运筹学

6. 线性规划的其它相关问题

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灵敏度分析

对于标准线性规划问题

$$\max \ C^T X$$
$$\text{s.t.} \quad AX = \vec{b}$$
$$X \geq 0$$

假定已求得最优可行基 B，并获得 B^{-1} 等有关数据

若某些参数发生变化，如 $C \to C + \Delta C, \vec{b} \to \vec{b} + \Delta\vec{b}$

如何利用已知数据确定新的最优解？

例1

$$\max\ z = 2x_1 + x_2$$

$$\text{s.t.}\quad 5x_2 + x_3 = 15$$

$$6x_1 + 2x_2 + x_4 = 24$$

$$x_1 + x_2 + x_5 = 5$$

$$x_i \geq 0,\ \forall 1 \leq i \leq 5$$

最终单纯型表

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_2	0	1	0	-0.25	1.5	1.5
x_3	0	0	1	1.25	-7.5	7.5
x_1	1	0	0	0.25	-0.5	3.5
	0	0	0	-0.25	-0.5	$z-8.5$

如果目标函数改变：$z=2x_1+x_2 \Rightarrow z=1.5x_1+2x_2$

最终单纯型表 $\Rightarrow$

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_2	0	1	0	−0.25	1.5	1.5
x_3	0	0	1	1.25	−7.5	7.5
x_1	1	0	0	0.25	−0.5	3.5
	1.5	2	0	0	0	z

$\Rightarrow$

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_2	0	1	0	−0.25	1.5	1.5
x_3	0	0	1	1.25	−7.5	7.5
x_1	1	0	0	0.25	−0.5	3.5
	0	0	0	0.125	−2.25	$z-8.25$

$\Rightarrow$ 继续迭代

如果常数向量改变： $\vec{b} = (15, 24, 5)^T \Rightarrow \vec{b}' = (15, 32, 5)^T$

最终单纯型表 $\Rightarrow$

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_2	0	1	0	−0.25	1.5	$\hat{x}_2$
x_3	0	0	1	1.25	−7.5	$\hat{x}_3$
x_1	1	0	0	0.25	−0.5	$\hat{x}_1$
	0	0	0	−0.25	−0.5	

其中新的常数向量为 $B^{-1}\vec{b}'$，如果 $B^{-1}\vec{b}' \geq 0$ ，已经得到最优解，否则可用对偶单纯型法继续迭代

如果增加一个变量，即将 $\sum_{i=1}^{n} c_i x_i$ 和 $\sum_{i=1}^{n} P_i x_i = \vec{b}$ 分别

变成 $\sum_{i=1}^{n+1} c_i x_i$ 和 $\sum_{i=1}^{n+1} P_i x_i = \vec{b}$

此时首先要确定 B^{-1} ，然后可算出

$$\hat{P}_{n+1} = B^{-1} P_{n+1},\quad \sigma_{n+1} = c_{n+1} - C_B^T \hat{P}_{n+1}$$

如果 $\sigma_{n+1} \leq 0$ ，原最优解不变，令 $\hat{x}_{n+1} = 0$

否则将 $\hat{P}_{n+1}$ 和 σ_{n+1} 加入最终单纯型表继续迭代

等式约束的系数矩阵发生变化，例如由

$$\sum_{i=1}^{n} P_i x_i = \vec{b} \quad \text{变成} \quad \sum_{\substack{i=1 \\ i \neq r}}^{n} P_i x_i + P_r' x_r = \vec{b}$$

如果 P_r 不在基中，计算 $\hat{P}_r' = B^{-1} P_r'$，$\sigma_r = c_r - C_B^T \hat{P}_r'$

然后类似增加一个变量的方法处理

否则要重新计算 B^{-1} ，根据基是否是原问题的可行基、是否是对偶问题的可行基、是否两者都不是进行适当处理，在第三种情况下要引入人工变量重新寻找可行基

如果增加约束条件，例如由

$$AX \le \vec{b} \quad 变成 \quad AX \le \vec{b},\ \vec{a}_{m+1}^{T} X \le b_{m+1}$$

或者由

$$AX \ge \vec{b} \quad 变成 \quad AX \ge \vec{b},\ \vec{a}_{m+1}^{T} X \ge b_{m+1}$$

如果当前最优解满足新增加的约束，那么仍然是新问题的最优解

否则要引入辅助变量或人工变量重新寻找可行解

小结：

无论怎么改变，首先看当前的最优可行基是否是新问题的可行解，甚至是最优可行基？

如果是，证明依然是最优解或者单纯形法操作

如果不是，或可用对偶单纯形法操作

参数线性规划

分析下述线性规划问题最优值随参数 λ 变化情况

$$\max\ (C+\lambda C')^T X \qquad \text{s.t.}\ AX=\vec{b},\quad X\ge 0$$

$$\max\ C^T X \qquad \text{s.t.}\ AX=\vec{b}+\lambda\vec{b}',\quad X\ge 0$$

处理方法

1）固定 λ 的数值解线性规划问题

2）确定保持当前最优基不变的 λ 的区间

3）确定 λ 在上述区间附近的最优基，回2）

例3

$$\max\ z=(2+\lambda)x_1+(1+2\lambda)x_2$$
$$\text{s.t.}\quad 5x_2+x_3=15$$
$$6x_1+2x_2+x_4=24$$
$$x_1+x_2+x_5=5$$
$$x_i\ge 0,\ i=1,2,\cdots,5$$

取 $\lambda=0$ 得到下述最优基

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_3	0	0	1	1.25	-7.5	7.5
x_1	1	0	0	0.25	-0.5	3.5
x_2	0	1	0	-0.25	1.5	1.5
	0	0	0	-0.25	-0.5	$z-8.5$

带入参数

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_3	0	0	1	1.25	–7.5	7.5
x_1	1	0	0	0.25	–0.5	3.5
x_2	0	1	0	–0.25	1.5	1.5
	$2+\lambda$	$1+2\lambda$	0	0	0	z

行变换

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_3	0	0	1	1.25	-7.5	7.5
x_1	1	0	0	0.25	-0.5	3.5
x_2	0	1	0	-0.25	1.5	1.5
	0	0	0	$0.25(\lambda-1)$	$-0.5-2.5\lambda$	$z-8.5-6.5\lambda$

由上表知最优目标值 $z(\lambda)=8.5+6.5\lambda, \forall -0.2\leq\lambda\leq 1$

对于 $\lambda > 1$，从下面的单纯型表可以看出，x_4 的检验数大于**0**，因此应该让其进基

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_3	0	0	1	1.25	-7.5	7.5
x_1	1	0	0	0.25	-0.5	3.5
x_2	0	1	0	-0.25	1.5	1.5
	0	0	0	$0.25(\lambda-1)$	$-0.5-2.5\lambda$	$z-8.5-6.5\lambda$

比较各行**RHS**和 x_4 的系数的比值，可以确定出基变量为 x_3

用单纯型迭代实现 x_4进基、x_3 出基，得到下面新的单纯型表

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_4	0	0	0.8	1	-6	6
x_1	1	0	-0.2	0	1	2
x_2	0	1	0.2	0	0	3
	0	0	$0.2(1-\lambda)$	0	$-2-\lambda$	$z-7-8\lambda$

由上表知最优目标值 $z(\lambda)=7+8\lambda,\ \forall\lambda>1$

对于 $\lambda < -0.2$，从以下单纯型表可以看出，x_5 的检验数大于0，因此应该让其进基

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_3	0	0	1	1.25	-7.5	7.5
x_1	1	0	0	0.25	-0.5	3.5
x_2	0	1	0	-0.25	1.5	1.5
	0	0	0	$0.25(\lambda-1)$	$-0.5-2.5\lambda$	$z-8.5-6.5\lambda$

比较各行**RHS**和 x_5 的系数的比值，可以确定出基变量为 x_2

用单纯型迭代实现 x_5 进基、x_2 出基，得到下面新的单纯型表

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_3	0	5	1	0	0	15
x_1	1	$1/3$	0	$1/6$	0	4
x_5	0	$2/3$	0	$-1/6$	1	1
	0	$(1+5\lambda)/3$	0	$-(2+\lambda)/6$	0	$z-8-4\lambda$

由上表知最优目标值 $z(\lambda)=8+4\lambda,\ \forall -2\le\lambda<-0.2$

对于 $\lambda < -2$ ，从以下单纯型表可以看出，x_4 的检验数大于**0**，因此应该让其进基

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_3	0	5	1	0	0	15
x_1	1	$1/3$	0	$1/6$	0	4
x_5	0	$2/3$	0	$-1/6$	1	1
	0	$(1+5\lambda)/3$	0	$-(2+\lambda)/6$	0	$z-8-4\lambda$

比较各行**RHS**和 x_4 的系数的比值，可以确定出基变量为 x_1

用单纯型迭代实现 x_4 进基、x_1 出基，得到下面新的单纯型表

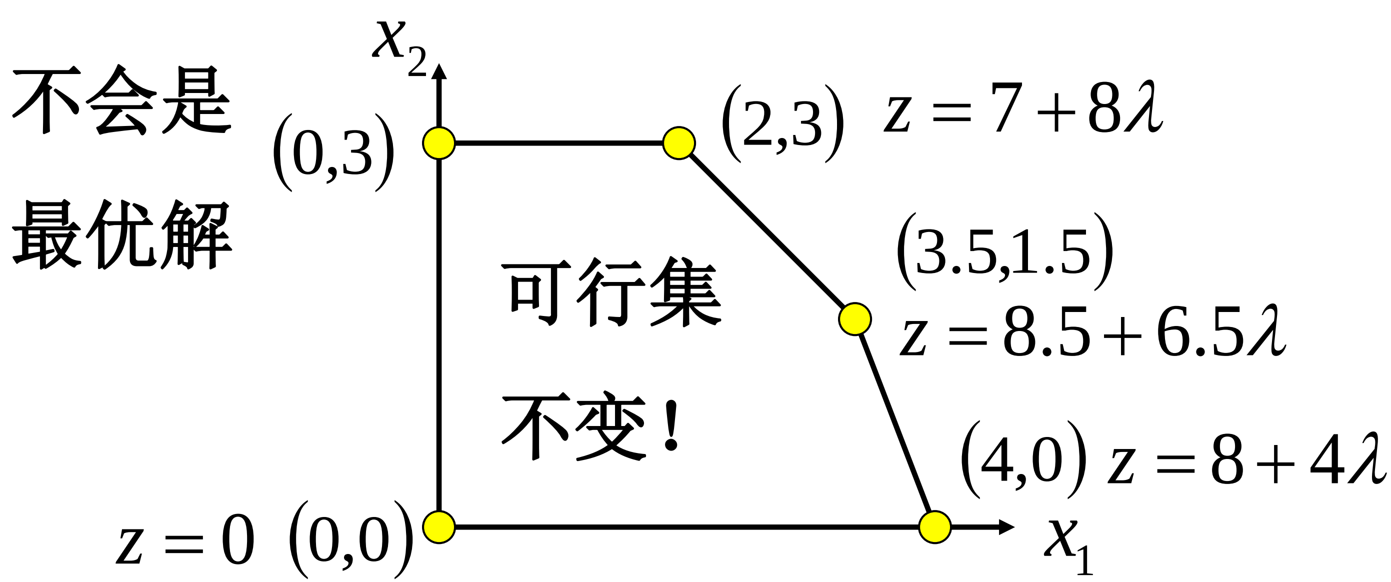
BV	x_1	x_2	x_3	x_4	x_5	RHS
x_3	0	5	1	0	0	15
x_4	6	2	0	1	0	24
x_5	1	1	0	0	1	5
	$2+\lambda$	$1+2\lambda$	0	0	0	z

由上表知最优目标值 $z(\lambda)=0,\ \forall\lambda<-2$

总结前面分析，最优目标函数值和 λ 的关系如下

$$z(\lambda)=\begin{cases}0, & \forall\ \lambda<-2\\ 8+4\lambda, & \forall -2\le\lambda<-.2\\ 8.5+6.5\lambda, & \forall -0.2\le\lambda\le 1\\ 7+8\lambda, & \forall\ \lambda>1\end{cases}$$

由于 $z=(2+\lambda)x_1+(1+2\lambda)x_2$ ，由下图容易理解 $z(\lambda)$



对于右边常数向量带参数的情况

$$\max\ C^T X$$
$$\text{s.t.}\quad AX = \vec{b} + \lambda \vec{b}'$$
$$X \geq 0$$

其对偶问题为

$$\min\ \left(\vec{b} + \lambda \vec{b}'\right)^T Y$$
$$\text{s.t.}\quad A^T Y \geq C$$

由于对偶问题的可行集不变，因此可用对偶单纯型法确定最优目标函数值和参数 λ 的关系

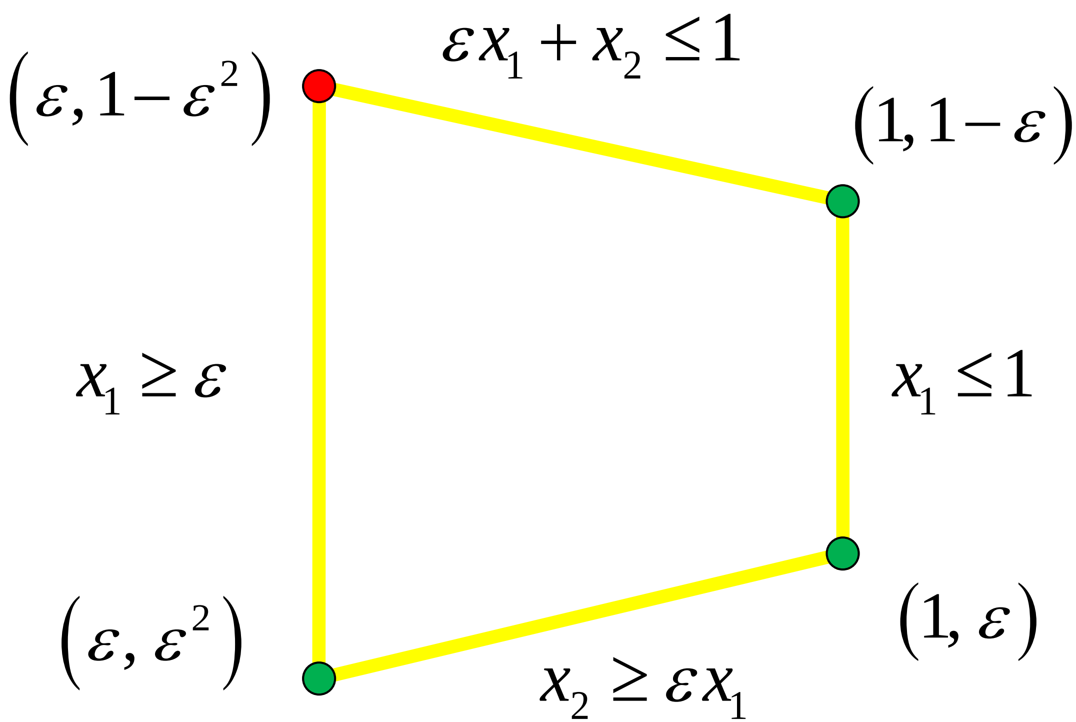
说明单纯型算法计算复杂性的例子

例、

$$\begin{aligned} \max\ & x_n \\ \text{s.t.}\ & \varepsilon \le x_1 \le 1 \\ & \varepsilon x_{j-1} \le x_j \le 1 - \varepsilon x_{j-1},\ \forall 2 \le j \le n \end{aligned}$$

其中 $0 < \varepsilon < 0.5$ （Klee-Minty，1971 ）

$n = 2$ 的可行集



对原问题进行可逆的线性变换，令

$$y_1 = x_1 - \varepsilon, \quad y_2 = \left(x_2 - \varepsilon x_1\right)/\alpha_2 \varepsilon$$

则

$$x_1 = y_1 + \varepsilon, \quad x_2 = \varepsilon y_1 + \alpha_2 \varepsilon y_2 + \varepsilon^2$$

原问题

$$\begin{aligned} \max \ & x_2 \\ \text{s.t.} \ & \varepsilon \le x_1 \le 1 \\ & \varepsilon x_1 \le x_2 \le 1 - \varepsilon x_1 \end{aligned}$$

$$\Rightarrow$$

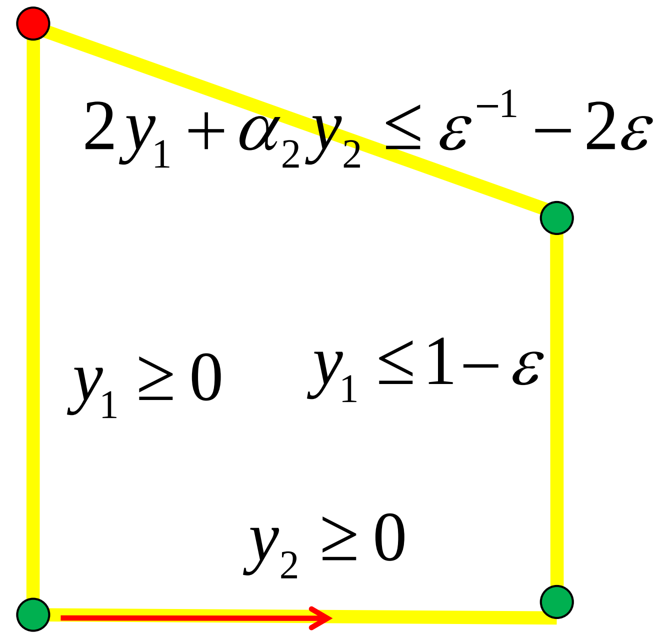
变换后的等价问题

$$\begin{aligned} \max \ & y_1 + \alpha_2 y_2 \\ \text{s.t.} \ & y_1 \le 1 - \varepsilon \\ & 2y_1 + \alpha_2 y_2 \le \varepsilon^{-1} - 2\varepsilon \\ & y_1 \ge 0, \ y_2 \ge 0 \end{aligned}$$

变换后问题的标准形式

$$\begin{aligned} \max\ & y_1 + \alpha_2 y_2 \\ \text{s.t.}\ & y_1 + s_1 = 1 - \varepsilon \\ & 2y_1 + \alpha_2 y_2 + s_2 = \varepsilon^{-1} - 2\varepsilon \\ & y_1 \ge 0,\ y_2 \ge 0, s_1 \ge 0,\ s_2 \ge 0 \end{aligned}$$

变换后的可行集



从 $(0, 0)$ 出发用单纯型法求解上述问题

如果选最大检验数进基，取 $\alpha_2 < 1$， y_1 进基

如果选最小正检验数进基，取 $\alpha_2 > 1$，还是 y_1 进基

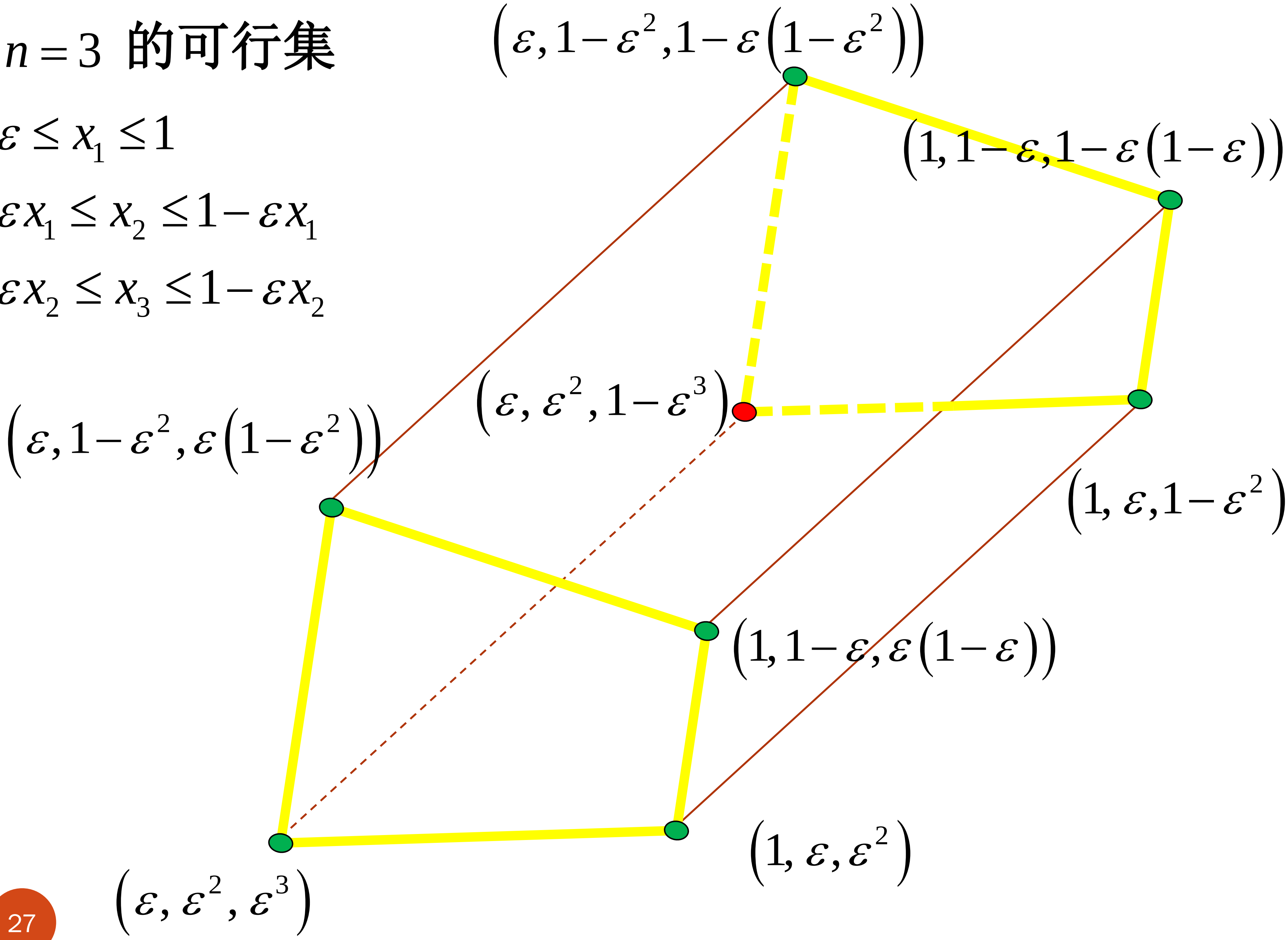
对任何给定算法，可选择参数经过所有 $2^2 - 1$ 个顶点！

$n=3$ 的可行集

$\varepsilon \le x_1 \le 1$

$\varepsilon x_1 \le x_2 \le 1-\varepsilon x_1$

$\varepsilon x_2 \le x_3 \le 1-\varepsilon x_2$



对原问题进行可逆的线性变换，令

$$y_1 = x_1 - \varepsilon, \quad y_2 = \left(x_2 - \varepsilon x_1\right) / \alpha_2 \varepsilon, \quad y_3 = \left(x_3 - \varepsilon x_2\right) / \alpha_3 \varepsilon^2$$

则

$$x_1 = y_1 + \varepsilon, \ x_2 = \varepsilon\left(y_1 + \alpha_2 y_2 + \varepsilon\right), \ x_3 = \varepsilon^2\left(y_1 + \alpha_2 y_2 + \alpha_3 y_3 + \varepsilon\right)$$

原问题

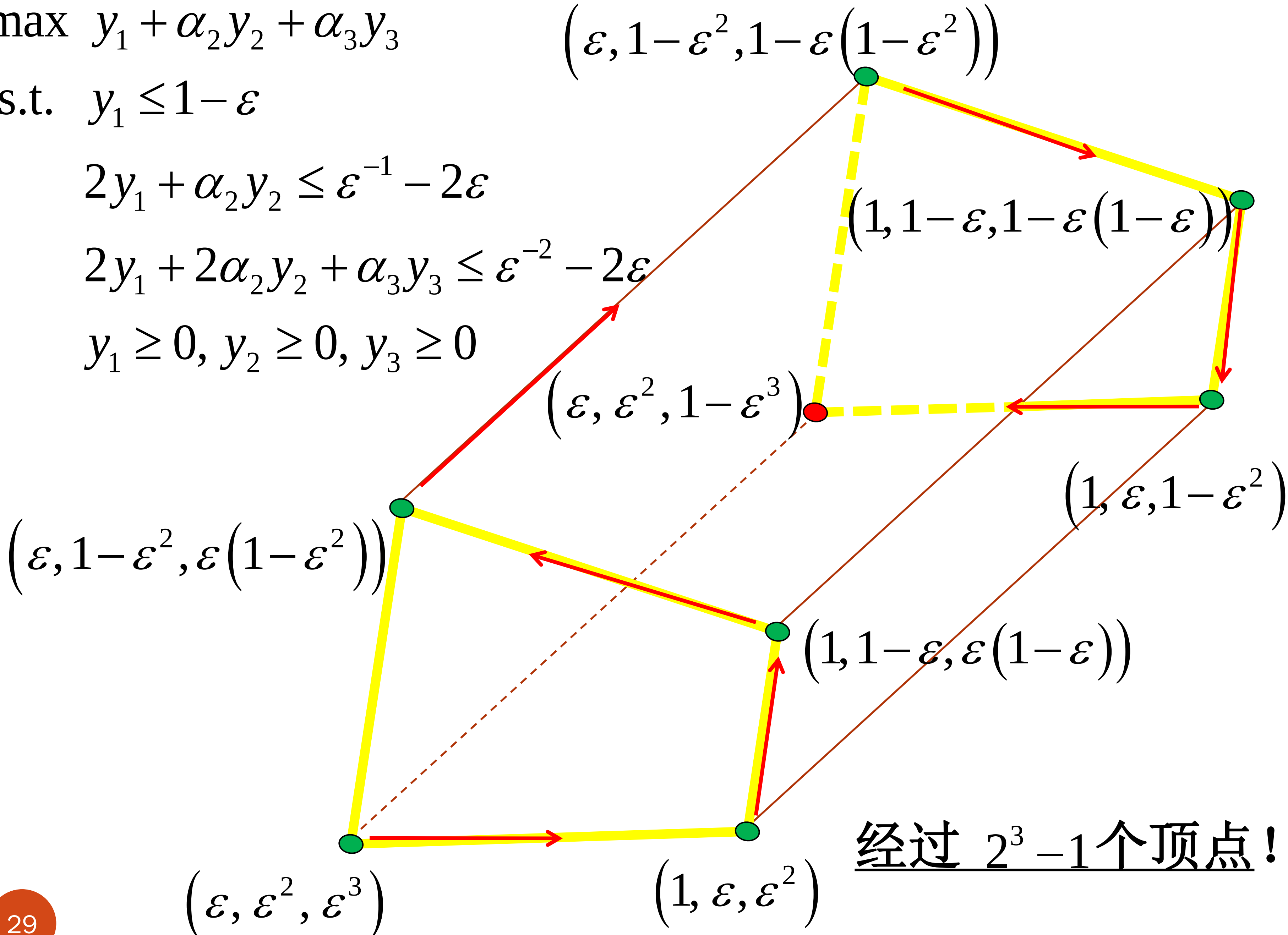
$$\begin{aligned} \max\ & x_3 \\ \text{s.t.}\ & \varepsilon \le x_1 \le 1 \\ & \varepsilon x_1 \le x_2 \le 1 - \varepsilon x_1 \\ & \varepsilon x_2 \le x_3 \le 1 - \varepsilon x_2 \end{aligned}$$

$$\Rightarrow$$

变换后的等价问题

$$\begin{aligned} \max\ & y_1 + \alpha_2 y_2 + \alpha_3 y_3 \\ \text{s.t.}\ & y_1 \le 1 - \varepsilon \\ & 2y_1 + \alpha_2 y_2 \le \varepsilon^{-1} - 2\varepsilon \\ & 2y_1 + 2\alpha_2 y_2 + \alpha_3 y_3 \le \varepsilon^{-2} - 2\varepsilon \\ & y_1 \ge 0,\ y_2 \ge 0,\ y_3 \ge 0 \end{aligned}$$

$$
\begin{aligned}
\max\quad & y_1+\alpha_2 y_2+\alpha_3 y_3\\
\text{s.t.}\quad & y_1\le 1-\varepsilon\\
& 2y_1+\alpha_2 y_2\le \varepsilon^{-1}-2\varepsilon\\
& 2y_1+2\alpha_2 y_2+\alpha_3 y_3\le \varepsilon^{-2}-2\varepsilon\\
& y_1\ge 0,\ y_2\ge 0,\ y_3\ge 0
\end{aligned}
$$



经过 2^3-1 个顶点！

Proof.

We prove this theorem by constructing the following LP problem

$$
\begin{aligned}
\max \quad & -x_n \\
\text{s.t.} \quad & x_1 \geq 0 \\
& x_1 \leq 1 \\
& x_i \geq \epsilon x_{i-1}, \; i = 2, 3, \cdots, n \\
& x_i \leq 1 - \epsilon x_{i-1}, \; i = 2, 3, \cdots, n
\end{aligned}
$$

where $\epsilon \in (0, 1/2)$. The feasible polytope is a perturbed n-dimensional cube. An example of the perturbed 3-dimensional cube.

It is easy to show that this LP problem has 2^n vertexes. We will show that there exists a most unfortunate sequence of choice at each selection step, so that the simplex algorithm uses 2^n pivots before coming to the optimal solution.

First, let us represent each vertex of the n-dimension perturb cube with an encoding in $\{0,1\}^n$, For example, in the perturbed 3-dimension cube shown in above, the vertices and their corresponding encoding are:

Vertex	**Encoding**
$(0,0,0)$	$(0,0,0)$
$(0,\epsilon,\epsilon^2)$	$(1,0,0)$
$(0,1-\epsilon,\epsilon-\epsilon^2)$	$(1,1,0)$
$(0,1,\epsilon)$	$(0,1,0)$
$(0,1,1-\epsilon)$	$(0,1,1)$
$(0,1-\epsilon,1-\epsilon+\epsilon^2)$	$(1,1,1)$
$(0,\epsilon,1-\epsilon^2)$	$(1,0,1)$
$(0,0,1)$	$(0,0,1)$

Second, let us define a Hamiltonian path in $\{0,1\}^n$ and show that corresponding vertices in the perturbed cube have increasing objective values. Indeed, the Hamiltonian path in the cube can be taken as a special type of gray code defined recursively as follows.

$$\Omega_n = \begin{array}{ll} & 0\cdots 00 \\ \Omega_{n-1} & 1\cdots 00 \\ & \cdots\cdots \\ & 1\cdots 10 \\ & \\ & 0\cdots 11 \\ \text{reverse} & \cdots\cdots \\ \Omega_{n-1} & 1\cdots 01 \\ & 0\cdots 01 \end{array}$$

For example, the Hamiltonian path defined on the perturbed 3-dimension cube is:

$$\Omega_3 = 000 \rightarrow 100 \rightarrow 110 \rightarrow 010 \rightarrow 011 \rightarrow 111 \rightarrow 101 \rightarrow 001$$

Third, we will show that the objective function of the linear program defined in (24) is decreasing, if we move from a basic feasible solution to another basic feasible solution by following the Hamiltonian path defined on binary reflected gray code. In the follows, we will prove by induction.

If n=2, the Hamiltonian path is $\Omega_3 = 00 \to 10 \to 11 \to 01$. The corresponding LP problem is

$$
\begin{aligned}
\min \quad & -x_2 \\
\text{s.t.} \quad & x_1 \geq 0 \\
& x_1 \leq 1 \\
& x_2 \geq \epsilon x_1 \\
& x_2 \leq 1 - \epsilon x_1
\end{aligned}
$$

Let $x^0 = (0,0)$ be the intital basic feasible solution. It is trivial to show the objective function is decreasing on the Hamiltonian path.

If $n = 3$, the LP problem is

$$\begin{aligned} \min \quad & -x_3 \\ \text{s.t.} \quad & x_1 \geq 0 \\ & x_1 \leq 1 \\ & x_2 \geq \epsilon x_1 \\ & x_2 \leq 1 - \epsilon x_1 \\ & x_3 \geq \epsilon x_2 \\ & x_3 \leq 1 - \epsilon x_2 \end{aligned}$$

Let us consider the first half of $\Omega_3 = 000 \to 100 \to 110 \to 010 \to 011 \to 111 \to 101 \to 001$. On this path, the last coordinate is always 0, which means we always take the equality

$$x_3 = \epsilon x_2$$

Let us focus on the first two bits. This path is the same as Ω_2. Further we notice x_3 doesnt appear in the first 4 inequalities in linear program (31) and these 4 inequalities are the same as those in linear program (30). We have shown that the objective function $\min -x_2$ is decreasing by following Ω_2 (In other words, x_2 increases). Similarly, we know that x_3 increases based on Equation (32) and the objective function $\min -x_2$ in the LP problem (31) decreases.

When we move from 010 to 011, x_3 changes from ϵx_2 to $1 - \epsilon x_2$. Since $\epsilon \in (0, 1/2)$, we have $\epsilon x_2 < 1 - \epsilon x_2$. So, this is an improving step.

Next, let us check the second half of $\Omega_3 : 011 \to 111 \to 101 \to 001$. On this path, the last coordinate is always 1, which means now we always take the equality

$$x_3 = 1 - \epsilon x_2$$

Let us check the first two bits on the path $011 \to 111 \to 101 \to 001$. This path is the reverse of Ω_2. So by following this path, x_2 decreases, then x_3 increases based on Equation (33).

Thus the objective function of linear program (31) continues to decrease.

Induction step

We assume the claim holds for $n-1$-cube and try to prove it for n-cube. The proof is similar as what we did to show the claim holds for the case of $n=3$ based on the result of $n=2$.

We start by moving from vertex $(0 \ \cdots \ 0\ 0)$ to $(0 \ \cdots \ 1\ 0)$ by following the first part of Ω_{n-1}. By induction, this is an decreasing path for $n-1$-cube. Since the last coordinate of the vertices on this part of path is always 0, we know we should always choose

$$x_n = \epsilon x_{n-1}$$

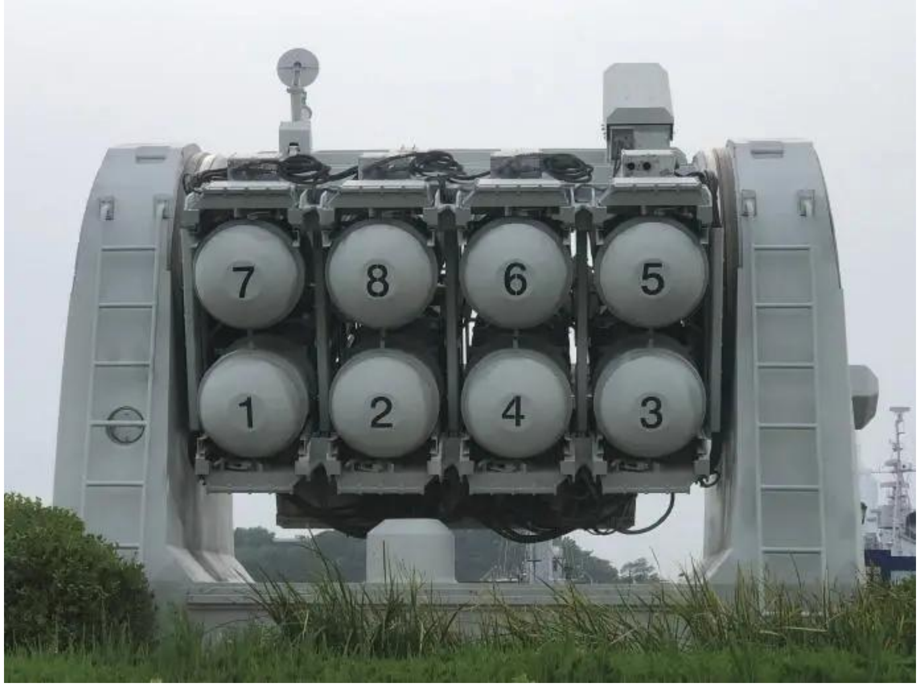
Thus, x_n is increasing and the objective function of linear program (24) is decreasing.
When we move from $(0 \ \cdots \ 1\ 0)$ to $(0 \ \cdots \ 1\ 1)$, x_n changes from ϵx_{n-1} to $1 - \epsilon x_{n-1}$. Since $\epsilon \in (0, 1/2)$, we have $\epsilon x_{n-1} < 1 - \epsilon x_{n-1}$. So, this is a decreasing step.
Then we follow the last part of Ω_n to move from $(0 \ \cdots \ 1\ 1)$ to $(0 \ \cdots \ 0\ 1)$. The first $n-1$ bits on this part of path is the reverse of Ω_{n-1}. So x_{n-1} is decreasing on the path. The last coordinate is always 1 here and so

$$x_n = 1 - \epsilon x_{n-1}$$

for the corresponding vertices. So, x_n continues to increase, which leads to continuous decreasing of the objective function.
Based on mathematical induction, we can conclude that the simplex algorithm takes exponential time by starting from a special vertex and choosing certain pivot rules.

格雷码，也称反射二进制码，是二进制数字系统的一种排序方式，使得两个连续值仅相差一位（二进制数字）。在数字电路中，格雷码每次的变换只会有一个二进制位的跳变，极大地减少了亚稳态的产生，保证了电路的稳定性。

十进制	格雷码	十进制	格雷码
0	0000	8	1100
1	0001	9	1101
2	0011	10	1111
3	0010	11	1110
4	0110	12	1010
5	0111	13	1011
6	0101	14	1001
7	0100	15	1000



对已经提出的进出基规则，均能设计出要经历的顶点个数是变量维数的指数函数的例子

但是，也不能证明无论采用什么进出基规则，均能设计出要经历的顶点个数是变量维数的指数函数的例子

天坑，不要跳

只要 n 比较大，搜索 2^n-1个顶点的计算量就不好完成！能否找到非指数时间的其他算法？

$\Rightarrow$ **椭球算法（Khachian，哈奇杨，1979）**

内点法（Karmarkar，1984，…）

线性规划问题求解顺序

原问题只包含两个决策变量	图解法
原问题决策变量较多，而约束条件较少	解对偶问题
原问题有明显的初始可行基	单纯型法
原问题没有明显的初始可行基，但有明显的对偶可行基	对偶单纯型法
原问题没有明显的初始可行基，同时也没有明显的对偶可行基	两阶段法

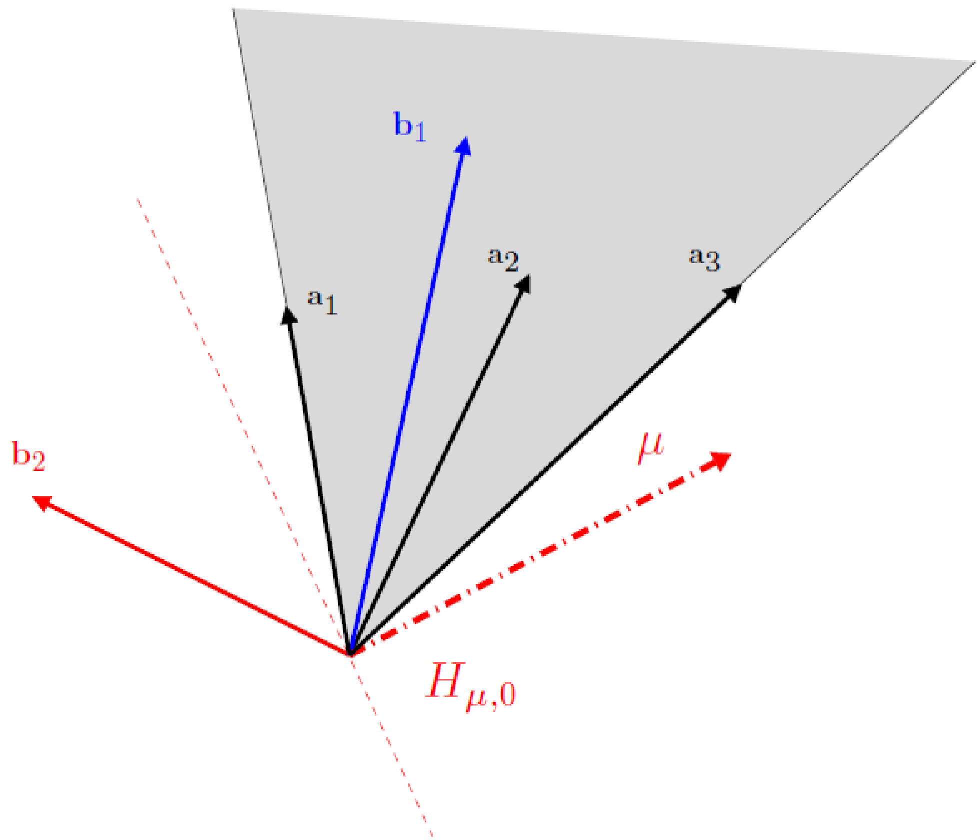
Farkas Lemma

Farkas Lemma

- Basically states the feasibility of two **different** problems, two **related** problems.

Theorem 1. *Let* $A \in \mathbf{R}^{m\times n}$ *and let* $\mathbf{b} \in \mathbf{R}^m$. *Then exactly one of the two alternatives holds*

1. *there exists* $\mathbf{x} \geq \mathbf{0}$ *such that* $A\mathbf{x} = \mathbf{b}$.
2. *there exists* $\boldsymbol{\mu}$ *such that* $\boldsymbol{\mu}^T A \geq 0$ *and* $\boldsymbol{\mu}^T \mathbf{b} < 0$.



Theorem (Farkas' Lemma)

Given $A \in \mathbb{R}^{m \times n}$ *is an* $m \times n$ *matrix,* $\boldsymbol{b} \in \mathbb{R}^m$ *is an m-dimensional column vector. Exactly one of the following linear system is feasible:*

I. There exists an $\boldsymbol{x} = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$ *such that* $A\boldsymbol{x} = \boldsymbol{b}$ *and* $\boldsymbol{x} \geq \boldsymbol{0}$*.*

II. There exists a $\boldsymbol{y} = [y_1, \ldots, y_m]^T \in \mathbb{R}^m$ *such that* $A^T\boldsymbol{y} \geq \boldsymbol{0}$ *and* $\boldsymbol{b}^T\boldsymbol{y} < 0$*.*

Proof.

First, we use contradiction method to show that both systems cannot simultaneously have feasible solutions.

If both system are simultaneously feasible, $\boldsymbol{b}^T\boldsymbol{y} < 0$ implies $\boldsymbol{y} \neq \mathbf{0}$ and $\boldsymbol{b} \neq \mathbf{0}$.

Meanwhile, if $\boldsymbol{b} \neq \mathbf{0}$, $A\boldsymbol{x} = \boldsymbol{b}$ implies $\boldsymbol{x} \neq \mathbf{0}$. If both systems holds, then we have

$$\boldsymbol{b}^T\boldsymbol{y} = (A\boldsymbol{x})^T\boldsymbol{y} = \boldsymbol{x}^T\left(A^T\boldsymbol{y}\right) \geq 0 \tag{16}$$

which contradicts $\boldsymbol{b}^T\boldsymbol{y} < 0$.

Second, we show that at least one of them has a feasible solution. If System (I) is feasible, we can finish right here. Otherwise, System (I) is infeasible, we have $\Omega = \{A\boldsymbol{x},\ \boldsymbol{x} \geq \boldsymbol{0}\}$ is a closed convex set. Moreover, $\boldsymbol{b} \notin \Omega$.
According to Separating Hyperplane Theorem, there exists a hyperplane $\boldsymbol{y}^T\boldsymbol{x} = z$ that separates $\boldsymbol{b}$ from Ω, where $\boldsymbol{y} = [y_1, \ldots, y_m]^T \in \mathbb{R}^m$ is an m-dimensional column vector. That is, $\boldsymbol{y}^T\boldsymbol{b} < z$ and $\forall\ \boldsymbol{s} \in \Omega$, $\boldsymbol{y}^T\boldsymbol{s} \geq z$.
Since $\boldsymbol{0} \in \Omega$, we have $z \leq 0$. As a result, $\boldsymbol{y}^T\boldsymbol{b} < 0$.
On the other hand, since $\boldsymbol{y}^T A\boldsymbol{x} > 0$ for all $\boldsymbol{x} \in \Omega$, we can see that $\boldsymbol{y}^T A > \boldsymbol{0}$, since each element of $\boldsymbol{x}$ can be arbitrarily large.
Therefore, we prove the whole statement.

Theorem (Strong Duality Theorem)

For LPP $\{\min \boldsymbol{c}^T\boldsymbol{x};\ s.t.\ A\boldsymbol{x} \geq \boldsymbol{b}\}$*, a feasible solution* $\boldsymbol{x}^*$ *to the primal problem is optimal if and only if there exists a feasible solution* $\boldsymbol{u}^*$ *to the dual LPP* $\{\max \boldsymbol{b}^T\boldsymbol{u};\ s.t.\ A^T\boldsymbol{u} = \boldsymbol{c}, \boldsymbol{u} \geq \boldsymbol{0}\}$ *such that*

$$\boldsymbol{c}^T\boldsymbol{x}^* = \boldsymbol{b}^T\boldsymbol{u}^* \tag{17}$$

Meanwhile, $\boldsymbol{u}^*$ *is an optimal solution to the dual.*

Proof.

First, We prove the sufficiency.
Based on weak duality theorem, for any feasible solution $\boldsymbol{x}$ of the primal problem, we have

$$\boldsymbol{c}^T\boldsymbol{x} \geq \boldsymbol{b}^T\boldsymbol{u}^* = \boldsymbol{c}^T\boldsymbol{x}^* \tag{18}$$

which shows that $\boldsymbol{x}^*$ is also the optimal solution of the primal problem.

Similarly, for any feasible solution $\boldsymbol{u}$ of the dual problem, we have

$$\boldsymbol{b}^T\boldsymbol{u} \leq \boldsymbol{c}^T\boldsymbol{x}^* = \boldsymbol{b}^T\boldsymbol{u}^* \tag{19}$$

which shows that $\boldsymbol{u}^*$ is also the optimal solution of the dual problem.
Next, we prove the necessariness based on Farkas' Lemma, since we do not introduce the simplex algorithm here.
Suppose $\boldsymbol{x}^*$ is an optimal solution. We will show that there exists a dual feasible solution $\boldsymbol{u}$ with $\boldsymbol{b}^T\boldsymbol{u} = \boldsymbol{c}^T\boldsymbol{x}^*$.
Let us define I as the set of constraint index that active at $\boldsymbol{x}^*$.
That is,

$$a_i^T\boldsymbol{x}^* = b_i, \quad i \in I \tag{20}$$

$$a_i^T\boldsymbol{x}^* > b_i, \quad i \notin I \tag{21}$$

$\boldsymbol{x}^*$ implies that, for any $\boldsymbol{d} \in \mathbb{R}^n$, the following set

$$a_i^T \boldsymbol{d} \geq 0, \ \boldsymbol{c}^T \boldsymbol{d} < 0, \ i \in I \tag{22}$$

is infeasible. Otherwise, we would have a small enough $\epsilon > 0$ such that

$$a_i^T (\boldsymbol{x}^* + \epsilon \boldsymbol{d}) \geq b_i, \ \boldsymbol{c}^T (\boldsymbol{x}^* + \epsilon \boldsymbol{d}) < \boldsymbol{c}^T \boldsymbol{x}^*, \ i = 1, \ldots, m \tag{23}$$

According to Farkas' Lemma, we know that the above inequality is infeasible if and only if there exists $\lambda_i, i \in I$ that

$$\lambda_i \geq 0, \ \sum_{i \in I} \lambda_i a_i = c \tag{24}$$

This yields a dual feasible solution $\boldsymbol{u}$ satisfying

$$u_i = \lambda_i, \quad i \in I \tag{25}$$

$$u_i = 0, \quad i \notin I \tag{26}$$

Finally, we show that $\boldsymbol{u}$ is the optimal solution for the dual problem. Indeed, we have

$$\boldsymbol{b}^T \boldsymbol{u} = \sum_{i \in I} b_i u_i = \sum_{i \in I} (a_i^T x_i^*) u_i = \boldsymbol{u}^* A \boldsymbol{x}^* = \boldsymbol{c}^T \boldsymbol{x}^* \tag{27}$$

Based on Weak Duality Theorem, we see $\boldsymbol{u}$ is the optimal solution for the dual problem. Thus comes our statement according to strong duality.

Matlab Codes for LP

https://zhuanlan.zhihu.com/p/61466360
https://zhuanlan.zhihu.com/p/61582750

```
function [x,z,ST,res_case] = SimplexMax(c,A,b,ind_B)
% 单纯形法求解标准形线性规划问题: max cx s.t. Ax=b x>=0
% 输入参数: c为目标函数系数, A为约束方程组系数矩阵, b为约束
% 输出参数: x最优解, z最优目标函数值, ST存储单纯形表数据, r

[m,n] = size(A);             %m约束条件个数, n决策变量数
ind_N = setdiff(1:n, ind_B);  %非基变量的索引
ST = [];
format rat
% 循环求解
while true
    x0 = zeros(n,1);
    x0(ind_B) = b;             %初始基可行解
    cB = c(ind_B);             %计算cB
    Sigma = zeros(1,n);
    Sigma(ind_N) = c(ind_N) - cB*A(:,ind_N);   %计算检验
    [~, k] = max(Sigma);       %选出最大检验数, 确定进基
    Theta = b ./ A(:,k);       %计算θ
    Theta(Theta<=0) = 10000;
    [~, q] = min(Theta);       %选出最小θ
    el = ind_B(q);             %确定出基变量索引el, 主元
    vals = [cB',ind_B',b,A,Theta];
    vals = [vals; NaN, NaN, NaN, Sigma, NaN];
    ST = [ST; vals];
    if ~any(Sigma > 0)         %此基可行解为最优解, any
        x = x0;
        z = c * x;
        res_case = 0;
        return
    end
    if all(A(:,k) <= 0)        %有无界解
        x = [];
        res_case = 1;
        break
    end
    % 换基
    ind_B(ind_B == el) = k;    %新的基变量索引
    ind_N = setdiff(1:n, ind_B); %非基变量索引
    % 更新A和b
    A(:,ind_N) = A(:,ind_B) \ A(:,ind_N);
    b = A(:,ind_B) \ b;
    A(:,ind_B) = eye(m,m);
end
```

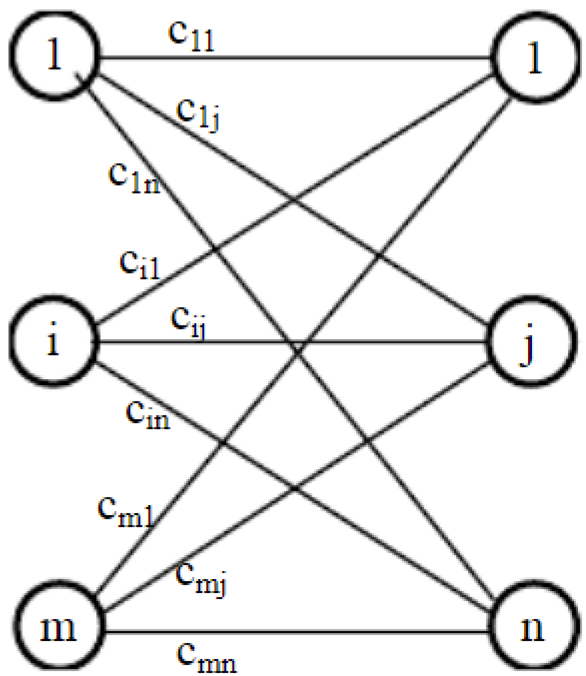
```
function [x,z,ST,res_case] = DualSimplexMax(c,A,b,ind_B)
% 对偶单纯形法求解标准形线性规划问题: max cx s.t. Ax <= b
% 输入参数: c为目标函数系数, A为约束方程组系数矩阵, b为约
% 输出参数: x最优解, z最优目标函数值, ST存储单纯形表数据, r

[m,n] = size(A);            %m约束条件个数, n决策变量数
ind_N = setdiff(1:n, ind_B);  %非基变量的索引
ST = [];
format rat
% 循环求解
while true
    x0 = zeros(n,1);
    x0(ind_B) = b;                  %初始基可行解
    cB = c(ind_B);                  %计算cB
    Sigma = zeros(1,n);
    Sigma(ind_N) = c(ind_N) - cB*A(:,ind_N);   %计算检验
    [~,q] = min(b);                 %选出最小的b
    r = ind_B(q);                   %确定出基变量索引r
    Theta = Sigma ./ A(q,:);        %计算θ
    Theta(Theta<=0) = 10000;
    [~,s] = min(Theta);             %确定进基变量索引s, 主元
    vals = [cB',ind_B',b,A];
    vals = [vals; NaN, NaN, NaN, Sigma];
    ST = [ST; vals];
    if ~any(b < 0)                 %此基可行解为最优解, any
        x = x0;
        z = c * x;
        res_case = 0;
        return
    end
    % 换基
    ind_B(ind_B == r) = s;      %新的基变量索引
    ind_N = setdiff(1:n, ind_B); %非基变量索引
    % 更新A和b
    A(:,ind_N) = A(:,ind_B) \ A(:,ind_N);
    b = A(:,ind_B) \ b;
    A(:,ind_B) = eye(m,m);
end
```

一些典型的线性规划问题

运输问题

设有同一种货物从 m 个发地 1，2，…，m 运往 n 个收地 1，2，…，n。第 i 个发地的供应量（Supply）为 a_i（$a_i \geqslant 0$），第 j 个收地的需求量（Demand）为 b_j（$b_j \geqslant 0$）。每单位货物从发地 i 运到收地 j 的运价为 c_{ij}。求一个使总运费最小的运输方案。我们假定从任一发地到任一收地都有道路通行。如果总供应量等于总需求量，这样的运输问题称为供求平衡的运输问题。我们先只考虑这一类问题。



运输问题

设有同一种货物从 m 个发地 1，2，…，m 运往 n 个收地 1，2，…，n。第 i 个发地的供应量（Supply）为 a_i（$a_i \geqslant 0$），第 j 个收地的需求量（Demand）为 b_j（$b_j \geqslant 0$）。每单位货物从发地 i 运到收地 j 的运价为 c_{ij}。求一个使总运费最小的运输方案。我们假定从任一发地到任一收地都有道路通行。如果总供应量等于总需求量，这样的运输问题称为供求平衡的运输问题。我们先只考虑这一类问题。

请写出原问题和对偶问题，

并解释对偶变量的物理意义

运输问题

当产销平衡时，其模型如下：

$$\min Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

$$\begin{cases} \sum x_{ij} = a_i \\ \sum x_{ij} = b_j \qquad (\sum a_i = \sum b_j) \\ x_{ij} \geq 0 \end{cases}$$

运输问题

当产大于销时，其模型如下：

$$\min Z=\sum_{i=1}^{m}\sum_{j=1}^{n}c_{ij}x_{ij}$$

$$\begin{cases}\sum x_{ij}\le a_i \\ \sum x_{ij}=b_j \qquad (\sum a_i>\sum b_j) \\ x_{ij}\ge 0\end{cases}$$

运输问题

当产小于销时，其模型如下：

$$\min Z = \sum\sum c_{ij} x_{ij}$$

$$\begin{cases} \sum x_{ij} = a_i \\ \sum x_{ij} \le b_j \quad (\sum a_i < \sum b_j) \\ x_{ij} \ge 0 \end{cases}$$

并假设：$a_i \ge 0, b_j \ge 0, c_{ij} \ge 0$

运输问题

由题给出的条件，数学模型可写为：

$$\min z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij}$$

$$st.\begin{cases} \sum_{j=1}^{n} x_{ij} \le a_i & (i = 1,\mathrm{L}\ ,m) \\ \sum_{i=1}^{m} x_{ij} \ge b_j & (j = 1,\mathrm{L}\ ,n) \\ x_{ij} \ge 0 \end{cases}$$

对偶问题可写为 ：

$$\max z' = \sum_{j=1}^{n} b_j v_j - \sum_{i=1}^{m} a_i u_i$$

$$st.\begin{cases} v_j - u_i \le c_{ij} & (i = 1,\mathrm{L}\ ,m; j = 1,\mathrm{L}\ ,n) \\ u_j, v_i \ge 0 \end{cases}$$

运输问题

对偶变量u_i的经济意义为在i产地单位物资的价格，v_j的经济意义为在第j销地单位物资的价格。

对偶问题的经济意义为：如该公司欲自己将该种物资运至各地销售，其差价不能超过两地之间的运价（否则买主将在i地购买自己运至j地），在此条件下，希望获利为最大。

最大流问题

给一个带收发点的网络（一般收点用 v_t 表示，发点用 v_s 表示，其余为中间点），其每条弧的权值称之为容量，在不超过每条弧的容量的前提下，要求确定每条弧的流量，使得从发点到收点的流量最大。

“流”，是指铁路线（弧）上的实际运输量。

每条弧旁的数字即为该弧的容量 c_{ij}，弧的方向就是允许流的方向。把标有弧容量的网络称为容量网络，记为 $D=(V,A,C)$. $c_{ij}\geq 0$ 。实际通过各弧的流量，记为 f_{ij} 。所有弧上流量的集 $F=\{f_{ij}\}$ 称为该网络D的一个流。

最大流问题

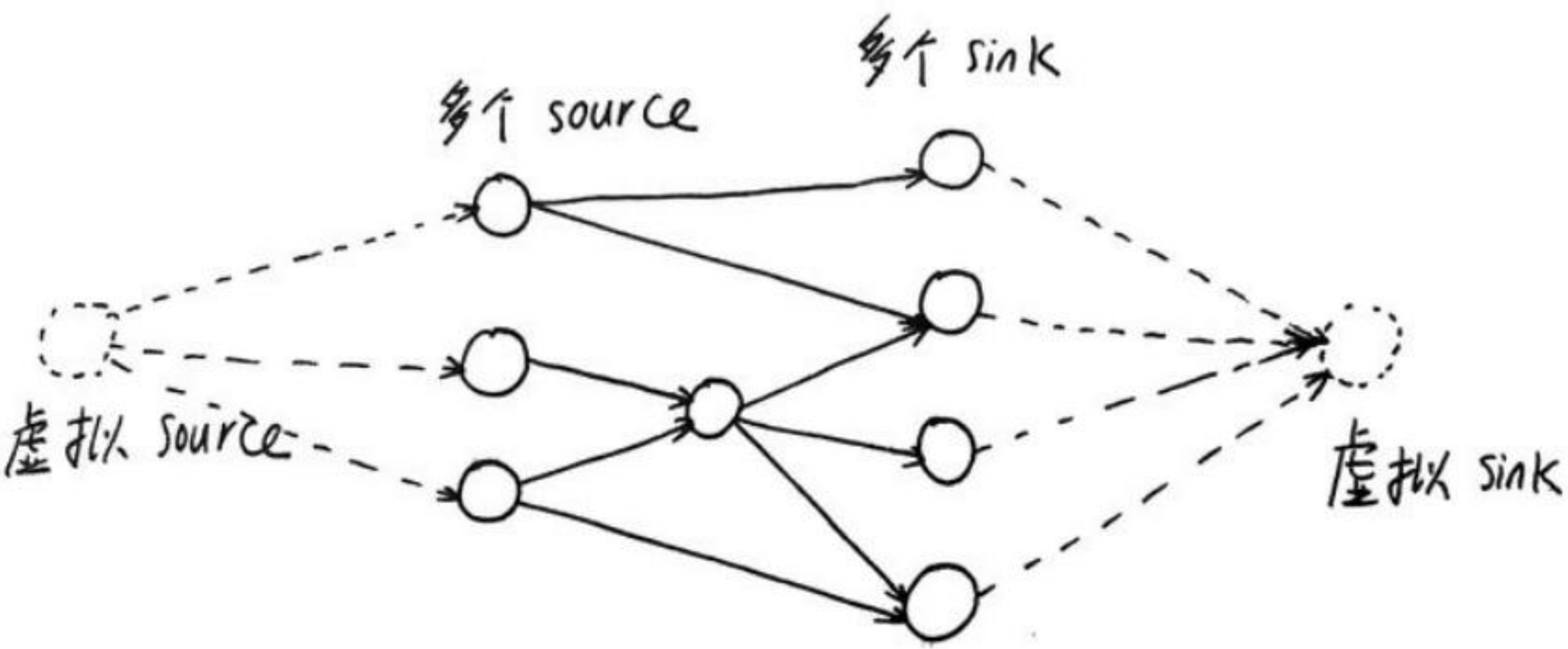
给一个带收发点的网络（一般收点用 v_t 表示，发点用 v_s 表示，其余为中间点），其每条弧的权值称之为容量，在不超过每条弧的容量的前提下，要求确定每条弧的流量，使得从发点到收点的流量最大。

请写出原问题和对偶问题，

并解释对偶变量的物理意义

最大流问题

*：发点和收点不唯一的情况用虚拟发/收点解决。示意图如下：



最大流问题

给一个带收发点的网络（一般收点用 v_t 表示，发点用 v_s 表示，其余为中间点），其每条弧的权值称之为容量，在不超过每条弧的容量的前提下，要求确定每条弧的流量，使得从发点到收点的流量最大。

$$\max v(f)$$

$$s.t.\quad 0 \le f_{ij} \le c_{ij} \qquad (v_i, v_j) \in A$$

$$\sum_{(v_i,v_j)\in A} f_{ij} - \sum_{(v_i,v_j)\in A} f_{ji} = \begin{cases} v(f) & i = s \\ 0 & i \ne s,t \\ -v(f) & i = t \end{cases}$$

最大流问题

$$
\begin{aligned}
&\text{maximize} && \sum_{v:(s,v)\in E} f(s,v) \\
&\text{subject to} && \\
& && \sum_{u:(u,v)\in E} f(u,v) = \sum_{w:(v,w)\in E} f(v,w) \quad && \forall v \in V - \{s,t\} \\
& && f(u,v) \leq c(u,v) && \forall (u,v) \in E \\
& && f(u,v) \geq 0 && \forall (u,v) \in E
\end{aligned}
$$

$$
\begin{aligned}
&\text{maximize} && \sum_{p\in P} x_p \\
&\text{subject to} && \\
& && \sum_{p\in P:(u,v)\in p} x_p \leq c(u,v) \quad && \forall (u,v) \in E \\
& && x_p \geq 0 && \forall p \in P
\end{aligned}
$$