

机器人运动学

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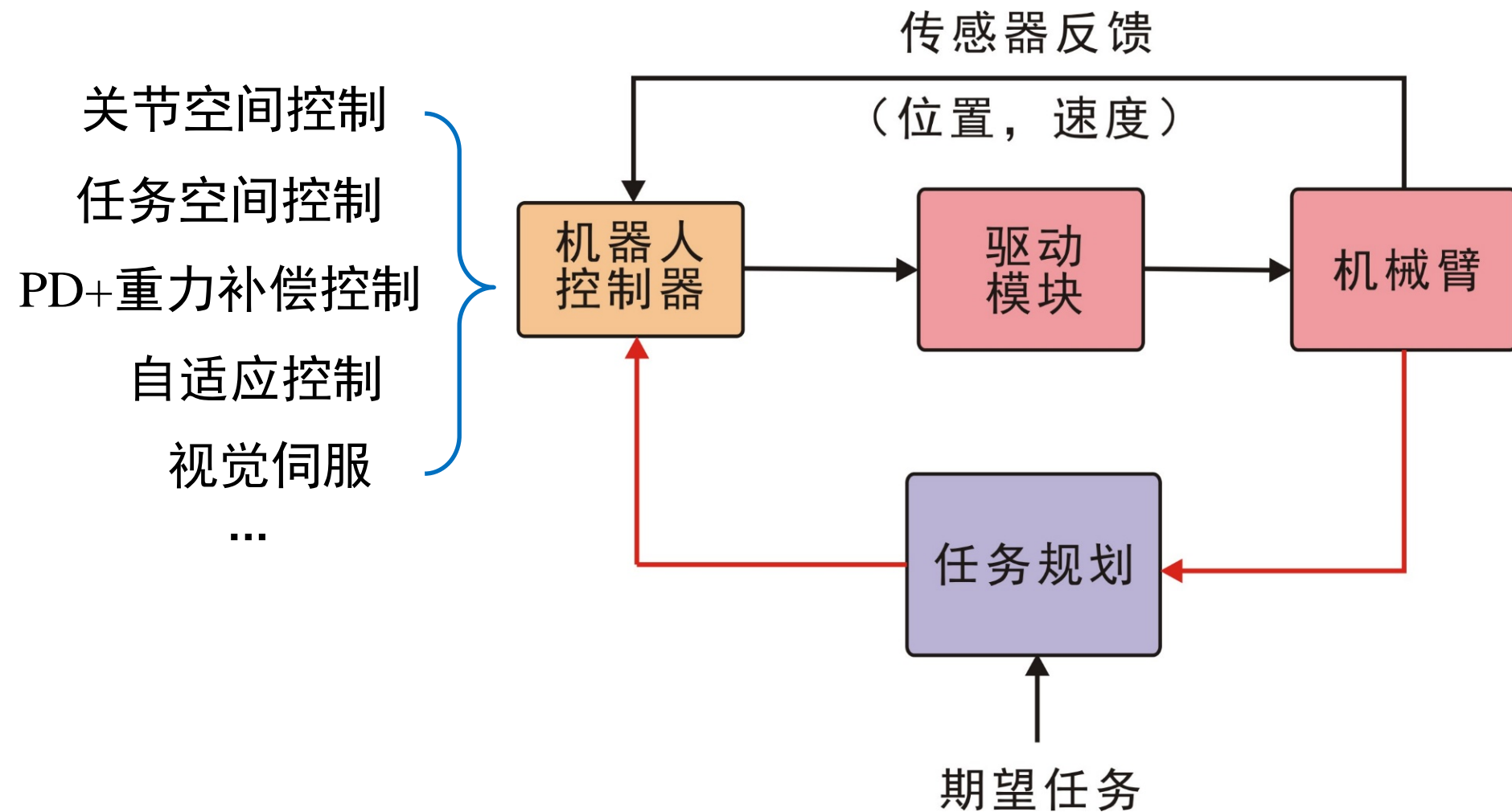
清华大学
Tsinghua University



内容提纲

- 坐标变化与D-H法则
- 末端执行器
- Jacobian矩阵与奇异性

机器人构成



图像空间

$$x = [u, v]^T \in R^2$$

The diagram illustrates a robotic arm system. The main arm is labeled "机械臂" (Mechanical Arm) and consists of a base with joint q_1 , a vertical column with joint q_2 , and a horizontal arm with joint q_3 . The end of the arm is connected to an "执行器" (Actuator), which is a flexible gripper. The gripper is shown in two states: a straight state and a curved state. The curved state is highlighted with a blue line and red stars, labeled "特征" (Feature). A camera, labeled "相机" (Camera), is positioned above the gripper, capturing its position. The camera's field of view is indicated by a green cone. The gripper's position is defined by a coordinate system with axes u and v . The gripper's end effector is shown with a coordinate system with axes r_1 , r_2 , and r_3 . The gripper is shown interacting with a brown surface, which is the target object.

运动学 - 研究机器人关节转角与末端执行器位置的关系

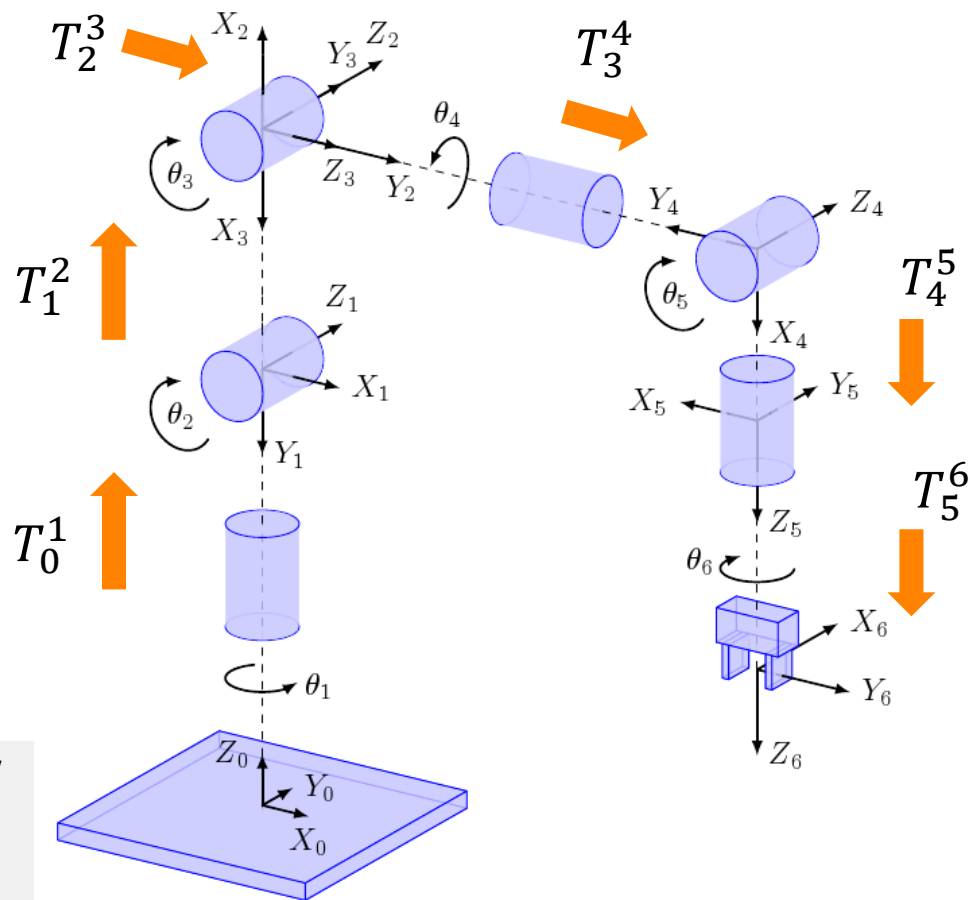
关节空间

关节
转角

运动学

末端
位置

任务空间
(如笛卡尔空间)



<https://robotics.stackexchange.com/questions/11372/any-free-or-paid-software-for-graphical-drawing-of-robotics-kinematic-scheme>

向量 p 在某坐标系可表示为：

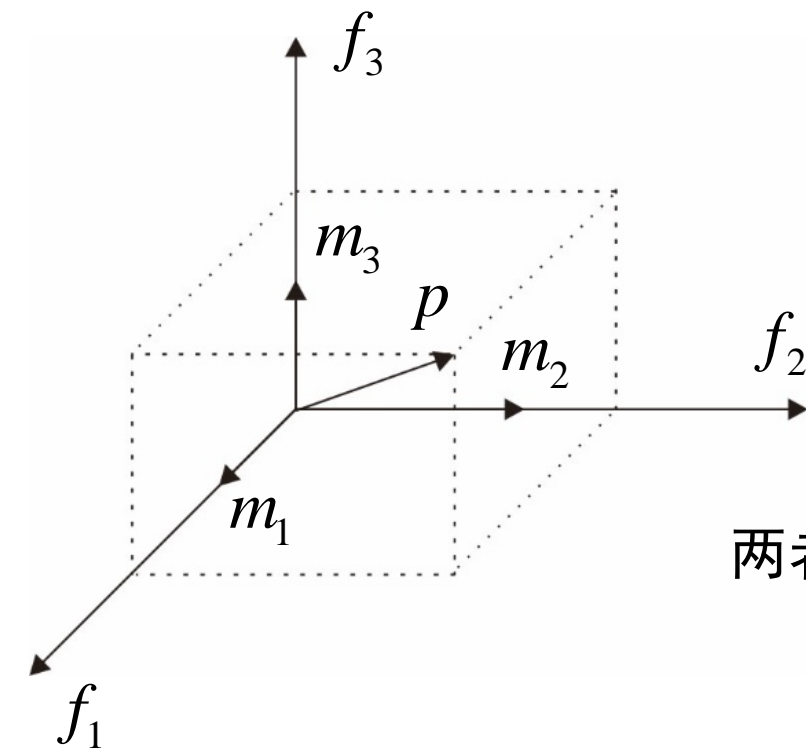
$$p = p_1^f f_1 + p_2^f f_2 + p_3^f f_3$$

$$\text{或者 } p = p_1^m m_1 + p_2^m m_2 + p_3^m m_3$$

标量 向量

两者转换关系： R - 旋转矩阵

$$\begin{bmatrix} p_1^f \\ p_2^f \\ p_3^f \end{bmatrix} = R \begin{bmatrix} p_1^m \\ p_2^m \\ p_3^m \end{bmatrix}$$



$$p = p_1^m m_1 + p_2^m m_2 + p_3^m m_3$$

$$\begin{cases} p_1^f = \underbrace{p \cdot f_1}_{\text{标量}} = p_1^m m_1 \cdot f_1 + p_2^m m_2 \cdot f_1 + p_3^m m_3 \cdot f_1 \\ p_2^f = p \cdot f_2 = p_1^m m_1 \cdot f_2 + p_2^m m_2 \cdot f_2 + p_3^m m_3 \cdot f_2 \\ p_3^f = p \cdot f_3 = p_1^m m_1 \cdot f_3 + p_2^m m_2 \cdot f_3 + p_3^m m_3 \cdot f_3 \end{cases}$$



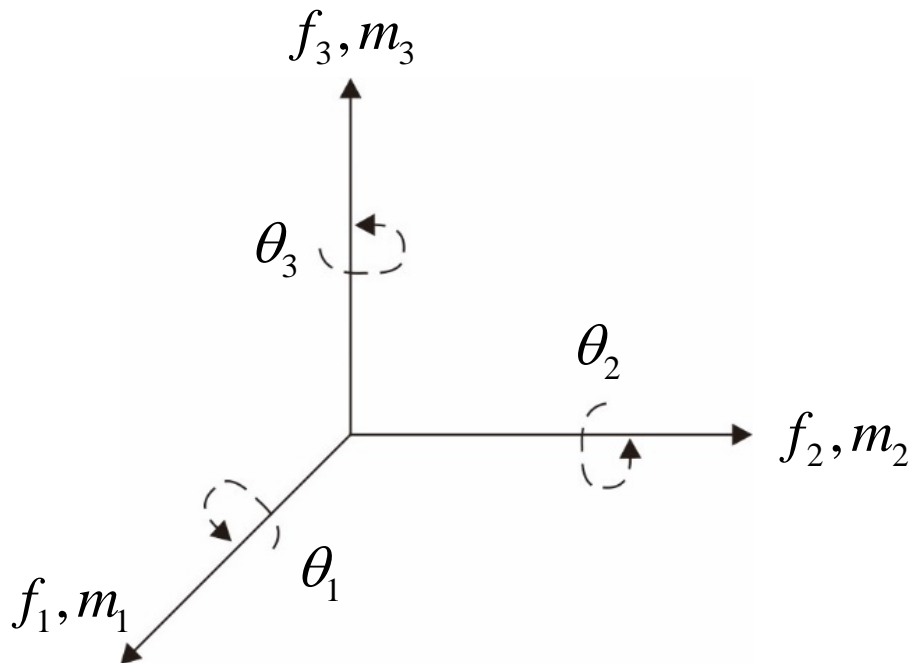
$$\begin{bmatrix} p_1^f \\ p_2^f \\ p_3^f \end{bmatrix} = \begin{bmatrix} m_1 \cdot f_1 & m_2 \cdot f_1 & m_3 \cdot f_1 \\ m_1 \cdot f_2 & m_2 \cdot f_2 & m_3 \cdot f_2 \\ m_1 \cdot f_3 & m_2 \cdot f_3 & m_3 \cdot f_3 \end{bmatrix} \begin{bmatrix} p_1^m \\ p_2^m \\ p_3^m \end{bmatrix}$$

$$p_F = R(\bullet) p_M$$

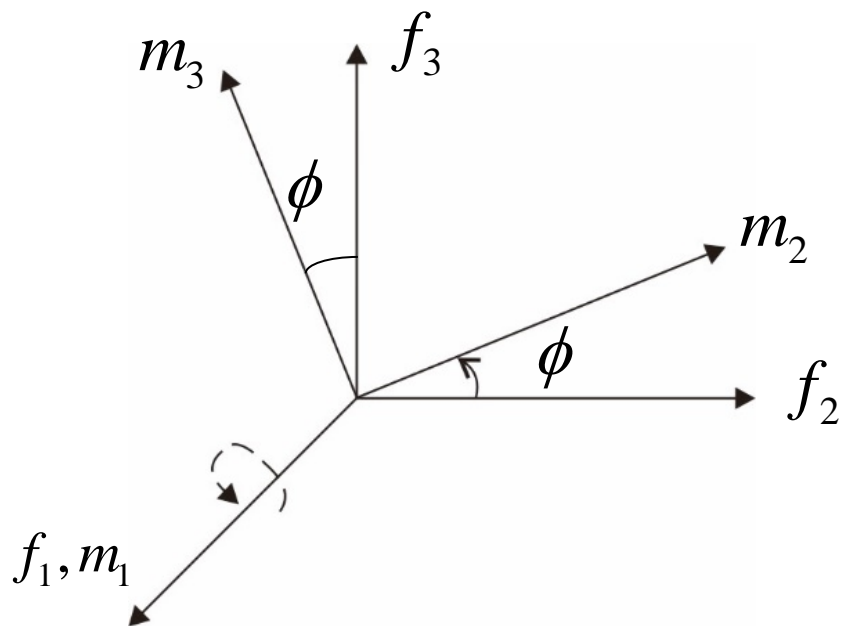
坐标变换



在笛卡尔空间的三种旋转方式：



将 M 坐标系统 f_1 旋转角度 ϕ



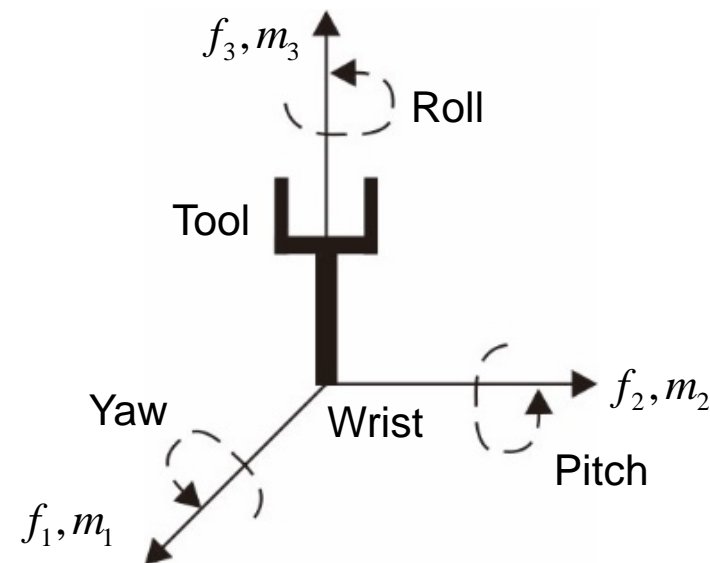
$$\begin{bmatrix} p_1^f \\ p_2^f \\ p_3^f \end{bmatrix} = \underbrace{\begin{bmatrix} m_1 \cdot f_1 & m_2 \cdot f_1 & m_3 \cdot f_1 \\ m_1 \cdot f_2 & m_2 \cdot f_2 & m_3 \cdot f_2 \\ m_1 \cdot f_3 & m_2 \cdot f_3 & m_3 \cdot f_3 \end{bmatrix}}_{R_1(\phi)} \begin{bmatrix} p_1^m \\ p_2^m \\ p_3^m \end{bmatrix}$$

$$\Rightarrow R_1(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{bmatrix}$$

将 M 坐标系 f_2 旋转角度 ϕ $\Rightarrow R_2(\phi) = \begin{bmatrix} \cos \phi & 0 & \sin \phi \\ 0 & 1 & 0 \\ -\sin \phi & 0 & \cos \phi \end{bmatrix}$

将 M 坐标系 f_3 旋转角度 ϕ $\Rightarrow R_3(\phi) = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$

复合旋转矩阵 - Yaw-Pitch-Roll矩阵



$$YPR(\theta) = R_3(\theta_3)R_2(\theta_2)R_1(\theta_1)$$

$$\Rightarrow \begin{bmatrix} C_2C_3 & S_1S_2C_3 - C_1S_3 & C_1S_2C_3 + S_1S_3 \\ C_2S_3 & S_1S_2S_3 + C_1C_3 & C_1S_2S_3 - S_1C_3 \\ -S_2 & S_1C_2 & C_1C_2 \end{bmatrix}$$

$$C_1 = \cos(\theta_1), S_1 = \sin(\theta_1), \dots$$

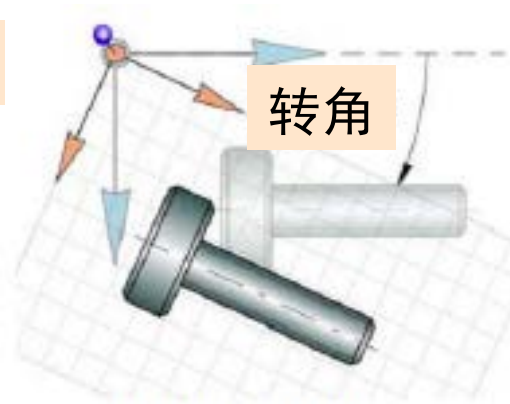
坐标变换



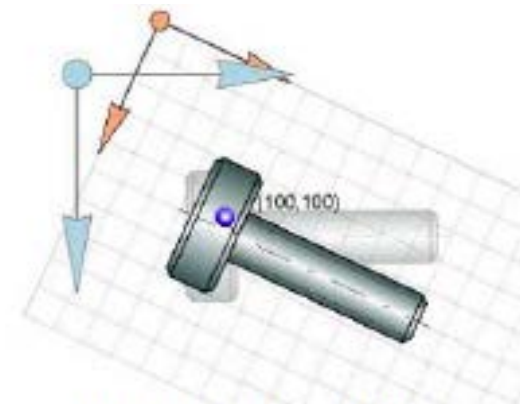
旋转矩阵

平移向量

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$



旋转



旋转 + 平移

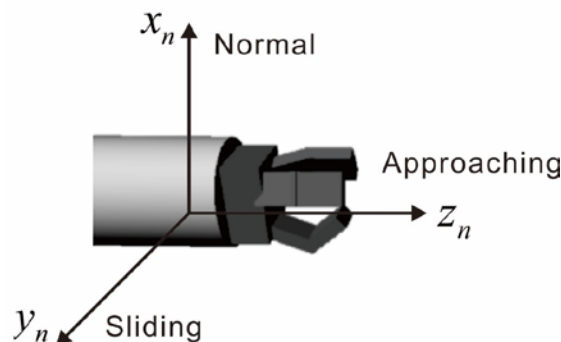
关节转角

$$T_{k-1}^k = \begin{bmatrix} \cos(\theta_k) & -\cos(\alpha_k)\sin(\theta_k) & \sin(\alpha_k)\sin(\theta_k) & a_k\cos(\theta_k) \\ \sin(\theta_k) & \cos(\alpha_k)\cos(\theta_k) & -\sin(\alpha_k)\cos(\theta_k) & a_k\sin(\theta_k) \\ 0 & \sin(\alpha_k) & \cos(\alpha_k) & d_k \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

θ_k α_k a_k d_k - 通过D-H算法描述的运动学参数

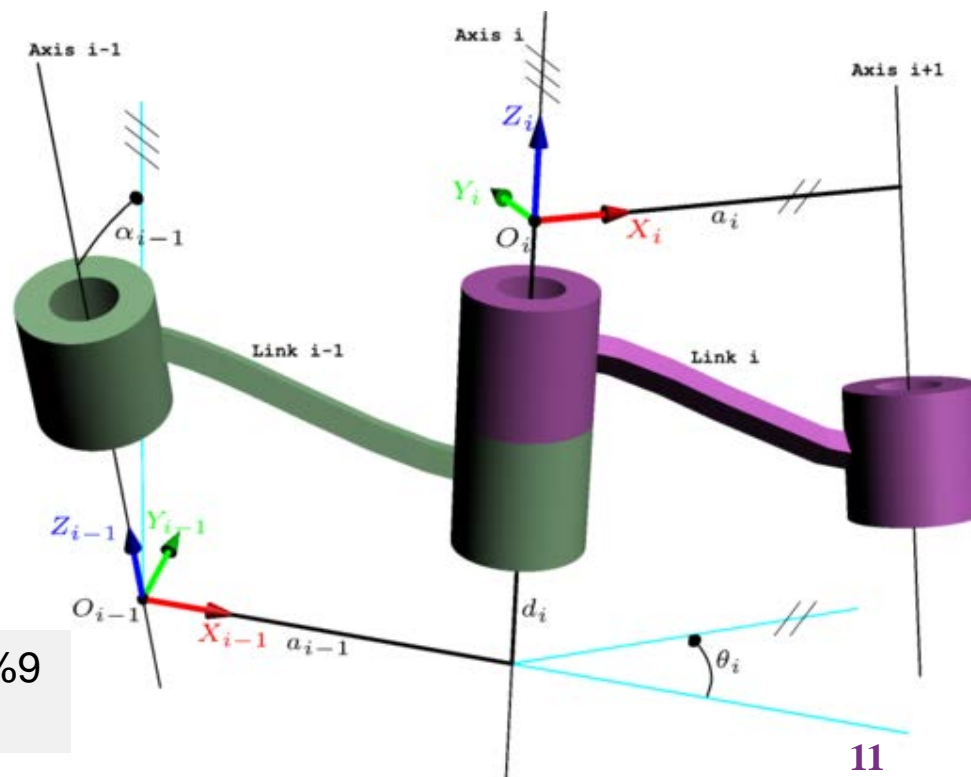
[Denavit and Hartenberg, 1955]

- (1) 从底座到末端，将每个关节依次记为 $1, 2, \dots, n$
- (2) 按照右手定则，将第0个坐标系分配至机器人底座，设定 z_0 ，**将其对齐于关节1的旋转轴**
- (3) 设置 $k=1$ ，按如下步骤进行：
 - 设定 z_k ，将其对齐于关节 $k+1$ 的旋转轴；
 - 将 z_k 与 z_{k-1} 轴的交点**设为第 k 个坐标系的原点**；如果两轴不相交，选择同时垂直于 z_k 和 z_{k-1} 的轴线，将 z_k 与该线交点设为第 k 个坐标系的原点；
 - 设定 x_k 为**同时垂直于** z_k 和 z_{k-1} 的轴线；如果 z_k 与 z_{k-1} 平行，选择远离 z_{k-1} 的方向为 x_k ；
 - 设定 y_k ，以**组成右手坐标系**
- (4) $k=k+1$ ，按照步骤(3)继续，直到 $k=n$
- (5) 设定第 n 个坐标系为末端执行器坐标系：
 - z_n – approaching vector
 - y_n – sliding vector
 - x_n – normal vector



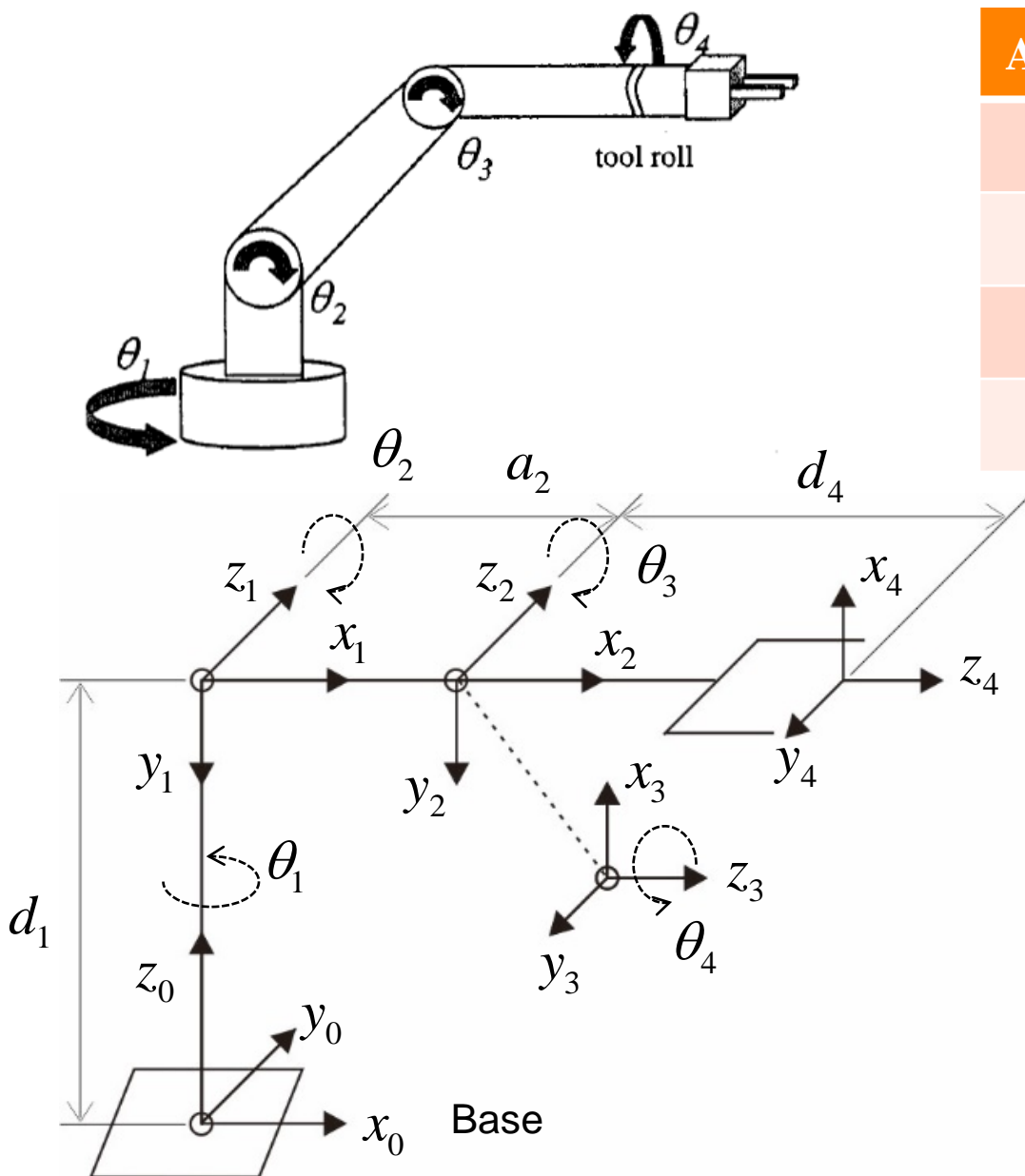
D-H参数定义为：

- θ_k 为 x_{k-1} 与 x_k 轴沿着 z_{k-1} 方向的夹角；
- d_k 为 x_{k-1} 与 x_k 轴沿着 z_{k-1} 方向的距离；
- a_k 为 z_{k-1} 与 z_k 轴沿着 x_k 方向的距离；
- α_k 为 z_{k-1} 与 z_k 轴沿着 x_k 方向的夹角。



https://en.wikipedia.org/wiki/Denavit%E2%80%933Hartenberg_parameters

D-H法则



Axis	θ	d	a	α
1	θ_1	d_1	0	$-\pi/2$
2	θ_2	0	a_2	0
3	θ_3	0	0	$-\pi/2$
4	θ_4	d_4	0	0

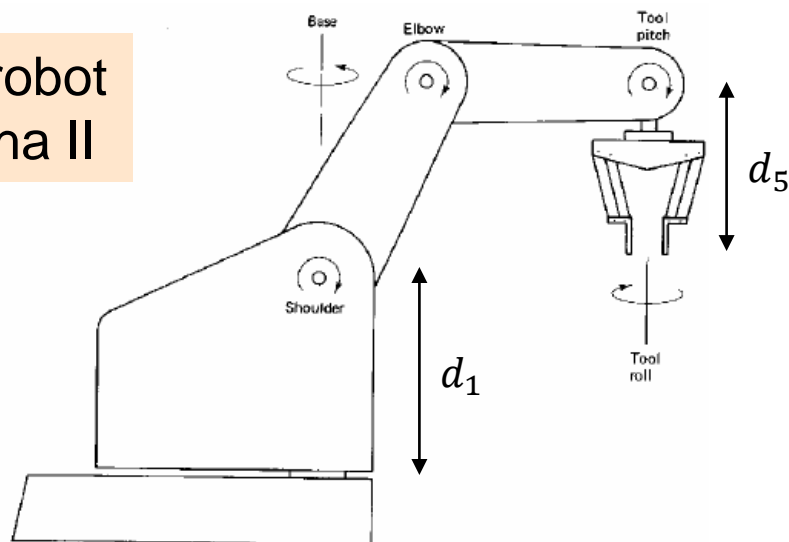
D-H参数定义为:

- θ_k 为 x_{k-1} 与 x_k 轴沿着 z_{k-1} 方向的夹角;
- d_k 为 x_{k-1} 与 x_k 轴沿着 z_{k-1} 方向的距离;
- a_k 为 z_{k-1} 与 z_k 轴沿着 x_k 方向的距离;
- α_k 为 z_{k-1} 与 z_k 轴沿着 x_k 方向的夹角。

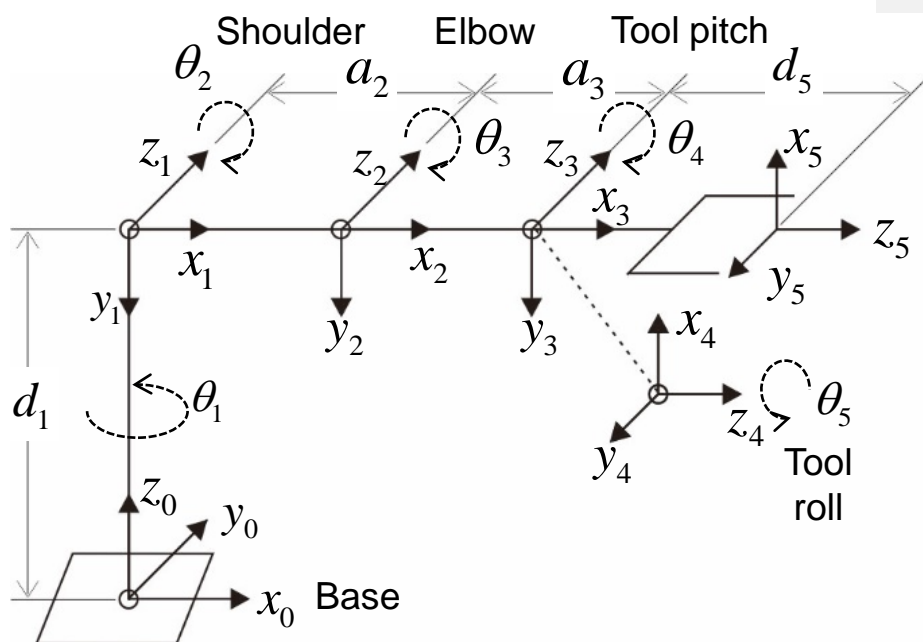
D-H法则



Microbot Alpha II

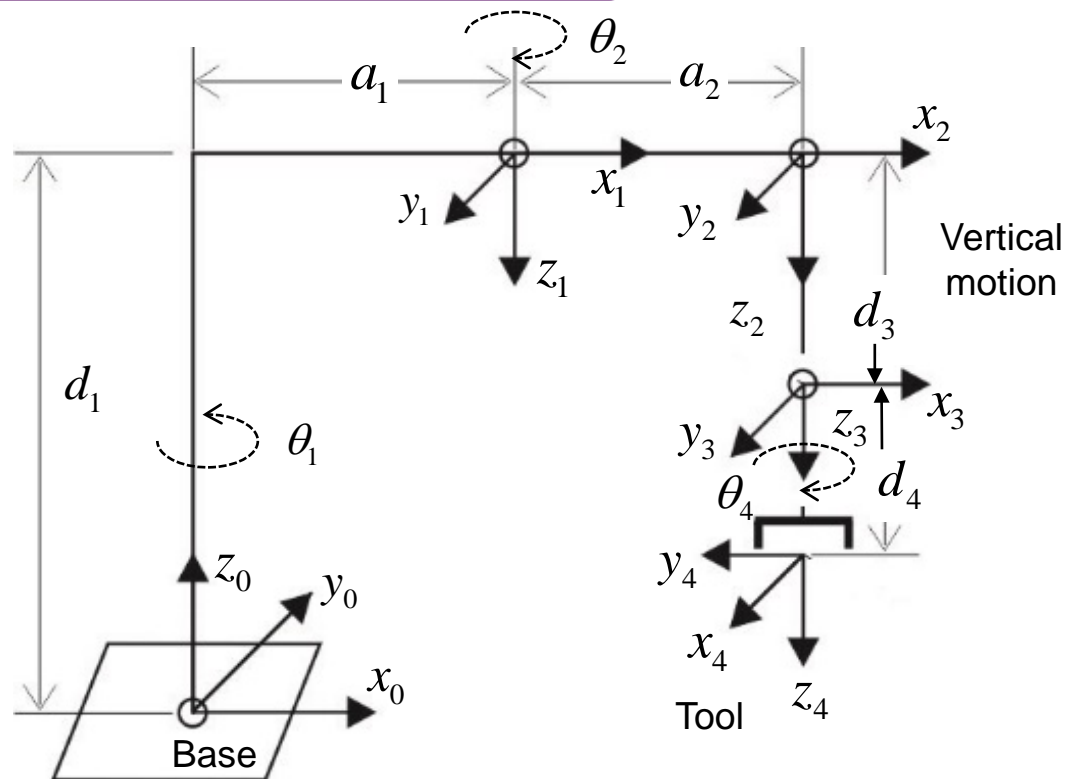
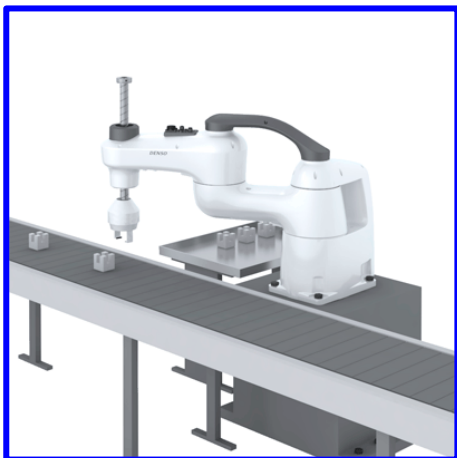


<https://www.youtube.com/watch?v=l2P9zETXmWo>



Axis	θ	d	a	α
1	θ_1	d_1	0	$-\pi/2$
2	θ_2	0	a_2	0
3	θ_3	0	a_3	0
4	θ_4	0	0	$-\pi/2$
5	θ_5	d_5	0	0

SCARA



<https://www.denso-wave.com/en/robot/product/function/Ctrack.html>

Axis	θ	d	a	α
1	θ_1	d_1	a_1	π
2	θ_2	0	a_2	0
3	0	d_3	0	0
4	θ_4	d_4	0	0

D-H参数定义为：

- θ_k 为 x_{k-1} 与 x_k 轴沿着 z_{k-1} 方向的夹角；
- d_k 为 x_{k-1} 与 x_k 轴沿着 z_{k-1} 方向的距离；
- a_k 为 z_{k-1} 与 z_k 轴沿着 x_k 方向的距离；
- α_k 为 z_{k-1} 与 z_k 轴沿着 x_k 方向的夹角。

SCARA

Axis	θ	d	a	α
1	θ_1	d_1	a_1	π
2	θ_2	0	a_2	0
3	0	d_3	0	0
4	θ_4	d_4	0	0

$$\Rightarrow T_{k-1}^k = \begin{bmatrix} \cos(\theta_k) & -\cos(\alpha_k)\sin(\theta_k) & \sin(\alpha_k)\sin(\theta_k) & a_k\cos(\theta_k) \\ \sin(\theta_k) & \cos(\alpha_k)\cos(\theta_k) & -\sin(\alpha_k)\cos(\theta_k) & a_k\sin(\theta_k) \\ 0 & \sin(\alpha_k) & \cos(\alpha_k) & d_k \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

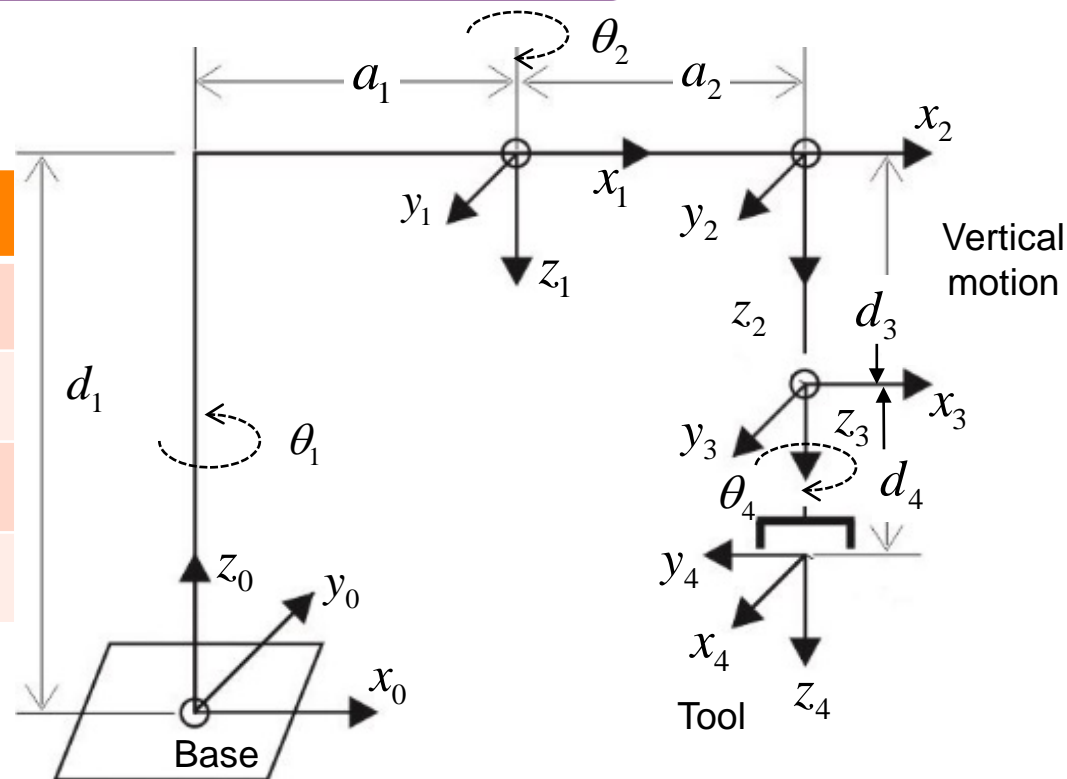
$$\Rightarrow T_o^1, T_1^2 \dots T_{n-1}^n$$

D-H法则



SCARA

Axis	θ	d	a	α
1	θ_1	d_1	a_1	π
2	θ_2	0	a_2	0
3	0	d_3	0	0
4	θ_4	d_4	0	0



→ T_{k-1}^k

→ $T_{base}^{tool} = T_o^1 \cdot T_1^2 \cdots T_{n-1}^n$

$$T_{base}^{tool} = \begin{bmatrix} C_{124} & S_{124} & 0 & a_1 C_1 + a_2 C_{12} \\ S_{124} & -C_{124} & 0 & a_1 S_1 + a_2 S_{12} \\ 0 & 0 & -1 & d_1 - d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$x_4 \quad y_4 \quad z_4$

Base

Tool

$$\begin{bmatrix} p_1^f \\ p_2^f \\ p_3^f \end{bmatrix} = \begin{bmatrix} m_1 \cdot f_1 & m_2 \cdot f_1 & m_3 \cdot f_1 \\ m_1 \cdot f_2 & m_2 \cdot f_2 & m_3 \cdot f_2 \\ m_1 \cdot f_3 & m_2 \cdot f_3 & m_3 \cdot f_3 \end{bmatrix} \begin{bmatrix} p_1^m \\ p_2^m \\ p_3^m \end{bmatrix}$$

$$C_{124} = \cos(\theta_1 + \theta_2 + \theta_4) \dots$$

旋转

平移

$$T_{base}^{tool} = \begin{bmatrix} \cos(q_1 + q_2) & \sin(q_1 + q_2) & l_1 \cos(q_1) + l_2 \cos(q_1 + q_2) \\ \sin(q_1 + q_2) & -\cos(q_1 + q_2) & l_1 \sin(q_1) + l_2 \sin(q_1 + q_2) \\ 0 & 0 & 1 \end{bmatrix}$$

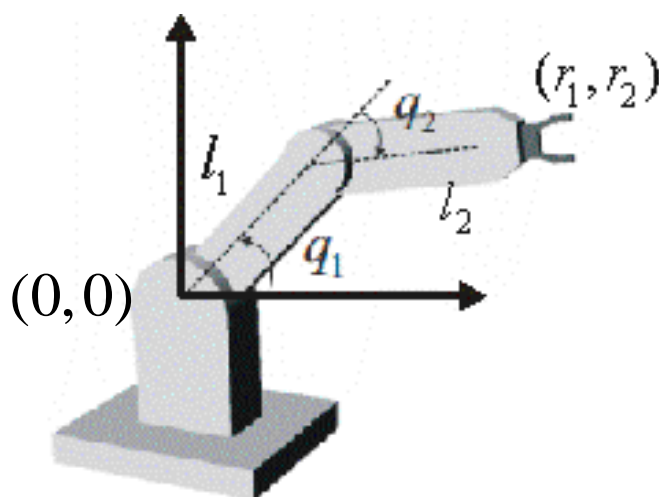
l_1 l_2 - 手臂长度

末端执行器相对于原点的坐标:

$$\begin{cases} r_1 = l_1 \cos(q_1) + l_2 \cos(q_1 + q_2) \\ r_2 = l_1 \sin(q_1) + l_2 \sin(q_1 + q_2) \end{cases}$$

表示为 $r = h(q)$

$h(\cdot)$ - 非线性函数



两自由度机械臂

速度关系:

$$\begin{cases} \dot{r}_1 = -l_1 \dot{q}_1 \sin(q_1) - l_2 (\dot{q}_1 + \dot{q}_2) \sin(q_1 + q_2) \\ \dot{r}_2 = l_1 \dot{q}_1 \cos(q_1) + l_2 (\dot{q}_1 + \dot{q}_2) \cos(q_1 + q_2) \end{cases}$$

矩阵表达式:

$$\begin{bmatrix} \dot{r}_1 \\ \dot{r}_2 \end{bmatrix} = \begin{bmatrix} -l_1 \sin(q_1) - l_2 \sin(q_1 + q_2) & -l_2 \sin(q_1 + q_2) \\ l_1 \cos(q_1) + l_2 \cos(q_1 + q_2) & l_2 \cos(q_1 + q_2) \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

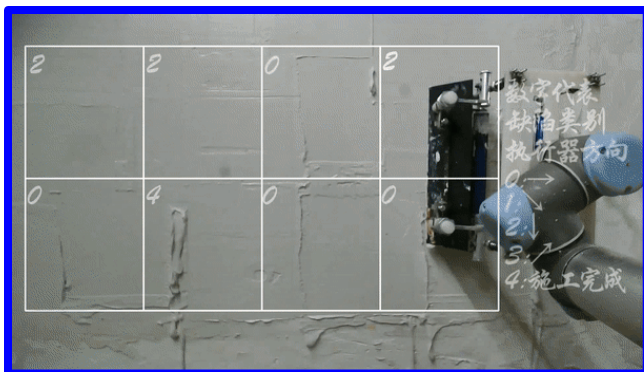
通用形式:

$$\dot{\mathbf{r}} = \mathbf{J}(\mathbf{q}) \dot{\mathbf{q}}$$

雅可比矩阵 (Jacobian Matrix)

- 代表关节空间到任务空间的映射

末端执行器



如何表征末端执行器的位置与姿态？

$$T_{base}^{tool} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & p_1 \\ R_{21} & R_{22} & R_{23} & p_2 \\ R_{31} & R_{32} & R_{33} & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

坐标系间平移

表征执行器位置？

OK

坐标系间旋转 → 表征执行器姿态？ → 信息冗余！

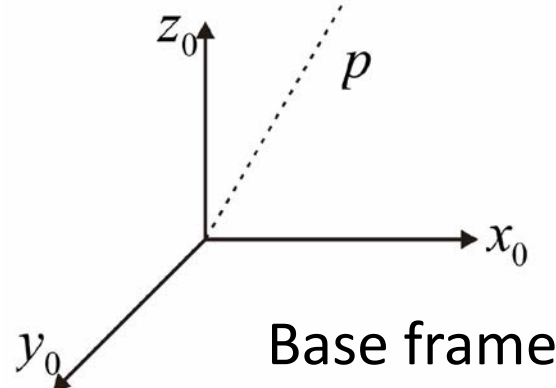
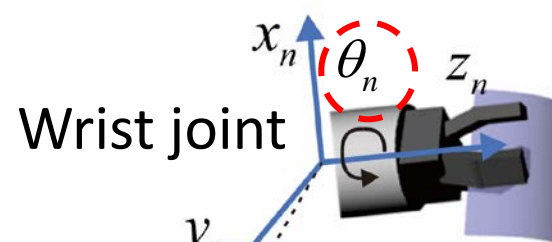
$$T_{base}^{tool} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & p_1 \\ R_{21} & R_{22} & R_{23} & p_2 \\ R_{31} & R_{32} & R_{33} & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Approaching vector

→ 用于表示末端执行器方向

→ 如何表示转角？

$$\begin{bmatrix} p_1^f \\ p_2^f \\ p_3^f \end{bmatrix} = \begin{bmatrix} m_1 \cdot f_1 & m_2 \cdot f_1 & m_3 \cdot f_1 \\ m_1 \cdot f_2 & m_2 \cdot f_2 & m_3 \cdot f_2 \\ m_1 \cdot f_3 & m_2 \cdot f_3 & m_3 \cdot f_3 \end{bmatrix} \begin{bmatrix} p_1^m \\ p_2^m \\ p_3^m \end{bmatrix}$$



前进方向 + 旋转角 → 末端姿态

$r_3 = [R_{13}, R_{23}, R_{33}]^T$ 单位向量，仅表示方向

➡ 将转角编码为**正值标量**与之相乘（不改变 r_3 表示的方向）

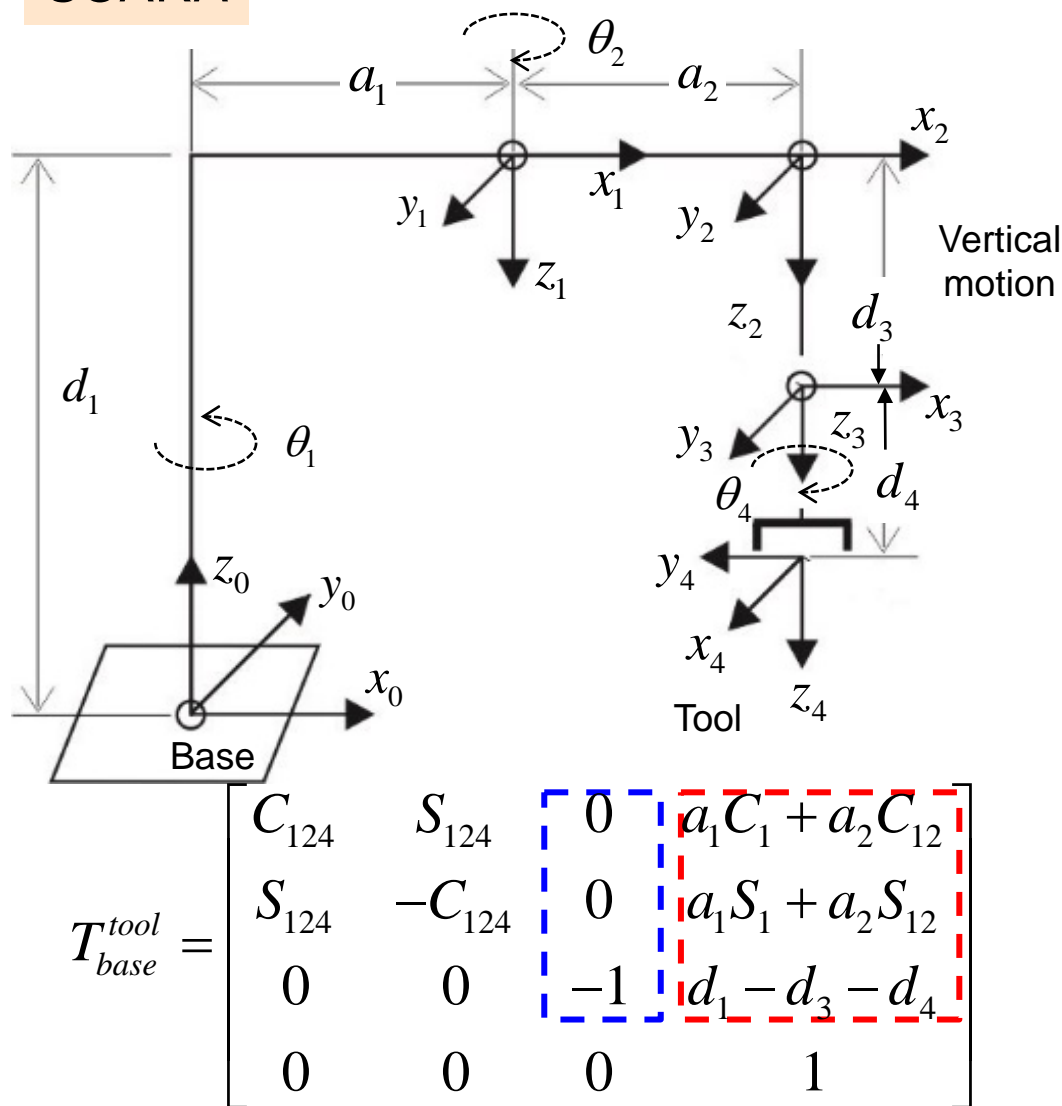
➡ $f(\theta_n) = e^{\theta_n/\pi}$

➡ $w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{bmatrix} = \begin{bmatrix} p \\ r_3 e^{\theta_n/\pi} \end{bmatrix} \left. \begin{array}{l} \} \mathbb{R}^3 \\ \} \mathbb{R}^3 \end{array} \right\} \text{最简表示}$

w_1, w_2, w_3 - 表示执行器的位置

w_4, w_5, w_6 - 表示执行器的姿态

SCARA



关节空间

$$q = [q_1, q_2, q_3, q_4]^T$$

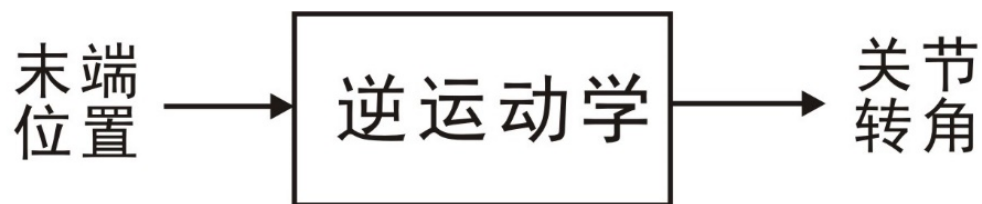
$$= [\theta_1, \theta_2, d_3, \theta_4]^T$$

末端执行器向量

$$w(q) = \begin{bmatrix} a_1 C_1 + a_2 C_{12} \\ a_1 S_1 + a_2 S_{12} \\ d_1 - q_3 - d_4 \\ 0 \\ 0 \\ -e^{q_4/\pi} \end{bmatrix}$$

$e^{q_4/\pi}$
末端转角

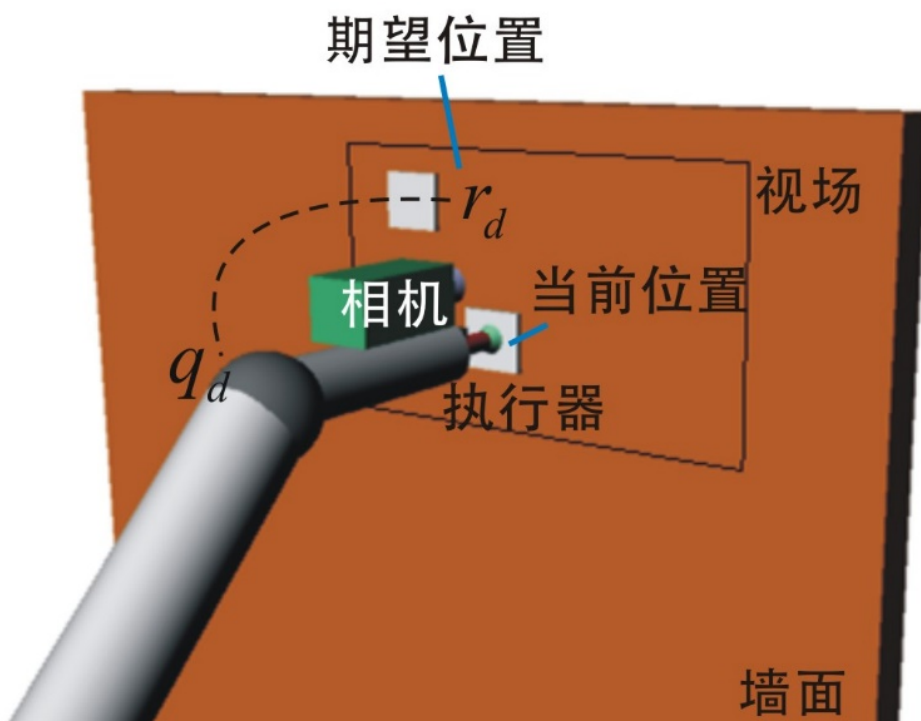
逆运动学 - 给定末端执行器期望位置，计算对应的机器人关节角度



正向: $r = h(q)$

逆向: $q = h^{-1}(r)$

$h^{-1}(\cdot)$ - 反函数



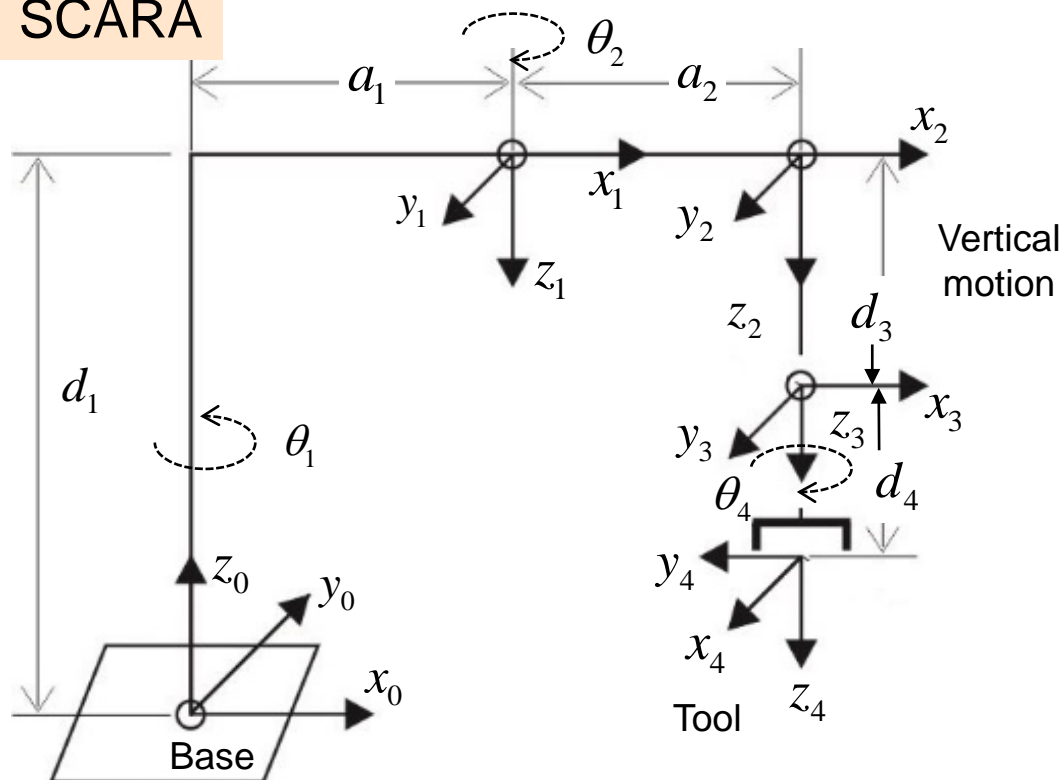
单关节机械臂运动学

$$\begin{cases} r_1 = l \cos(q) \\ r_2 = l \sin(q) \end{cases}$$

逆运动学

➔ $q = \cos^{-1}\left(\frac{r_1}{l}\right)$

SCARA



末端执行器位姿向量

$$w(q) = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{bmatrix} = \begin{bmatrix} a_1 C_1 + a_2 C_{1-2} \\ a_1 S_1 + a_2 S_{1-2} \\ d_1 - q_3 - d_4 \\ 0 \\ 0 \\ -e^{q_4/\pi} \end{bmatrix}$$

给定 $w(q)$

求解 q_1, q_2, q_3, q_4

$$\Rightarrow w_1^2 + w_2^2 = a_1^2 + 2a_1a_2C_2 + a_2^2$$

$$\Rightarrow q_2 = \pm \cos^{-1} \frac{w_1^2 + w_2^2 - a_1^2 - a_2^2}{2a_1a_2}$$

第二关节

$$\begin{cases} w_1 = (a_1 + a_2 C_2) C_1 + a_2 S_2 S_1 \\ w_2 = (a_1 + a_2 C_2) S_1 - a_2 S_2 C_1 \end{cases}$$

因 q_2 已求出



$$\begin{cases} S_1 = \frac{a_2 S_2 w_1 + (a_1 + a_2 C_2) w_2}{(a_2 S_2)^2 + (a_1 + a_2 C_2)^2} \\ C_1 = \frac{(a_1 + a_2 C_2) w_1 - a_2 S_2 w_2}{(a_2 S_2)^2 + (a_1 + a_2 C_2)^2} \end{cases}$$

➡ 第一关节

$$q_1 = \tan^{-1} \frac{a_2 S_2 w_1 + (a_1 + a_2 C_2) w_2}{(a_1 + a_2 C_2) w_1 - a_2 S_2 w_2}$$

$$w_3 = d_1 - q_3 - d_4 \quad \Rightarrow \quad q_3 = d_1 - w_3 - d_4 \quad \text{第三关节}$$

$$w_6 = -e^{q_4/\pi} \quad \Rightarrow \quad q_4 = \pi \ln(-w_6) \quad \text{第四关节}$$

解析法求解

- 无固定流程,
- 对于高自由度
- 对于未知的运

数值法求解

$$q = \arg \min \| w$$

启发式算法

Comparison of four different **heuristic** optimization algorithms for the **inverse kinematics** solution of a real 4-DOF serial **robot** manipulator

[M Ayyıldız](#), [K Çetinkaya](#) - Neural Computing and Applications, 2016 - Springer

In this study, a 4-degree-of-freedom (DOF) serial **robot** manipulator was designed and developed for the pick-and-place operation of a flexible manufacturing system. The solution of the **inverse kinematics** equation, one of the most important parts of the control process of ...

☆ 被引用次数: 43 相关文章 所有 6 个版本

A new **heuristic** approach for **inverse kinematics** of **robot** arms

[T Çavdar](#), [M Mohammad...](#) - Advanced Science Letters, 2013 - ingentaconnect.com

Inverse kinematics of a **robot** arm has become very important research area in recent decades. Also, the use of bio-inspired algorithms such as Genetic Algorithm (GA), Particle Swarm Optimization (PSO) and Harmony Search (HS) has been expeditiously increasing to ...

☆ 被引用次数: 17 相关文章 所有 6 个版本

IK-FA, a new **heuristic inverse kinematics** solver using firefly algorithm

[N Rokbani](#), [A Casals](#), [AM Alimi](#) - Computational Intelligence Applications ..., 2015 - Springer

... needed to produce the motion to $(q = (\theta_1, \dots, \theta_n))$; the **robot** position in ... to the end-segment of link (3). The fitness function used for both **heuristics** is given ... In this paper a new **heuristic** method for **inverse kinematics** based on Firefly Algorithm is proposed, IK-FA ...

☆ 被引用次数: 19 相关文章 所有 7 个版本

A **heuristic** approach to the **inverse differential kinematics** problem

[U Beyer](#), [F Śmieja](#) - Journal of Intelligent and Robotic Systems, 1997 - Springer

Inversion of the **kinematics** of manipulators is one of the central problems in the field of **robot** arm control. The iterative use of **inverse differential kinematics** is a popular method of solving this task. Normally the solution of the problem requires a complex mathematical apparatus. It ...

☆ 被引用次数: 13 相关文章 所有 10 个版本

Trajectory optimization for redundant **robots** using genetic algorithms with **heuristic** operators

前向运动学 对时间求导

$$r = h(q)$$



$$\underset{m \times 1}{\dot{r}} = \underset{m \times n}{J(q)} \underset{n \times 1}{\dot{q}}$$

$m = n$ (非冗余机器人)



逆矩阵 $J^{-1}(q)$

$m < n$ (冗余机器人)



广义矩阵

$$\underset{n \times m}{J^+(q)} = \underset{n \times m}{J^T(q)} (\underset{m \times m}{J(q) J^T(q)})^{-1}$$

单位矩阵

$$\longrightarrow J(q) J^+(q) = J(q) J^T(q) (J(q) J^T(q))^{-1} = \boxed{I_m}$$

- 在某些关节角度， $J(q)$ 不满秩，导致逆矩阵（或广义逆阵）不存在
- 即笛卡尔空间的微小运动需要关节无穷大的转速

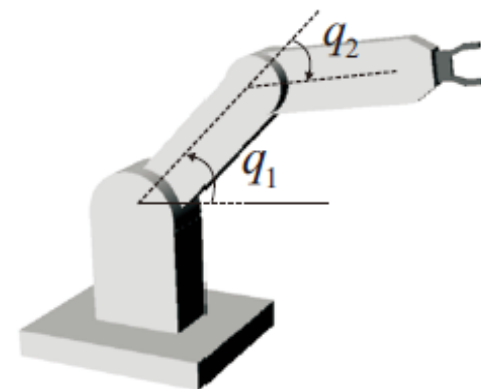
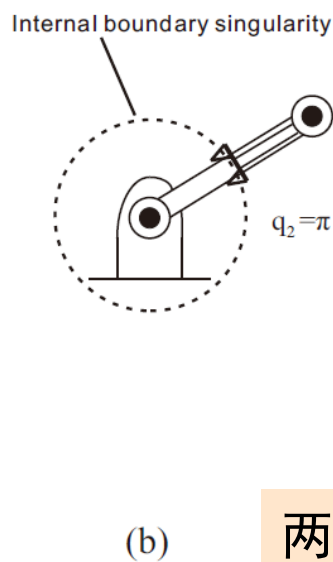
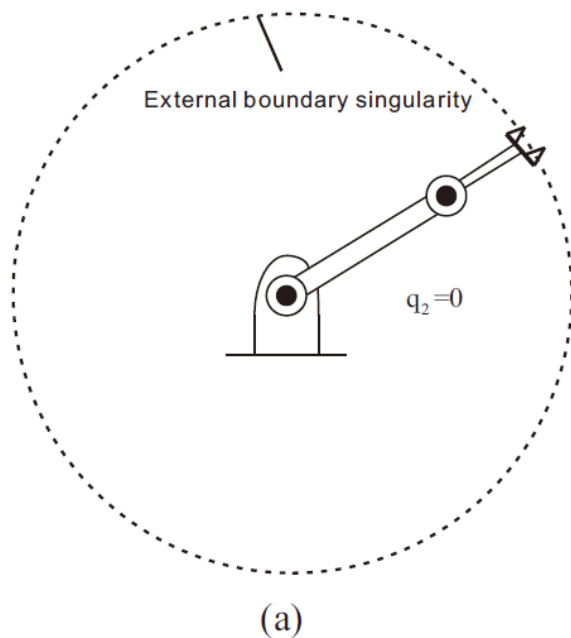
$$J^{-1}(q) \qquad J^{+}(q)=J^{T}(q)(J(q)J^{T}(q))^{-1}$$

- 通过求解 $\det(J(q))=0$ （无冗余） $\det(J(q)J^{T}(q))=0$ （有冗余）
- 机器人奇异问题可以被分为外部奇异问题和内部奇异问题。外部奇异发生在工作空间外部边界，内部奇异包括内边界奇异和内奇异。

当 $\det[J(\mathbf{q})] = l_1 l_2 \sin(q_2) = 0$ 时该矩阵是奇异的。

奇异问题出现在 $q_2 = 0$ (外边界奇异点) 和 $q_2 = \pi$ (内边界奇异点)

$$J(\mathbf{q}) = \begin{bmatrix} -l_1 \sin(q_1) - l_2 \sin(q_1 + q_2) & -l_2 \sin(q_1 + q_2) \\ l_1 \cos(q_1) + l_2 \cos(q_1 + q_2) & l_2 \cos(q_1 + q_2) \end{bmatrix} \in \mathbb{R}^{2 \times 2}$$



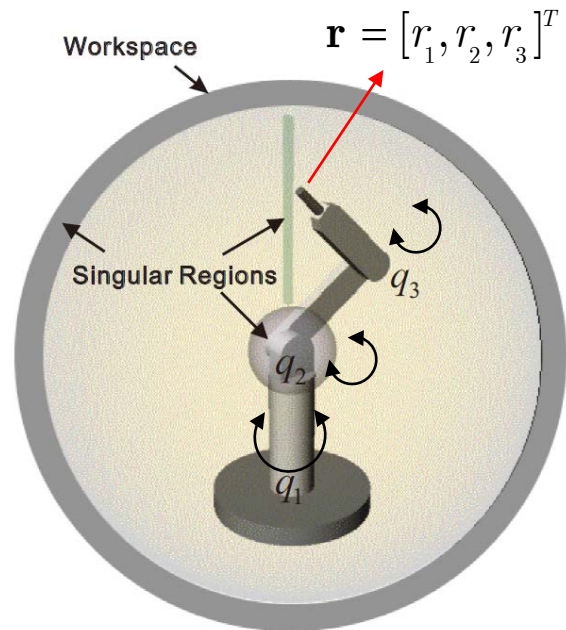
考虑一个3自由度机械臂。用 $\mathbf{r} = [r_1, r_2, r_3]^T$ 表示笛卡尔空间中末端执行器的位置，得到前向运动学方程

$$\mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} \cos(q_1) [l_2 \cos(q_2) + l_3 \cos(q_2 + q_3)] \\ \sin(q_1) [l_2 \cos(q_2) + l_3 \cos(q_2 + q_3)] \\ l_1 - l_2 \sin(q_2) - l_3 \sin(q_2 + q_3) \end{bmatrix} \in \mathbb{R}^3$$

得到雅可比矩阵：

$$\mathbf{J}(\mathbf{q}) = \begin{bmatrix} -s_1 (l_2 c_2 + l_3 c_{23}) & -s_2 c_1 l_2 - s_{23} l_3 c_1 & -s_{23} l_3 c_1 \\ -c_1 (l_2 c_2 + l_3 c_{23}) & -s_2 s_1 l_2 - s_{23} l_3 s_1 & -s_{23} l_3 s_1 \\ 0 & -c_2 l_2 - c_{23} l_3 & -c_{23} l_3 \end{bmatrix} \in \mathbb{R}^{3 \times 3}$$

其中 $s_1 = \sin(q_1)$, $s_{23} = \sin(q_2 + q_3)$, $c_{23} = \cos(q_2 + q_3)$...



考虑一个3自由度机械臂。

$$J(\mathbf{q}) = \begin{bmatrix} -s_1(l_2 c_2 + l_3 c_{23}) & -s_2 c_1 l_2 - s_{23} l_3 c_1 & -s_{23} l_3 c_1 \\ -c_1(l_2 c_2 + l_3 c_{23}) & -s_2 s_1 l_2 - s_{23} l_3 s_1 & -s_{23} l_3 s_1 \\ 0 & -c_2 l_2 - c_{23} l_3 & -c_{23} l_3 \end{bmatrix} \in \mathbb{R}^{3 \times 3}$$

雅可比矩阵在 $\det[\mathbf{J}(\mathbf{q})] = [l_2 \cos(q_2) + l_3 \cos(q_2 + q_3)] l_3 l_2 \sin(q_3) = 0$ 时奇异。

即 $q_3 = 0$ (外边界奇异点)

$q_3 = \pi$ (内边界奇异点)

$l_2 \cos(q_2) + l_3 \cos(q_2 + q_3) = 0$ (内奇异点)

