运筹学

3. 线性规划的代数观点

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线性规划的表述形式

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线性多变量方程组的基本解Basic Solution

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线性规划问题的基本可行解Basic Feasible Solution，对应可行域多面体的顶点Extreme Point

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将问题的目标函数和约束条件用基本可行解表述

↓

单纯形法在顶点间转移的过程中，需要借助基本可行解的性质来保证转移的可行性

3.1. 基本知识点

线性规划问题的表述形式有较大的灵活性

1. 目标函数可以max，也可以min

2. 约束条件可以是大于等于，小于等于，也可以是等于

3. 决策变量可以是非负的，也可以无此要求

为了方便证明线性规划的性质，我们需要建立一些标准形式的线性规划问题；同时，我们还需要考虑如何将其它形式的线性规划问题转化为标准形式

线性规划问题的标准形式Standard Form

目标函数是max是min都行

$$
\begin{array}{lll}
\text{minimize} & c_1x_1 + c_2x_2 + \ldots + c_nx_n & \\
\text{subject to} & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & = b_1 \\
& a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & = b_2 \\
& \vdots & \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = b_m \\
\text{and} & x_1 \geqslant 0,\ x_2 \geqslant 0,\ \ldots,\ x_n \geqslant 0, &
\end{array}
$$

有些教材约定b非负

where the b_i's, c_i's and a_{ij}'s are fixed real constants, and the x_i's are real numbers to be determined. In more compact vector notation, this standard problem becomes

$$
\begin{array}{l}
\text{minimize} \quad \mathbf{c}^T\mathbf{x} \\
\text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{b} \quad \text{and} \quad \mathbf{x} \geqslant \mathbf{0}.
\end{array}
$$

Here $\mathbf{x}$ is an n-dimensional column vector, $\mathbf{c}^T$ is an n-dimensional row vector, $\mathbf{A}$ is an $m \times n$ matrix, and $\mathbf{b}$ is an m-dimensional column vector. The vector inequality $\mathbf{x} \geqslant \mathbf{0}$ means that each component of $\mathbf{x}$ is nonnegative.

线性规划问题的规范形式Canonical Form

$$\min \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j \geq b_i, \quad \forall 1 \leq i \leq m$$

$$x_j \geq 0, \quad \forall 1 \leq j \leq n$$

$$\Leftrightarrow$$

$$\min \vec{c}^T X$$

$$\text{s.t.} \quad AX \geq \vec{b}$$

$$X \geq 0$$

A minimization LPP with objective function min $\sum_{i=1}^{n} c_i x_i$ is equivalent to a maximization LPP with with objective function max $\sum_{i=1}^{n} -c_i x_i$; and vice verse.

线性规划问题的其它形式，与标准形式Standard Form和规范形式Canonical Form之间的转换方法

$$a_{i1}x_1+\cdots+a_{in}x_n\ge b_i \quad\Leftrightarrow\quad \begin{aligned}&a_{i1}x_1+\cdots+a_{in}x_n-x_{n+1}=b_i\\&x_{n+1}\ge 0\end{aligned}$$

$$a_{i1}x_1+\cdots+a_{in}x_n=b_i \quad\Leftrightarrow\quad \begin{aligned}&a_{i1}x_1+\cdots+a_{in}x_n\ge b_i\\&-a_{i1}x_1-\cdots-a_{in}x_n\ge -b_i\end{aligned}$$

$$\begin{aligned}&a_{i1}x_1+\cdots+a_{in}x_n\ge b_i\\&\infty>x_1>-\infty\end{aligned} \quad\Leftrightarrow\quad \begin{aligned}&a_{i1}\left(x_1^+-x_1^-\right)+\cdots+a_{in}x_n\ge b_i\\&x_1^+\ge 0,\ x_1^-\ge 0\end{aligned}$$

3.3. 线性方程组的基本解

为了解线性规划问题的解的特性，我们首先回忆线性方程组的基本解Basic Solution

Generally, $A\boldsymbol{x} = \boldsymbol{b}$ behaves in three possible ways: 1) The system has no solution; 2) The system has a single unique solution; 3) The system has infinitely many solutions.

studying the *augmented matrix* A_b constructed from A and $\boldsymbol{b}$

$$A_b = [A, \boldsymbol{b}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n, \boldsymbol{b}]$$

where $\boldsymbol{a}_i$ is the ith column of matrix A.

线性方程组的基本解Basic Solution

1) when $\mathrm{rank}(A) < \mathrm{rank}(A_b)$, the columns of A and $\boldsymbol{b}$ are linearly independent. In other words, system $A\boldsymbol{x} = \boldsymbol{b}$ has no solution $\boldsymbol{x}$ that satisfies

$$\sum_{i=1}^{n} x_i \boldsymbol{a}_i = \boldsymbol{b}$$

Since $\mathrm{rank}(\boldsymbol{b}) = 1$, we have $\mathrm{rank}(A_b) = \mathrm{rank}(A) + 1$.

2) when $\mathrm{rank}(A) = \mathrm{rank}(A_b)$, the columns of A and $\boldsymbol{b}$ are linearly dependent. In other words, system $A\boldsymbol{x} = \boldsymbol{b}$ has at least one solution $\boldsymbol{x}$

2.1) when $m = n = \mathrm{rank}(A)$, the solution is unique

$$\boldsymbol{x} = A^{-1}\boldsymbol{b}$$

2.2) when $m = \mathrm{rank}(A) < n$, we have more than one solutions. Indeed, suppose $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are two different solutions say $\boldsymbol{x}_1 \neq \boldsymbol{x}_2$. For any $\lambda \in [0, 1]$, we have

$$A\left[\lambda\boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2\right] = \lambda A\boldsymbol{x}_1 + (1-\lambda)A\boldsymbol{x}_2 = \lambda\boldsymbol{b} + (1-\lambda)\boldsymbol{b} = \boldsymbol{b}$$

which indicates that $\lambda\boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2$ is also a solution of $A\boldsymbol{x} = \boldsymbol{b}$. This proves that $A\boldsymbol{x} = \boldsymbol{b}$ has an infinite number of solutions.

3.3. 线性方程组的基本解

Suppose $\operatorname{rank}(A) = m < n$, in order to further describe the solutions of $A\boldsymbol{x} = \boldsymbol{b}$, we rewrite it as

$$B\boldsymbol{x}_B + N\boldsymbol{x}_N = \begin{bmatrix} B & N \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_B \\ \boldsymbol{x}_N \end{bmatrix} = \boldsymbol{b}$$

where B is the selected basis matrix and N is the associated nonbasis matrix

$$B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m],$$

$$N = \begin{bmatrix} a_{1m+1} & a_{1m+2} & \cdots & a_{1n} \\ a_{2m+1} & a_{2m+2} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{mm+1} & a_{mm+2} & \cdots & a_{mn} \end{bmatrix} = [\boldsymbol{a}_{m+1}, \boldsymbol{a}_{m+2}, \ldots, \boldsymbol{a}_n]$$

Since $\mathrm{rank}(A) = m$, we can always reorder x_i to find an B that is not singular. Then, we have

$$\boldsymbol{x}_B = B^{-1}(\boldsymbol{b} - N\boldsymbol{x}_N) = B^{-1}\boldsymbol{b} - B^{-1}N\boldsymbol{x}_N$$

Given any $\boldsymbol{x}_N$, we can always uniquely solve $A\boldsymbol{x} = \boldsymbol{b}$ as

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_B \\ \boldsymbol{x}_N \end{bmatrix} = \begin{bmatrix} B^{-1}\boldsymbol{b} - B^{-1}N\boldsymbol{x}_N \\ \boldsymbol{x}_N \end{bmatrix}$$

标准模型顶点的**数学描述**

标准模型可行集 $\Omega = \left\{ Y, Y \in R^n \middle| \sum_{j=1}^{n} P_j y_j = \vec{b},\ Y \geq 0 \right\}$

其中 $A = (P_1\ P_2 \cdots P_n),\ P_j \in R^m, \forall j = 1, 2, \cdots, n$

对任意 $X \in \Omega$ 可进行如下划分

$$x_j > 0,\ \ j = k(1), \cdots, k(\hat{m}),\quad x_j = 0,\ \ j = k(\hat{m}+1), \cdots, k(n)$$

当且仅当 $\sum_{t=1}^{\hat{m}} P_{j(t)} y_t = \vec{b}$ 的**解唯一**时，X 是顶点

标准模型顶点的**等价数学描述之一**

如果把 $X \in \Omega = \left\{ Y, Y \in R^n \left| \sum_{j=1}^{n} P_j y_j = \vec{b},\ Y \geq 0 \right. \right\}$ 的非零分量

称为正分量，那么任何可行解是顶点的充要条件为：

<u>其正分量对应的系数向量（ P_j ）线性无关</u>

即，如果 $X \in \Omega$ 划分为

$$x_j > 0,\ \ j = k(1), \cdots, k(\hat{m}),\ \ \ x_j = 0,\ \ j = k(\hat{m}+1), \cdots, k(n)$$

其为顶点的充要条件是 $P_{k(t)}, t = 1, 2, \cdots, m$ 线性无关

标准模型顶点的等价数学描述之二

如果 $(P_1,\cdots,P_n)$ **是行满秩矩阵，那么** $X=(x_1,\cdots,x_n)^T$ **是**

$$\Omega=\left\{Y,Y\in R^n\,\middle|\,\sum_{j=1}^{n}P_j y_j=\vec{b},Y\geq 0\right\}$$

顶点的充要条件是：存在 $(k(1),\cdots,k(m))\subseteq(1,\cdots,n)$ **满足**

$$\begin{pmatrix} x_{k(1)} \\ \vdots \\ x_{k(m)} \end{pmatrix}=\left(P_{k(1)},\cdots,P_{k(m)}\right)^{-1}\vec{b}\geq 0,\quad x_{k(j)}=0,\forall m+1\leq j\leq n$$

（注意：前面结论中的 $\hat{m}$ **可以小于** m **）**

称满足 $\left(P_{k(1)},\cdots,P_{k(m)}\right)^{-1}\vec{b}\geq 0$ 的矩阵为<u>可行基矩阵</u>

关于标准模型顶点的两点说明

1）假定标准模型 $(P_1,\cdots,P_n)$ 是行满秩矩阵不失一般性

不满足该假定只有以下两种可能：

无可行解（不用考虑）　　或　　有多余约束（删除）

2）给定可行基矩阵可唯一确定一个顶点，反之不一定

若给定顶点有 m 个非零分量（非退化顶点），只有一个可行基矩阵可确定该顶点，否则（称为退化顶点），可能有多个可行基矩阵确定同一个顶点

3. 3. 线性方程组的基本解

线性规划标准形式的基矩阵、基本解和基本可行解

称可逆矩阵 $\left(P_{k(1)},\cdots,P_{k(m)}\right)$ 为基矩阵

称其分量由下式决定的 X 为基本解

$$\begin{pmatrix} x_{k(1)} \\ \vdots \\ x_{k(m)} \end{pmatrix} = \left(P_{k(1)},\cdots,P_{k(m)}\right)^{-1}\vec{b},\ \ x_{k(j)} = 0,\ \forall m+1 \le j \le n$$

称可行基本解为基本可行解

称基矩阵对应变量为基变量，其余变量为非基变量

标准线性规划的基本可行解就是可行集的顶点

标准线性规划的可行集的顶点个数总是有限的

Clearly, the number of basic solutions is equivalent to the number of nonsingular matrices B that can possibly be formed from A. This number is obviously not bigger than $C_n^m = \frac{n!}{m!(n-m)!}$.

线性规划问题的**基本可行解Basic Feasible Solution**是针对**标准形式Standard Form**而言。和线性方程组的差别在于多了约束 $X \geq 0$

题1

$$\max \quad z = x_1 + x_2 + x_3 + x_4$$

$$\text{s.t.} \quad \begin{pmatrix} 1 \\ 2 \end{pmatrix} x_1 + \begin{pmatrix} 2 \\ 2 \end{pmatrix} x_2 + \begin{pmatrix} 3 \\ 1 \end{pmatrix} x_3 + \begin{pmatrix} 4 \\ 2 \end{pmatrix} x_4 = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$$

$$x_j \geq 0, \forall 1 \leq j \leq 4$$

该问题至多有下面 $C_4^2 = 6$ 个可能的求顶点的矩阵

$$\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$$

对每个矩阵 B ，计算 $B^{-1}\vec{b}$ ，可得

$$\begin{pmatrix}1&2\\2&2\end{pmatrix}^{-1}\begin{pmatrix}7\\3\end{pmatrix}=\begin{pmatrix}-4\\5.5\end{pmatrix},\quad \boxed{\begin{pmatrix}1&3\\2&1\end{pmatrix}^{-1}\begin{pmatrix}7\\3\end{pmatrix}=\begin{pmatrix}0.4\\2.2\end{pmatrix}}$$

$$\begin{pmatrix}1&4\\2&2\end{pmatrix}^{-1}\begin{pmatrix}7\\3\end{pmatrix}=\begin{pmatrix}-1/3\\11/6\end{pmatrix},\quad \boxed{\begin{pmatrix}2&3\\2&1\end{pmatrix}^{-1}\begin{pmatrix}7\\3\end{pmatrix}=\begin{pmatrix}0.5\\2\end{pmatrix}}$$

$$\begin{pmatrix}2&4\\2&2\end{pmatrix}^{-1}\begin{pmatrix}7\\3\end{pmatrix}=\begin{pmatrix}-0.5\\2\end{pmatrix},\quad \boxed{\begin{pmatrix}3&4\\1&2\end{pmatrix}^{-1}\begin{pmatrix}7\\3\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}}$$

该线性规划只有3个顶点（满足 $B^{-1}\vec{b}\geq 0$ ），很容易判断其中谁是最优解

3. 4. 线性规划问题的基本定理

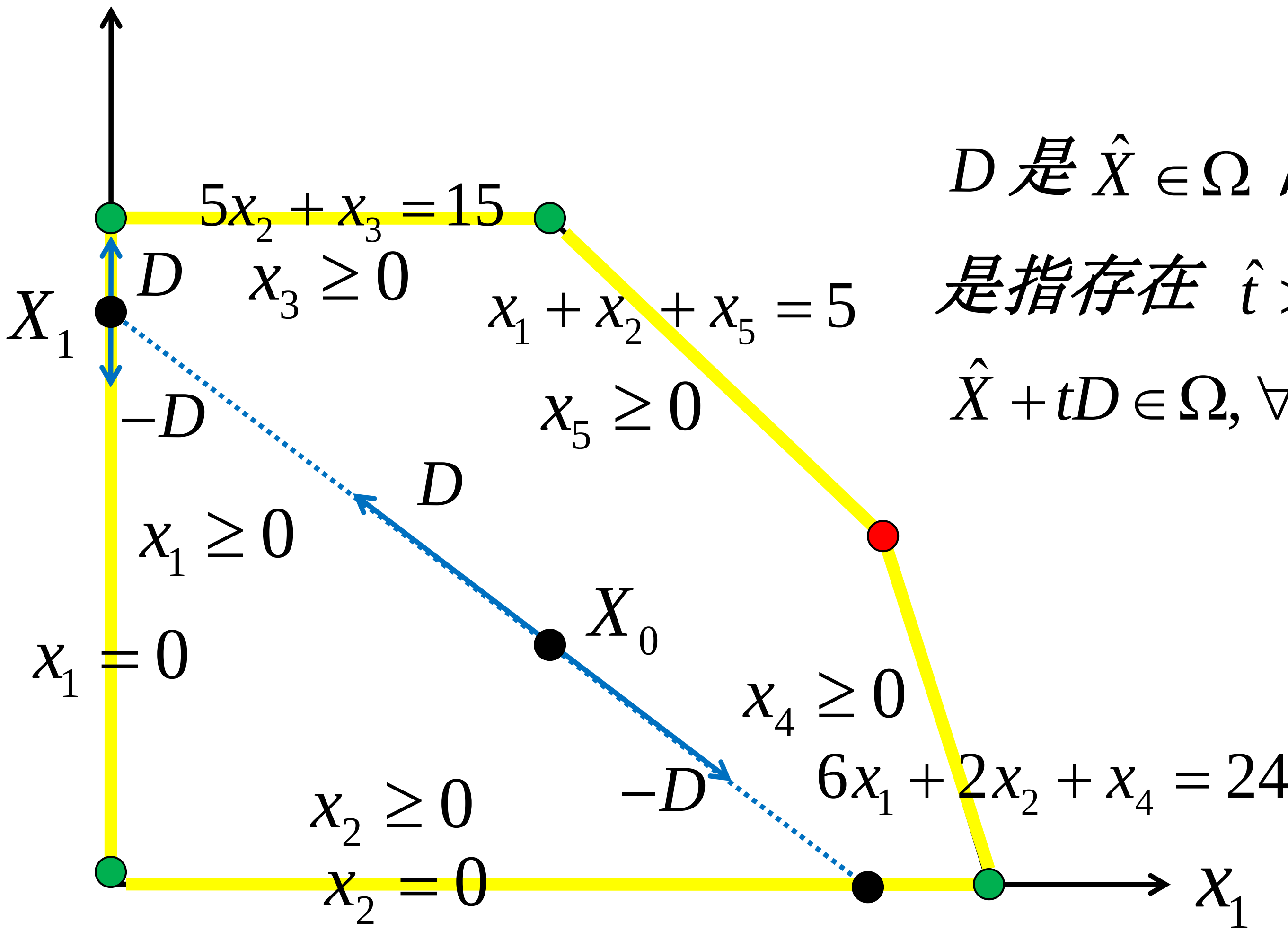
线性规划问题的基本定理

1. 一个标准模型的线性规划问题若有可行解，则至少存在一个基本可行解

2. 一个标准模型的线性规划问题若有有限的最优目标值，则一定存在一个基本可行解是最优解

3. 4.　线性规划问题的基本定理

1.　一个标准模型的线性规划问题若有可行解，则至少存在一个基本可行解

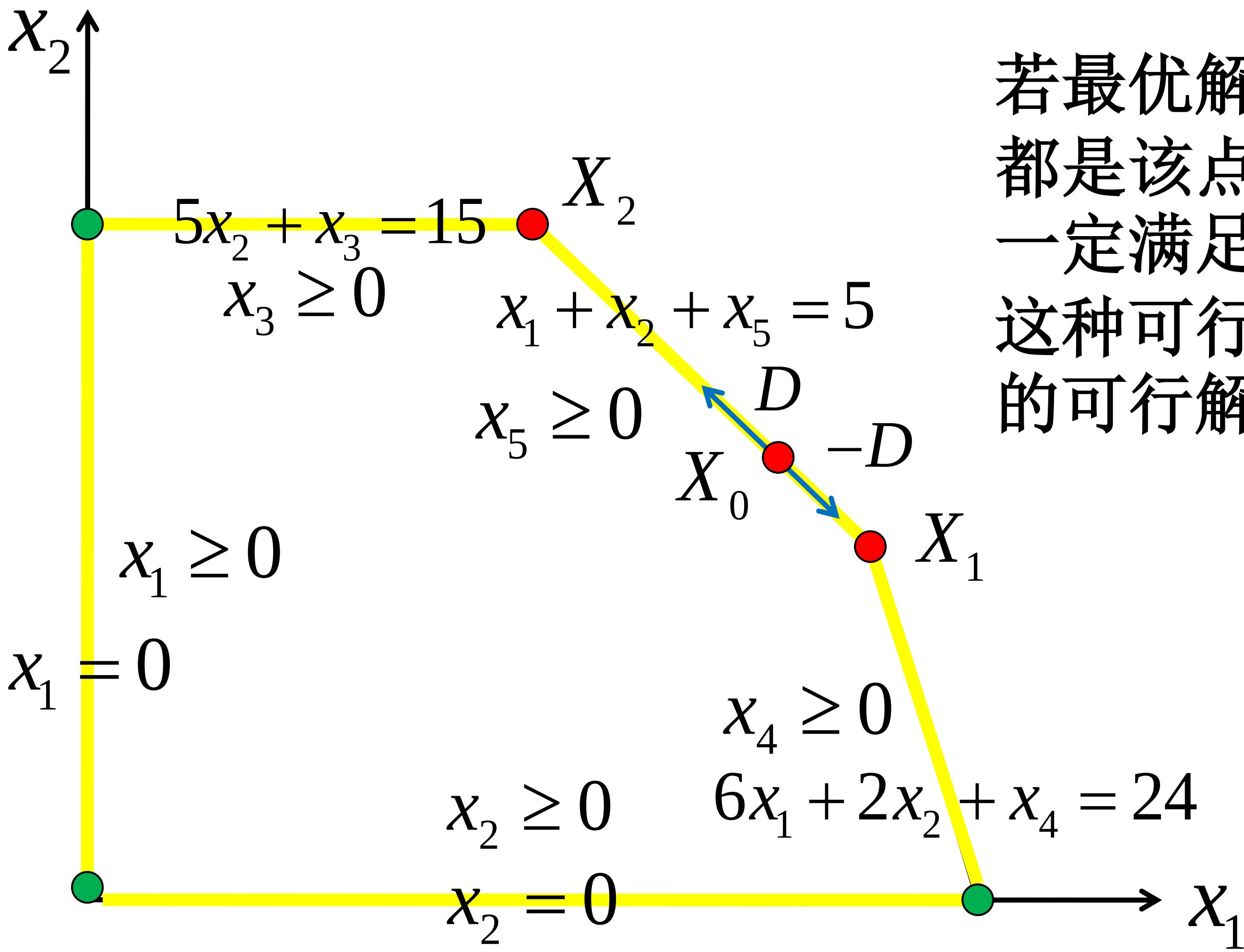


D 是 $\hat{X} \in \Omega$ 的可行方向

是指存在 $\hat{t} > 0$ 满足

$\hat{X} + tD \in \Omega, \forall t \in \left[0, \hat{t}\right]$

2. 一个标准模型的线性规划问题若有有限的最优目标值，则一定存在一个基本可行解是最优解



若最优解不是顶点，$\pm D$ 都是该点的可行方向，一定满足 $C^T D=0$，沿这种可行方向前进得到的可行解都是最优解

3. 4.　线性规划问题的基本定理

线性规划问题的基本定理

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2.　一个标准模型的线性规划问题若有有限的最优目标值，则一定存在一个基本可行解是最优解

只要可行解不是顶点，一定有 $\pm D$ 都是可行方向，这个结果对任何线性规划模型（实际上对任何可行集是凸集的优化模型）都成立。但获得上述基本定理还需要用到所有变量的非负约束。

3. 4. 线性规划问题的基本定理

一个标准模型的线性规划问题若有可行解，则至少存在一个基本可行解

Assume that there is a feasible solution $\boldsymbol{x}$ with p positive variables, $p \leq n$. For convenience, Let us reorder the variables so that the first p variables are positive. Then the feasible solution can be written as $\boldsymbol{x} = [x_1, \ldots, x_p, 0, \ldots, 0]^T$, such that

$$\sum_{j=1}^{p} x_j \boldsymbol{a}_j = \boldsymbol{b}$$

3. 4. 线性规划问题的基本定理

We can category the solutions according to the first p column vectors $\{\boldsymbol{a}_j\}_{j=1}^p$.

If $\{\boldsymbol{a}_j\}_{j=1}^p$ is linearly independent, we have $p \leq m$, because m is the largest number of linearly independent vectors in A. If $p = m$, the solution is basic by definition and non-degenerate. If $p < m$, there exist $\boldsymbol{a}_{p+1}, \ldots, \boldsymbol{a}_m$ such that the set $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$ is linearly independent. Since $x_{p+1}, \ldots, x_m$ are all zero, it follows that

$$\sum_{j=1}^{m} x_j \boldsymbol{a}_j = \sum_{j=1}^{p} x_j \boldsymbol{a}_j = \boldsymbol{b}$$

So the solution is basic and degenerate.

If $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_p\}$ is linearly dependent. Without loss of generality, we assume that $\boldsymbol{a}_j \neq \mathbf{0}$. Otherwise, we can set x_j to zeros and hence reduce p by 1, since $\boldsymbol{a}_j = \mathbf{0}$. Under this assumption, there exists $\{\alpha_j\}_{j=1}^{p}$ not all zero such that

$$\sum_{j=1}^{p} \alpha_j \boldsymbol{a}_j = \mathbf{0}$$

Let $\alpha_r \neq 0$, we have

$$\boldsymbol{a}_r = \sum_{\substack{j=1 \\ j \neq r}}^{p} \left(-\frac{\alpha_j \boldsymbol{a}_j}{\alpha_r} \right)$$

Substitute this into Eq. , we have

$$\sum_{\substack{j=1 \\ j\neq r}}^{p}\left(x_j - x_r\frac{\alpha_j}{\alpha_r}\right)\boldsymbol{a}_j = \boldsymbol{b}$$

Hence we have a new vector

$$\hat{\boldsymbol{x}} = \left[x_1 - x_r\frac{\alpha_1}{\alpha_r}, \cdots, x_{r-1} - x_r\frac{\alpha_{r-1}}{\alpha_r}, 0, x_{r+1} - x_r\frac{\alpha_{r+1}}{\alpha_r}, \cdots, x_p - x_r\frac{\alpha_p}{\alpha_r}, 0, \cdots, 0\right]^T$$

which has no more than $(p-1)$ non-zero variables.

3. 4. 线性规划问题的基本定理

Next, we will show that by appropriately choosing α_r, $\hat{\mathbf{x}}$ above is still a feasible solution to the LPP. To reach this goal, we need to prove that $\hat{\mathbf{x}} \geq \mathbf{0}$.

In order to do that, we choose our α_r such that

$$x_j - x_r \frac{\alpha_j}{\alpha_r} \geq 0, \ j = 1, \ldots, p$$

For those $\alpha_j = 0$, Ineq. obviously holds as $x_j \geq 0$ for all $j = 1, 2, \ldots, p$. For those $\alpha_j \neq 0$, the inequality becomes

$$\frac{x_j}{\alpha_j} - \frac{x_r}{\alpha_r} \geq 0, \ \text{for } \alpha_j > 0$$

$$\frac{x_j}{\alpha_j} - \frac{x_r}{\alpha_r} \leq 0, \ \text{for } \alpha_j < 0$$

If we choose our $\alpha_r > 0$, Ineq. will automatically hold.
Moreover if α_r is chosen as

$$\frac{x_r}{\alpha_r} = \min_j \left\{ \frac{x_j}{\alpha_j} : \alpha_j > 0 \right\}$$

Ineq. will also be satisfied.
Thus, by choosing α_r appropriately as in Eq.(19), we can make Ineq. hold.
So, $\hat{\mathbf{x}}$ is a feasible solution, which has no more than $p-1$ non-zero variables.

We now check whether the corresponding $p-1$ column vectors of $\{\boldsymbol{a}_j\}_{j=1}^{p}$ are linearly independent or not. If it is, then we have a basic feasible solution. If it is not, we repeat the above process to reduce the number of non-zero variables in a feasible solution to $p-2$.

Since p is finite, such a process must stop after at most $p-1$ operations, at which we only have one non-zero variable. The corresponding column of A is clearly linearly independent. Then, the finally found $\boldsymbol{x}$ is a basic feasible solution of the original LPP.

构造法+递降法

3.4. 线性规划问题的基本定理

The basic feasible solutions of a standard form LPP are extreme points of the corresponding feasible region.

Proof.

Suppose $\boldsymbol{x}$ is a basic feasible solution. Without loss of generality, we assume that $\boldsymbol{x}$ has the form $\boldsymbol{x} = \begin{bmatrix} \mathbf{x}_\alpha \\ \mathbf{0} \end{bmatrix}$, where $\boldsymbol{x}_\alpha = A_\alpha^{-1}\boldsymbol{b}$ is an $m \times 1$ vector.

On the contrary, suppose there exist two feasible solutions $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, different from $\boldsymbol{x}$, such that $\boldsymbol{x} = \lambda\boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2$ for certain $\lambda \in (0,1)$. We then have

$$\boldsymbol{x}_1 = \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{v}_1 \end{bmatrix}, \quad \boldsymbol{x}_2 = \begin{bmatrix} \boldsymbol{u}_2 \\ \boldsymbol{v}_2 \end{bmatrix}$$

where $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ are m-vectors and $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ are $(n-m)$-vectors.

So, we have

$$\mathbf{0} = \lambda \boldsymbol{v}_1 + (1-\lambda)\boldsymbol{v}_2.$$

As $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ are feasible, $\boldsymbol{v}_1$, $\boldsymbol{v}_2 \geq \mathbf{0}$. Since $\lambda, (1-\lambda) > 0$, this means $\boldsymbol{v}_1 = \mathbf{0}$ and $\boldsymbol{v}_2 = \mathbf{0}$. Moreover, we have

$$\boldsymbol{b} = A\boldsymbol{x}_1 = [A_\alpha, A_\beta] \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{v}_1 \end{bmatrix} = A_\alpha \boldsymbol{u}_1$$

and similarly, we have $\boldsymbol{b} = A\boldsymbol{x}_2 = A_\alpha \boldsymbol{u}_2$ and

$$A_\alpha \boldsymbol{u}_1 = A_\alpha \boldsymbol{u}_2 = \boldsymbol{b} = A_\alpha \boldsymbol{x}_\alpha$$

Since A_α is non-singular, this implies that $\boldsymbol{u}_1 = A_\alpha^{-1} A_\alpha \boldsymbol{u}_2 = \boldsymbol{x}_\alpha$. Hence, we have $\boldsymbol{x} = \boldsymbol{x}_1 = \boldsymbol{x}_2$, which is a contradiction. So, $\boldsymbol{x}$ must be an extreme point of the feasible region.

Conversely, the extreme points of a feasible region are basic feasible solutions of the corresponding standard form LPP

Proof.

Suppose $\boldsymbol{x}_0 = [x_1, \ldots, x_n]^T$ is an extreme point of feasible region. Assume that there are r components of $\boldsymbol{x}_0$ which are non-zero. Without loss of generality, we assume $x_i > 0$ for $i = 1, \ldots, r$ and $x_i = 0$ for $i = r + 1, \ldots, n$. Then, we have

$$\sum_{i=1}^{r} x_i \boldsymbol{a}_i = \boldsymbol{b}$$

We first prove that $\{\mathbf{a}_1, \dots, \mathbf{a}_r\}$ is linearly independent. Suppose on contrary that there exist α_i, $i = 1, 2, \dots, r$, not all zero, such that

$$\sum_{i=1}^{r} \alpha_i \mathbf{a}_i = \mathbf{0}$$

Let ϵ be such that $0 < \epsilon < \min_{\alpha_i \neq 0} \frac{x_i}{|\alpha_i|}$, we have

$$x_i \pm \epsilon \cdot \alpha_i > 0, \ \forall i = 1, \dots, r$$

We can show that $\boldsymbol{x}_1 = \boldsymbol{x}_0 + \epsilon \cdot \boldsymbol{\alpha}$ and $\boldsymbol{x}_2 = \boldsymbol{x}_0 - \epsilon \cdot \boldsymbol{\alpha}$ are feasible solutions, where $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_r, 0, \ldots, 0]^T$. Clearly, $\boldsymbol{x}_1, \boldsymbol{x}_2 \geq \mathbf{0}$. Since $\sum_{i=1}^{r} \alpha_i \boldsymbol{a}_i = \mathbf{0}$, we have

$$A\boldsymbol{x}_1 = A\boldsymbol{x}_0 + \epsilon A\boldsymbol{\alpha} = A\boldsymbol{x}_0 + \mathbf{0} = \boldsymbol{b},\ A\boldsymbol{x}_2 = A\boldsymbol{x}_0 - \epsilon A\boldsymbol{\alpha} = A\boldsymbol{x}_0 - \mathbf{0} = \boldsymbol{b}$$

Therefore, $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are also a feasible solutions. Since $\boldsymbol{x}_0 = \frac{1}{2}\boldsymbol{x}_1 + \frac{1}{2}\boldsymbol{x}_2$, $\boldsymbol{x}_0$ is not an extreme point. This leads to a contradiction. Therefore the set $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r\}$ must be linearly independent, which indicates $\boldsymbol{x}_0$ is a basic feasible solution.

The optimal solution of a standard form LPP (7)-(9) occurs at an extreme point of the feasible region.

Proof.
Suppose that $\Omega = \{x \mid \mathbf{c}^T\mathbf{x} = z\}$ is an optimal hyperplane that gives the optimal objective value z. We will first show that no interior point of the feasible region corresponds to the optimal value.
Suppose that, on contrary, the optimal value is achieved at point $\mathbf{x}_0$ in the interior of the feasible region. Then there exist an $\epsilon > 0$ such that the open ball $B_{2\epsilon}(\mathbf{x}_0) = \{\mathbf{x} \mid |\mathbf{x} - \mathbf{x}_0|_2 < 2\epsilon\}$ is in the feasible region.

So, the point $\boldsymbol{x}_1 = \boldsymbol{x}_0 + \epsilon \frac{\boldsymbol{c}}{|\boldsymbol{c}|_2^2} \in B_{2\epsilon}(\boldsymbol{x}_0)$ is a feasible solution and

$$\boldsymbol{c}^T \boldsymbol{x}_1 = \boldsymbol{c}^T \boldsymbol{x}_0 + \boldsymbol{c}^T \epsilon \frac{\boldsymbol{c}}{|\boldsymbol{c}|_2^2} = z + \epsilon > z$$

This leads to a contradiction to the optimality of z. Thus $\boldsymbol{x}_0$ has to be a boundary point.
Since $\boldsymbol{c}^T \boldsymbol{x} \leq z$ holds for all feasible solutions, the optimal hyperplane is a supporting hyperplane of the feasible region at the point $\boldsymbol{x}_0$. By Theorem 4, the feasible region is bounded from below. Based on previous theorem, the supporting hyperplane $z = \boldsymbol{c}^T \boldsymbol{x}$ contains at least one extreme point of the feasible region. Clearly that extreme point must also be an optimal solution to the LPP.

3. 4. 线性规划问题的基本定理

Definition . A set Ω is said to be bounded from below if for every $\boldsymbol{x} = [x_1, \ldots, x_n] \in \Omega$, we have

$$\inf\{x_j\} > -\infty, j = 1, \ldots, n$$

Clearly, any bounded set is bounded from below.

Example 1.4. $\mathbb{R}^{n+}$ is bounded from below.

Theorem . *If a closed convex set Ω is bounded from below, every supporting hyperplane of Ω contains an extreme point of Ω.*

利用基本定理求解线性规划问题 简单粗暴

题1 $\max \quad z = x_1 + x_2 + x_3 + x_4$

$$\text{s.t.} \quad \begin{pmatrix} 1 \\ 2 \end{pmatrix} x_1 + \begin{pmatrix} 2 \\ 2 \end{pmatrix} x_2 + \begin{pmatrix} 3 \\ 1 \end{pmatrix} x_3 + \begin{pmatrix} 4 \\ 2 \end{pmatrix} x_4 = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$$

$$x_j \geq 0, \forall 1 \leq j \leq 4$$

$$\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 7 \\ 3 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 2.2 \end{pmatrix} \Rightarrow X_1 = (0.4, 0, 2.2, 0)^T, \quad \boxed{z_1 = 2.6 \quad \text{最优顶点}}$$

$$\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 7 \\ 3 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 2 \end{pmatrix} \Rightarrow X_2 = (0, 0.5, 2, 0)^T, \quad z_2 = 2.5$$

$$\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 7 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow X_3 = (0, 0, 1, 1)^T, \quad z_3 = 2$$