单元1.7 哈密顿图

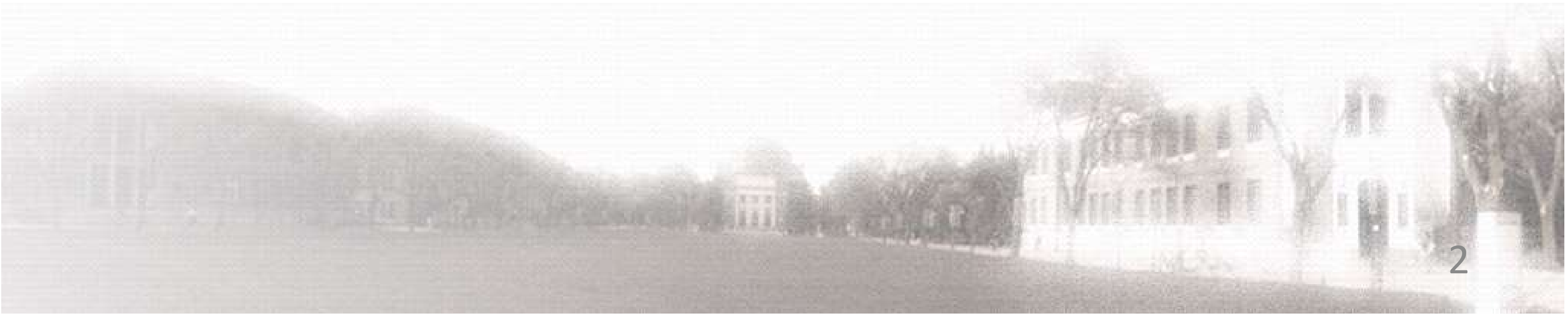
第15章 欧拉图与哈密顿图

15.2 哈密顿图

讲义参考北京大学《离散数学》及电子科技大学《离散数学》讲义

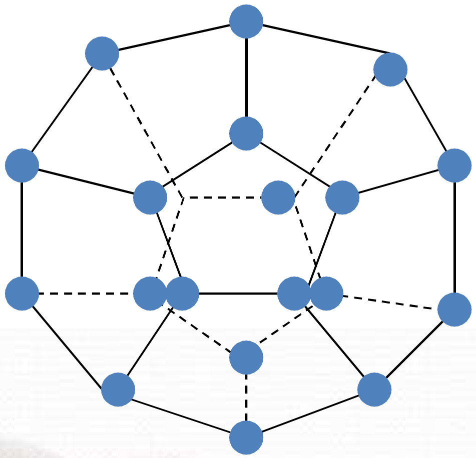
内容提要

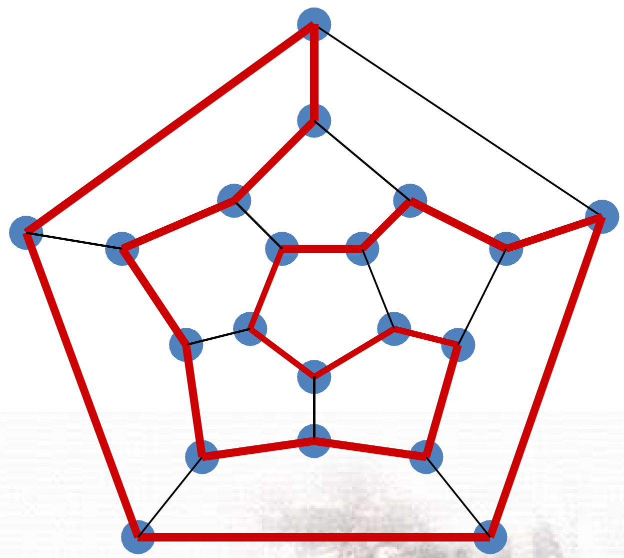
- 哈密顿回路、哈密顿通路
- 哈密顿图、半哈密顿图
- 哈密顿图的必要条件
- 半哈密顿图的必要条件
- 哈密顿图的充分条件
- 半哈密顿图的充分条件



周游世界

- **Sir William Rowan Hamilton, 1857, Icosian game:**



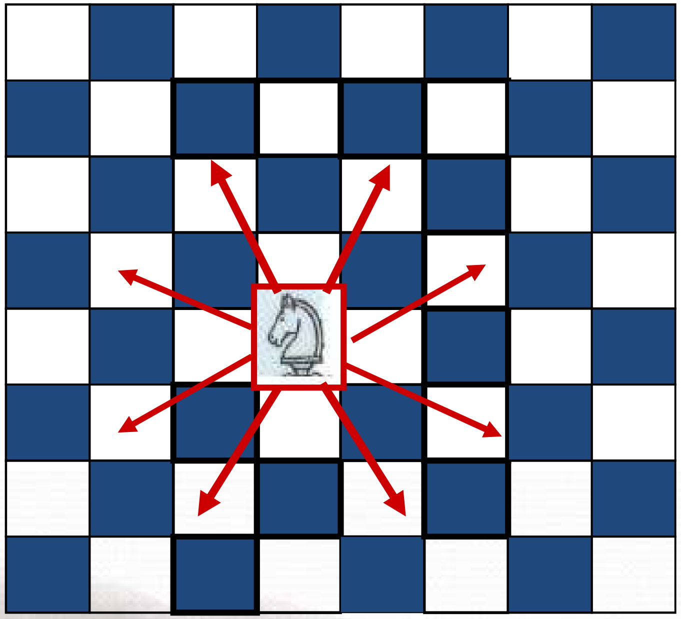


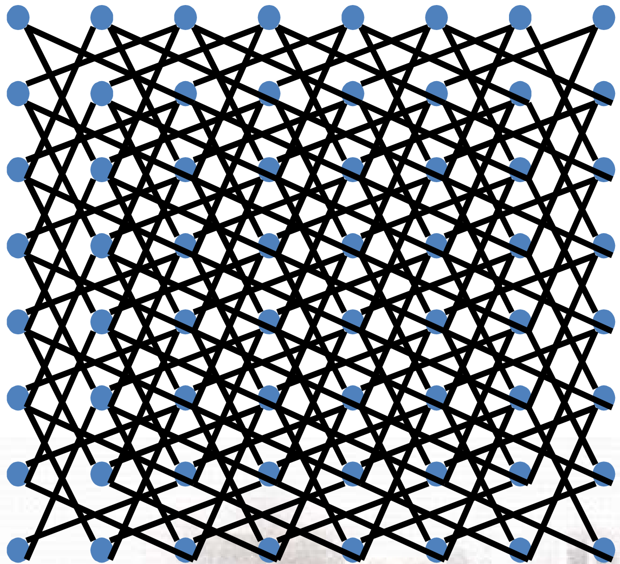
Willam Rowan Hamilton (1805~1865)

- 爱尔兰神童**(child prodigy)**
- 三一学院**(Trinity college Dublin)**
- 光学**(optics)**
- **1827, Astronomer Royal of Ireland.**
- **1837,** 复数公理化**, a+bi**
- 四元数**(quaternion): a+bi+cj+dk,**
  放弃乘法交换律**!**

马的周游路线(knight’s tour)

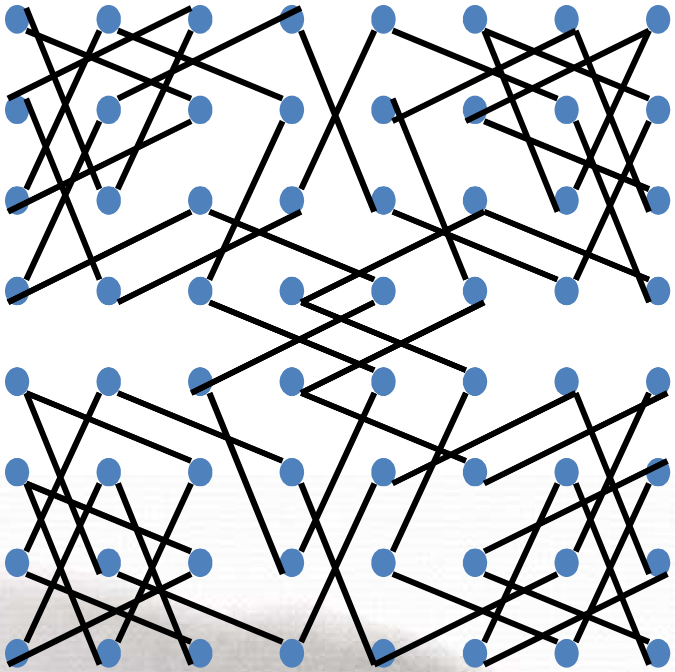
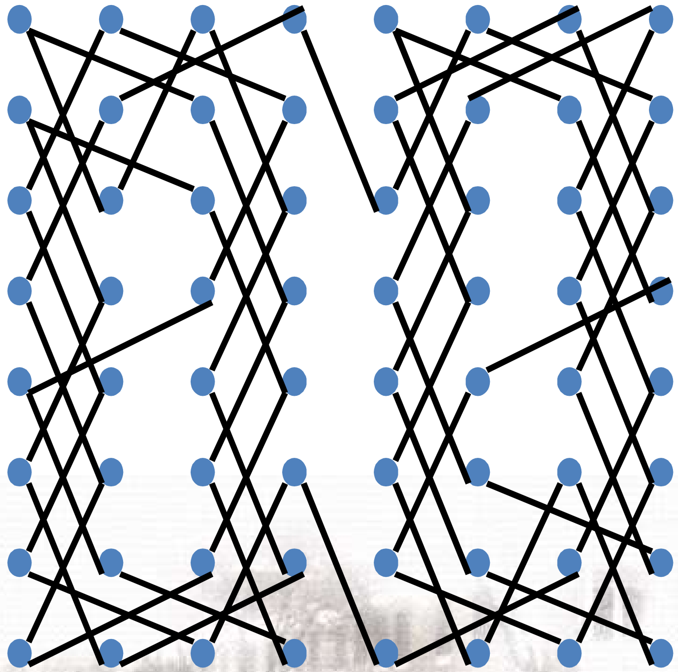
- **Leohard Euler, 1759**





马的周游路线(knight’s tour)

- **Leohard Euler, 1759,** 详细分析

哈密顿通(回)路、(半)哈密顿图

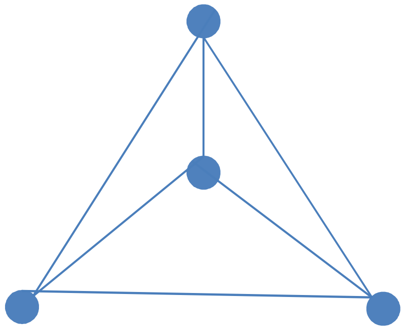
- 哈密顿通路：经过图中所有顶点的初级通路
- 半哈密顿图：有哈密顿通路的图

- 哈密顿回路：经过图中所有顶点的初级回路
- 哈密顿图：有哈密顿回路的图
  （定义平凡图为哈密顿图）

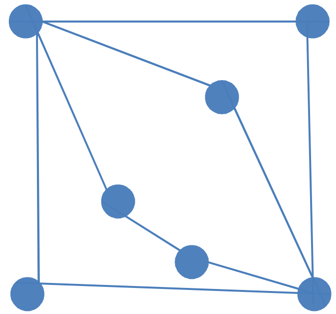
- 如果仅用点来描述，哈密顿通路就是图中所有结点的一种全排列

哈密顿通(回)路、(半)哈密顿图

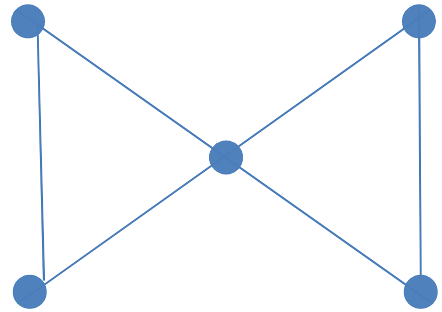
- 判断如下各图是否为（半）哈密顿图



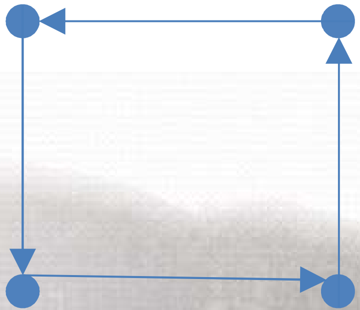
G1



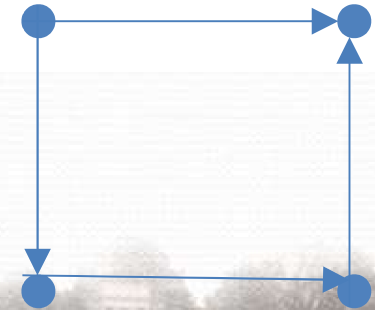
G2



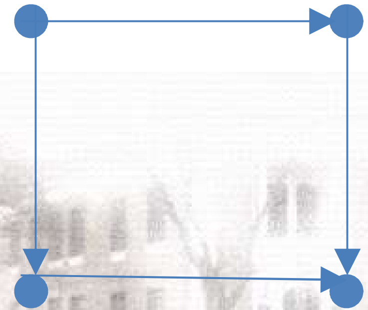
G3



D1



D2

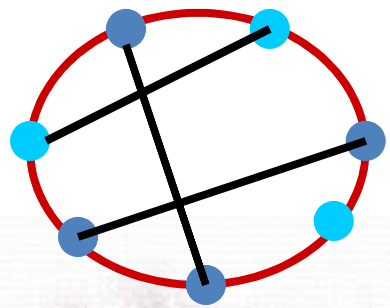


D3

无向哈密顿图的必要条件

- 定理15.6: 设G=<V,E>是无向哈密顿图, 则对V的任意非空真子集V_1有 $p(G-V_1) \leq |V_1|$。
- 证明：设C是G中任意哈密顿回路, 当V_1中顶点在C中都不相邻时, $p(C-V_1)=|V_1|$最大;

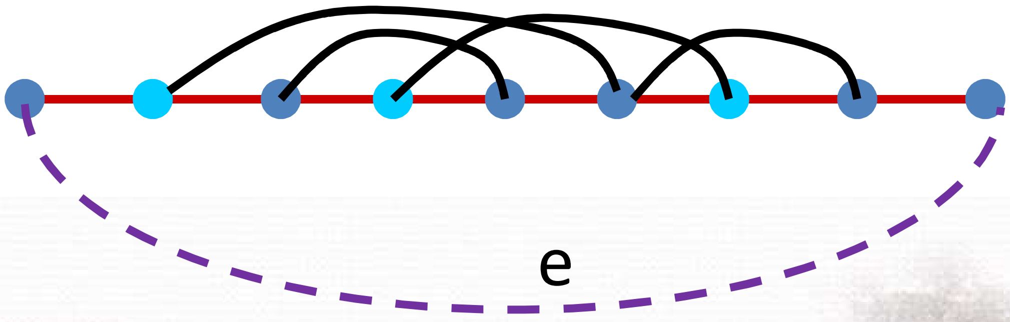
  若V_1中顶点相邻, $p(C-V_1)<|V_1|$. C是G的生成子图, 所以$p(G-V_1) \leq P(C-V_1) \leq |V_1|$. #



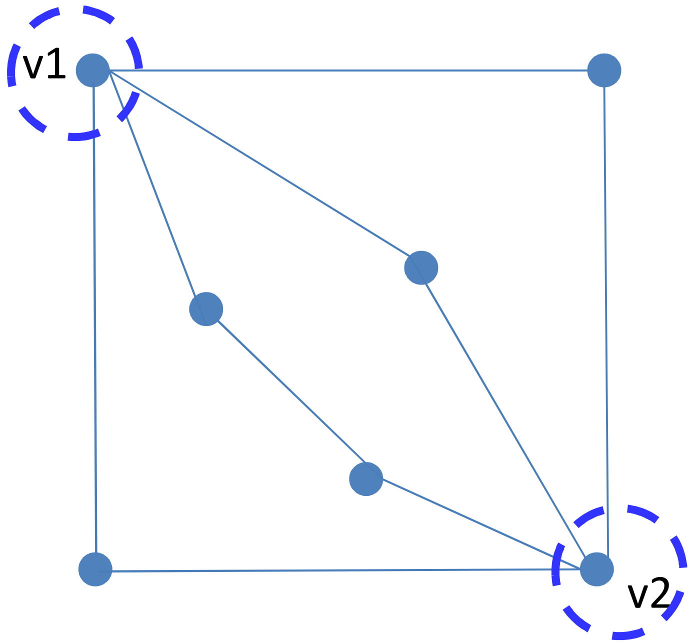
无向半哈密顿图的必要条件

- 推论: 设$G=\langle V,E\rangle$是无向半哈密顿图,则对V的任意非空真子集V_1有

$$p(G-V_1)\le|V_1|+1 \quad \#$$

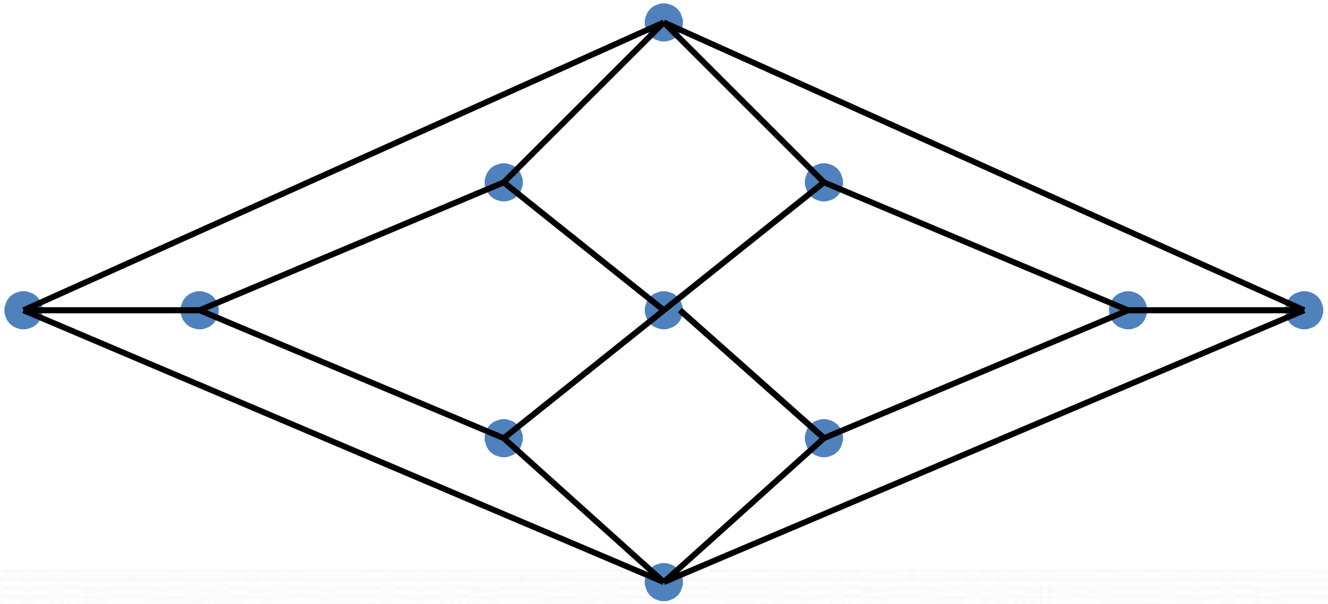


判断是否哈密顿图

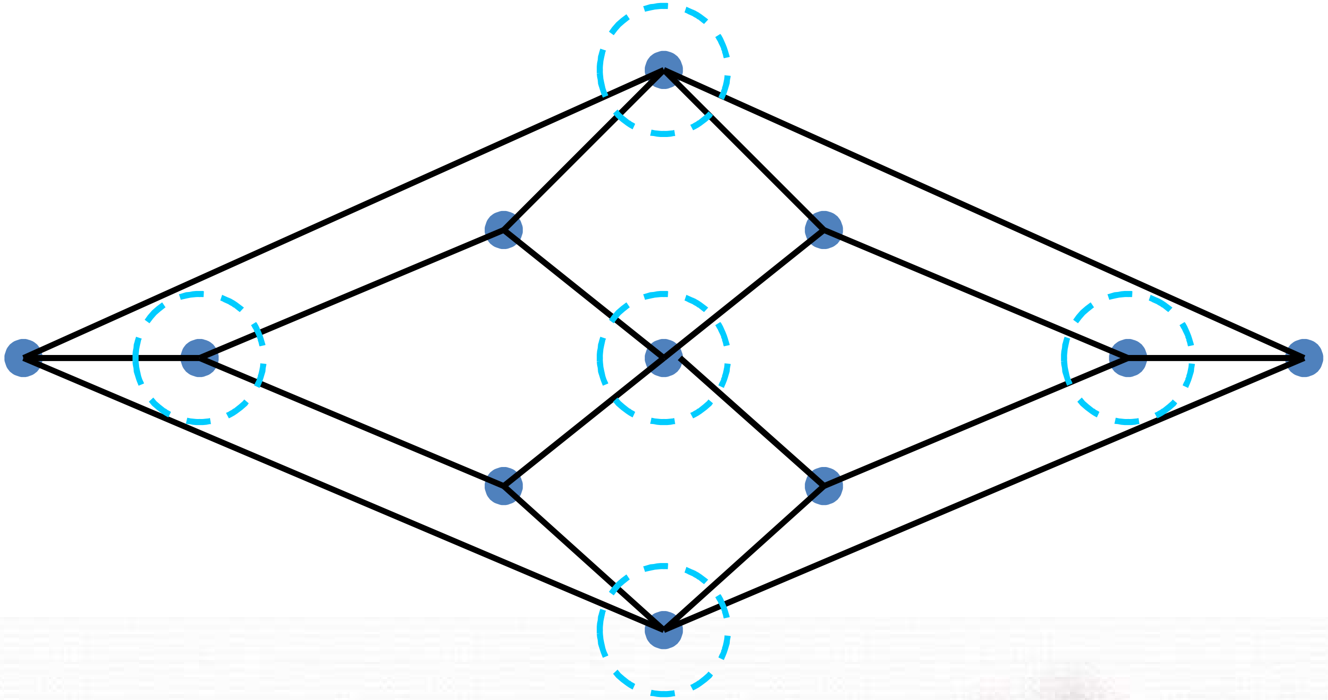


寻找子集V_1使得 $p(G-V_1) > |V_1|$

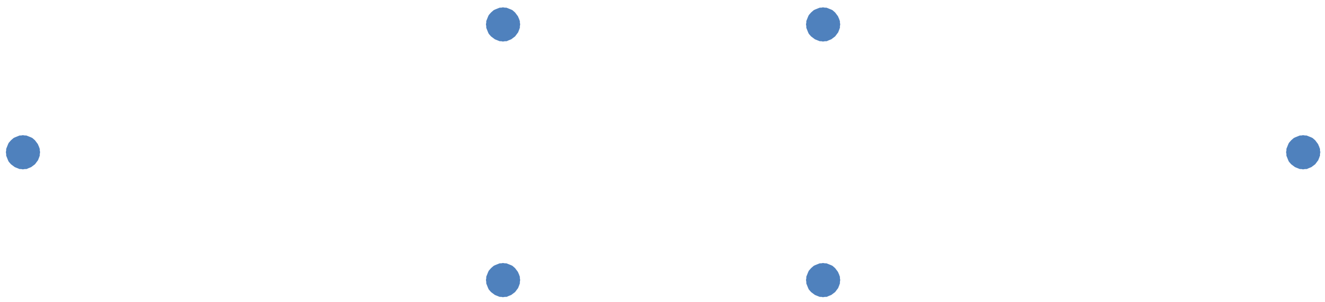
判断是否哈密顿图



选择V_1

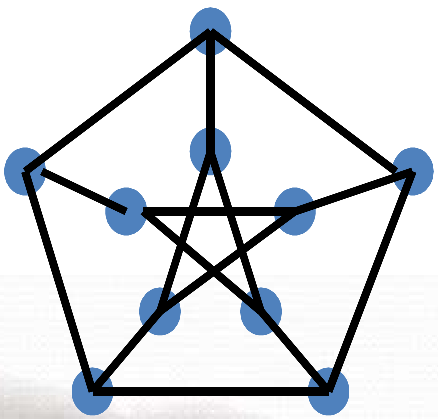


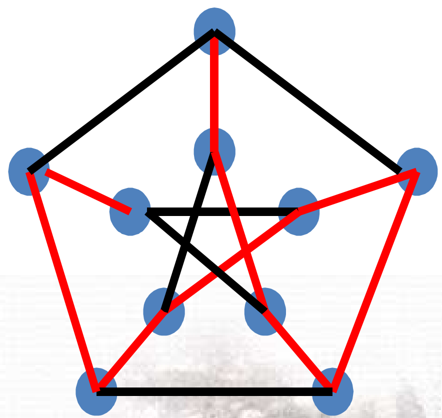
$p(G-V_1)=6 > 5=|V_1|$



非充分条件的反例

- **Petersen图**
  - $\forall V_1 \neq \varnothing$, $p(G-V_1) \leq |V_1|$
  - 不是哈密顿图, 是半哈密顿图





https://www.zhihu.com/question/39223140

无向半哈密顿图的充分条件

- 定理15.7: 设G是n(≥2)阶无向简单图, 若对G中任意不相邻顶点u与v有

$$d(u)+d(v)\geq n-1$$

则G是半哈密顿图.

- 证: 只需证明

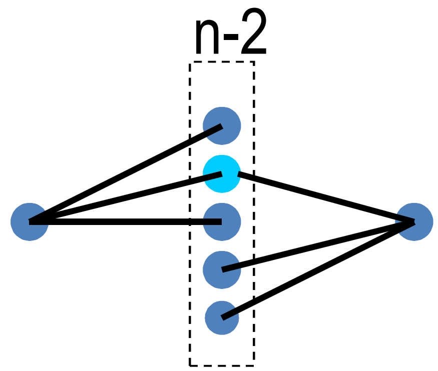
(1) G连通

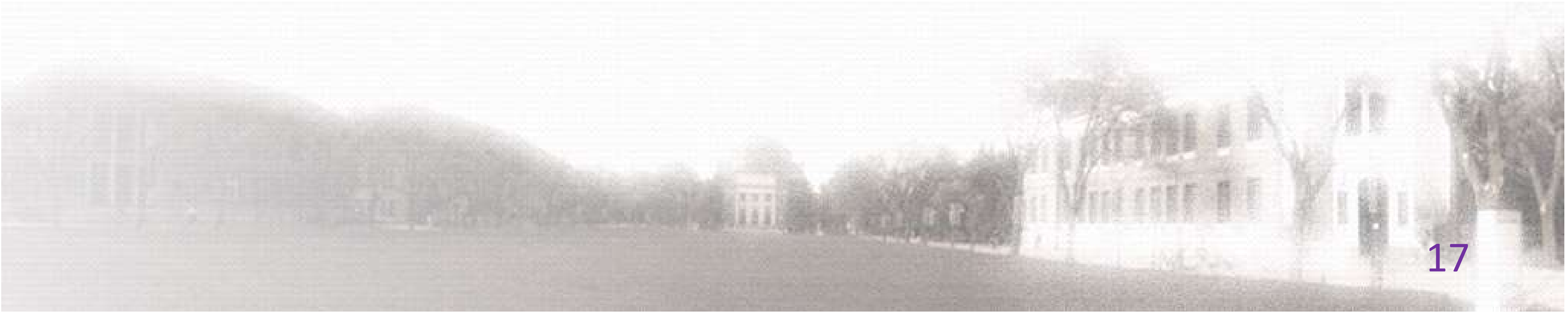
(2) 由极大路径可得圈

(3) 由圈可得更长路径

定理15.7证明(1)

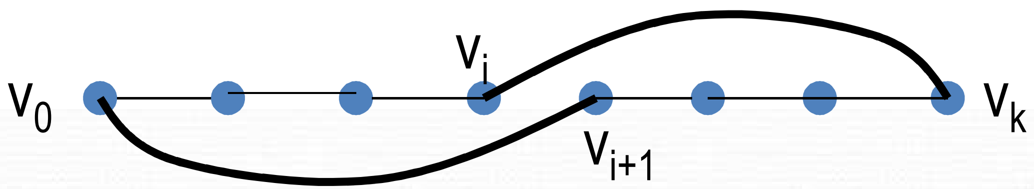
- **(1) G连通: $\forall u \forall v(\ (u,v)\notin E \rightarrow \exists w((u,w)\in E \wedge (w,v)\in E\ )$**





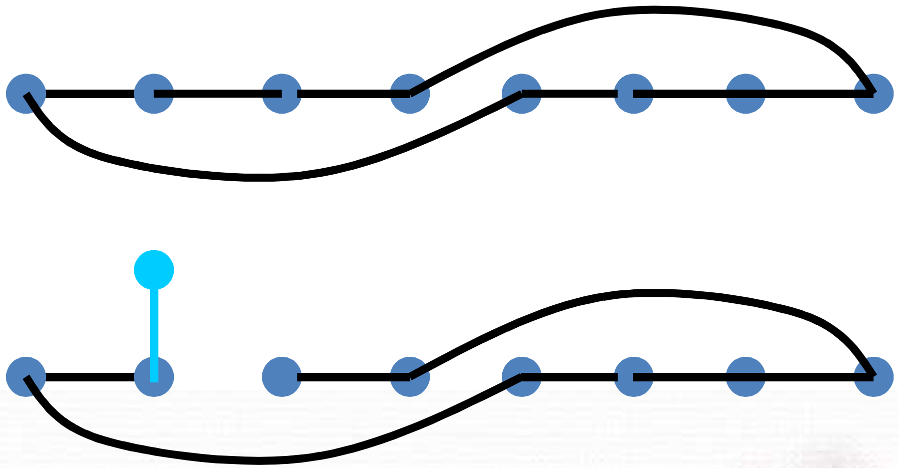
定理15.7证明(2)

- 设极大路径 $\Gamma=v_0v_1\ldots v_k$, $k\le n-2$. 若$(v_0,v_k)\notin E$, 则 $\exists i(\ 1\le i\le k-1 \wedge (v_i,v_k)\in E \wedge (v_0,v_{i+1})\in E\ )$,
否则, $d(v_0)+d(v_k)$
$\le d(v_0)+k-1-(d(v_0)-1)=k\le n-2$(矛盾).
于是得圈$C=v_0\ldots v_iv_kv_{k-1}\ldots v_{i+1}v_0$.



定理15.7证明(3)

- **(3)** 由圈得更长路径**:** 连通**. #**



无向哈密顿图的充分条件一

- 推论1: 设G是n(≥3)阶无向简单图,若对G中任意不相邻顶点u与v有

$$d(u)+d(v)\geq n$$

则G是哈密顿图.

- 证: 由定理15.7，G连通且有哈密顿通路 $\Gamma=v_0v_1\ldots v_{n-1}$.

若$(v_0,v_{n-1})\in E$, 则得哈密顿回路

$$C=v_0v_1\ldots v_{n-1}v_0.$$

若$(v_0,v_k)\notin E$, 则与定理15.7证明(2)类似, 也存在哈密顿回路. #

无向哈密顿图的充分条件二

- 推论2: 设G是n(≥3)阶无向简单图,若对G中任意顶点u有

$$d(u) \ge n/2$$

则G是哈密顿图. #

- 定理15.8: 设u,v是无向n阶简单图G中两个不相邻顶点,且d(u)+d(v)≥n, 则

$$G\text{是哈密顿图} \Leftrightarrow$$

$$G \cup (u,v)\text{是哈密顿图}. \ \#$$

Open Question Points: 10

Setting

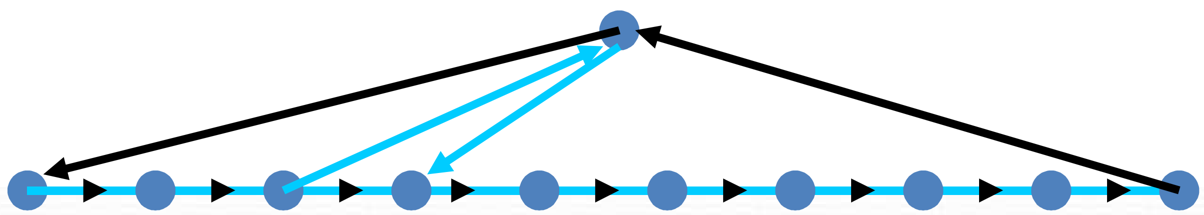
某个景区有5个景点，每个景点均有2条路与其他景点相连。问游人是否可以经过每个景点恰好一次而走完所有5个景点?

Open Question is only supported on Version 2.0 or newer.

Answer

有向半哈密顿图的充分条件

- 定理15.9: 设D是n(≥2)阶竞赛图, 则D是半哈密顿图. #
- 推论:设D是n阶有向图, 若D含n阶竞赛图作为生成子图, 则D是半哈密顿图. #

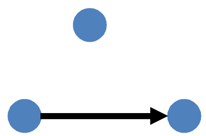


有向哈密顿图的充分条件

- 定理**:** 强连通的竞赛图是哈密顿图.
- 证**: n=1**时**,**平凡图是哈密顿图.

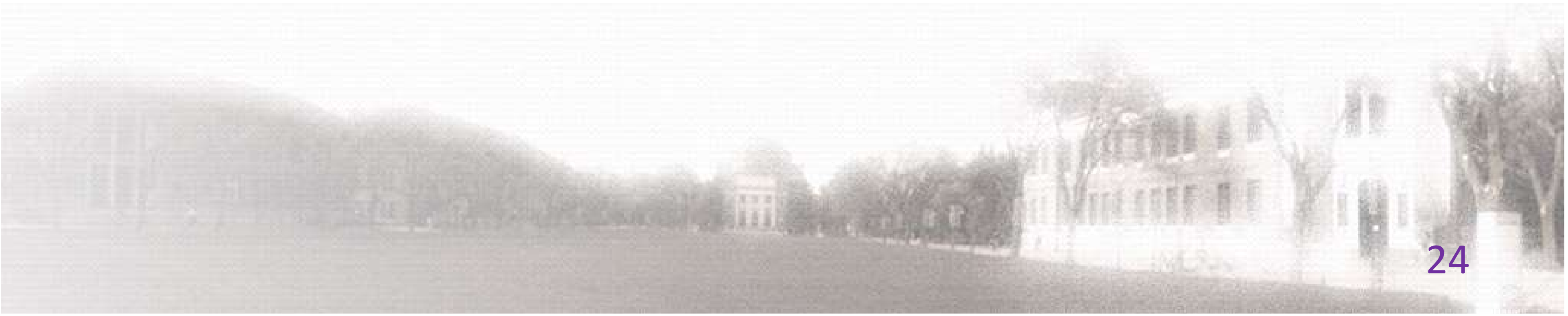
  n=2时**,**不可能强连通.

  下面设**n**$\geq$**3.** 只需证明**:**



(1) D中存在长度为**3**的圈**.**

(2) D中存在长度为**k**的圈 $\Rightarrow$ **D**中存在长度为**k+1**的圈**.**



定理证明(1)

- 证： $\forall v \in V(D)$,

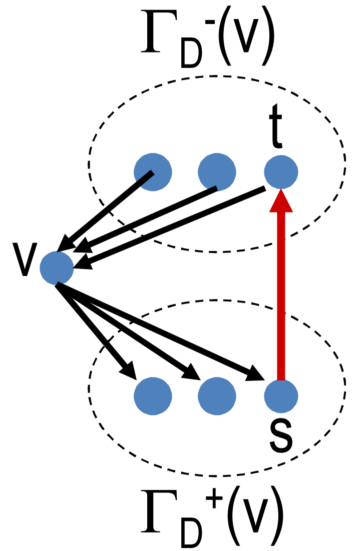
D竞赛图$\Rightarrow \Gamma_D^-(v) \cup \Gamma_D^+(v) = V(D)-\{v\}$

D强连通 $\Rightarrow \Gamma_D^-(v) \neq \varnothing$, $\Gamma_D^+(v) \neq \varnothing$.

D强连通 $\Rightarrow \exists s \in \Gamma_D^+(v)$, $\exists t \in \Gamma_D^-(v)$,

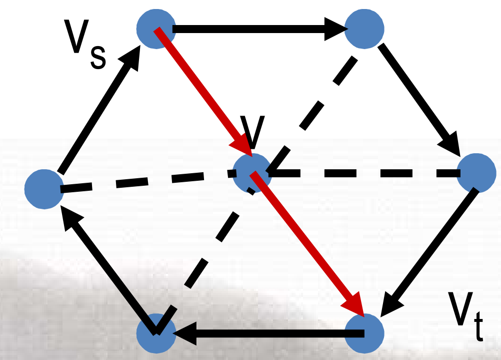
$\langle s,t \rangle \in E(D)$.

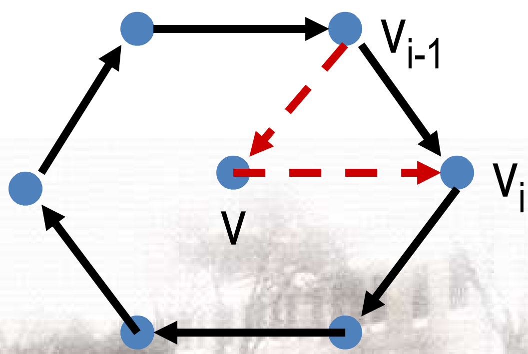
于是C=vstv是长度为3的圈.



定理证明(2)

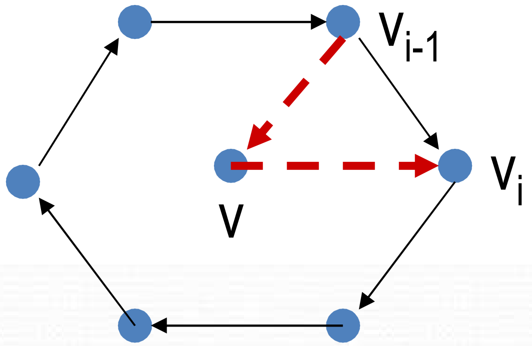
- 设D中有圈$C=v_1v_2\ldots v_kv_1$, $(3\le k\le n)$
  若$\exists v\in V(D-C)$, $\exists v_s,v_t\in V(C)$,
  使得 $\langle v_s v\rangle\in E(D)$, $\langle v,v_t\rangle\in E(D)$,
  则 $\exists v_{i-1},v_i\in V(C)$,
  使得$\langle v_{i-1},v\rangle\in E(D)$,$\langle v,v_i\rangle\in E(D)$.

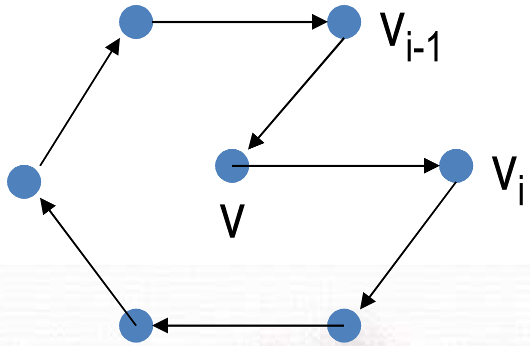




定理证明(2)

- 则 $C'=v_1v_2\ldots v_{i-1}vv_i\ldots v_kv_1$ 是长度为$k+1$的圈.





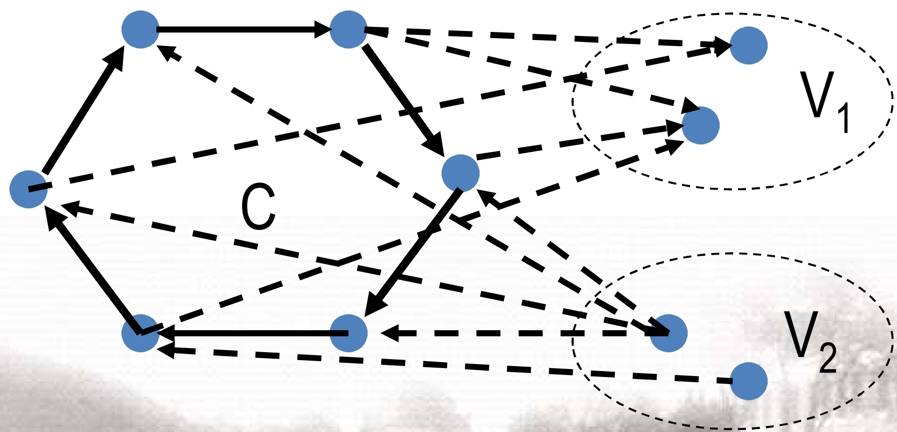
定理证明(2)

- 否则, 令

$V_1=\{v\in V(D-C) \mid \forall u\in V(C), \langle u,v\rangle\in E(D) \}$

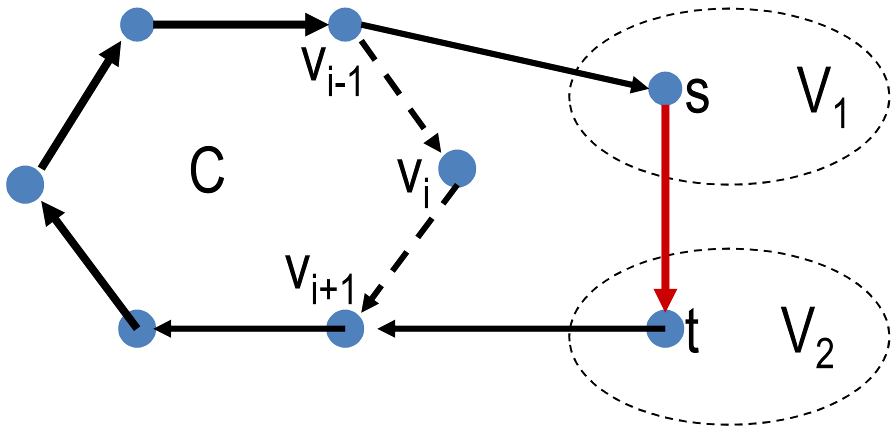
$V_2=\{v\in V(D-C) \mid \forall u\in V(C), \langle v,u\rangle\in E(D) \}$

则 $V_1\neq\varnothing, V_2\neq\varnothing, V_1\cap V_2 = \varnothing$.



定理证明(2)

- 于是$\exists s\in V_1$, $\exists t\in V_2$, $<s,t>\in E(D)$. 在C上任取相邻3点v_{i-1},v_i,v_{i+1}, 则$C'=v_1v_2\ldots v_{i-1}stv_{i+1}\ldots v_kv_1$ 是长度为k+1的圈. #

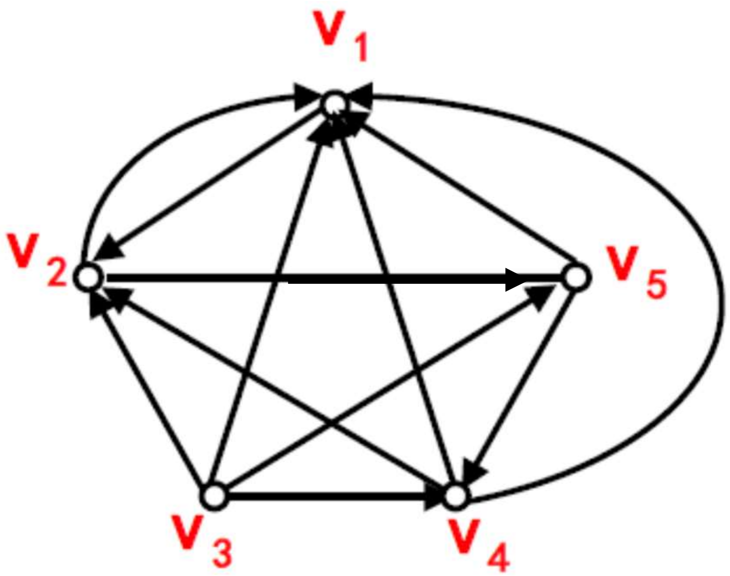


- 推论：设D是n阶有向图, 若D含n阶强连通竞赛图作为生成子图, 则D是哈密顿图. #

Open Question Points: 10

Setting

下图是否有哈密顿通路?



Open Question is only supported on Version 2.0 or newer.

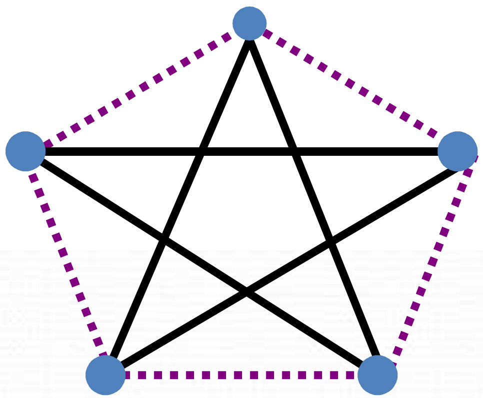
Answer

边不重的哈密顿回路

（非考试范围内）

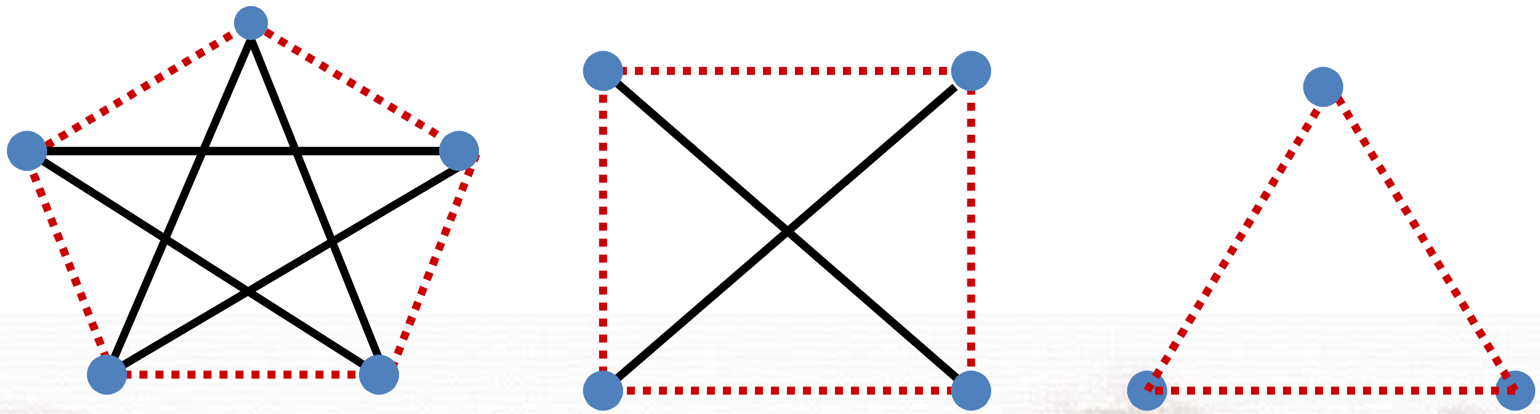
C_1与C_2都是图G的哈密顿回路

$E(C_1) \cap E(C_2)=\varnothing$



问题

- $K_n(\geq 3)$中同时存在多少条边不重的哈密顿回路?



定理

- 完全图$K_{2k+1}(k\geq 1)$中同时有k条边不重的哈密顿回路,且这k条边不重的哈密顿回路含有K_{2k+1}中所有边
- 证: 设$V(K_{2k+1})=\{v_1,v_2,\ldots,v_{2k+1}\}$,

  对 $i=1,2,\ldots,k$, 令 $P_i=v_iv_{i-1}v_{i+1}v_{i-2}v_{i+2}\ldots v_{i-(k-1)}v_{i+(k-1)}v_{i-k}$,

  下标mod(2k)转换到$\{1,2,\ldots,2k+1\}$中，0转换成2k.

  令$C_i=v_{2k+1}P_iv_{2k+1}$. 可以证明:

  (1) C_i都是哈密顿回路,

  (2) $E(C_i)\cap E(C_j)=\varnothing$ $(i\neq j)$,

  (3) $\cup_{i=1}^{n}E(C_i)=E(K_{2k+1})$. #

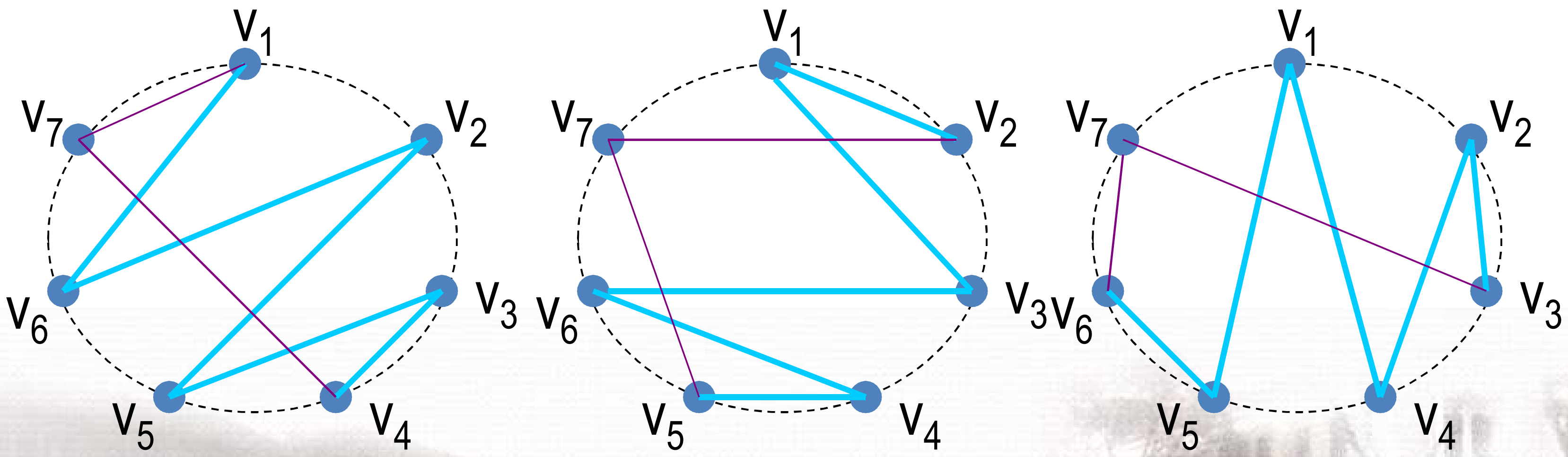
定理举例: K_7

$V(K_7)=\{v_1,v_2,\ldots,v_7\}$, $k=3$, mod 6

$P_1=v_1v_0v_2v_{-1}v_3v_{-2}=v_1v_6v_2v_5v_3v_4$,

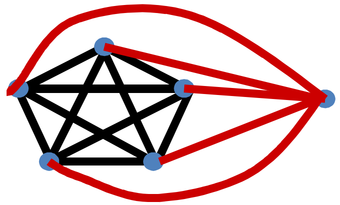
$P_2=v_2v_1v_3v_0v_4v_{-1}=v_2v_1v_3v_6v_4v_5$,

$P_3=v_3v_2v_4v_1v_5v_0=v_3v_2v_4v_1v_5v_6$,



定理推论

- 完全图K_{2k}($k\ge 2$)中同时有**k-1**条边不重的哈密顿回路, 除此之外, 剩下的是**k**条彼此不相邻的边
- 证: $k=2$时, K_4显然. 下面设$k\ge 3$.



$K_{2k} = K_{2(k-1)+1} + K_1$ (联图)

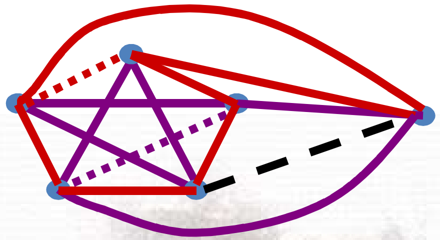
设$V(K_{2(k-1)+1}) = \{v_1,v_2,\ldots,v_{2k-1}\}$, $V(K_1)=\{v_{2k}\}$.

K_{2k-1}中有$k-1$条边不重的哈密顿回路,

设为$C'_1, C'_2,\ldots,C'_{k-1}$,

依次把v_{2k} "加入" C'_i,

得到满足要求的C_i. #



小结

- 欧拉图 **Easy**
  - 充要条件
- 哈密顿图 **Hard**
  - 必要条件
  - 充分条件