运筹学

2. 线性规划的几何解释

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2.1. 基本知识点

凸集Convex Set

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球Ball、超平面Hyperplane、半空间Half Space都是凸集

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作为超平面、半空间的交集的可行域是多面体Polyhedron，多面体是凸集

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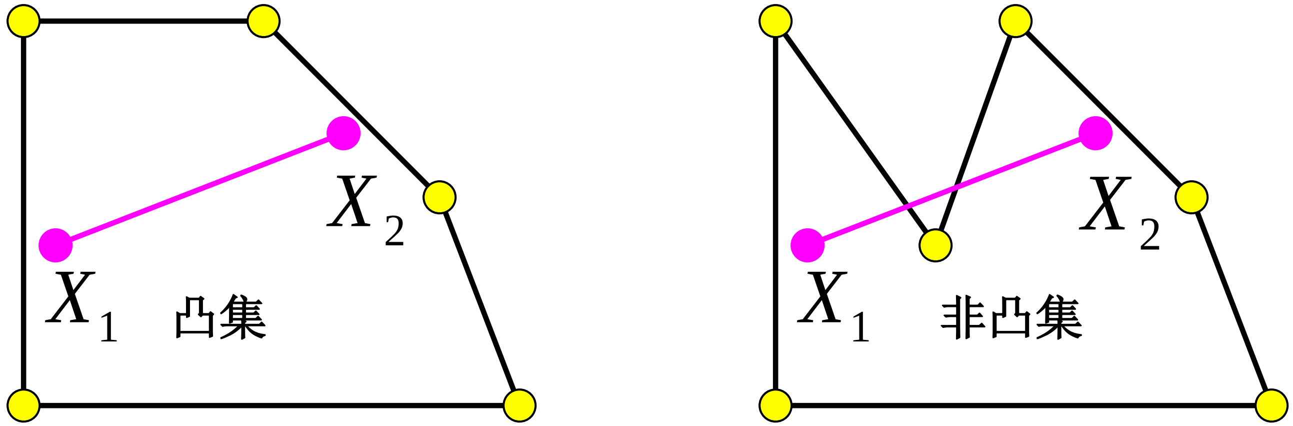
基于球的概念定义边界点Boundary Point，顶点Extreme Point(也称极点)

2.2. 凸集

凸集Convex Set：如果某个集合中任意两点连起来的直线都属于该集合，则称其为凸集，否则为非凸集



Ω是凸集的数学描述：对任意实数$0<\alpha<1$和任意的

$X_1, X_2 \in \Omega$ 均成立 $\alpha X_1 + (1-\alpha) X_2 \in \Omega$

球Ball

Definition

A *open ball* centered at a point $\boldsymbol{x}^*$ with radius $r \in \mathbb{R}^+$ is defined as

$$B_r(\boldsymbol{x}^*) = \{\boldsymbol{x} \mid |\boldsymbol{x} - \boldsymbol{x}^*|_2 < r\}$$

Definition

A *closed ball* centered at a point $\boldsymbol{x}^*$ with radius $r \in \mathbb{R}^+$ is defined as

$$\bar{B}_r(\boldsymbol{x}^*) = \{\boldsymbol{x} \mid |\boldsymbol{x} - \boldsymbol{x}^*|_2 \leq r\}$$

球Ball

Both open balls and closed balls are convex.
We only show that the statement holds for open balls, since the other proof is similar. Suppose there are two points $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ in a open ball $B_r(\boldsymbol{x}^*)$. For any $\lambda \in [0,1]$, we have

$$
\begin{aligned}
|(\lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2) - \boldsymbol{x}^*|_2 &= |\lambda(\boldsymbol{x}_1 - \boldsymbol{x}^*) + (1-\lambda)(\boldsymbol{x}_2 - \boldsymbol{x}^*)|_2 \\
&\leq \lambda|\boldsymbol{x}_1 - \boldsymbol{x}^*|_2 + (1-\lambda)|\boldsymbol{x}_2 - \boldsymbol{x}^*|_2 \\
&< \lambda r + (1-\lambda) r = r
\end{aligned}
$$

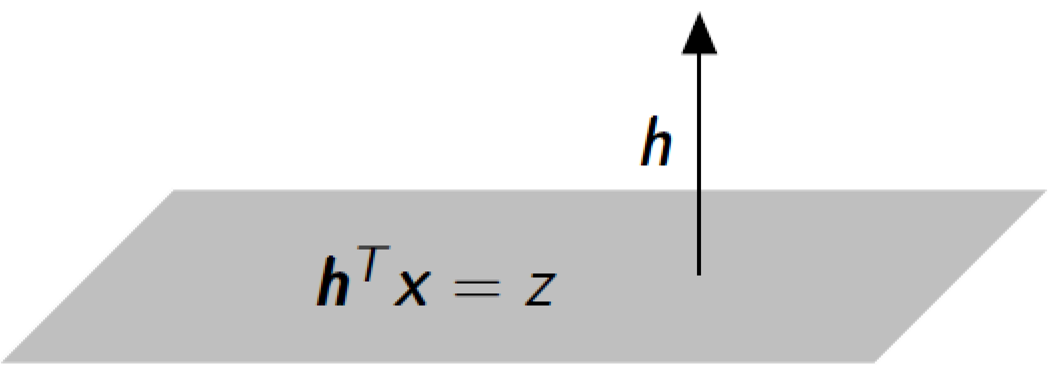
By definition, the open balls are convex.

超平面Hyperplane

A *hyperplane* in $\mathbb{R}^n$ is a set of all points satisfying $\{\boldsymbol{x} \mid \boldsymbol{h}^T\boldsymbol{x} = z\}$, where $\boldsymbol{x}$ is an n-dimensional column vector in $\mathbb{R}^n$, $\boldsymbol{h}$ is a non-zero n-dimensional column vector in $\mathbb{R}^n$ and $z \in \mathbb{R}$. $\boldsymbol{h}$ is said to be the *normal* of hyperplane $H = \{\boldsymbol{x} \mid \boldsymbol{h}^T\boldsymbol{x} = z \neq 0\}$.

Apparently, if and only if $z = 0$, the hyperplane $\{\boldsymbol{x} \mid \boldsymbol{h}^T\boldsymbol{x} = z\}$ passes through the origin.

Moreover, $\boldsymbol{h}$ is orthogonal to any vectors lying in the hyperplane H. Indeed, for $\boldsymbol{x}_1, \boldsymbol{x}_2 \in H$, we have $\boldsymbol{h}^T(\boldsymbol{x}_1 - \boldsymbol{x}_2) = z - z = 0$.



半空间Half Space

A *closed half-space* in $\mathbb{R}^n$ is the set of all points satisfying $\{\boldsymbol{x} \mid \boldsymbol{h}^T\boldsymbol{x} \leq z\}$, where $\boldsymbol{x}$ is an n-dimensional column vector in $\mathbb{R}^n$, $\boldsymbol{h}$ is an n-dimensional column vector in $\mathbb{R}^n$ and $z \in \mathbb{R}$.

A *open half-space* in $\mathbb{R}^n$ is the set of all points satisfying $\{\boldsymbol{x} \mid \boldsymbol{h}^T\boldsymbol{x} < z\}$, where $\boldsymbol{x}$ is an n-dimensional column vector in $\mathbb{R}^n$, $\boldsymbol{h}$ is an n-dimensional column vector in $\mathbb{R}^n$ and $z \in \mathbb{R}$.

Some literatures use $\geq$ $(>)$ instead of $\leq$ $(<)$, but their meaning are indeed the same.

半空间Half Space

1. *Any a hyperplane is convex.*
2. *The closed half-space* $\{\mathbf{x} \mid \boldsymbol{h}^T\mathbf{x} \le z\}$ *and* $\{\mathbf{x} \mid \boldsymbol{h}^T\mathbf{x} \ge z\}$ *are convex.*
3. *The open half-space* $\{\mathbf{x} \mid \boldsymbol{h}^T\mathbf{x} < z\}$ *and* $\{\mathbf{x} \mid \boldsymbol{h}^T\mathbf{x} > z\}$ *are convex.*

For statement 2, let $\Omega = \{\mathbf{x} \mid \boldsymbol{h}^T\mathbf{x} \ge z\}$ be the studied half-space. For all $\mathbf{x}_1, \mathbf{x}_2 \in \Omega$ and $\lambda \in [0, 1]$, we have

$$\boldsymbol{h}^T[\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2] = \lambda\boldsymbol{h}^T\mathbf{x}_1 + (1-\lambda)\boldsymbol{h}^T\mathbf{x}_2 \ge \lambda z + (1-\lambda)z = z. \tag{4}$$

So $\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 \in \Omega$, which means Ω is convex. Similarly we can prove the other half-space is convex.

多面体Polyhedron

The intersection of convex sets is still convex.

Let us define the intersection of k convex sets as $\Omega = \bigcap_{i=1,\ldots,k} \Omega_i$. For any $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \Omega_i$, $i = 1, \ldots, k$, we have $\lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2 \in \Omega_i$, $i = 1, \ldots, k$. So, $\lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2 \in \Omega$. Therefore, Ω is convex.

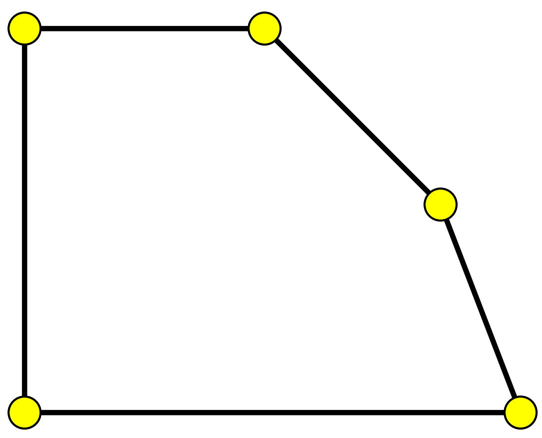
A *polyhedron* is defined as the solution set of a finite number of linear equalities and inequalities:
$\Omega = \left\{ \boldsymbol{x} \mid \boldsymbol{a}_i^T \boldsymbol{x} \leq b_i, i = 1, \ldots, m, \boldsymbol{c}_j^T \boldsymbol{x} = d_j, j = 1, \ldots, p \right\}$. A polyhedron is thus the intersection of a finite number of halfspaces and hyperplanes.

2.2. 凸集

大前提：线性规划问题的约束条件为若干个线性等式或者不等式，而这些集合都是凸集

小前提：凸集的交集也是凸集

结论：作为这些凸集交集的线性规划问题定义域也是凸集（我们一般约定空集为凸集）



边界点Boundary Point

Let Ω be a subset of $\mathbb{R}^n$. A point $\boldsymbol{x}$ is a *boundary point* of Ω if every open ball centered at $\boldsymbol{x}$ contains both a point in Ω and a point in $\mathbb{R}^n - \Omega$. The set of all boundary points of Ω, denoted by $\partial\Omega$, is the *boundary* of Ω.

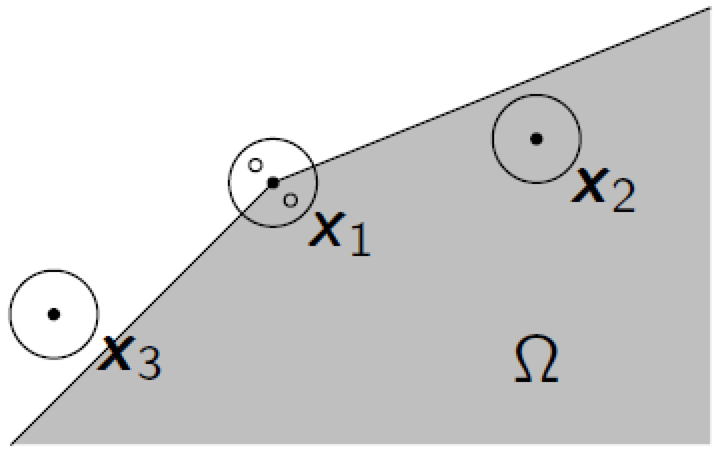


Figure: An illustration of the boundary points, where $\boldsymbol{x}_1$ is a boundary point and $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ are not.

顶点Extreme Point（也称极点）

A point $\boldsymbol{x}$ is an *extreme point* of a convex set C if there exist no two distinct points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2 \in C$ such that $\boldsymbol{x} = \lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2$ for some $\lambda \in (0,1)$.

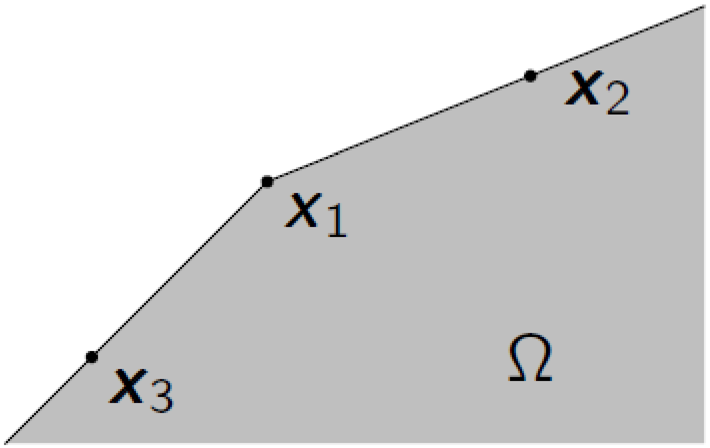
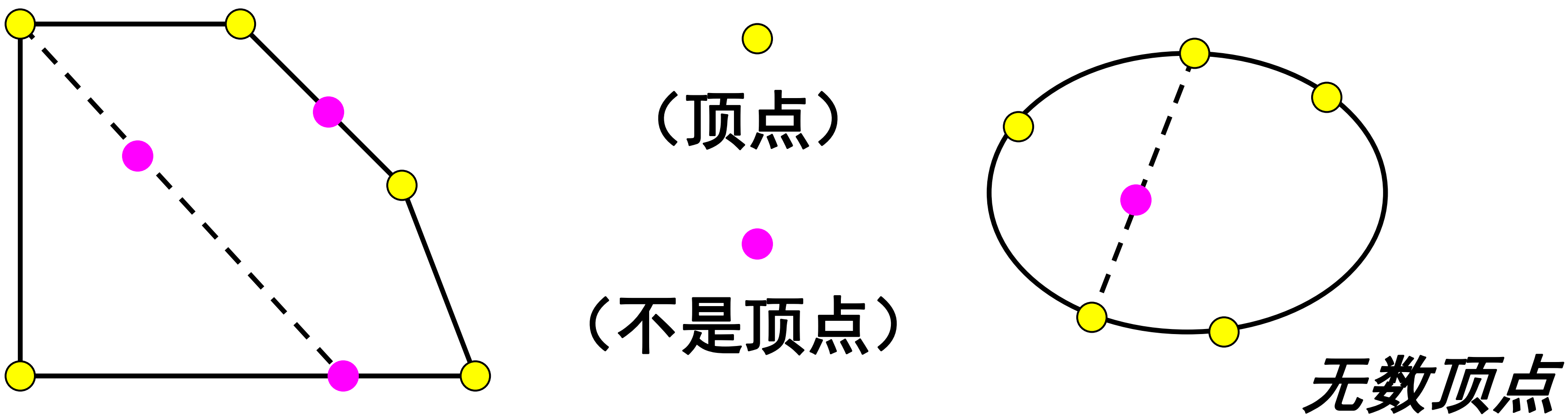


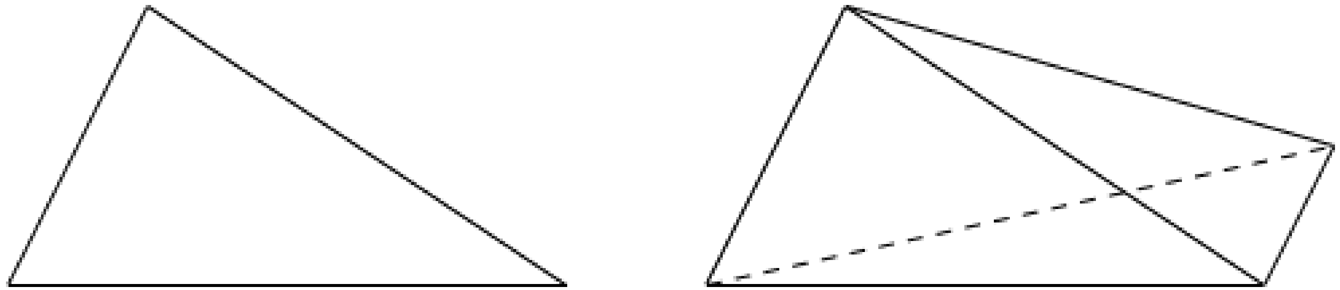
Figure: An illustration of the extreme points, where $\boldsymbol{x}_1$ is an extreme point and $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ are not.

2.2. 凸集

顶点Extreme Point，如果凸集内的一点不在凸集内任何不同的两点的连线上，则称该点为该凸集的顶点



单纯形Simplex：We call the convex hull of any set of n+1 points in Rn which do not lie on a hyperplane a simplex.



A simplex in $\mathbb{R}^2$ and a simplex in $\mathbb{R}^3$.

极线Ray

For a nonempty polyhedron $\Omega = \{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} \geq \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\}$, the recession cone is the set of all vectors $\boldsymbol{d}$ satisfying

$$A\boldsymbol{d} = \boldsymbol{0},\ \boldsymbol{d} \geq \boldsymbol{0}$$

There may exist infinite number of rays in a polyhedron. To specify them using a finite number of elements, we need to further consider the so called extreme rays.

Definition . The nonzero elements of the recession cone are called the *rays* of the polyhedron Ω. A nonzero element $\boldsymbol{x}$ of a polyhedral cone $\Omega \subset \mathbb{R}^n$ is called an *extreme ray* of Ω, if there are $n-1$ linearly independent constraints that are active (that is, are equalities) at $\boldsymbol{x}$.

It should be pointed out that a positive multiple of an extreme ray is also an extreme ray. So, we say two extreme rays are non-equivalent if one is not a positive multiple of the other; otherwise, we call them equivalent. A *complete set of extreme rays* is a collection of extreme rays that contains exactly one form for each set of equivalent extreme rays.