Review

- $S: x = x(u,v), y = y(u,v), z = z(u,v), (u,v) \in D$,

  简记为$\mathrm{r} = \mathrm{r}(u,v), (u,v) \in D$. 则曲面面积为

$$\iint_D \|\mathrm{r}'_u \times \mathrm{r}'_v\| \mathrm{d}u\mathrm{d}v.$$

- $S: z = f(x,y), (x,y) \in D$,曲面面积为

$$\iint_D \sqrt{1 + f_x'^2 + f_y'^2}\, \mathrm{d}x\mathrm{d}y.$$

- 微元法

§3. 第一型曲面积分

1. 光滑曲面

Def. 点(x, y, z)在曲面S上变化时,若S的单位法向量 $\vec{n}(x, y, z)$与$-\vec{n}(x, y, z)$都连续变化,则称S为光滑曲面.

Remark: 设曲面$S: z = f(x, y)$,则S的法向量

$$\vec{n}(x, y, z) = \pm \frac{(f_x', f_y', -1)}{\sqrt{1 + f_x'^2 + f_y'^2}}.$$

因此,S为光滑曲面 $\Leftrightarrow$ f连续可微.

Remark: 设曲面S由隐函数$F(x,y,z)=0$表示,则

$$\vec{n}=\pm\frac{(F_x',F_y',F_z')}{\sqrt{F_x'^2+F_y'^2+F_z'^2}}.$$

因此,$F(x,y,z)$连续可微$\Rightarrow S$为光滑曲面.

Remark: 设曲面$S: x=x(u,v), y=y(u,v), z=z(u,v)$,

简记为$S: \mathrm{r}=\mathrm{r}(u,v)$,则$S$的法向量为:

$$\vec{n}(x,y,z)=\pm\mathrm{r}_u'\times\mathrm{r}_v'/\|\mathrm{r}_u'\times\mathrm{r}_v'\|,$$

其中$\mathrm{r}_u'=(x_u',y_u',z_u'), \mathrm{r}_v'=(x_v',y_v',z_v')$. 于是

$$\vec{n}(x,y,z)=\pm\frac{(A,B,C)}{\sqrt{A^2+B^2+C^2}},$$

其中, $A=\det\frac{\partial(y,z)}{\partial(u,v)}, B=\det\frac{\partial(z,x)}{\partial(u,v)}, C=\det\frac{\partial(x,y)}{\partial(u,v)}.$

因此,

$x(u,v), y(u,v), z(u,v)$都连续可微

$\Rightarrow S$为光滑曲面.

2.第一型曲面积分的物理背景及定义

曲面S上任一点$P(x,y,z)$处的密度为$\mu(x,y,z)$，求S的质量.将S分割成$\Delta S_1,\cdots,\Delta S_n$,在$\Delta S_i$上任取点$P_i(\xi_i,\eta_i,\delta_i)$,仍以$\Delta S_i$表示曲面$\Delta S_i$的面积,则$S$的质量$m \approx \sum_{i=1}^{n} \mu(P_i)\Delta S_i$.记$\lambda = \max_i \{d(\Delta S_i)\}$,若极限$\lim_{\lambda \to 0} \sum_{i=1}^{n} \mu(P_i)\Delta S_i$存在,则该极限就是$S$的质量.

Def. 设函数$f(x,y,z)$在空间曲面S上有定义, 将S任意分割成$\Delta S_1,\Delta S_2,\cdots,\Delta S_n$, 记$\lambda$为分割的直径, 仍以$\Delta S_i$表示曲面$\Delta S_i$的面积, 在$\Delta S_i$上任意取点$P_i(\xi_i,\eta_i,\delta_i)$, 若极限 $\lim\limits_{\lambda\to 0}\sum_{i=1}^{n} f(P_i)\Delta S_i$存在, 则称该极限为函数$f(x,y,z)$在曲面$S$上的第一型曲面积分, 记作$\iint_S f(x,y,z)\mathrm{d}S$, 即

$$\iint_S f(x,y,z)\mathrm{d}S=\lim_{\lambda\to 0}\sum_{i=1}^{n} f(P_i)\Delta S_i,$$

当S为封闭曲面时, 记作$\oiint_S f(x,y,z)\mathrm{d}S$.

Remark: 定义中极限值 $\lim_{\lambda\to 0}\sum_{i=1}^{n} f(P_i)\Delta S_i$ 与对 S 所做的分割无关, 与 P_i 的选取无关.

Remark: $\iint_S \mathrm{d}S$ 表示曲面 S 的面积.

2.第一型曲面积分 $\iint_S f(x,y,z)\mathrm{d}S$ 的计算

设曲面 S 有参数方程

$$x = x(u,v), y = y(u,v), z = z(u,v), (u,v)\in D,$$

简记为

$$S : \mathrm{r} = \mathrm{r}(u,v), (u,v)\in D.$$

•Step1.分划:

在ouv平面上,用平行于坐标轴的直线

$$u=u_i(i=1,2,\cdots,n),\ v=v_j(j=1,2,\cdots,m)$$

对D进行分划$\{\Delta D_{ij}\}$. 在映射$\mathrm{r}=\mathrm{r}(u,v),(u,v)\in D$下,

曲面S上有分划$T=\{\Delta S_{ij}\}$,其中ΔD_{ij}与ΔS_{ij}对应.

•Step2.取点:

$$P_{ij}=(x(u_i,v_j),y(u_i,v_j),z(u_i,v_j))\in\Delta S_{ij}.$$

•Step3.求和:

面积$\Delta S_{ij}\approx\sqrt{A^2+B^2+C^2}\Delta u_i\Delta v_j$,其中

$$(A,B,C)=(\mathrm{r}'_u\times\mathrm{r}'_v)\Big|_{(u_i,v_j)}$$

$$=\left(\det\frac{\partial(y,z)}{\partial(u,v)},\det\frac{\partial(z,x)}{\partial(u,v)},\det\frac{\partial(x,y)}{\partial(u,v)}\right)\Bigg|_{(u_i,v_j)}.$$

于是, $\sum_{i,j} f(P_{ij})\Delta S_{ij}$

$$\approx\sum_{i,j} f(x(u_i,v_j),y(u_i,v_j),z(u_i,v_j))\cdot\sqrt{A^2+B^2+C^2}\,\Delta u_i\Delta v_j$$

•Step4.取极限:

$$\iint_S f(x,y,z)\mathrm{d}S$$

$$=\iint_D f(x(u,v),y(u,v),z(u,v))\cdot\sqrt{A^2+B^2+C^2}\,\mathrm{d}u\mathrm{d}v$$

$$=\iint_D f(x(u,v),y(u,v),z(u,v))\cdot\|\mathrm{r}'_u\times\mathrm{r}'_v\|\mathrm{d}u\mathrm{d}v$$

Remark: 若曲面S的方程为$z=f(x,y),(x,y)\in D$,

则$S: x=x, y=y, z=f(x,y),(x,y)\in D$.于是

$$A=\det\begin{pmatrix}0 & f'_x \\ 1 & f'_y\end{pmatrix}=-f'_x, B=-f'_y, C=1.$$

$$\iint_S g(x,y,z)\,\mathrm{d}S$$

$$=\iint_D g(x,y,f(x,y))\sqrt{1+f_x'^2+f_y'^2}\,\mathrm{d}x\mathrm{d}y.$$

4. 第一型曲面积分的性质

(1) (可积的充分条件) S为光滑曲面, $f(x,y,z)$在S上连续, 则$\iint_S f\,\mathrm{d}S$存在.

(2)(线性性质)若$\iint_S f\,\mathrm{d}S$与$\iint_S g\,\mathrm{d}S$都存在, 则$\forall\alpha,\beta\in\mathbb{R}$, $\iint_S(\alpha f+\beta g)\mathrm{d}S$存在, 且

$$\iint_S(\alpha f+\beta g)\mathrm{d}S=\alpha\iint_S f\,\mathrm{d}S+\beta\iint_S g\,\mathrm{d}S.$$

(3)(对曲面的可加性) S由$S_1, S_2, \cdots, S_n$拼接而成,则

$$\iint_S f\,\mathrm{d}S = \iint_{S_1} f\,\mathrm{d}S + \iint_{S_2} f\,\mathrm{d}S + \cdots + \iint_{S_n} f\,\mathrm{d}S.$$

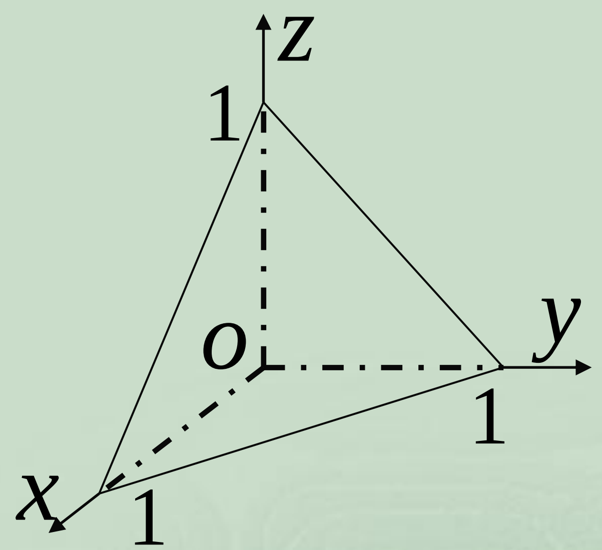
(4)(轮换不变性) 若S关于x, y轮换对称,即

$$(x, y, z) \in S \Leftrightarrow (y, x, z) \in S,$$

则

$$\iint_S f(x, y, z)\mathrm{d}S = \iint_S f(y, x, z)\mathrm{d}S.$$

例: $I=\iint\limits_{S}\frac{\mathrm{d}S}{(1+x+y)^2}$, 其中$S$为平面 $x+y+z=1$在第一卦限中的部分.



解: S的方程为

$$z=1-x-y,(x,y)\in D,$$

其中$D=\{0\le x\le 1,0\le y\le 1-x\}$.

故

$$I=\iint_D\frac{\sqrt{1+z_x'^2+z_y'^2}\,\mathrm{d}x\mathrm{d}y}{(1+x+y)^2}=\iint_D\frac{\sqrt{3}\mathrm{d}x\mathrm{d}y}{(1+x+y)^2}$$

$$=\int_0^1\mathrm{d}x\int_0^{1-x}\frac{\sqrt{3}}{(1+x+y)^2}\mathrm{d}y=\frac{\sqrt{3}}{2}(2\ln 2-1).\square$$

例:设$a^2+b^2+c^2=1$,S是平面$ax+by+cz=d$上的有界区域.求S在三个坐标平面上的投影面积.

解:S的单位法向量为$\vec{n}=(a,b,c)$.设S在oxy平面的投影为D_{xy}.分别以$\sigma(S)$和$\sigma(D_{xy})$表示S和D_{xy}的面积.

- 当$c\neq 0$时,$S: z=(d-ax-by)/c$,$(x,y)\in D_{xy}$,则

$$\sigma(S)=\iint_{D_{xy}}\sqrt{1+z_x'^2+z_y'^2}\,\mathrm{d}x\mathrm{d}y$$

$$=\iint_{D_{xy}}\sqrt{1+(a/c)^2+(b/c)^2}\,\mathrm{d}x\mathrm{d}y=\sigma(D_{xy})/|c|,$$

•当$c=0$时，S所在平面与oxy平面垂直，

$$\sigma(D_{xy})=0.$$

综上，$\sigma(D_{xy})=|c|\sigma(S)$.

同理，S在oyz平面和ozx平面的投影面积分别为

$$\sigma(D_{yz})=|a|\sigma(S),$$

$$\sigma(D_{zx})=|b|\sigma(S).\square$$

例. 求密度$\mu \equiv 1$的锥面$S: \dfrac{z^2}{b^2} = \dfrac{x^2}{a^2} + \dfrac{y^2}{a^2} (0 \le z \le b)$关于直线$\dfrac{x}{1} = \dfrac{y}{0} = \dfrac{z-b}{0}$的转动惯量.

解:转动惯量为$\iint_S \left[y^2 + (z-b)^2 \right] \mathrm{d}S.$

$$S: \mathrm{r} = (a\rho\cos t, a\rho\sin t, b\rho), (t, \rho) \in D = [0, 2\pi] \times [0,1].$$

$$\mathrm{r}_t' = (-a\rho\sin t, a\rho\cos t, 0)$$

$$\mathrm{r}_\rho' = (\ \ a\cos t, \quad a\sin t, \quad b)$$

$$\mathrm{r}_t' \times \mathrm{r}_\rho' = (ab\rho\cos t, ab\rho\sin t, -a^2\rho),$$

$$\mathrm{d}S = \left\| \mathrm{r}'_t \times \mathrm{r}'_\rho \right\| \mathrm{d}t\mathrm{d}\rho = a\sqrt{a^2 + b^2}\,\rho\mathrm{d}t\mathrm{d}\rho$$

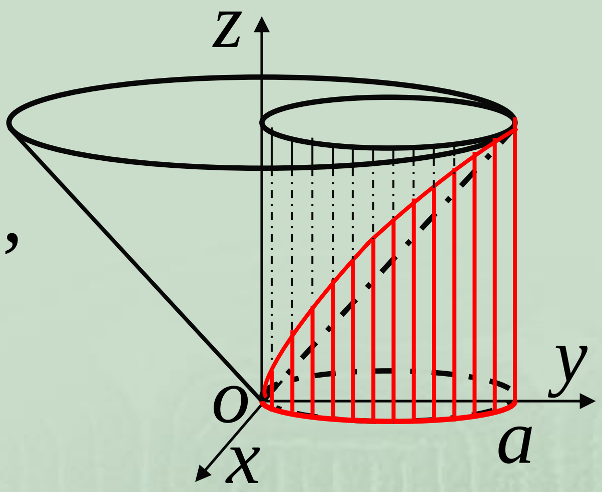
$$\iint_S \left[y^2 + (z-b)^2 \right] \mathrm{d}S$$

$$= a\sqrt{a^2 + b^2} \iint_D \left[(a\rho\sin t)^2 + (b\rho - b)^2 \right] \rho\mathrm{d}t\mathrm{d}\rho$$

$$= a\sqrt{a^2 + b^2} \int_0^1 \mathrm{d}\rho \int_0^{2\pi} \left[a^2\rho^3\sin^2 t + b^2\rho(\rho-1)^2 \right] \mathrm{d}t$$

$$= \frac{a(3a^2 + 2b^2)\sqrt{a^2 + b^2}\,\pi}{12}. \square$$

例:求圆柱面$x^2+y^2=ay$界于锥面$z=\sqrt{x^2+y^2}$和平面$z=0$之间部分S的面积$\sigma(S)$.



解:S: $x=\dfrac{a}{2}\cos\theta,\ y=\dfrac{a}{2}(1+\sin\theta),\ z=t,$

$$\theta\in[0,2\pi],\ \ 0\le t\le\frac{a}{\sqrt{2}}\sqrt{1+\sin\theta}.$$

$$\mathrm{r}'_\theta=(-\frac{a}{2}\sin\theta,\frac{a}{2}\cos\theta,0),$$

$$\mathrm{r}'_t=(\quad 0\quad,\quad 0\quad,1),$$

$$\mathrm{r}'_\theta\times\mathrm{r}'_t=(\frac{a}{2}\cos\theta,\frac{a}{2}\sin\theta,0),$$

$$\|\mathrm{r}'_\theta\times\mathrm{r}'_t\|=\frac{a}{2}.$$

$$\sigma(S)$$

$$=\int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{a}{\sqrt{2}}\sqrt{1+\sin\theta}}\|\mathrm{r}'_\theta\times\mathrm{r}'_t\|\mathrm{d}t$$

$$=\int_0^{2\pi}\frac{a^2}{2\sqrt{2}}\sqrt{1+\sin\theta}\mathrm{d}\theta$$

$$=2a^2.\square$$

例:设$f(x)\in C[0,1]$,则

$$(1)\int_0^{\pi/2}\int_0^{\pi/2}\sin\varphi f(\sin\varphi\sin\theta)\mathrm{d}\varphi\mathrm{d}\theta$$

$$=\frac{\pi}{2}\int_0^{\pi/2}\sin\varphi f(\cos\varphi)\mathrm{d}\varphi,$$

(2)计算$I=\iint\limits_{0\le\varphi,\theta\le\pi/2}\sin\varphi e^{\sin\varphi\sin\theta}\mathrm{d}\varphi\mathrm{d}\theta.$

解:(1)令S为$x^2+y^2+z^2=1, x\ge 0, y\ge 0, z\ge 0$. S 关于 y,z 轮换对称性, 故

$$\iint_S f(y)\mathrm{d}S=\iint_S f(z)\mathrm{d}S.$$

往证上式两边分别等于(1) 式两边.

$$S: \mathrm{r} = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi), 0 \le \varphi, \theta \le \pi/2.$$

$$\mathrm{r}'_\varphi = (\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi),$$

$$\mathrm{r}'_\theta = (-\sin\varphi\sin\theta, \sin\varphi\cos\theta, \quad 0 \ ),$$

$$\mathrm{r}'_\varphi \times \mathrm{r}'_\theta = (\sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta, \sin\varphi\cos\varphi),$$

$$\left\|\mathrm{r}'_\varphi \times \mathrm{r}'_\theta\right\| = \sin\varphi$$

$$\iint_S f(y)\mathrm{d}S = \int_0^{\pi/2}\int_0^{\pi/2} \sin\varphi f(\sin\varphi\sin\theta)\mathrm{d}\varphi\mathrm{d}\theta,$$

$$\iint_S f(\mathrm{z})\mathrm{d}S = \int_0^{\pi/2}\int_0^{\pi/2} \sin\varphi f(\cos\varphi)\mathrm{d}\varphi\mathrm{d}\theta$$

$$= \frac{\pi}{2}\int_0^{\pi/2} \sin\varphi f(\cos\varphi)\mathrm{d}\varphi.$$

(2)计算$I = \iint\limits_{0\le\varphi,\theta\le\pi/2} \sin\varphi e^{\sin\varphi\sin\theta}\mathrm{d}\varphi\mathrm{d}\theta.$

解:由(1),

$$I = \iint\limits_{0\le\varphi,\theta\le\pi/2} \sin\varphi e^{\sin\varphi\sin\theta}\mathrm{d}\varphi\mathrm{d}\theta$$

$$= \frac{\pi}{2}\int_0^{\pi/2}\sin\varphi e^{\cos\varphi}\mathrm{d}\varphi$$

$$= -\frac{\pi}{2}e^{\cos\varphi}\Big|_0^{\pi/2} = \frac{\pi}{2}(e-1).\square$$

例*. $S:(x-1)^2+(y-2)^2+(z-3)^2=1,$

求$\displaystyle\oint\!\!\!\oint_S (x+y+z)\mathrm{d}S.$

解: 作平移变换$u=x-1, v=y-2, w=z-3,$ 则

$$\oint\!\!\!\oint_S (x+y+z)\mathrm{d}S \overset{?}{=} \oint\!\!\!\oint_{u^2+v^2+w^2=1} (u+v+w+6)\mathrm{d}S$$

$$= \oint\!\!\!\oint_{u^2+v^2+w^2=1} 6\mathrm{d}S = 24\pi. \square$$

Remark. 物理意义.

作业:习题4.3

No.1(4),6,10.

例* $S=\{(x,y,z)\mid x^2+y^2+z^2=1\}$.证明Poisson公式

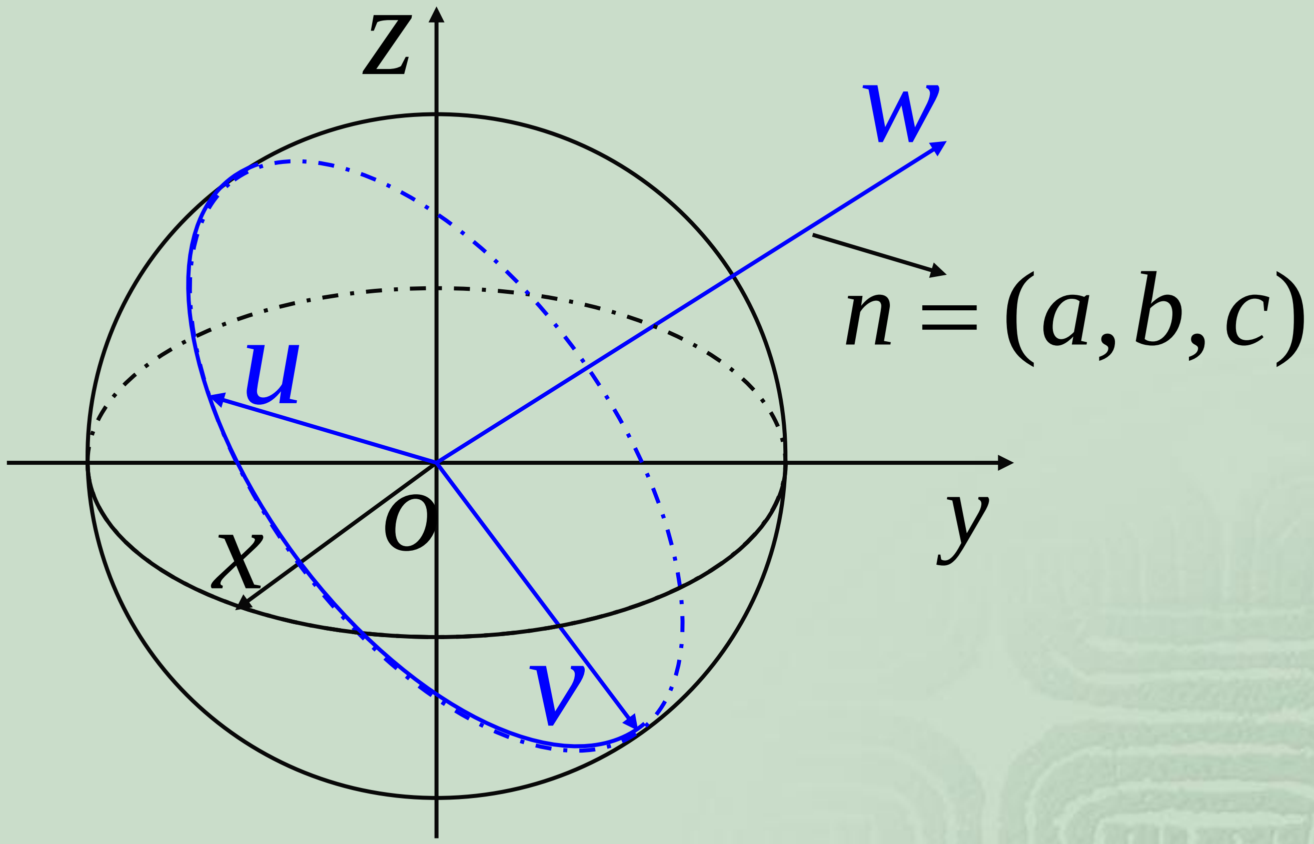
$$\iint_S f(ax+by+cz)\mathrm{d}S = 2\pi\int_{-1}^{1} f(\sqrt{a^2+b^2+c^2}\,t)\mathrm{d}t.$$

解:(1)若$a=b=c=0$,则

$$\iint_S f(ax+by+cz)\mathrm{d}S = \iint_S f(0)\mathrm{d}S$$

$$=4\pi f(0) = 2\pi\int_{-1}^{1} f(\sqrt{a^2+b^2+c^2}\,t)\mathrm{d}t.$$

(2)若$(a,b,c)\neq(0,0,0)$.作正交变换(旋转+反射),将$oxyz$坐标系变为$ouvw$坐标系,使坐标原点保持不变,并取$\vec{n}=(a,b,c)$为新坐标系的w轴正方向.



在该变换下,$oxyz$坐标系下的单位球面变成$ouvw$坐标系下的单位球面.$oxyz$坐标系下向量(x,y,z)在$ouvw$坐标系下w方向的分量为

$$w=(x,y,z)\cdot\frac{(a,b,c)}{\sqrt{a^2+b^2+c^2}}=\frac{ax+by+cz}{\sqrt{a^2+b^2+c^2}}.$$

于是, $ax+by+cz=\sqrt{a^2+b^2+c^2}\,w.$

旋转变换不改变曲面面积的大小,因此在该变换下,面积微元dS保持不变.故

$$\iint_S f(ax+by+cz)\mathrm{d}S$$

$$=\iint_{u^2+v^2+w^2=1} f(\sqrt{a^2+b^2+c^2}\,w)\mathrm{d}S$$

(物理解释：不同坐标系下计算曲面质量)

$$= \iint\limits_{\substack{0\le\varphi\le\pi \\ 0\le\theta\le 2\pi}} f(\sqrt{a^2+b^2+c^2}\cos\varphi)\sin\varphi \mathrm{d}\varphi \mathrm{d}\theta$$

$$= 2\pi\int_0^{\pi} f(\sqrt{a^2+b^2+c^2}\cos\varphi)\sin\varphi \mathrm{d}\varphi$$

$$= -2\pi\int_0^{\pi} f(\sqrt{a^2+b^2+c^2}\cos\varphi)\,\mathrm{d}\cos\varphi$$

$$= 2\pi\int_{-1}^{1} f(\sqrt{a^2+b^2+c^2}\,t)\mathrm{d}t. \square$$

Remark. 正交变换下第一型曲面积分的计算.

$$\mathrm{p} \triangleq (x, y, z)^{\mathrm{T}} = \mathrm{A}(u, v, w)^{\mathrm{T}} \triangleq \mathrm{Aq}, \qquad \mathrm{AA}^{\mathrm{T}} = \mathrm{I}.$$

$$(t, \tau) \in D \quad \leftrightarrow \quad (x, y, z) \in S \quad \leftrightarrow \quad (u, v, w) \in \Sigma$$

$$\frac{\partial(x, y, z)}{\partial(t, \tau)} = \mathrm{A}\frac{\partial(u, v, w)}{\partial(t, \tau)}, \qquad \mathrm{p}'_t = \mathrm{Aq}'_t, \mathrm{p}'_\tau = \mathrm{Aq}'_\tau,$$

$$\|\mathrm{p}'_t \times \mathrm{p}'_\tau\| = \|\mathrm{p}'_t\| \cdot \|\mathrm{p}'_\tau\| \sqrt{1 - \cos^2 \langle \mathrm{p}'_t, \mathrm{p}' \rangle_\tau}$$

$$= \|\mathrm{p}'_t\| \cdot \|\mathrm{p}'_\tau\| \sqrt{1 - \frac{(\mathrm{p}'_t \cdot \mathrm{p}'_\tau)^2}{\|\mathrm{p}'_t\|^2 \cdot \|\mathrm{p}'_\tau\|^2}} = \sqrt{\|\mathrm{p}'_t\|^2 \cdot \|\mathrm{p}'_\tau\|^2 - (\mathrm{p}'_t \cdot \mathrm{p}'_\tau)^2}$$

$$= \sqrt{\|\mathrm{q}'_t\|^2 \cdot \|\mathrm{q}'_\tau\|^2 - (\mathrm{q}'_t \cdot \mathrm{q}'_\tau)^2} = \|\mathrm{q}'_t \times \mathrm{q}'_\tau\|$$

$$\iint_S f(x,y,z)\mathrm{d}S$$

$$=\iint_D f(x(t,\tau),y(t,\tau),z(t,\tau))\left\|\mathrm{p}_t'\times\mathrm{p}_\tau'\right\|\mathrm{d}t\mathrm{d}\tau$$

$$=\iint_D f(x(t,\tau),y(t,\tau),z(t,\tau))\left\|\mathrm{q}_t'\times\mathrm{q}_\tau'\right\|\mathrm{d}t\mathrm{d}\tau$$

$$=\iint_\Sigma f(x(u,v,w),y(u,v,w),z(u,v,w))\mathrm{d}S.$$

Question. 正交变换下第一型曲线积分的计算?