An brief introduction of

Passive Dynamic Walking

Part II: Analysis Methodology

Mingguo Zhao
Dept. of Automation, Tsinghua University

Basic Definitions:

Generalized Coordinate: $q = [q_1, q_2, \cdots, q_m]^T$

Coordinate: $x = [x_1, x_2, \cdots, x_n]^T$

Constrain: $x = x(q) = \left[x_1(q_1, q_2, \cdots, q_m), x_2(q_1, q_2, \cdots, q_m), \cdots, x_n(q_1, q_2, \cdots, q_m)\right]^T$

Jacobin Matrix:
$$\dot{x} = \frac{\partial x}{\partial q}\dot{q} = J\dot{q} = \begin{bmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_1}{\partial q_2} & \cdots & \frac{\partial x_1}{\partial q_m} \\ \frac{\partial x_2}{\partial q_1} & \frac{\partial x_2}{\partial q_2} & \cdots & \frac{\partial x_2}{\partial q_m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_n}{\partial q_1} & \frac{\partial x_n}{\partial q_2} & \cdots & \frac{\partial x_n}{\partial q_m} \end{bmatrix}\dot{q} \qquad J = J(x, q)$$

Accleration: $\ddot{x} = J\ddot{q} + \dot{J}\dot{q} = J\ddot{q} + D \qquad D = D(q, \dot{q})$

EOM: equation of motion

by Virtual Power: $J^T \cdot F^c = 0$

EOM: $M\ddot{x} = F^c + F^a$

$$\begin{cases} MJ\ddot{q} + MD = F^c + F^a \\ J^T F^c = 0 \end{cases}$$

$J^T MJ\ddot{q} + J^T MD = J^T F^a$ when $(J^T MJ)$ is singular:

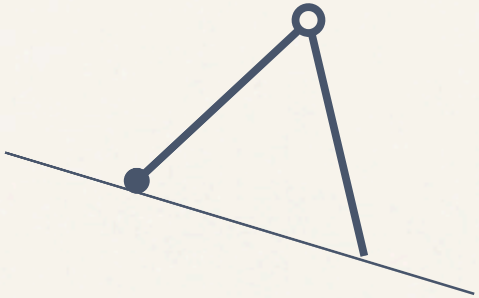
$$\boxed{\ddot{q} = (J^T MJ)^{-1}(J^T F^a - J^T MD)}$$

Constrain Force: $F^c = MJ\ddot{q} - MD - F^a$

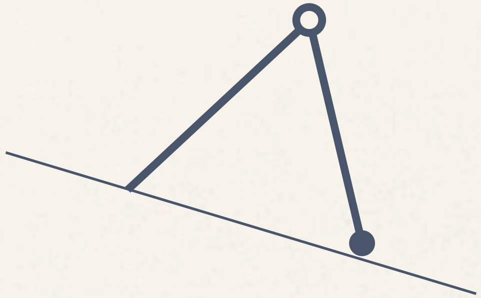
Impact

Conservation of Momentum: $M\dot{x}^{+} - M\dot{x}^{-} = \rho$

Constrains:



$x^{st} = x^{st}(q) \Rightarrow J^{st}$



$x^{sw} = x^{sw}(q) \Rightarrow J^{sw}$

$$\dot{x}^{+} = J^{sw}\dot{q}^{+}$$
$$\dot{x}^{-} = J^{st}\dot{q}^{-}$$

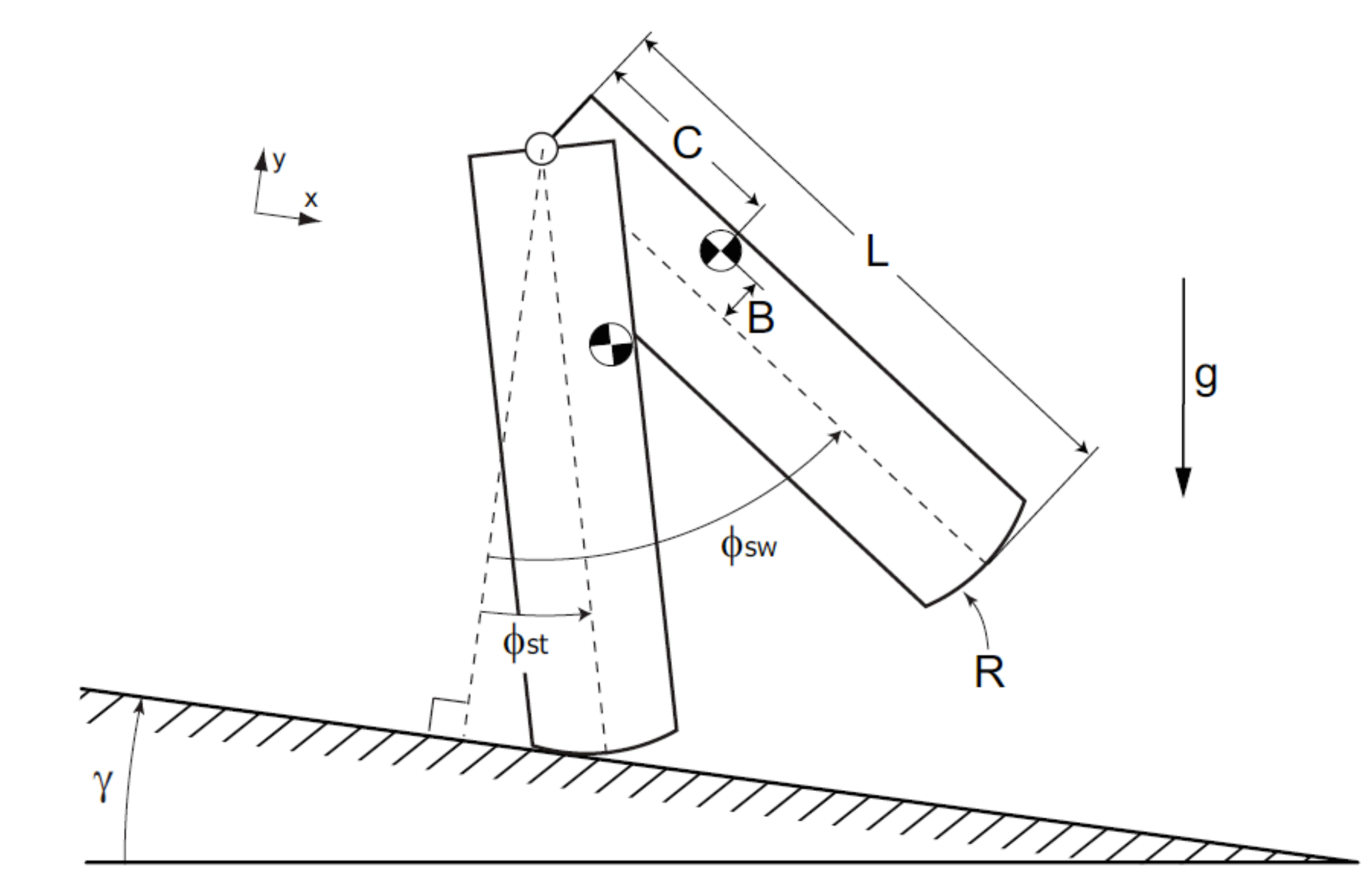
Impact: $MJ^{sw}\dot{q}^{+} - MJ^{st}\dot{q}^{-} = \rho$ impact force can be calculated later

$$\begin{cases} (J^{sw})^T M(J^{sw})\dot{q}^{+} - (J^{sw})^T M(J^{st})\dot{q}^{-} = (J^{sw})^T \rho \\ (J^{sw})^T \rho = 0 \end{cases}$$

$$(J^{sw})^T M(J^{sw})\dot{q}^{+} = (J^{sw})^T M(J^{st})\dot{q}^{-}$$

when $(J^{sw})^T M(J^{sw})$ is singular: $\boxed{\dot{q}^{+} = [(J^{sw})^T M(J^{sw})]^{-1}(J^{sw})^T M(J^{st})\dot{q}^{-}}$

Experiment model



参数：

L －腿长

m －腿部质量

B －质心水平偏移

C －质心竖直偏移

R －脚半径

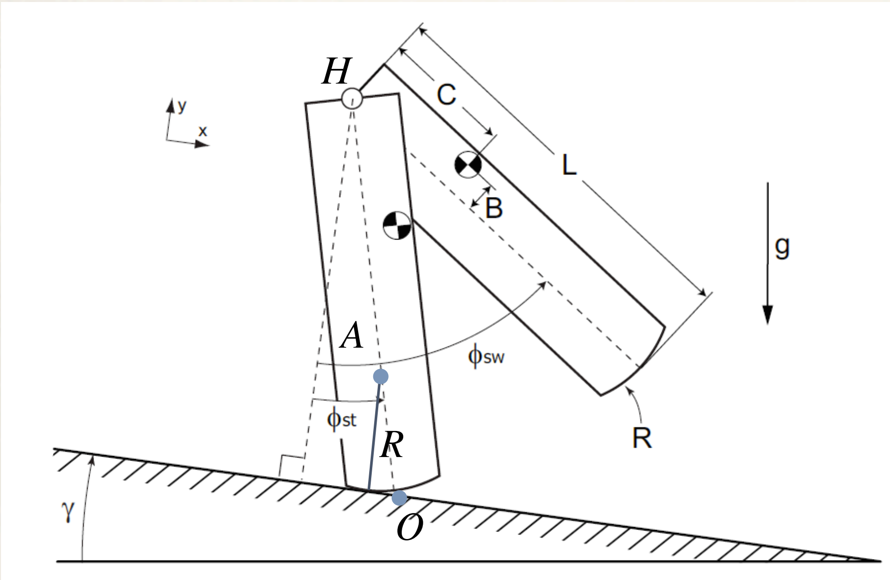
γ －斜坡角度

变量：

ϕ_{st} －支撑腿角度

ϕ_{sw} －摆动腿角度

Experiment Model:EOM



$$
\left\{
\begin{aligned}
P_h &= \begin{bmatrix} -R\phi_{st} - (L-R)\sin(\phi_{st}) \\ R + (L-R)\cos(\phi_{st}) \end{bmatrix} \\
R_{st} &= \begin{bmatrix} \cos(\phi_{st}) & -\sin(\phi_{st}) \\ \sin(\phi_{st}) & \cos(\phi_{st}) \end{bmatrix} \\
R_{sw} &= \begin{bmatrix} \cos(\phi_{sw}) & -\sin(\phi_{sw}) \\ \sin(\phi_{sw}) & \cos(\phi_{sw}) \end{bmatrix}
\end{aligned}
\right.
\Rightarrow
\left\{
\begin{aligned}
P_{st} &= P_h + R_{st}\begin{bmatrix} B \\ -C \end{bmatrix} \\
P_{sw} &= P_h + R_{sw}\begin{bmatrix} B \\ -C \end{bmatrix}
\end{aligned}
\right.
$$

$$
\left\{
\begin{aligned}
X &= \left[P_{st}(1), P_{st}(2), \phi_{st}, P_{sw}(1), P_{sw}(2), \phi_{sw}\right]^T \\
q &= [\phi_{st}, \phi_{sw}]^T
\end{aligned}
\right.
\Rightarrow J = \frac{\partial X}{\partial q}
$$

$$
\left\{
\begin{aligned}
M &= diag[m, m, I, m, m, I] \\
F^a &= mg\left[\sin(\gamma), -\cos(\gamma), 0, \sin(\gamma), -\cos(\gamma), 0\right]^T
\end{aligned}
\right.
$$

$$
\Rightarrow \boxed{\ddot{q} = (J^T M J)^{-1}(J^T F^a - J^T M D)}
$$

Experiment Model:Impact

$$P_h^{st} = \begin{bmatrix} -R\phi_{st} - (L-R)\sin(\phi_{st}) \\ R + (L-R)\cos(\phi_{st}) \end{bmatrix}$$

$$\begin{cases} P_{st}^{st} = P_h^{st} + R_{st}\left[B,-C\right]^T \\ P_{sw}^{st} = P_h^{st} + R_{sw}\left[B,-C\right]^T \end{cases}$$

$$\begin{cases} X^{st} = \left[P_{st}^{st}(1), P_{st}^{st}(2), \phi_{st}, P_{sw}^{st}(1), P_{sw}^{st}(2), \phi_{sw}\right]^T \\ q = [\phi_{st}, \phi_{sw}]^T \end{cases}$$

$$J^{st} = \frac{\partial X^{st}}{\partial q}$$

$$P_h^{sw} = \begin{bmatrix} -R\phi_{sw} - (L-R)\sin(\phi_{sw}) \\ R + (L-R)\cos(\phi_{sw}) \end{bmatrix}$$

$$\begin{cases} P_{st}^{sw} = P_h^{sw} + R_{st}\left[B,-C\right]^T \\ P_{sw}^{sw} = P_h^{sw} + R_{sw}\left[B,-C\right]^T \end{cases}$$

$$\begin{cases} X^{sw} = \left[P_{st}^{sw}(1), P_{st}^{sw}(2), \phi_{st}, P_{sw}^{sw}(1), P_{sw}^{sw}(2), \phi_{sw}\right]^T \\ q = [\phi_{st}, \phi_{sw}]^T \end{cases}$$

$$J^{sw} = \frac{\partial X^{sw}}{\partial q}$$

$$\boxed{\dot{q}^+ = [(J^{sw})^T M (J^{sw})]^{-1} (J^{sw})^T M (J^{st}) \dot{q}^-}$$

END

Thank You!