3.1 解：$H(k)=\frac{1}{2}u^2(k)+\lambda^T(k+1)\left\{\begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}x_1(k)\\x_2(k)\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}u(k)\right\}$

$$\frac{\partial H(k)}{\partial u(k)}=u(k)+\lambda^T(k+1)\cdot\begin{bmatrix}1\\1\end{bmatrix}=0 \quad 得 \quad u(k)=-\lambda^T(k+1)\cdot\begin{bmatrix}1\\1\end{bmatrix}$$

正则方程 $\begin{cases} x(k+1)=\begin{bmatrix}1&1\\1&0\end{bmatrix}x(k)+\begin{bmatrix}1\\1\end{bmatrix}u(k) \\ \lambda(k)=\frac{\partial L}{\partial x(k)}+\left[\frac{\partial f}{\partial x(k)}\right]^T\lambda(k+1)=\begin{bmatrix}\lambda_1(k+1)+\lambda_2(k+1)\\ \lambda_1(k+1)\end{bmatrix}\end{cases}$

由 $X(0)=\begin{bmatrix}1\\1\end{bmatrix}$，则 $\lambda(2)=\frac{\partial \varphi}{\partial x(2)}=\begin{bmatrix}x_1(2)\\0\end{bmatrix}$

故 $\lambda(1)=\begin{bmatrix}\lambda_1(2)+\lambda_2(2)\\ \lambda_1(2)\end{bmatrix}=\begin{bmatrix}x_1(2)\\x_1(2)\end{bmatrix}$

代入状态方程 $\begin{bmatrix}x_1(2)\\x_2(2)\end{bmatrix}=\begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}x_1(1)\\x_2(1)\end{bmatrix}-\begin{bmatrix}1\\1\end{bmatrix}\cdot\lambda^T(2)\cdot\begin{bmatrix}1\\1\end{bmatrix}$

$$=\begin{bmatrix}x_1(1)+x_2(1)\\x_1(1)\end{bmatrix}-\begin{bmatrix}x_1(2)\\x_1(2)\end{bmatrix}$$

$$\begin{bmatrix}x_1(1)\\x_2(1)\end{bmatrix}=\begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}x_1(0)\\x_2(0)\end{bmatrix}-\begin{bmatrix}1\\1\end{bmatrix}\lambda^T(1)\begin{bmatrix}1\\1\end{bmatrix}$$

$$=\begin{bmatrix}x_1(0)+x_2(0)\\x_1(0)\end{bmatrix}-2\begin{bmatrix}x_1(2)\\x_1(2)\end{bmatrix}$$

$$=\begin{bmatrix}2-2x_1(2)\\1-2x_1(2)\end{bmatrix}$$

解得 $\begin{bmatrix}x_1(2)\\x_2(2)\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}$，故 $u(1)=-\frac{1}{2}$

$\begin{bmatrix}x_1(1)\\x_1(1)\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$，故 $u(0)=-1$

3.2 解：(1) $\delta J = \frac{\partial}{\partial a} J[x(t) + a\delta x(t)] \Big|_{a=0}$

$$= \frac{\partial}{\partial a} \int_0^1 (x(t) + a\delta x(t))^3 \sin(t)\, dt \Big|_{a=0}$$

$$= \int_0^1 \frac{\partial}{\partial a} (x(t) + a\delta x(t))^3 \sin(t) \Big|_{a=0} dt$$

$$= 3\int_0^1 x^2(t) \sin(t)\, \delta x(t)\, dt$$

(2) $\delta J = \int_0^1 \frac{\partial F}{\partial y(t)} \delta y(t) + \frac{\partial F}{\partial x(t)} \delta x(t)\, dt$

$$= \int_0^1 3y^2(t)\, x^2(t)\, \delta x(t) + 2x(t) y(t)\, \delta y(t)\, dt$$

3.3 解： $\frac{\partial T}{\partial y} - \frac{d}{dx}\left(\frac{\partial T}{\partial y'}\right) = 0$

即 $\int_0^a \frac{\partial}{\partial y(x)} \frac{\sqrt{1+y'^2(x)}}{\sqrt{2y(x)}} dx - \frac{d}{dx} \int_0^a \frac{\partial}{\partial y'(x)} \frac{\sqrt{1+y'^2(x)}}{\sqrt{2y(x)}} dx = 0.$

$$-\frac{1}{2}\int_0^a \frac{\sqrt{1+y'^2(x)}}{\sqrt{2y^3(x)}} dx - \frac{d}{dx}\int_0^a \frac{\sqrt{2y(x)}}{\sqrt{1+y'^2(x)}} dx = 0.$$

即 $\int_0^a \frac{\sqrt{1+y'^2(x)}}{\sqrt{8y^3(x)}} dx + \frac{\sqrt{2y(x)}}{\sqrt{1+y'^2(x)}} \Big|_0^a = 0$

3.4 解：此问题为末状态固定、末时间固定问题

因此 $z(T)=0, z(0)=z_0$ 终端条件

$J=\int_0^T u^2(t)\,dt$ 优化目标.

$\dot{z}(t)=\dfrac{u(t)-z^3(t)-z(t)}{1-z^2(t)}$ 状态方程

3.5 解：$J=\frac{1}{2}\int_0^1 [3x^2(t)+u^2(t)]\,dt$

则 $H(x,u,\lambda,t)=\lambda(t)[-x(t)+u(t)]+\frac{3}{2}x^2(t)+\frac{1}{2}u^2(t)$

因为 $\frac{\partial H}{\partial u}=0$，所以 $\lambda(t)+u(t)=0$ 即 $u(t)=-\lambda(t)$

$$\begin{cases}\lambda(1)=\dfrac{\partial\varphi}{\partial x(1)}=0 \\ x(0)=x_0 \\ \dot{\lambda}(t)=-\dfrac{\partial H}{\partial x}=\lambda(t)-3x(t) \\ \dot{x}(t)=-x(t)+u(t)=-x(t)-\lambda(t)\end{cases}$$

解得 $u(t)=\dfrac{3e^{4t}-3e^{4}}{e^{6t}+3e^{4}}x(t)$

3.6 解：$H(x,u,\lambda)=x^2(t)+4u^2(t)+4\lambda(t)u(t)$

$$\frac{\partial H}{\partial u}=0 \Rightarrow u(t)=-\frac{1}{2}\lambda(t)$$

正则方程
$$\begin{cases}\dot{x}(t)=4u(t)=-2\lambda(t)\\ \dot{\lambda}(t)=-\frac{\partial H}{\partial x}=-2x(t)\end{cases}$$

边界条件 $x(0)=x_0,\ x(T)=x_T$

$$\lambda(T)=\frac{\partial \varphi}{\partial x(T)}=0$$

拉普拉斯变换解方程可得
$$\begin{cases}x(t)=\frac{-e^{-2T}x_0+x_T}{e^{2T}-e^{-2T}}e^{2t}+\frac{e^{2T}x_0-x_T}{e^{2T}-e^{-2T}}e^{-2t}\\ \lambda(t)=\frac{e^{-2T}x_0-x_T}{e^{2T}-e^{-2T}}e^{2t}+\frac{e^{2T}x_0-x_T}{e^{2T}-e^{-2T}}e^{-2t}\end{cases}$$

故 $u(t)=-\frac{1}{2}\lambda(t)=-\frac{1}{2}\frac{e^{-2T}x_0-x_T}{e^{2T}-e^{-2T}}e^{2t}-\frac{1}{2}\frac{e^{2T}x_0-x_T}{e^{2T}-e^{-2T}}e^{-2t}$

3.7 解：$H(x,u,\lambda)=u^2(t)+\lambda(t)u(t)$

$$\frac{\partial H}{\partial u}=0 \Rightarrow u(t)=-\frac{1}{2}\lambda(t)$$

正则方程
$$\begin{cases}\dot{x}(t)=u(t)=-\frac{1}{2}\lambda(t)\\ \dot{\lambda}(t)=-\frac{\partial H}{\partial x}=0\end{cases}$$

边界条件
$$\begin{cases}x(0)=1,\ x(t_f)=0\\ H(t_f)=-\frac{\partial \varphi}{\partial t_f}\Rightarrow \lambda^2(t_f)=8t_f\Rightarrow \lambda(t_f)=2\sqrt{2t_f}\end{cases}$$

故 $x(t)=-\sqrt{2t_f}\,t+1$

即 $x(t)=-4^{\frac{1}{3}}t+1$

$u(t)=-4^{\frac{1}{3}}$

3.8 解：$\dot{x}=\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}u(t)$

$$H(x,u,\lambda)=u^2(t)+\lambda^T(t)\cdot\left\{\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}x(t)+\begin{bmatrix}0\\0\\1\end{bmatrix}u(t)\right\}$$
$$=u^2(t)+\lambda_3u(t)+\lambda_1x_2(t)+\lambda_2x_3(t)$$

$$\frac{\partial H}{\partial u}=2u(t)+\lambda^T(t)\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}=0$$

即 $[\lambda_1\ \ \lambda_2\ \ \lambda_3]\begin{bmatrix}0\\0\\1\end{bmatrix}=-2u(t)$

$$u(t)=-\frac{1}{2}\lambda_3(t)$$

正则方程 $\begin{cases}\dot{x}=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}x(t)+\begin{bmatrix}0\\0\\1\end{bmatrix}\cdot\left[-\frac{1}{2}\lambda_3(t)\right]\\ \dot{\lambda}=-\frac{\partial H}{\partial x}=\begin{bmatrix}0\\-\lambda_1(t)\\-\lambda_2(t)\end{bmatrix}\\ x(0)=\begin{bmatrix}0\\0\\0\end{bmatrix}\end{cases}$

定义 $g(x(t_f),t_f)=x_2(t_f)+\mu_1[x_1^2(t_f)-t_f^2]+\mu_2[x_2(t_f)-x_3^2(t_f)]$

则 $\lambda_1(t_f)=2\mu_1x_1(t_f)$

$\lambda_2(t_f)=1+\mu_2$

$\lambda_3(t_f)=-2\mu_2x_3(t_f)$

$$u^*=\arg\min_{u\in[-1,1]}H(x,u,\lambda)=\begin{cases}-\frac{\lambda_3}{2}, & |\lambda_3|\le 2\\ -\operatorname{sign}(\lambda_3), & |\lambda_3|>2\end{cases}$$

3.8 解：由 $\dot{x}=\begin{bmatrix}0&1\\0&0\end{bmatrix}x+\begin{bmatrix}0\\1\end{bmatrix}u$

故 $AB=\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$

则 $Q_k=[B\quad AB]=\begin{bmatrix}0&1\\1&0\end{bmatrix}$，$rank=2=n$

因此系统完全可控

故无限时间状态调节器问题存在最优解

求解 Riccati 方程

$$PA+A^TP-PBR^{-1}B^TP+Q=0$$

又 $A=\begin{bmatrix}0&1\\0&0\end{bmatrix}$，$B=\begin{bmatrix}0\\1\end{bmatrix}$，$R=1$，$Q=\begin{bmatrix}1&0\\0&0\end{bmatrix}$

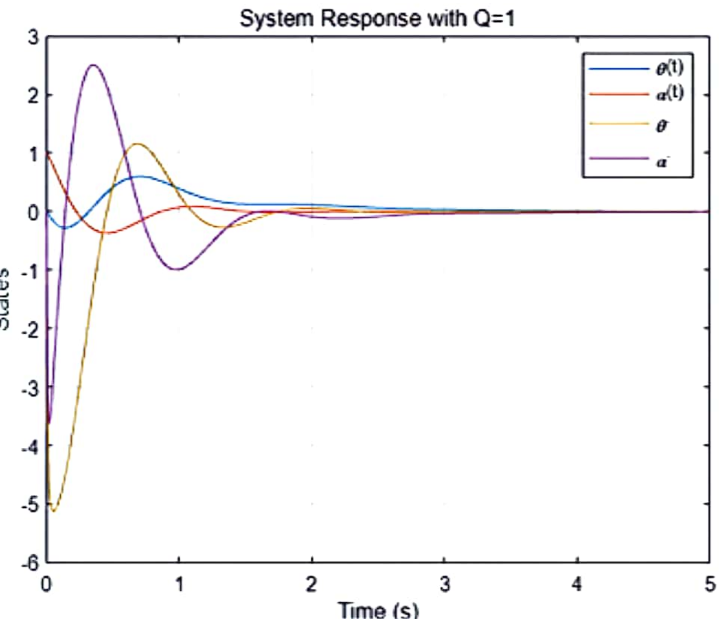
故设 $P=\begin{bmatrix}p_1&p_2\\p_3&p_4\end{bmatrix}$，有 $\begin{bmatrix}0&p_1\\0&p_3\end{bmatrix}+\begin{bmatrix}0&0\\p_1&p_2\end{bmatrix}-\begin{bmatrix}p_2p_3&p_2p_4\\p_3p_4&p_4^2\end{bmatrix}+\begin{bmatrix}1&0\\0&0\end{bmatrix}=0$

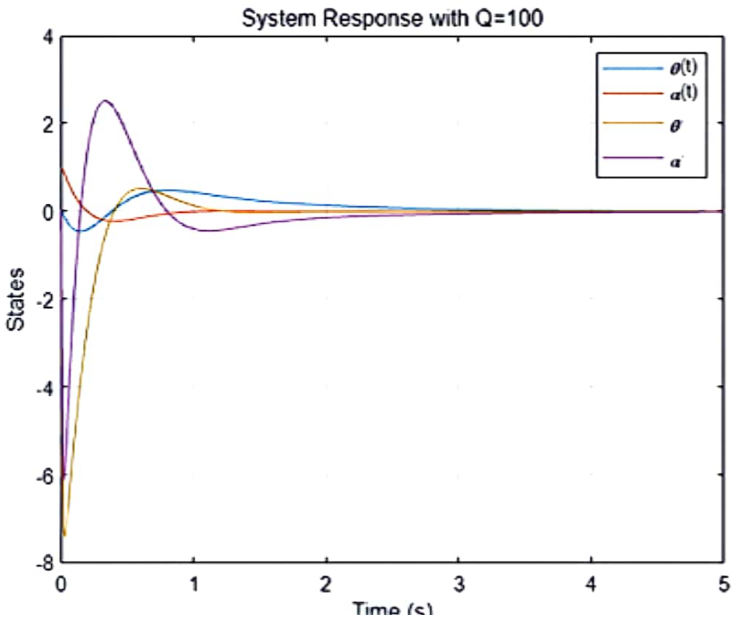
解得 $P=\begin{bmatrix}\sqrt{2}&1\\1&\sqrt{2}\end{bmatrix}$

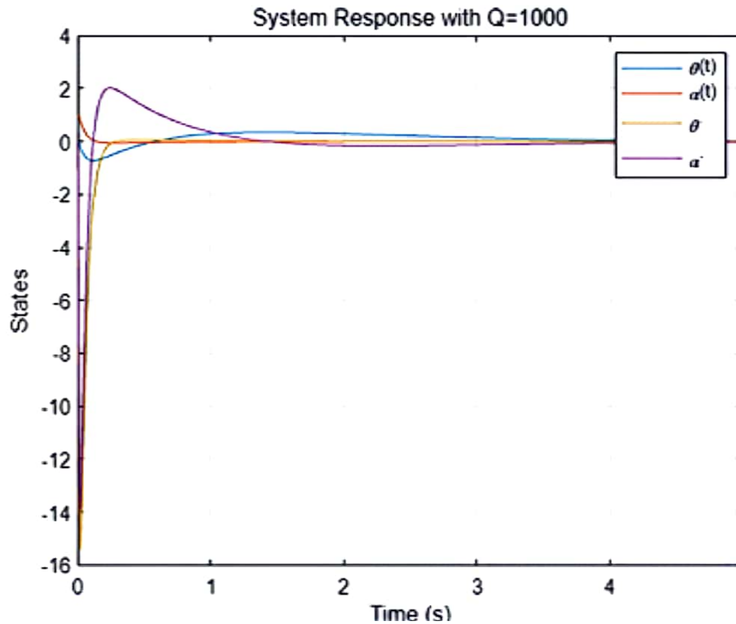
故 $u^*(t)=-R^{-1}B^TPx(t)=-[0\quad 1]\cdot\begin{bmatrix}\sqrt{2}&1\\1&\sqrt{2}\end{bmatrix}\cdot\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}=x_1(t)+\sqrt{2}x_2(t)$

$J^*(t)=\frac{1}{2}x^T(t_0)Px(t_0)=\frac{1}{2}[2\quad 2]\begin{bmatrix}\sqrt{2}&1\\1&\sqrt{2}\end{bmatrix}\begin{bmatrix}2\\2\end{bmatrix}=4\sqrt{2}+4$

3.10 解：随 ξ 增加，系统的超调降低


System Response with Q=1
States
Time (s)
θ(t)
α(t)
θ̇
α̇


System Response with Q=100
States
Time (s)
θ(t)
α(t)
θ̇
α̇


System Response with Q=1000
States
Time (s)
θ(t)
α(t)
θ̇
α̇