Review

- 三重积分化累次积分

（先一后二）

$$\Omega:\begin{cases}(x,y)\in D_{xy},\\ z_1(x,y)\le z\le z_2(x,y),\end{cases}$$

$$\iiint\limits_{\Omega} f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z=\iint\limits_{D_{xy}}\mathrm{d}x\mathrm{d}y\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\mathrm{d}z.$$

（先二后一）

$$\Omega:\begin{cases} c \le z \le d, \\ (x,y) \in \Omega_z, \end{cases}$$

$$\iiint_{\Omega} f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_c^d \mathrm{d}z \iint_{\Omega_z} f(x,y,z)\mathrm{d}x\mathrm{d}y.$$

- 投影法确定积分区域

- 三重积分的变量替换

$$u = u(x, y, z), v = v(x, y, z), w = w(x, y, z)$$

$$(x, y, z) \in \Omega \leftrightarrow (u, v, w) \in \Omega^*.$$

$$\iiint_{\Omega} f(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= \iiint_{\Omega^*} f\left(x(u, v, w), y(u, v, w), z(u, v, w)\right)$$

$$\cdot \left|\det \frac{\partial(x, y, z)}{\partial(u, v, w)}\right| \mathrm{d}u\mathrm{d}v\mathrm{d}w.$$

§5. 重积分的应用

- 曲面面积
- 质心
- 转动惯量
- 万有引力

原则：微元法

1. 曲面的面积

设曲面S的参数方程为

$$x=x(u,v), y=y(u,v), z=z(u,v), (u,v)\in D,$$

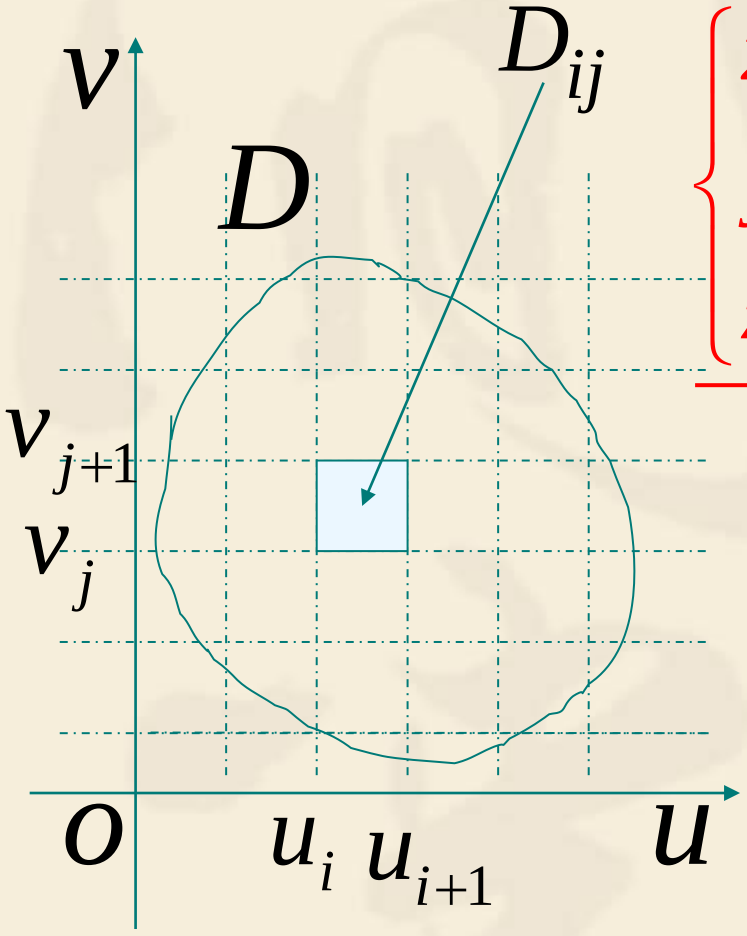
简记为

$$S: \mathrm{r}=\mathrm{r}(u,v), (u,v)\in D.$$

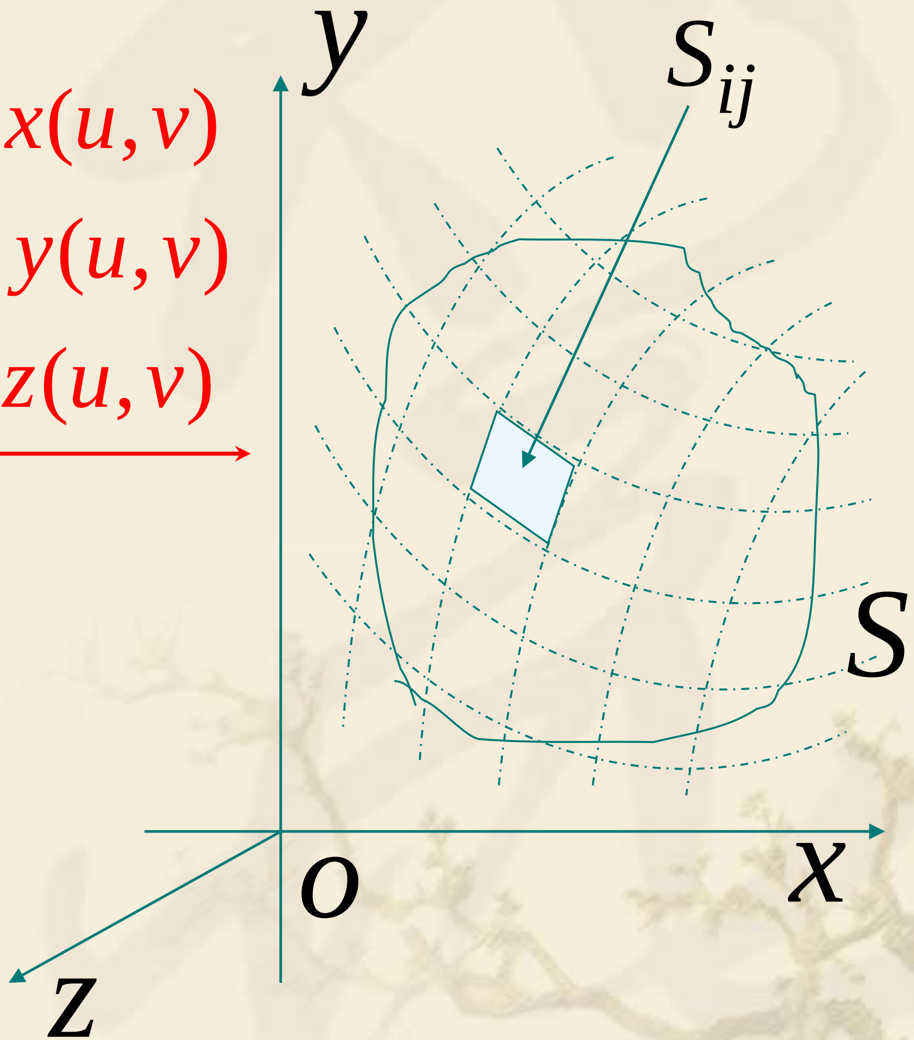
在ouv平面上,用平行于坐标轴的直线

$$u=u_i(i=1,2,\cdots,n), v=v_j(j=1,2,\cdots,m)$$

将区域D分割成若干小矩形D_{ij}.



$$\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$$



D_{ij}的顶点为$(u_i, v_j),(u_{i+1}, v_j),(u_i, v_{j+1}),(u_{i+1}, v_{j+1})$.

对应地,空间曲边四边形S_{ij}的四个顶点为

$$P_{ij}(x(u_i, v_j), y(u_i, v_j), z(u_i, v_j)),$$

$$P_{i+1,j}(x(u_{i+1}, v_j), y(u_{i+1}, v_j), z(u_{i+1}, v_j)),$$

$$P_{i,j+1}(x(u_i, v_{j+1}), y(u_i, v_{j+1}), z(u_i, v_{j+1})),$$

$$P_{i+1,j+1}(x(u_{i+1}, v_{j+1}), y(u_{i+1}, v_{j+1}), z(u_{i+1}, v_{j+1})).$$

$$\overrightarrow{P_{ij}P_{i+1,j}} \approx (x_u'(u_i, v_j), y_u'(u_i, v_j), z_u'(u_i, v_j))\Delta u_i$$

$$= \mathrm{r}_u'(u_i, v_j)\Delta u_i$$

$$\overrightarrow{P_{ij}P_{i,j+1}} \approx \mathrm{r}'_v(u_i, v_j)\Delta v_j.$$

当分划很细时,空间曲面S_{ij}可近似地看成以线段

$P_{ij}P_{i+1,j}, P_{ij}P_{i,j+1}$为邻边的平行四边形, 其面积

$$\Delta S_{ij} \approx \left\| \mathrm{r}'_u(u_i, v_j) \times \mathrm{r}'_v(u_i, v_j) \right\| \Delta u_i \Delta v_j$$

$$= \left\| \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ x'_u & y'_u & z'_u \\ x'_v & y'_v & z'_v \end{pmatrix} \right\|_{(u_i, v_j)} \Delta u_i \Delta v_j$$

即 $\Delta S_{ij} \approx \sqrt{A^2+B^2+C^2}\,\Delta u_i \Delta v_j$,其中

$$A=\det\left.\frac{\partial(y,z)}{\partial(u,v)}\right|_{(u_i,v_j)},\quad B=\det\left.\frac{\partial(z,x)}{\partial(u,v)}\right|_{(u_i,v_j)},$$

$$C=\det\left.\frac{\partial(x,y)}{\partial(u,v)}\right|_{(u_i,v_j)}.$$

•曲面 $S:\vec{r}=\vec{r}(u,v)=(x(u,v),y(u,v),z(u,v))$, $(u,v)\in D$,的面积为

$$\iint_D \|\vec{r}'_u \times \vec{r}'_v\| \mathrm{d}u\mathrm{d}v=\iint_D \sqrt{A^2+B^2+C^2}\,\mathrm{d}u\mathrm{d}v.$$

- 若曲面S的方程为$z=f(x,y),(x,y)\in D$,则

$$S: x=x, y=y, z=f(x,y),(x,y)\in D.$$

$$\mathrm{r}'_x \times \mathrm{r}'_y = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f'_x \\ 0 & 1 & f'_y \end{pmatrix} = (-f'_x, -f'_y, 1)$$

$$A=\det\begin{pmatrix} 0 & f'_x \\ 1 & f'_y \end{pmatrix} = -f'_x, B=-f'_y, C=1.$$

曲面S的面积为$\iint_D \sqrt{1+f'^2_x+f'^2_y}\,\mathrm{d}x\mathrm{d}y.$

例: 求球面$S: x^2+y^2+z^2=R^2$的面积.

解: 球面S的参数方程为

$$x=R\sin\varphi\cos\theta, y=R\sin\varphi\sin\theta, z=R\cos\varphi,$$

$$(0\le\varphi\le\pi, 0\le\theta\le2\pi).$$

则

$$\vec{\mathrm{r}}'_\varphi\times\vec{\mathrm{r}}'_\theta=\det\begin{pmatrix}\vec{i} & \vec{j} & \vec{k}\\ R\cos\varphi\cos\theta & R\cos\varphi\sin\theta & -R\sin\varphi\\ -R\sin\varphi\sin\theta & R\sin\varphi\cos\theta & 0\end{pmatrix}$$

$$=(R^2\sin^2\varphi\cos\theta,\ R^2\sin^2\varphi\sin\theta,\ R^2\sin\varphi\cos\varphi)$$

$$\left\|\vec{\mathrm{r}}'_\varphi\times\vec{\mathrm{r}}'_\theta\right\|=R^2\sin\varphi,$$

球面S的面积为

$$\iint\limits_{\substack{0\le\varphi\le\pi\\0\le\theta\le2\pi}} \left\|\vec{\mathrm{r}}'_{\varphi}\times\vec{\mathrm{r}}'_{\theta}\right\|\mathrm{d}\varphi\mathrm{d}\theta$$

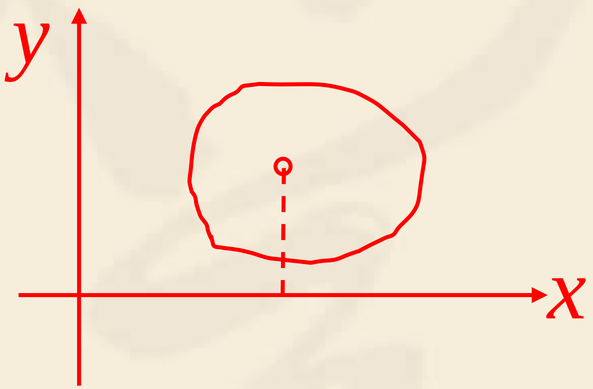
$$=\iint\limits_{\substack{0\le\varphi\le\pi\\0\le\theta\le2\pi}} R^2\sin\varphi\mathrm{d}\varphi\mathrm{d}\theta$$

$$=R^2\int_0^{\pi}\sin\varphi\mathrm{d}\varphi\int_0^{2\pi}\mathrm{d}\theta=4\pi R^2 \square$$

2. 物体的质心

关于x轴的力矩微元为

$y\mu(x,y)\mathrm{d}x\mathrm{d}y$



• 平板D的质心$(\overline{x},\overline{y})$

平板密度$\mu(x,y)$

平板质量$\mathrm{M}=\iint_D \mu(x,y)\mathrm{d}x\mathrm{d}y$

平板关于x轴的静力矩为$\mathrm{M}\overline{y}=\iint_D y\mu(x,y)\mathrm{d}x\mathrm{d}y$

$$故\ \overline{y}=\frac{\iint_D y\mu(x,y)\mathrm{d}x\mathrm{d}y}{\iint_D \mu(x,y)\mathrm{d}x\mathrm{d}y},同理\ \overline{x}=\frac{\iint_D x\mu(x,y)\mathrm{d}x\mathrm{d}y}{\iint_D \mu(x,y)\mathrm{d}x\mathrm{d}y}.$$

•空间物体Ω的质心$(\bar{x}, \bar{y}, \bar{z})$

密度$\mu(x, y, z)$，质量$\mathrm{M} = \iiint_\Omega \mu(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$

Ω关于yz平面的静力矩为

$$\mathrm{M}\bar{x} = \iiint_\Omega x\mu(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

故

$$\bar{x} = \frac{\iiint_\Omega x\mu(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z}{\iiint_\Omega \mu(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z},$$

$$\bar{y} = \frac{\iiint_\Omega y\mu(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z}{\iiint_\Omega \mu(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z}, \bar{z} = \frac{\iiint_\Omega z\mu(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z}{\iiint_\Omega \mu(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z}.$$

3. 转动惯量

- 位于(x, y, z)处质量为m的质点，绕x, y, z轴的转动惯量分别为$m(y^2+z^2), m(z^2+x^2), m(x^2+y^2)$.
- $\Omega \subset \mathbb{R}^3$，密度$\rho(x, y, z)$，绕坐标轴的转动惯量为

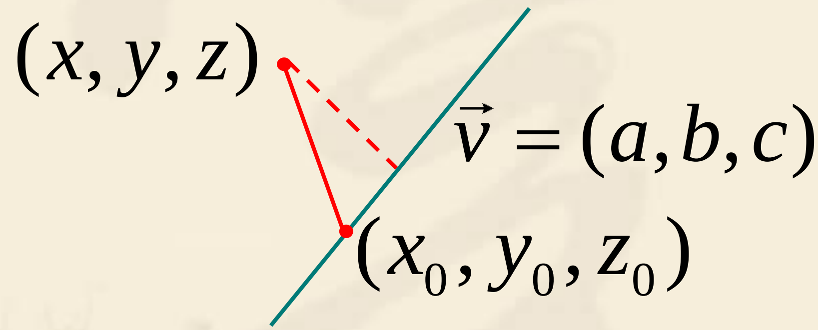
$$J_x = \iiint_\Omega (y^2+z^2)\rho(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$J_y = \iiint_\Omega (z^2+x^2)\rho(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

$$J_z = \iiint_\Omega (x^2+y^2)\rho(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

Question.直线l过点(x_0, y_0, z_0)沿方向(a,b,c), Ω绕l的转动惯量?

$$\iiint_{\Omega} d^2(x,y,z)\,\rho(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$



其中, $d(x,y,z)$

$$= \frac{1}{\sqrt{a^2+b^2+c^2}} \left\| \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ x-x_0 & y-y_0 & z-z_0 \end{pmatrix} \right\|.$$

4. 万有引力

- 位于$P(x,y,z)$, $P_0(x_0,y_0,z_0)$的两质点,质量分别为m, m_0.记$r=\|PP_0\|$, $\overrightarrow{P_0P}$与x,y,z正半轴夹角为α,β,γ, m对m_0的万有引力的大小为$\dfrac{kmm_0}{r^2}$, 引力沿x,y,z轴的分量分别为

$$F_x=\frac{kmm_0}{r^2}\cos\alpha=\frac{kmm_0(x-x_0)}{r^3},$$

$$F_y=\frac{kmm_0}{r^2}\cos\beta=\frac{kmm_0(y-y_0)}{r^3},$$

$$F_z=\frac{kmm_0}{r^2}\cos\gamma=\frac{kmm_0(z-z_0)}{r^3}.$$

● 密度为$\rho(x,y,z)$的物体Ω对$P_0(x_0,y_0,z_0)\notin\Omega$处质量为$m_0$的质点的万有引力:

$$F_x=\iiint_\Omega \frac{km_0(x-x_0)\rho(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z}{\left(\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}\right)^3},$$

$$F_y=\iiint_\Omega \frac{km_0(y-y_0)\rho(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z}{\left(\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}\right)^3},$$

$$F_z=\iiint_\Omega \frac{km_0(z-z_0)\rho(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z}{\left(\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}\right)^3},$$

例: 半径为R,质量为M的均匀球体$x^2+y^2+z^2 \le R^2$对点$P(0,0,a)\ (a>R)$处质量为m的质点的引力.

解: $$F_x=\iiint_\Omega \frac{km x\rho \mathrm{d}x\mathrm{d}y\mathrm{d}z}{\left(\sqrt{x^2+y^2+(a-z)^2}\right)^3}=0,\ F_y=0.$$

$$-F_z=\iiint_\Omega \frac{km(a-z)\rho \mathrm{d}x\mathrm{d}y\mathrm{d}z}{\left(\sqrt{x^2+y^2+(a-z)^2}\right)^3},\quad \frac{4}{3}\pi R^3\rho=M.$$

在柱坐标系$x=r\cos\theta, y=r\sin\theta, z=z$下,

$$-F_z=\int_{-R}^{R}\mathrm{d}z\int_0^{2\pi}\mathrm{d}\theta\int_0^{\sqrt{R^2-z^2}}\frac{km(a-z)\rho r\mathrm{d}r}{(\sqrt{r^2+(a-z)^2})^3}$$

$$-F_z = \int_{-R}^{R} \mathrm{d}z \int_0^{2\pi} \mathrm{d}\theta \int_0^{\sqrt{R^2-z^2}} \frac{km(a-z)\rho r \mathrm{d}r}{(\sqrt{r^2+(a-z)^2})^3}$$

$$= k\pi m\rho \int_{-R}^{R} (a-z)\mathrm{d}z \int_0^{\sqrt{R^2-z^2}} \frac{\mathrm{d}r^2}{(\sqrt{r^2+(a-z)^2})^3}$$

$$= 2k\pi m\rho \int_{-R}^{R} \left(1 - \frac{a-z}{\sqrt{R^2+a^2-2az}}\right) \mathrm{d}z$$

$$= \frac{4k\pi m\rho R^3}{3a^2} = \frac{kMm}{a^2}. \square$$

(分部积分)

Question. 密度分别为$\rho_1(x,y,z),\rho_2(x,y,z)$的两物体$\Omega_1,\Omega_2$之间的万有引力？

作业：习题3.5 No.1(3),8