_0沿$\vec{v}$方向的射线,若

$$g(t)=f(\mathrm{x}_0+\frac{\vec{v}}{\|\vec{v}\|}t)=f(\mathrm{x}_0^{(1)}+\frac{v_1}{\|\vec{v}\|}t,\cdots,\mathrm{x}_0^{(n)}+\frac{v_n}{\|\vec{v}\|}t)$$

在$t=0$存在右导数,即极限

$$\lim_{\substack{\mathrm{x}\to\mathrm{x}_0\\ \mathrm{x}\in l}}\frac{f(\mathrm{x})-f(\mathrm{x}_0)}{\|\mathrm{x}-\mathrm{x}_0\|}=\lim_{s\to0^+}\frac{f(\mathrm{x}_0+s\vec{v})-f(\mathrm{x}_0)}{\|\vec{v}\|s}$$

$$=\lim_{t\to0^+}\frac{g(t)-g(0)}{t}=g'_+(0)$$

存在,则称该极限为$f(\mathrm{x})$在x_0沿方向$\vec{v}$的方向导数, 记作

$$\frac{\partial f(\mathrm{x}_0)}{\partial\vec{v}},\left.\frac{\partial f}{\partial\vec{v}}\right|_{\mathrm{x}_0}\text{ 或}f'_{\vec{v}}(\mathrm{x}_0).$$

Remark. $\dfrac{\partial f(\mathrm{x}_0)}{\partial \vec{v}}$ 是函数 $f(\mathrm{x})$ 在点 x_0 沿方向 $\vec{v}$ 的变化率.

Remark. 记 $\vec{e}_i=(0,\cdots 0,1,0,\cdots 0)$(第 i 个分量为1), 则

$$\frac{\partial f(\mathrm{x}_0)}{\partial x_i}=\frac{\partial f(\mathrm{x}_0)}{\partial \vec{e}_i}=-\frac{\partial f(\mathrm{x}_0)}{\partial (-\vec{e}_i)}.$$

Def. n 元函数 f 在 $\mathrm{x}_0\in\mathbb{R}^n$ 可微, 称

$$\mathrm{grad} f(\mathrm{x}_0)\triangleq\nabla f(\mathrm{x}_0)\triangleq\left(\frac{\partial f(\mathrm{x}_0)}{\partial x_1},\frac{\partial f(\mathrm{x}_0)}{\partial x_2},\cdots,\frac{\partial f(\mathrm{x}_0)}{\partial x_n}\right)$$

为数量场 $u=f(\mathrm{x})$ 在点 x_0 的梯度.

Thm. 设f在$\mathrm{x}_0 \in \mathbb{R}^n$可微, $\vec{v}=(v_1, v_2, \cdots, v_n) \in \mathbb{R}^n$为非零向量,

则方向导数$\dfrac{\partial f(\mathrm{x}_0)}{\partial \vec{v}}$存在, 且$\dfrac{\partial f(\mathrm{x}_0)}{\partial \vec{v}} = \nabla f(\mathrm{x}_0) \cdot \dfrac{v}{\|\vec{v}\|}$.

Proof. f在$\mathrm{x}_0 \in \mathbb{R}^n$可微, 则

$$f(\mathrm{x}_0 + \frac{\vec{v}}{\|\vec{v}\|} t) - f(\mathrm{x}_0) = \sum_{i=1}^{n} \frac{\partial f(\mathrm{x}_0)}{\partial x_i} \frac{v_i}{\|\vec{v}\|} t + o(t), t \to 0^+.$$

$$\frac{\partial f(\mathrm{x}_0)}{\partial \vec{v}} = \lim_{t \to 0^+} \frac{f(\mathrm{x}_0 + \frac{\vec{v}}{\|\vec{v}\|} t) - f(\mathrm{x}_0)}{t} = \sum_{i=1}^{n} \frac{\partial f(\mathrm{x}_0)}{\partial x_i} \frac{v_i}{\|\vec{v}\|}. \square$$

例. 求$f=x^2+y^2$在$M_0(2,1)$沿$\vec{w}=(3,-4)$的方向导数.

解法一. $\vec{w}/\|\vec{w}\|=(3/5,-4/5)^{\mathrm{T}}$,

$$g(t)=f(2+\frac{3}{5}t,1-\frac{4}{5}t)=\cdots,\quad \frac{\partial f(2,1)}{\partial\vec{w}}=g'_+(0)=\frac{4}{5}.$$

解法二. $\dfrac{\partial f(2,1)}{\partial\vec{w}}=\lim\limits_{t\to0^+}\dfrac{f(2+3t,1-4t)-f(2,1)}{5t}=\cdots=\dfrac{4}{5}.$

解法三. $\vec{w}/\|\vec{w}\|=(3/5,-4/5)^{\mathrm{T}}$, $f'_x(2,1)=4$, $f'_y(2,1)=2$.

$$\frac{\partial f(2,1)}{\partial\vec{w}}=f'_x(2,1)\cdot\frac{3}{5}+f'_y(2,1)\cdot\frac{-4}{5}=\frac{4}{5}.\square$$

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Thm. f 在 $\mathrm{x}_0 \in \mathbb{R}^n$ 可微, 记 $\vec{w} = \mathrm{grad}f(\mathrm{x}_0)$ 则

$$\max_{\vec{v}\in\mathbb{R}^n,\vec{v}\neq 0} \frac{\partial f(\mathrm{x}_0)}{\partial \vec{v}} = \frac{\partial f(\mathrm{x}_0)}{\partial \vec{w}} = \|\mathrm{grad}f(\mathrm{x}_0)\|,$$

即 f 在 x_0 沿梯度方向的方向导数最大, 且最大方向导数的值为梯度的模.

Proof. f 在 x_0 可微, 则 $\dfrac{\partial f(\mathrm{x}_0)}{\partial \vec{v}} = \mathrm{grad}f(\mathrm{x}_0)\cdot\dfrac{\vec{v}}{\|\vec{v}\|}$

$$= \|\mathrm{grad}f(\mathrm{x}_0)\|\cdot\cos < \mathrm{grad}f(\mathrm{x}_0),\vec{v} > \quad \leq \|\mathrm{grad}f(\mathrm{x}_0)\|,$$

当且仅当 $\vec{v} = k\cdot\mathrm{grad}f(\mathrm{x}_0), k > 0$ 时 "=" 成立.□

Remark: 即使$f(x,y)$在某点存在所有的方向导数, 也不能推断f在该点连续.

例. $f(x,y)=\begin{cases}1, & y=x^2, x>0,\\ 0, & \text{其它},\end{cases}$在原点不连续,但沿任何非零向量$\vec{v}\in\mathbb{R}^2$, $\dfrac{\partial f(0,0)}{\partial \vec{v}}=0.$

作业：习题1.4

No. 2(1)(3), 4(5)(8),

8, 9,13,15(2)(3)