Review

- 向量值函数在一点可微的定义

$$f(x_0+\Delta x)-f(x_0)=\mathrm{A}\Delta x+\alpha,\ \lim_{\Delta x\to 0}\frac{\|\alpha\|_m}{\|\Delta x\|_n}=0$$

- $f=\left(f_1,f_2,\cdots,f_m\right)^{\mathrm{T}}:\Omega\left(\subset\mathbb{R}^n\right)\to\mathbb{R}^m$在$x_0$可微

$\Leftrightarrow n$元函数$f_i:\Omega\left(\subset\mathbb{R}^n\right)\to\mathbb{R}$在$x_0$可微,$i=1,2,\cdots,m.$

- $$\mathrm{A}=\frac{\partial f}{\partial x}(x_0)=\begin{pmatrix}\dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n}\\ \vdots & \ddots & \vdots\\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_n}\end{pmatrix}$$

- Chain Rule

$$u = g(x): \Omega \subset \mathbb{R}^n \to \mathbb{R}^m,\ y = f(u): g(\Omega) \subset \mathbb{R}^m \to \mathbb{R}^k,$$

$g(x)$在$x_0 \in \Omega$可微, $f(u)$在$u_0 = g(x_0)$可微,则

$$J(f \circ g)\big|_{x_0} = J(f)\big|_{u_0} \cdot J(g)\big|_{x_0},$$

即$$\left.\frac{\partial(y_1, y_2, \cdots, y_k)}{\partial(x_1, x_2, \cdots, x_n)}\right|_{x_0} = \left.\frac{\partial(y_1, y_2, \cdots, y_k)}{\partial(u_1, u_2, \cdots, u_m)}\right|_{u_0} \cdot \left.\frac{\partial(u_1, u_2, \cdots, u_m)}{\partial(x_1, x_2, \cdots, x_n)}\right|_{x_0},$$

简记为$$\left.\frac{\partial y}{\partial x}\right|_{x_0} = \left.\frac{\partial y}{\partial u}\right|_{u_0} \cdot \left.\frac{\partial u}{\partial x}\right|_{x_0}.$$

$k = 1$时, $$\frac{\partial y}{\partial x_i} = \frac{\partial y}{\partial u_1}\frac{\partial u_1}{\partial x_i} + \frac{\partial y}{\partial u_2}\frac{\partial u_2}{\partial x_i} + \cdots + \frac{\partial y}{\partial u_m}\frac{\partial u_m}{\partial x_i}, i = 1, 2, \cdots, n.$$

§ 6.　隐函数定理与反函数定理

曲线$x^2+y^2=1$在$(0,1)$的某个邻域中可表示为

$y=\sqrt{1-x^2}$，且$y'(x)=\dfrac{-x}{\sqrt{1-x^2}}$；在$(1,0)$的某个邻域

中可表示为$x=\sqrt{1-y^2}$，且$x'(y)=\dfrac{-y}{\sqrt{1-y^2}}$.

Question: (1) $f(x,y)=0$何时确定隐函数$y=y(x)$？

(2)如何通过$f(x,y)$的性质研究隐函数$y=y(x)$的性质，如连续性，可微性？

(3)如何计算隐函数的（偏）导数和（全）微分？

1. 一个方程确定的隐函数

设$f(x,y)=0$, $f(x_0,y_0)=0$.若存在连续可微的隐函数$y=y(x)$, $y(x_0)=y_0$,满足$f(x,y(x))\equiv 0$, 两边对x求导,有

$$f_1'(x,y(x))+f_2'(x,y(x))\cdot y'(x)=0.$$

若$f_2'(x_0,y_0)\neq 0$,则在x_0的某个邻域中,

$$y'(x)=-\frac{f_1'(x,y(x))}{f_2'(x,y(x))}=-\frac{\partial f(x,y(x))}{\partial x}\Big/\frac{\partial f(x,y(x))}{\partial y}.$$

(这里求偏导函数时x,y相互独立!)

Thm. 设F在$(x_0, y_0) \in \mathbb{R}^2$的某个邻域W中有定义,且

(1)$F(x_0, y_0) = 0$,

(2) $F(x, y) \in C^1(\mathrm{W})$,即$F_x', F_y'$在W中连续,

(3)$F_y'(x_0, y_0) \neq 0$.

则存在$\delta > 0$以及$I = (x_0 - \delta, x_0 + \delta)$上定义的函数

$y = y(x)$,满足

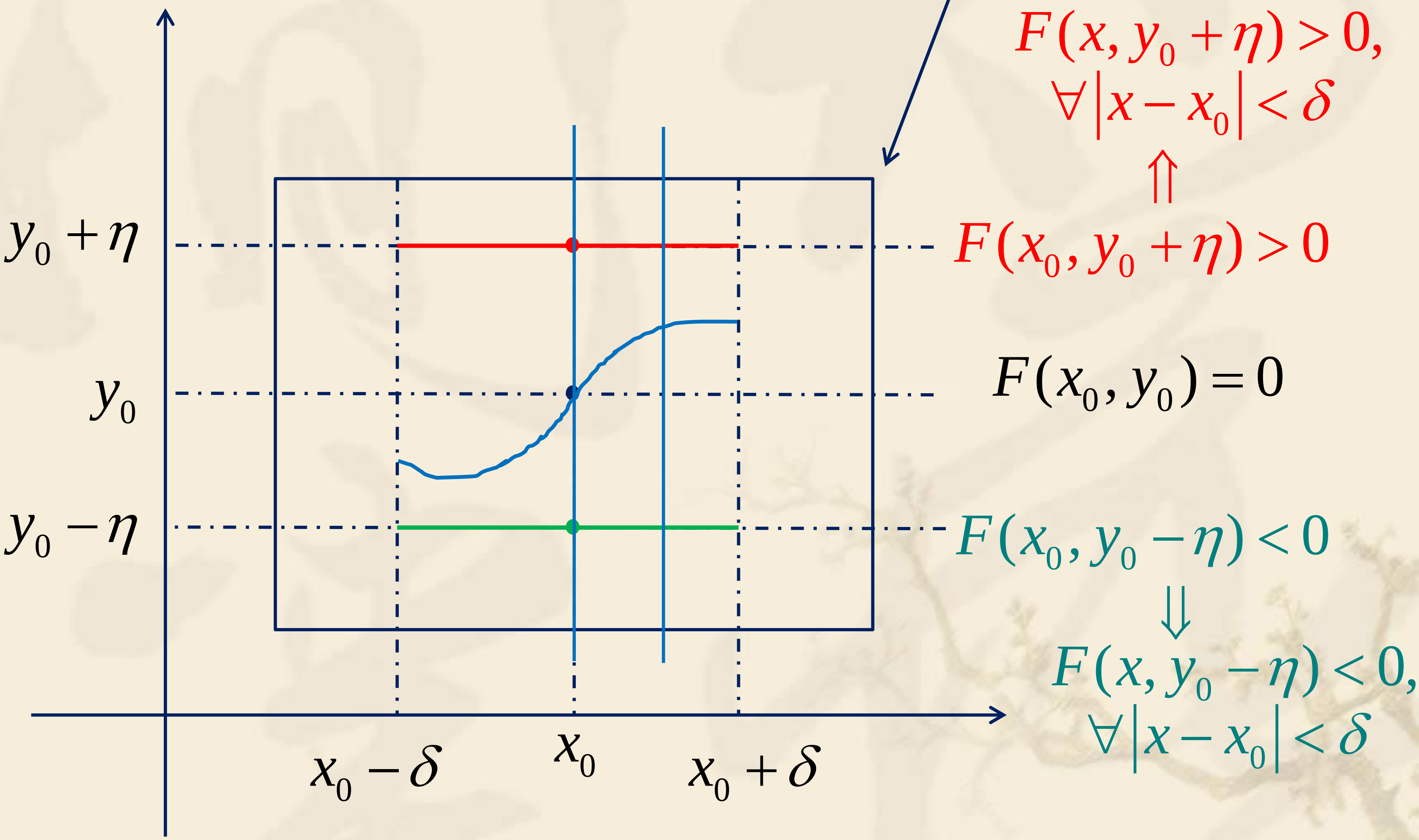
(1)$y(x_0) = y_0$,且$F(x, y(x)) \equiv 0, \forall x \in I$,

(2)$y = y(x) \in C^1(I)$,即$y'(x)$在I上连续,

(3)$\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{\partial F(x, y(x))}{\partial x} \Big/ \dfrac{\partial F(x, y(x))}{\partial y}, \forall x \in I.$

(这里求偏导函数时x, y相互独立!)

$$F_y'(x_0, y_0) > 0 \Rightarrow F_y'(x, y) > 0, \forall (x, y) \in W_1$$



Proof. (1)先证隐函数的存在性.

因$F_y'(x_0,y_0)\neq 0$,不妨设$F_y'(x_0,y_0)>0.F\in C^1(\mathrm{W})$,则
$\exists a,b>0,s.t.$ $F_y'(x,y)>0,\forall|x-x_0|<a,|y-y_0|<b.$ (*)
$F(x_0,y)$对y连续,由(*)及$F(x_0,y_0)=0$,给定$\eta\in(0,b)$,有

$$F(x_0,y_0-\eta)<0<F(x_0,y_0+\eta).$$

由F的连续性,$\exists\delta\in(0,a),s.t.$

$$F(x,y_0-\eta)<0<F(x,y_0+\eta),\ \forall|x-x_0|<\delta.$$

由(*)知,任意给定$|x-x_0|<\delta,F(x,y)$是y的增函数.结合连续函数的介值定理,$\forall|x-x_0|<\delta,\exists!\,y=y(x)\in(y_0-\eta,y_0+\eta),s.t.\ F(x,y)=0.$

(2)记(1)中构造的隐函数为$y = f(x)$, 下证其连续性.

由(1)中证明知, $\forall 0 < \eta_0 < b, \exists \delta_0 > 0$, 当$|x - x_0| < \delta_0$时, 必有$|y - y_0| < \eta_0$. 因此 $y = f(x)$在x_0连续.

任给$x_1 \in (x_0 - \delta, x_0 + \delta)$, 记$y_1 = f(x_1)$, 则$|y_1 - y_0| < \eta$, $F(x_1, y_1) = 0, F_y'(x_1, y_1) > 0$.即$F$在$(x_1, y_1)$与$(x_0, y_0)$满足相同的条件.由前面的证明, F在(x_1, y_1)的充分小邻域中确定了同一个隐函数$y = f(x)$, 且f在x_1连续.

(3)最后证隐函数$y = y(x)$ 的可导公式及连续可微性.

任意给定$x \in (x_0 - \delta, x_0 + \delta)$,由隐函数的连续性, 当$\Delta x \to 0$时,$\Delta y = y(x + \Delta x) - y(x) \to 0$.由隐函数的定义及$F$的连续可微性知,

$$0 = F(x + \Delta x, y(x) + \Delta y) - F(x, y(x))$$

$$= F_x'(x, y(x))\Delta x + F_y'(x, y(x))\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

其中, $\lim\limits_{(\Delta x, \Delta y) \to (0,0)} \varepsilon_i = 0, i = 1, 2.$

而$F_y'(x, y(x)) > 0$, 于是有

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=-\lim_{\Delta x\to 0}\frac{F_x'(x,y(x))+\varepsilon_1}{F_y'(x,y(x))+\varepsilon_2}$$

$$=-\frac{F_x'(x,y(x))}{F_y'(x,y(x))},\qquad \forall|x-x_0|<\delta.$$

即 $$y'(x)=-\frac{F_x'(x,y(x))}{F_y'(x,y(x))},\qquad \forall|x-x_0|<\delta.$$

由F的连续可微性知,$y'(x)$在$(x_0-\delta,x_0+\delta)$上连续.□

Remark:$F_y{}'(x_0,y_0)\neq 0$不是隐函数存在的必要条件.

设$f(x_1,\cdots,x_n,y)=0$, $f(x_1^0,\cdots,x_n^0,y_0)=0$ 确定了连续可微的隐函数 $y=y(x_1,\cdots,x_n)$, $y_0=y(x_1^0,\cdots,x_n^0)$, 满足

$$f(x_1,\cdots,x_n,y(x_1,\cdots,x_n))\equiv 0,$$

两边对x_i求偏导, 有

$$f_i'(x_1,\cdots,x_n,y(x_1,\cdots,x_n))\cdot 1+f_{n+1}'\cdot y_{x_i}'=0.$$

若$f_{n+1}'(x_1^0,\cdots,x_n^0,y_0)\neq 0$, 则在$(x_1^0,\cdots,x_n^0)$的某邻域中,

$$y_{x_i}'(x_1,x_2,\cdots,x_n)=-\frac{f_i'(x_1,\cdots,x_n,y(x_1,\cdots,x_n))}{f_{n+1}'(x_1,\cdots,x_n,y(x_1,\cdots,x_n))}$$

$$=-\frac{f_{x_i}'(x_1,\cdots,x_n,y(x_1,\cdots,x_n))}{f_y'(x_1,\cdots,x_n,y(x_1,\cdots,x_n))}.$$

右端求偏导函数时 $x_1,\cdots,x_n,y$相互独立

Thm. 设函数$F(x_1, x_2, \cdots, x_n, y)$在点$(x_1^0, x_2^0, \cdots, x_n^0, y_0)$ $\in \mathbb{R}^{n+1}$的某个邻域W中有定义, 且

$$(1) F(x_1^0, x_2^0, \cdots, x_n^0, y_0) = 0,$$

$$(2) F(x_1, x_2, \cdots, x_n, y) \in C^1(\mathrm{W}),$$

$$(3) \left.\frac{\partial F}{\partial y}\right|_{(x_1^0, x_2^0, \cdots, x_n^0, y_0)} \neq 0.$$

则存在点$(x_1^0, x_2^0, \cdots, x_n^0) \in \mathbb{R}^n$的一个邻域U,以及定义在U上的$n$元函数$y = y(x_1, x_2, \cdots, x_n)$,满足

(1) $y_0 = y(x_1^0, x_2^0, \cdots, x_n^0)$, 且当 $(x_1, x_2, \cdots, x_n) \in U$ 时,

$$F\left(x_1, x_2, \cdots, x_n, y(x_1, x_2, \cdots, x_n)\right) \equiv 0;$$

(2) $y = y(x_1, x_2, \cdots, x_n) \in C^1(\mathrm{U})$, 即 y'_{x_i} 在U中连续,

$i = 1, 2, \ldots \cdots, n;$

(3) $$y'_{x_i}(x_1, \cdots, x_n) = -\frac{F'_{x_i}(x_1, \cdots, x_n, y(x_1, \cdots, x_n))}{F'_y(x_1, \cdots, x_n, y(x_1, \cdots, x_n))}.$$

右端求偏导函数时 $x_1, \cdots, x_n, y$ 相互独立!

Remark: $F'_y(x_1^0, x_2^0, \cdots, x_n^0, y_0) \neq 0$ 不是隐函数存在的必要条件.

2. 方程组确定的隐函数

设 $F_i(x_1,x_2,\cdots,x_n,y_1,y_2,\cdots,y_m)=0, i=1,2,\cdots,m.$

$$F_i(x_1^0,x_2^0,\cdots,x_n^0,y_1^0,y_2^0,\cdots,y_m^0)=0, i=1,2,\cdots,m.$$

若存在连续可微的隐函数

$$y_i=y_i(x_1,x_2,\cdots,x_n), i=1,2,\cdots,m.$$

满足

$$y_i(x_1^0,x_2^0,\cdots,x_n^0)=y_i^0, i=1,2,\cdots,m.$$

$$F_i(x_1,x_2,\cdots,x_n,y_1(x_1,x_2,\cdots,x_n),y_2(x_1,x_2,\cdots,x_n),\cdots,y_m(x_1,x_2,\cdots,x_n))=0,\quad i=1,2,\cdots,m$$

简记为 $F(x,y)=0, F(x_0,y_0)=0,$ $\left(\begin{array}{l} x\in\mathbb{R}^n, y\in\mathbb{R}^m, \\ F:\mathbb{R}^{n+m}\to\mathbb{R}^m. \end{array}\right)$

$F(x,y(x))=0, y(x_0)=y_0.$

由复合隐射的链式法则, 有 $\dfrac{\partial F}{\partial(x,y)}\dfrac{\partial(x,y)}{\partial x}=0,$

(求 $\dfrac{\partial F}{\partial(x,y)}$ 时x, y相互独立!)

即 $\begin{pmatrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial y} \end{pmatrix}\begin{pmatrix} \dfrac{\partial x}{\partial x} \\ \dfrac{\partial y}{\partial x} \end{pmatrix}=0,$ $\dfrac{\partial F}{\partial x}I_n+\dfrac{\partial F}{\partial y}\dfrac{\partial y}{\partial x}=0,$

若 $\dfrac{\partial F}{\partial y}$ 可逆,则 $\dfrac{\partial y}{\partial x}=-\left(\dfrac{\partial F}{\partial y}\right)^{-1}\dfrac{\partial F}{\partial x}.$

Thm. $F(x,y):\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^m$在$(x_0,y_0)$的邻域W中有定义,且满足 (1)$F(x_0,y_0)=0$, (2)$F\in \mathrm{C}^q(\mathrm{W})$,即$F$的各分量函数在W中$q$阶连续可微, (3) $\dfrac{\partial F}{\partial y}(x_0,y_0)$可逆,

则存在x_0的某个邻域$\mathrm{U}\in\mathbb{R}^n$,以及定义在U上的向量值函数$y=y(x)$,满足

(1)$y(x_0)=y_0, F(x,y(x))=0, \forall x\in \mathrm{U}$;

(2)$y(x)$在U上q阶连续可微;

(3) $\dfrac{\partial y}{\partial x}=-\left(\dfrac{\partial F}{\partial y}\right)^{-1}\dfrac{\partial F}{\partial x}$. 求$\dfrac{\partial F}{\partial x},\dfrac{\partial F}{\partial y}$时 x,y 相互独立!

Remark: $\dfrac{\partial F}{\partial y}(x_0, y_0)$可逆不是隐函数存在的必要条件.

Remark: $F:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^m,(x,y)\mapsto F(x,y)$,若$\dfrac{\partial F}{\partial y}$可逆,

则$F(x,y)=0$确定隐"函数"$y=y(x)$,求$\dfrac{\partial y}{\partial x}$有两种方法:

- 套用定理:$\dfrac{\partial y}{\partial x}=-\left(\dfrac{\partial F}{\partial y}\right)^{-1}\dfrac{\partial F}{\partial x}$.

这里求Jaccobi矩阵时 x, y 相互独立!

- 将$F(x,y)=0$中y视为$y=y(x)$,利用复合映射的链式法则,方程组 $F(x,y(x))=0$两边对x求Jaccobi矩阵.

Remark: 对具体的例子, 不必死记硬背隐函数定理中的公式, 只要将某些变量视为其它变量的隐函数, 再利用复合函数的求导法则即可.

Remark: m个方程确定m个隐函数, 将某m个变量看成函数, 其它变量相互独立.

例. φ可微, $x^2+z^2=y\varphi\left(\dfrac{z}{y}\right)$确定隐函数$z=z(x,y)$.求$z'_x, z'_y$.

解: 视$x^2+z^2=y\varphi\left(z/y\right)$中$z=z(x,y)$为隐函数. 两边分别对$x, y$求偏导, 有

$$2x+2zz'_x=y\varphi'\left(z/y\right)\cdot\frac{1}{y}z'_x,$$

$$2zz'_y=\varphi\left(z/y\right)+y\varphi'\left(z/y\right)\cdot\frac{1}{y^2}(yz'_y-z).$$

求解得

$$z'_x=\frac{2x}{\varphi'\left(z/y\right)-2z},\ z'_y=\frac{y\varphi\left(z/y\right)-\varphi'\left(z/y\right)}{2yz-y\varphi'\left(z/y\right)}.\square$$

例. $u=f(x,y,z)$有连续偏导数, 且$z=z(x,y)$由方程

$xe^x-ye^y=ze^z$所确定, 求du.

解: 方程$xe^x-ye^y=ze^z$两边分别对x,y求偏导, 有

$$\left.\begin{aligned} e^x+xe^x&=z'_xe^z+zz'_xe^z \\ -e^y-ye^y&=z'_ye^z+zz'_ye^z \end{aligned}\right\}\Rightarrow\begin{cases} z'_x=\dfrac{1+x}{1+z}e^{x-z}, \\ z'_y=\dfrac{-(1+y)}{1+z}e^{y-z}. \end{cases}$$

于是,

$$\mathrm{d}u=u'_x\mathrm{d}x+u'_y\mathrm{d}y=\left(f'_x+f'_zz'_x\right)\mathrm{d}x+\left(f'_y+f'_zz'_y\right)\mathrm{d}y$$

$$=\left(f'_x+\frac{1+x}{1+z}e^{x-z}f'_z\right)\mathrm{d}x+\left(f'_y-\frac{1+y}{1+z}e^{y-z}f'_z\right)\mathrm{d}y.\square$$

Remark:

$$
\begin{aligned}
\mathrm{d}u &= f'_x \mathrm{d}x + f'_y \mathrm{d}y + f'_z \mathrm{d}z \\
&= f'_x \mathrm{d}x + f'_y \mathrm{d}y + f'_z (z'_x \mathrm{d}x + z'_y \mathrm{d}y) \\
&= \left(f'_x + f'_z z'_x \right) \mathrm{d}x + \left(f'_y + f'_z z'_y \right) \mathrm{d}y.
\end{aligned}
$$

一阶微分的形式不变性

例. $u=f(x-ut,y-ut,z-ut),g(x,y,z)=0$,求$u'_x,u'_y$.

分析: 五个变量x,y,z,t,u,两个方程, 确定两个隐函数 $z=z(x,y,t)=z(x,y),u=u(x,y,t)$.

解法一: 视$u=f(x-ut,y-ut,z-ut)$中$z=z(x,y)$为隐函数, 两边分别对x,y求偏导, 有

$$u'_x=(1-tu'_x)f'_1+(-tu'_x)f'_2+(z'_x-tu'_x)f'_3,$$

$$u'_y=(-tu'_y)f'_1+(1-tu'_y)f'_2+(z'_y-tu'_y)f'_3.$$

其中 f'_1,f'_2,f'_3 在$(x-ut,y-ut,z-ut)$处取值.

视$g(x,y,z)=0$中$z=z(x,y)$, 两边对x,y求偏导,有

$$\begin{cases} g'_x + g'_z z'_x = 0, \\ g'_y + g'_z z'_y = 0, \end{cases} \Rightarrow \begin{cases} z'_x = -g'_x / g'_z \,, \\ z'_y = -g'_y / g'_z \,. \end{cases}$$

代入前两式,求解得

$$u'_x = \frac{f'_1 + f'_3 z'_x}{1 + t(f'_1 + f'_2 + f'_3)} = \frac{f'_1 g'_z - f'_3 g'_x}{[1 + t(f'_1 + f'_2 + f'_3)] g'_z}$$

$$u'_y = \frac{f'_2 + f'_3 z'_y}{1 + t(f'_1 + f'_2 + f'_3)} = \frac{f'_2 g'_z - f'_3 g'_y}{[1 + t(f'_1 + f'_2 + f'_3)] g'_z}.$$

解法二：套用隐函数定理.

$$h(x,y,z,u,t) \triangleq f(x-ut, y-ut, z-ut) - u = 0,$$

$$g(x,y,z) = 0.$$

$$\frac{\partial(u,z)}{\partial(x,y,t)} = -\left(\frac{\partial(h,g)}{\partial(u,z)}\right)^{-1} \frac{\partial(h,g)}{\partial(x,y,t)}$$

$$= \begin{pmatrix} 1+t(f_1'+f_2'+f_3') & -f_3' \\ 0 & -g_z' \end{pmatrix}^{-1} \begin{pmatrix} f_1' & f_2' & -u(f_1'+f_2'+f_3') \\ g_x' & g_y' & 0 \end{pmatrix}$$

$$= \frac{-1}{[1+t(f_1'+f_2'+f_3')]g_z'} \begin{pmatrix} -g_z' & f_3' \\ 0 & 1+t(f_1'+f_2'+f_3') \end{pmatrix} \frac{\partial(h,g)}{\partial(x,y,t)}$$

于是 $(u_x', u_y') = \dfrac{(g_z' \quad -f_3')}{[1+t(f_1'+f_2'+f_3')]g_z'} \begin{pmatrix} f_1' & f_2' \\ g_x' & g_y' \end{pmatrix}$. □

3. 逆映射定理

Thm. (逆映射的微分) $f:\Omega\left(\subset\mathbb{R}^n\right)\to\mathbb{R}^n$连续可微, $x_0\in\Omega$.若 $J(f)\big|_{x_0}$ 可逆,则存在$y_0=f(x_0)$的某个邻域U,使得U上定义了映射$y=f(x)$的逆映射$x=f^{-1}(y)$, $x_0=f^{-1}(y_0)$, 且$x=f^{-1}(y)$在y_0可微,

$$\frac{\partial\left(x_1,x_2,\cdots,x_n\right)}{\partial\left(y_1,y_2,\cdots,y_n\right)}=\left(\frac{\partial\left(y_1,y_2,\cdots,y_n\right)}{\partial\left(x_1,x_2,\cdots,x_n\right)}\right)^{-1},$$

即$J(f^{-1})=\left(J(f)\right)^{-1}$.

Proof: 考虑方程组$F(x,y)\triangleq f(x)-y=0$,有

$$F(x_0,y_0)=0,\text{且}\left.\frac{\partial F}{\partial x}\right|_{(x_0,y_0)}=\left.\frac{\partial f}{\partial x}\right|_{x_0}\text{可逆}.$$

由隐函数定理,存在$y_0=f(x_0)$的邻域U及U上定义的函数$x=x(y)\triangleq f^{-1}(y)$,满足

$$f(x(y))-y\equiv 0,x(y_0)=x_0,$$

由复合映射的链式法则,有

$$\frac{\partial f}{\partial x}(x(y))\cdot\frac{\partial x}{\partial y}(y)-I=0,\quad \forall y\in U.$$

即$J(f)\cdot J(f^{-1})=\mathrm{I}$, $J(f^{-1})=\left(J(f)\right)^{-1}$.□

作业： 习题1.6 No. 4,5,7,10(1)