对状态方程求解:

(1)

$$\dot{x}=\begin{bmatrix}-1&0\\0&-2\end{bmatrix}x+\begin{bmatrix}1\\1\end{bmatrix}u$$

$u(t)=1(t)$

$\Downarrow$ Laplace

$u(s)=\frac{1}{s}$

有 $x(s)=(sI-A)^{-1}[x(0)+B\,u(s)]$

$$x(s)=\begin{bmatrix}s+1&0\\0&s+2\end{bmatrix}^{-1}\left[[1,1]^T+\begin{bmatrix}1\\1\end{bmatrix}\frac{1}{s}\right]$$

$$x(s)=\begin{bmatrix}\frac{1}{s+1}&0\\0&\frac{1}{s+2}\end{bmatrix}_{2\times2}\begin{bmatrix}\frac{s+1}{s}\\\frac{s+1}{s}\end{bmatrix}_{2\times1}$$

$$x(s)=\begin{bmatrix}\frac{1}{s}\\\frac{s+1}{s(s+2)}\end{bmatrix}=\begin{bmatrix}\frac{1}{s}\\\frac{\frac{1}{2}}{s}+\frac{\frac{1}{2}}{s+2}\end{bmatrix}$$

$$x(t)=\begin{bmatrix}1\\\frac{1}{2}+\frac{1}{2}e^{-2t}\end{bmatrix}$$

$$\alpha_1=\lim_{s\to0}s\cdot\frac{s+1}{s(s+2)}=\frac{1}{2}$$

$$\alpha_2=\lim_{s\to-2}(s+2)\frac{s+1}{s(s+2)}=\frac{-1}{-2}=\frac{1}{2}$$

(2)

$$x(t)=e^{At}x(0)+A^{-1}(e^{At}-I)B$$

(k=1, 单位阶跃)

$$=e^{At}\begin{bmatrix}-1\\0\end{bmatrix}+\begin{bmatrix}-\frac{3}{2}&-\frac{1}{2}\\1&0\end{bmatrix}\begin{bmatrix}-e^{-2t}+e^{-t}-1\\2e^{-2t}-e^{-t}-1\end{bmatrix}$$

$$=\begin{bmatrix}e^{-2t}-2e^{-t}\\-2e^{-2t}+2e^{-t}\end{bmatrix}+\begin{pmatrix}-\frac{3}{2}&-\frac{1}{2}\\1&0\end{pmatrix}\begin{bmatrix}-e^{-2t}+e^{-t}-1\\2e^{-2t}-e^{-t}-1\end{bmatrix}$$

$$=\begin{bmatrix}\frac{2}{3}e^{-2t}-\frac{13}{6}e^{-t}+\frac{7}{6}\\-3e^{-2t}+3e^{-t}-1\end{bmatrix}$$

$\det(A-\lambda I)=0$

$$\begin{vmatrix}-\lambda&1\\-2&-\lambda-3\end{vmatrix}=0$$

$\lambda(\lambda+3)+2=0$

$\lambda^2+3\lambda+2=0$

$\lambda_1=-1$ $\lambda_2=-2$

$$\begin{bmatrix}1&1\\-2&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=0$$

$x_1+x_2=0$

$$\begin{matrix}x_1\\x_2\end{matrix}=\begin{bmatrix}-1\\1\end{bmatrix}x_2$$

$AP_2=\lambda_2P_2$

$$\begin{bmatrix}2&1\\-2&-1\end{bmatrix}x=0$$

$2x_1+x_2=0$

$$\begin{matrix}x_1\\x_2\end{matrix}=\begin{bmatrix}1\\-2\end{bmatrix}x_1$$

$$P=\begin{bmatrix}-1&1\\1&-2\end{bmatrix}$$

故 $$e^{At}=\begin{bmatrix}-1&1\\1&-2\end{bmatrix}\begin{bmatrix}e^{-t}&0\\0&e^{-2t}\end{bmatrix}\begin{bmatrix}-2&-1\\-1&-1\end{bmatrix}$$

$$e^{At}=\begin{bmatrix}-e^{-2t}+2e^{-t}&-e^{-2t}+e^{-t}\\2e^{-2t}-2e^{-t}&2e^{-2t}-e^{-t}\end{bmatrix}$$

3、 $\Phi(s) = (sI-A)^{-1}$

$$\Phi(s) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & \theta \\ 0 & -\theta & s \end{bmatrix}^{-1}$$

$$\Phi(s) = \begin{bmatrix} \frac{1}{s} & 0 & 0 \\ 0 & \frac{s}{s^2+\theta^2} & -\frac{\theta}{s^2+\theta^2} \\ 0 & \frac{\theta}{s^2+\theta^2} & \frac{s}{s^2+\theta^2} \end{bmatrix}$$

$$e^{At} = \Phi(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta t & -\sin\theta t \\ 0 & \sin\theta & \cos\theta t \end{bmatrix}$$

对于齐次方程有

4、 $$\Phi(s) = \begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ 0 & \frac{1}{s+2} - \frac{2}{(s+2)^2} & \frac{4}{(s+2)^2} \\ 0 & -\frac{1}{(s+2)^2} & \frac{1}{s+2} + \frac{2}{(s+2)^2} \end{bmatrix} = (sI-A)^{-1}$$

$$\begin{bmatrix} s+1 & 0 & 0 \\ 0 & s+4 & -4 \\ 0 & 1 & s \end{bmatrix} = sI-A$$

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 4 \\ 0 & -1 & 0 \end{bmatrix}$$

5、对于齐次方程有: $X(s) = (sI-A)^{-1}X_0$

$$x_1(t) = \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix} \Rightarrow X(s) = \begin{bmatrix} \frac{1}{s+2} \\ -\frac{1}{s+2} \end{bmatrix}$$

$$x_2(t) = \begin{bmatrix} 2e^{-t} \\ -e^{-t} \end{bmatrix} \Rightarrow X(s) = \begin{bmatrix} \frac{2}{s+1} \\ -\frac{1}{s+1} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{s+2} & \frac{2}{s+1} \\ -\frac{1}{s+2} & -\frac{1}{s+1} \end{bmatrix} = (sI-A)^{-1} \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{s+2} & \frac{2}{s+1} \\ -\frac{1}{s+2} & -\frac{1}{s+1} \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} = (sI-A)^{-1}$$

$$\begin{bmatrix} s & -2 \\ 1 & s+3 \end{bmatrix} = sI-A$$

$$A = \begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix}$$

易得 $\lambda_1 = -1$ $\lambda_2 = -2$

$$\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} X = 0 \qquad \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} x = 0$$

$$\begin{matrix} x_1 \\ x_2 \end{matrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} x_2 \qquad \begin{matrix} x_1 \\ x_2 \end{matrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} x_1$$

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix}$$ (特征值互异)

$$\Phi(t) = e^{At} = \begin{bmatrix} -e^{-2t}+e^{-t} & -2e^{-2t}+2e^{-t} \\ e^{-2t}-e^{-t} & 2e^{-2t}-e^{-t} \end{bmatrix}$$