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Swing-up and stabilization of a cart–pendulum system under restricted cart track length

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Abstract

This paper describes the swing-up and stabilization of a cart–pendulum system with a restricted cart track length and restricted control force using generalized energy control methods. Starting from a pendant position, the pendulum is swung up to the upright unstable equilibrium configuration using energy control principles. An "energy well" is built within the cart track to prevent the cart from going outside the limited length. When sufficient energy is acquired by the pendulum, it goes into a "cruise" mode when the acquired energy is maintained. Finally, when the pendulum is close to the upright configuration, a stabilizing controller is activated around a linear zone about the upright configuration. The proposed scheme has worked well both in simulation and a practical setup and the conditions for stability have been derived using the multiple Lyapunov functions approach.

Keywords: Multi-mode control; Control of nonlinear systems; Control of hybrid systems

1. Introduction

The swing-up and stabilization of a cart–pendulum system is a ubiquitous experiment in most control laboratories around the world. This control problem involves swinging up a pendulum from its normal pendant configuration by the application of a control force to the cart; when the pendulum approaches the upright configuration, the control is switched to a stabilizing controller (usually linear) which maintains it about the unstable equilibrium configuration.

The inverted pendulum on a cart is an under-actuated mechanical system with two degrees of freedom and one control input. Many methods for achieving swing-up and stabilization of this system have been proposed in literature. In [11], a non-linear control law has been applied to a pendulum with restricted travel by decomposing the control law into a sequence of steps. In [9], a conservative law is derived from Lyapunov functions having a certain zone of non-convergence. Fuzzy [8] and neural [1] controllers have also been applied to this problem. However, it is difficult to ensure the stability of the control systems based on these approaches. Recently, the method of controlled Lagrangian [4,5] has been proposed. A hybrid control strategy has been proposed in [12], which is essentially of bang–bang nature, with local stabilization effected once the

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pendulum reaches close to the upright. A variable structure system version of the energy-speed-gradient method has been proposed in [10], which guarantees convergence of all solutions to the upright equilibrium condition for a cart-less pendulum. However, the technique has not been tried out on a practical cart-pendulum system.

A fundamental method of swing-up using energy methods for a cart-less pendulum has been proposed by Åström and Furuta [2,3]. The method has its advantage of a hierarchical nature, first controlling the pendulum angle, then the hinge position. This scheme works when the available cart track length is unlimited. However, in practical setups, there is an inherent restriction on the cart track length and the magnitude of control force that can be applied. An extension of this algorithm which takes care of restricted cart track length with switched force was developed in [6], but the method was not very general. This gives the motivation to find out energy-based methods for controlling the cart position with restricted cart track length and restricted force applicable to the cart.

In this paper, a control strategy in such a spirit is developed. We introduce "potential wells" for the cart position, for effective control within the cart track length restrictions, so that the cart never goes out of bounds. Within that well, energy is injected into the system in such a way as to drive the potential and rotational kinetic energy towards a value that is equal to the potential energy of the upright configuration. In the process, the oscillations of the cart can be kept under control by introducing penalties on the cart velocity, since some practical setups have a limitation of the cart velocity. The energy required to keep the pendulum at the upright position needs to be maintained after it is acquired since we do not have direct control on the configuration at the instant when this energy is reached. The system is controlled now by the energy maintenance mode, christened the "cruise" mode. Thereafter, once the system reaches the vicinity of the upright configuration, it is "caught" by the stabilizing controller which maintains it at and near the upright.

This scheme allows total control over the restricted dynamics of the cart, and any level of desired performance can be achieved (e.g., the number of swings of the cart can be adjusted) subject to the capacity of the system components.

In contrast to the existing approaches to the control of a cart-pendulum system, this work has the following distinguishing features:

- The rate of energy injection during the swing-up mode can be explicitly controlled. This rate can be maintained approximately linear for the entire swing-up period, leading to a faster transient response compared to the usual approaches where the rate of rise is exponential and tapers off in the end.
- The control laws have a clear intuitive interpretation. The sufficient conditions for stability derived in the paper are also quite logical. It has been observed that violations of these conditions usually lead to instability, indicating that these are not very conservative. Most of the stability-based approaches lead to highly conservative designs as evidenced by sluggish responses [9].
- Certain approaches assuring global stability require that the initial state does not lie in some "forbidden regions" [9]. The present work does not have this limitation.
- The control system can be viewed as a hybrid automaton with clear separation among the various "modes", leading to a modular design. It is expected that this feature would be helpful in the generalization of the control law to multiple inverted pendulums.
- The method has been validated by applying it to a practical cart-pendulum system. In [10], a modified form of the energy-speed-gradient method has been shown to be globally attractive, but a practical implementation has not been reported.

The paper is organized in the following fashion. Section 2 briefly describes the mathematical model of the cart–pendulum system derived using the Lagrangian principles. Then we discuss pure energy control in Section 3.1 and its limitations in Section 3.2. We improve upon it using the concept of the "cart potential well" in Section 4.1 and the "cart velocity well" in Section 4.2. We dwell upon the necessity of three distinct locations or modes where the system is allowed to evolve in time and what these modes should be. The first one is the swing-up or energy injection mode, followed by the "cruise" or energy maintenance stage incorporating the above energy wells. Lastly, we take up the issue of stabilization in

Section 4.3. We thus have three distinct modes—which essentially makes the system hybrid in nature. Results of MATLAB-based simulation and experimental implementation are presented in Section 5. In Section 6 the stability analysis is carried out based on multiple Lyapunov functions approach. Finally, Section 7 concludes with the discussion of some open issues.

2. Lagrangian modeling

As shown in Fig. 1, we consider a cart of mass M on which a pendulum of length $2l$ and mass m is hinged. Let the position of the cart be denoted by x and the angle of the pendulum with respect to the vertical axis be denoted by θ.

The kinetic energy ($\mathscr{K}$) of the system is

$$\mathscr{K} = \frac{1}{2}m\left(\frac{\mathrm{d}}{\mathrm{d}t}(x + l\sin\theta)\right)^2 + \frac{1}{2}m\left(\frac{\mathrm{d}}{\mathrm{d}t}(l\cos\theta)\right)^2 + \frac{1}{2}M\dot{x}^2 + \frac{1}{2}J\dot{\theta}^2. \tag{1}$$

The Potential energy ($\mathscr{V}$) of the system with the upright configuration of the pendulum at zero is

$$\mathscr{V} = mgl\cos\theta. \tag{2}$$

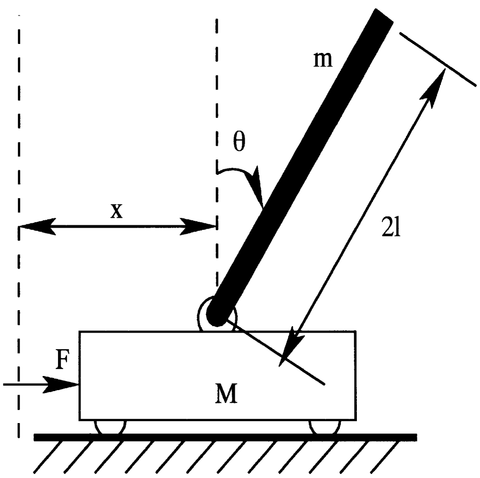

Fig. 1. The setup.

The Lagrangian of the system is constructed and the differential equations are obtained as

$$\mathscr{L} = \mathscr{K} - \mathscr{V},$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{L}}{\partial\dot{x}}\right) - \frac{\partial\mathscr{L}}{\partial x} = \mathscr{F},$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathscr{L}}{\partial\theta} = 0,$$

whence with $[x_1, x_2, x_3, x_4]^{\mathrm{T}} = [x, \theta, \dot{x}, \dot{\theta}]^{\mathrm{T}}$, we obtain the state-space model as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ \dfrac{\dfrac{\mathscr{F}}{M+m} - \dfrac{3mg\sin x_2\cos x_2}{4(M+m)} + \dfrac{mlx_4^2\sin x_2}{M+m}}{1 - \dfrac{3m\cos^2 x_2}{4(M+m)}} \\ \dfrac{\dfrac{3g\sin x_2}{4l} - \dfrac{3\mathscr{F}\cos x_2}{4l(M+m)} - \dfrac{3mx_4^2\sin x_2\cos x_2}{4(M+m)}}{1 - \dfrac{3m\cos^2 x_2}{4(M+m)}} \end{bmatrix}. \tag{3}$$

It may be observed from the state-space model obtained in this section that the subsystem $[\theta, \dot{\theta}]^{\mathrm{T}}$ is independent of the $[x, \dot{x}]^{\mathrm{T}}$ subsystem. In this sense, there is only one-way coupling between the subsystems. This helps us in obtaining a simple formulation of the Lyapunov functions of the pendulum and the cart subsystems for stability analysis.

3. Derivation of energy control law

Our objective is to take the pendulum to the upright position and maintain it there despite small disturbances, with the cart being the only means of providing actuation to the pendulum. The objective can be realized by injecting energy into the system so that the sum of rotational kinetic energy and potential energy of the pendulum reaches a desired level. This in turn means that the rotational energy of the pendulum must initially increase, whatever be its translational kinetic energy. The translational kinetic energy of the pendulum does not play any important role, since the pendulum translation and rotation equations are decoupled.

3.1. The pure energy control law

We denote the sum of the rotational kinetic energy of the pendulum and its potential energy by $\mathscr{E}_{\mathrm{rp}}$. This is expressed as

$$\mathscr{E}_{\mathrm{rp}} = \tfrac{1}{2} J_{\mathrm{H}} \dot{\theta}^2 + mgl\cos\theta, \tag{4}$$

where J_{H} is the moment of inertia of the pendulum about the hinge. This energy must increase, i.e.,

$$\frac{\mathrm{d}\mathscr{E}_{\mathrm{rp}}}{\mathrm{d}t} = -m\ddot{x}l\dot{\theta}\cos\theta \geqslant 0, \tag{5}$$

after substituting for $\ddot{\theta}$ from the state-space model (Eq. (3)). This necessarily means that the acceleration $\ddot{x}$ and $\dot{\theta}\cos\theta$ must be of opposite sign. A simple law for the acceleration can be formulated as [3]

$$\ddot{x} = k_{\mathrm{su}}(\mathscr{E}_{\mathrm{rp}} - \mathscr{E}_{\mathrm{up}})\,\mathrm{sgn}(\dot{\theta}\cos\theta), \tag{6}$$

where $\mathscr{E}_{\mathrm{up}}$ is the potential energy of the pendulum in the upright configuration.

3.2. Difficulties of pure energy control

Eq. (6) provides for continuous energy injection into the pendulum. However, there is no restriction explicitly imposed on the cart track. It works well for unlimited cart track length, or for systems like the whirling pendulum where rotation simulates an infinite cart track.

Another fact noticeable in Eq. (6) is the lethargic nature of energy injection into the system. The rate of injection is dependent on the difference of $\mathscr{E}_{\mathrm{rp}}$ and $\mathscr{E}_{\mathrm{up}}$, and as this difference decreases the rate of energy injection also decreases. In fact, the desired energy level $\mathscr{E}_{\mathrm{up}}$ will be theoretically reached in infinite time. Ideally, the control law must be able to take the pendulum to the upright configuration in finite time, with the rate of energy injection as a parameter. Further, it must provide a guided path to the final destination of the pendulum despite continuous energy injection and cart track length restrictions.

4. The proposed control scheme

4.1. The cart potential well

To introduce the cart track length restriction, we introduce the concept of the cart potential well. The well is constructed in such a way that the cart experiences a repulsive force as it approaches the boundaries in the neighborhood of the limitations. We define a potential function $\Phi(x)$ for the cart position x, such that the desired restrictions can be imposed during evolution of the system, which restricts the dynamics without interfering with the mechanical energies of the system. We can derive

$$\ddot{x} = -\nabla\Phi, \tag{7}$$

where $\ddot{x}$ is the normalized field, in this case, the acceleration of the hinge. Further, it is better if the cart swings with a larger amplitude within the bounds of the cart track length. It is therefore necessary that the penalty on acceleration should be small close to the center of the cart and higher as the cart track limits are reached. Such an acceleration function can be realized by the logarithmic component in the following equation:

$$u_{\mathrm{cart\ well}} = k_{\mathrm{cw}}\,\mathrm{sgn}(x)\log\left(1 - \frac{|x|}{L}\right), \tag{8}$$

where the center of the cart track is at 0, with L length of track available on both sides.

It may be mentioned that in Eq. (6), no term which changes sign during evolution of the system can be inserted as a multiplier of the right-hand side. The equation demands that always a negative quantity should be a multiplier of $\mathrm{sgn}(\dot{\theta}\cos\theta)$, like $(\mathscr{E}_{\mathrm{rp}} - \mathscr{E}_{\mathrm{up}})$. Hence an acceleration term like Eq. (8) cannot be multiplied with the right-hand side of Eq. (6). Thus we introduce this term as an additive component.

One drawback of the pure energy control equation (6) as discussed in Section 3.2 is the lethargic rise of energy towards the desired potential energy level. If the term $(\mathscr{E}_{\mathrm{rp}} - \mathscr{E}_{\mathrm{up}})$ is removed from Eq. (6), it would ensure a rate of injection of energy solely specified by the constant k_{su}. With this modification, the final equation after introduction of the “cart potential well”

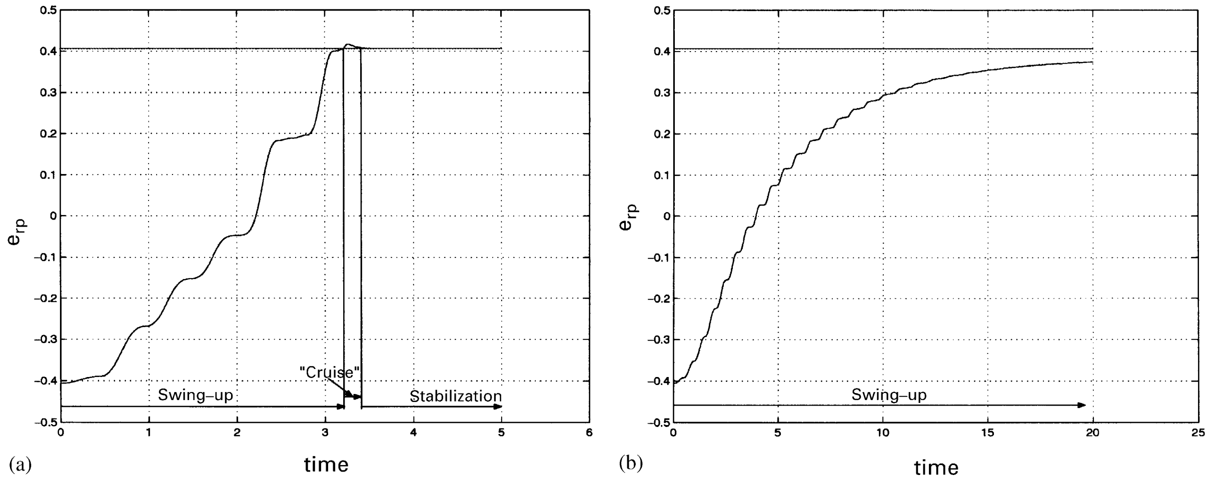

Fig. 2. Energy plots: (a) energy plot for three swings corresponding to Eq. (9) and (b) energy plot for three swings corresponding to Eq. (6).

stands as

$$\ddot{x} = -k_{su}\,\mathrm{sgn}(\dot{\theta}\cos\theta) + k_{cw}\,\mathrm{sgn}(x)\log\left(1 - \frac{|x|}{L}\right), \qquad (9)$$

where k_{su}, $k_{cw} > 0$. Fig. 2 shows a comparison of the energy profiles (in a typical case) during swing-up corresponding to Eqs. (6) and (9). The proposed controller leads to a much faster energy rise, as expected.

4.2. Cruise mode

The purpose of swing-up is to inject enough $\mathscr{E}_{rp}$ to the pendulum so that it is able to swing-up to the desired upright (unstable) equilibrium position. But it cannot be guaranteed that the energy injection reaches the desired value of $\mathscr{E}_{rp} = \mathscr{E}_{up}$ when the pendulum is close to the upright configuration so that the stabilizing controller acts directly. It may even be the case, depending on the values of k_{su} and k_{cw}, that the energy injection has to stop close to the pendant configuration of the pendulum.

In order to avoid the loss of $\mathscr{E}_{rp}$ due to further evolution of the system in time from frictional losses, it is desirable to maintain the value of $\mathscr{E}_{rp}$ till the instant when the system is caught by the stabilizing strategy. Additionally, it is desired that when the control is switched to the stabilizing strategy (which is most effective close to the linear region, at the origin of the state space considered), the cart position and the cart velocity should also be found close to their zero values.

This is not difficult to achieve for the cart position; just a suitable increase in the constant k_{cw} in Eq. (8) would provide a more constrained cart potential well. Choice of a suitable function for the acceleration ($u_{\text{cart well}}$), which is differentiable at all points in the interval $]-L, L[$ and whose first differential is continuous (not necessarily smooth), is not unique. Thus, the function has to be chosen based on trial and error to achieve the desired level of performance.

Similar requirement for the cart velocity may be achieved using a cart velocity well defined like the "cart potential well". However, negative feedback in the closed loop dynamics must be ensured for all such acceleration terms. One possible acceleration function for constraining the cart velocity may be

$$u_{\text{velocity well}} = k_{vw}\,\mathrm{sgn}(\dot{x})\log\left(1 - \frac{|\dot{x}|}{\dot{x}_{max}}\right), \qquad (10)$$

where $\dot{x}_{max}$ depends on the maximum cart velocity that the system is capable of withstanding, or it may depend on the desired zone of velocity that is easier for the stabilizing controller to handle. The steepness of this cart velocity well determines the region in $\dot{x}$ space where the system is allowed to execute its dynamics.

A control law which incorporates maintenance of $\mathscr{E}_{\mathrm{rp}}$ close to $\mathscr{E}_{\mathrm{up}}$ may be formulated as

$$u_{\text{energy-maint}} = k_{\mathrm{em}}(\exp|\mathscr{E}_{\mathrm{rp}} - \mathscr{E}_{\mathrm{up}}| - 1) \times \operatorname{sgn}(\mathscr{E}_{\mathrm{rp}} - \mathscr{E}_{\mathrm{up}}) \operatorname{sgn}(\dot{\theta}\cos\theta), \qquad (11)$$

since the term $\operatorname{sgn}(\dot{\theta}\cos\theta)$ ensures change of $\mathscr{E}_{\mathrm{rp}}$ according to the sign of $(\mathscr{E}_{\mathrm{rp}} - \mathscr{E}_{\mathrm{up}})$ (by virtue of Eq. (5)). The exponential effectively maintains $\mathscr{E}_{\mathrm{rp}}$ close to $\mathscr{E}_{\mathrm{up}}$, thus forming a kind of hysteresis band with soft limits.

After incorporation of all the three terms $u_{\text{cart well}}$, $u_{\text{velocity well}}$ and $u_{\text{energy-maint}}$, the total acceleration for the "cruise" mode comes to

$$\ddot{x} = u_{\text{cart well}} + u_{\text{velocity well}} + u_{\text{energy-maint}} \qquad (12)$$

from Eqs. (8), (10), and (11). But there is an inherent drawback of this control law. The energy maintenance component in the above equation functions only if it is strong enough, i.e., only if the $\mathscr{E}_{\mathrm{rp}}$ is substantially different from $\mathscr{E}_{\mathrm{up}}$ since at $\mathscr{E}_{\mathrm{rp}} = \mathscr{E}_{\mathrm{up}}$, the component's contribution is zero. The energy therefore would not remain close to $\mathscr{E}_{\mathrm{up}}$ at any time, but will decrease to a lower level. Functioning alone, without the cart-well and the velocity-well components, it would however maintain the energy constant around $\mathscr{E}_{\mathrm{up}}$. Therefore Eq. (11) needs to be modified to, for instance,

$$u_{\text{energy-maint}} = k_{\mathrm{em}}(\exp|\mathscr{E}_{\mathrm{rp}} - \eta\mathscr{E}_{\mathrm{up}}| - 1)\operatorname{sgn}(\mathscr{E}_{\mathrm{rp}} - \mathscr{E}_{\mathrm{up}}) \times \operatorname{sgn}(\dot{\theta}\cos\theta), \quad \eta > 1. \qquad (13)$$

The significance of the parameter η will be better understood in the context of stability, discussed in Section 6.

Eq. (13) ensures availability of the system near the origin of the cart track and the cart velocity when control is switched to the stabilization mode. Desired performance can be extracted in the "cruise" mode by tuning the constants k_{cw}, k_{vw} and k_{em} (k_{cw} may be same as the previous mode) for specific choices of the "cart potential" and the "cart velocity well" functions.

4.3. Stabilization

Stabilization is carried out using the LQR after linearizing the state-space equations about the upright (unstable) equilibrium configuration ($[0,0,0,0]^{\mathrm{T}}$). This yields the approximation, with u as the input (force $\mathscr{F}$ in this case):

$$\dot{\mathscr{X}} = A\mathscr{X} + Bu,$$

where

$$\mathscr{X} = [x_1, x_2, x_3, x_3]^{\mathrm{T}}$$

as defined in Section 2,

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{-m^2gl^2}{(J+ml^2)(m+M-(m^2l^2/J+ml^2))} & 0 & 0 \\ 0 & \dfrac{mgl(m+M)}{(J+ml^2)(m+M-(m^2l^2/J+ml^2))} & 0 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{(m+M-(m^2l^2/J+ml^2))} \\ \dfrac{-ml}{(J+ml^2)(m+M-(m^2l^2/J+ml^2))} \end{bmatrix}.$$

Addition of state feedback control $u = -K\mathscr{X}$ leads to

$$\dot{\mathscr{X}} = (A - BK)\mathscr{X}.$$

K is derived from minimization of the integral

$$J = \int (\mathscr{X}^{\mathrm{T}}Q\mathscr{X} + u^{\mathrm{T}}Ru)\,\mathrm{d}t$$

where Q and R are positive semi-definite and positive definite matrices, respectively. With the choice

$$Q = \begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 50 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad R = 5,$$

we obtain

$$K = [-4.4721, -75.9573, -7.1427, -11.5948].$$

5. Simulation and implementation results

Simulation was performed using the following model of the cart pendulum:

Mass of the cart (M): 2.4 kg.
Mass of the pendulum (m): 0.23 kg.
Length of the pendulum ($2l$): 0.36 m.
Length of the cart track (L): $\pm$ 0.5 m.

Implementation of the algorithm was carried out on an experimental setup and the relevant plots are shown in Figs. 3–5. The implementation plot in Fig. 4(b) has the first one swing removed from all axis (including time). Figs. 3(a) and (b) show the simulation and implementation results, respectively, for three swings. It is seen that the pendulum angle and cart position during experimental swing-up bear a strong resemblance with the simulation results. However, the practical

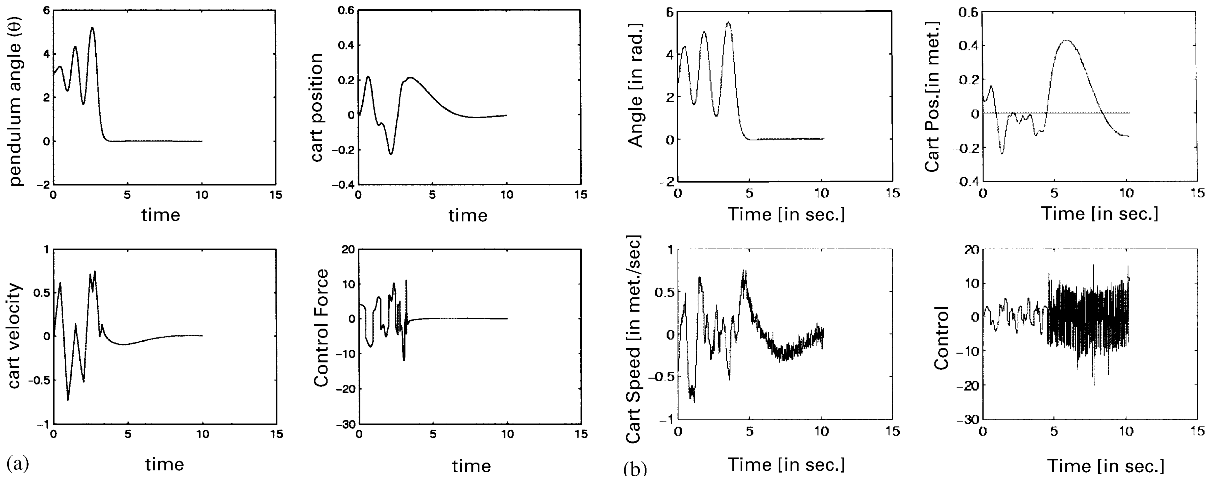

Fig. 3. Simulation and implementation results for three swings, $k_{su} = 1.64$, $k_{cw} = 2.25$, $k_{vw} = 5$, $k_{em} = 6$, $\eta = 1.05$: (a) simulation for three swings and (b) implementation for three swings.

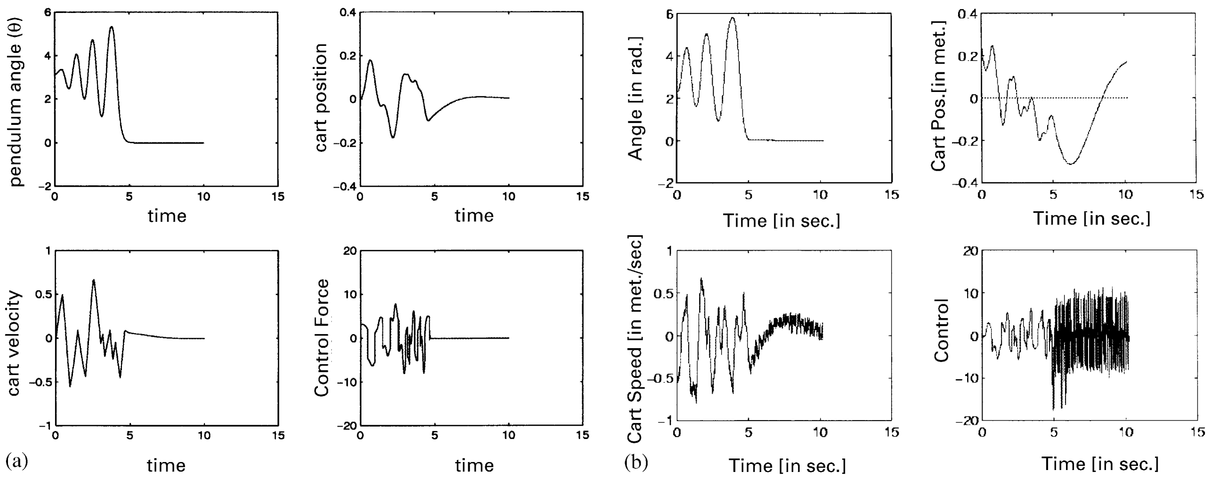

Fig. 4. Simulation and implementation results for four swings, $k_{su} = 1.3$, $k_{cw} = 2.25$, $k_{vw} = 5$, $k_{em} = 6$, $\eta = 1.05$: (a) simulation for four swings and (b) implementation for four swings.

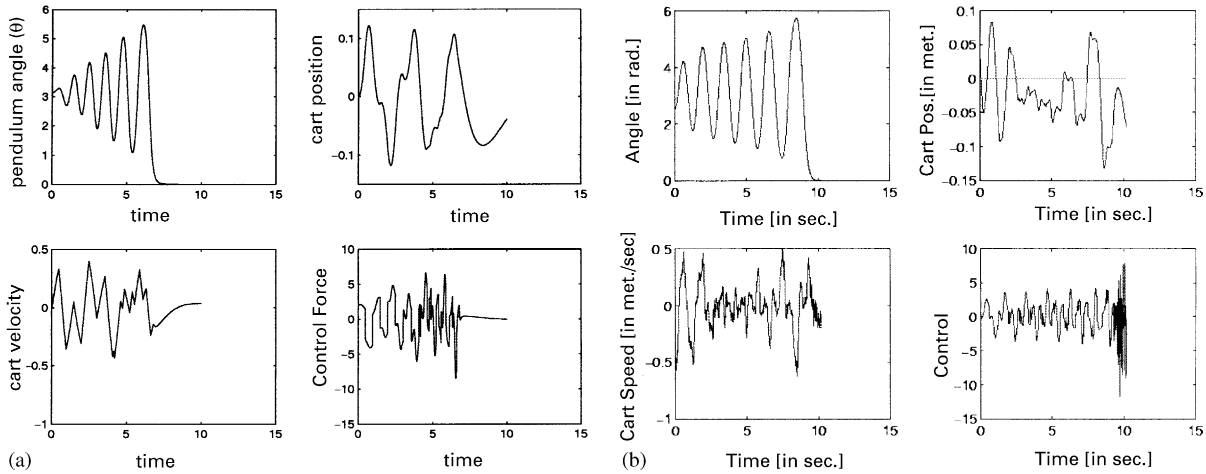

Fig. 5. Simulation and implementation results for six swings, $k_{su} = 0.87$, $k_{cw} = 2.25$, $k_{vw} = 5$, $k_{em} = 6$, $\eta = 1.05$: (a) simulation for six swings and (b) implementation for six swings.

system is found to be noisy and it takes a longer time to reach the steady state. The noise occurs primarily due to the computation of the velocity terms in the state by differentiation of position variables in the experimental setup. It may also be mentioned that friction was not taken into account in the mathematical model used for control design. Figs. 4 and 5 show the corresponding results for four and six swings, respectively. In all cases there is quite good correspondence between theory and experiments.

6. Stability analysis

As the system under study is a hybrid control system with three distinct modes of operation, it has been found necessary to consider multiple Lyapunov functions, one for each of the modes. Following [7], we formulate distinct Lyapunov functions for each of the locations of operation which are continuous, but not necessarily smooth, for the overall stability of a hybrid system. We then derive sufficient conditions under which the system will be stable.

The stability of the control system is primarily determined by the swing-up mode and the "cruise" modes in which energy is actively injected into the pendulum. The third mode is stabilizing by design around the local operating point of the upright unstable equilibrium position. We therefore analyze the stability of the swing-up mode first by finding a Lyapunov function and determine the conditions under which it satisfies conditions of stability.

6.1. The swing-up mode

Before carrying out the formal proof of stability let us note the following:

- In view of the nature of subsystem dynamics given by Eq. (3), the $[\theta, \dot{\theta}]^{\mathrm{T}}$ subsystem does not depend on the $[x, \dot{x}]^{\mathrm{T}}$ subsystem.
- The existence of the system in the swing-up mode depends on the energy imparted to the $[\theta, \dot{\theta}]^{\mathrm{T}}$ subsystem.
- If it can be shown that the $[\theta, \dot{\theta}]^{\mathrm{T}}$ subsystem acquires sufficient energy in a finite time, then the stability of the swing-up mode will be guaranteed, since all the associated state variables will be bounded.

We prove the latter by defining a Lyapunov function for the $[\theta, \dot{\theta}]^{\mathrm{T}}$ subsystem, based on the energy function $\mathscr{E}_{\mathrm{rp}}$ as follows.

For the swing-up process to be stable in $[\theta, \dot{\theta}]^{\mathrm{T}}$ subspace, there must exist a function $\mathscr{V}_{\text{sw-up}}(\mathscr{X}_{[\theta,\dot{\theta}]^{\mathrm{T}}})$ such that $\mathscr{V}_{\text{sw-up}} > 0\ \forall \mathscr{X}_{[\theta,\dot{\theta}]^{\mathrm{T}}} \neq 0$, $\mathscr{V}_{\text{sw-up}}(0) = 0$ and $\dot{\mathscr{V}}_{\text{sw-up}} \leqslant 0$, $\forall t > 0$. A candidate for such a function is $\mathscr{V}_{\text{sw-up}} = (\mathscr{E}_{\text{up}} - \mathscr{E}_{\text{rp}})$, since it is positive during the swing-up mode, $\theta \in]0, 2\pi[$. We examine $\dot{\mathscr{V}}_{\text{sw-up}}$ below:

$$\dot{\mathscr{V}}_{\text{sw-up}} = -\dot{\mathscr{E}}_{\text{rp}} = m\ddot{x}l\dot{\theta}\cos\theta.$$

Substituting the acceleration $\ddot{x}$ from Eq. (9), we obtain Eq. (14) as the condition for stability in $[\theta, \dot{\theta}]^{\mathrm{T}}$ subspace in the sense of Lyapunov:

$$\dot{\mathscr{V}}_{\text{sw-up}} = ml\dot{\theta}\cos\theta\left(-k_{\text{su}}\,\text{sgn}(\dot{\theta}\cos\theta) - \frac{\mathrm{d}\Phi}{\mathrm{d}x}\right) \leqslant 0, \tag{14}$$

which necessarily means that

$$k_{\text{su}} \geqslant \left(-\frac{\mathrm{d}\Phi}{\mathrm{d}x}\right)\text{sgn}\,(\dot{\theta}\cos\theta).$$

Since $k_{\text{su}} > 0$, a sufficient condition is $k_{\text{su}} \geqslant |\mathrm{d}\Phi/\mathrm{d}x|$.

In the present context, where a logarithmic cart potential well has been chosen, the above result means that

$$\frac{|x|}{L} \leqslant \left(1 - \exp\left(-\frac{k_{\text{su}}}{k_{\text{cw}}}\right)\right).$$

The above is a sufficient condition for stability in the $[\theta, \dot{\theta}]^{\mathrm{T}}$ subsystem. It is clear that the ratio $k_{\text{su}}/k_{\text{cw}}$ plays a very important role in this regard. In physical terms, it ensures that the energy injection component of the acceleration is the dominating component so that the sign of $\mathrm{d}\mathscr{E}_{\text{rp}}/\mathrm{d}t$ in Eq. (5) is maintained negative. This is intuitively quite logical. Once this condition is satisfied, the values of k_{su} and k_{cw} can be chosen as a trade off between the rate of energy rise and the track length restrictions.

Once stability in $[\theta, \dot{\theta}]^{\mathrm{T}}$ space is ensured, we claim that the swing-up mode exists for a finite time. This follows from the condition in Eq. (14) that $\mathscr{V}_{\text{sw-up}}$ is a non-increasing function. In fact, the equality is valid only instantaneously, when the pendulum switches swinging direction.

6.2. The "cruise" mode

In a similar spirit to the swing-up mode, we establish that the "cruise" mode exists for a finite time. For the "cruise" mode to be stable in $[\theta, \dot{\theta}]^{\mathrm{T}}$ subspace, there must exist a function $\mathscr{V}_{\text{cruise}}(\mathscr{X}_{[\theta,\dot{\theta}]^{\mathrm{T}}})$ such that $\mathscr{V}_{\text{cruise}}(\mathscr{X}_{[\theta,\dot{\theta}]^{\mathrm{T}}}) > 0\ \forall \mathscr{X}_{[\theta,\dot{\theta}]^{\mathrm{T}}} \neq 0$, $\mathscr{V}_{\text{cruise}}(0) = 0$ and $\dot{\mathscr{V}}_{\text{cruise}} \leqslant 0$, $\forall t > 0$. A candidate for a Lyapunov function is $\mathscr{V}_{\text{cruise}} = \frac{1}{2}(\mathscr{E}_{\text{rp}} - \mathscr{E}_{\text{up}})^2$ since it is positive for all values of $\mathscr{E}_{\text{rp}}$, and at the origin of the state space, it is 0 since $\mathscr{E}_{\text{rp}} = 0$. We examine $\dot{\mathscr{V}}_{\text{cruise}}$ below:

$$\begin{aligned}\dot{\mathscr{V}}_{\text{cruise}} &= (\mathscr{E}_{\text{rp}} - \mathscr{E}_{\text{up}})\dot{\mathscr{E}}_{\text{rp}}\\ &= (\mathscr{E}_{\text{rp}} - \mathscr{E}_{\text{up}})(-m\ddot{x}l\dot{\theta}\cos\theta).\end{aligned}$$

For the system to be stable, with $(\mathscr{E}_{\text{rp}} - \mathscr{E}_{\text{up}}) = \Delta\mathscr{E}_{\text{rp}}$, $\Psi = \Psi(\dot{x})$ the cart-velocity potential function $\Phi = \Phi(x)$ the "cart potential function", $f(\mathscr{E}_{\text{rp}})$ the energy maintenance component (function),

$$\begin{aligned}&\mathscr{V}_{\text{cruise}} = \text{sgn}(\Delta\mathscr{E}_{\text{rp}})\dot{\mathscr{E}}_{\text{rp}} \leqslant 0\\ &\quad\Rightarrow \text{sgn}(\Delta\mathscr{E}_{\text{rp}})\,\text{sgn}(\dot{\theta}\cos\theta)\ddot{x} \geqslant 0\\ &\quad\Rightarrow k_{\text{em}}\,f(\mathscr{E}_{\text{rp}}) \geqslant \text{sgn}(\Delta\mathscr{E}_{\text{rp}})\text{sgn}(\dot{\theta}\cos\theta)\left[\frac{\mathrm{d}\Phi}{\mathrm{d}x} + \frac{\mathrm{d}\Psi}{\mathrm{d}\dot{x}}\right]\\ &\quad\Rightarrow k_{\text{em}}\,|f(\mathscr{E}_{\text{rp}})| \geqslant \left|\frac{\mathrm{d}\Phi}{\mathrm{d}x} + \frac{\mathrm{d}\Psi}{\mathrm{d}\dot{x}}\right|.\end{aligned}$$

In the present context, where logarithmic cart potential and cart velocity potentials have been chosen, the above result means that, from Eq. (13),

$$\begin{aligned}&k_{\text{em}}\,|\exp|\mathscr{E}_{\text{rp}} - \eta\mathscr{E}_{\text{up}}| - 1|\\ &\quad\geqslant \left|k_{\text{cw}}\ln\left(1 - \frac{|x|}{L}\right)\text{sgn}(x)\right.\\ &\qquad\left. + k_{\text{vw}}\ln\left(1 - \frac{|\dot{x}|}{|\dot{x}_{\max}|}\right)\text{sgn}(\dot{x})\right|\\ &\quad\Rightarrow k_{\text{em}}\,|\exp|\mathscr{E}_{\text{rp}} - \eta\times\mathscr{E}_{\text{up}}| - 1|\\ &\quad\geqslant \left|k_{\text{cw}}\ln\left(1 - \frac{|x|}{L}\right)\right.\\ &\qquad\left. + k_{\text{vw}}\ln\left(1 - \frac{|\dot{x}|}{|\dot{x}_{\max}|}\right)\right|, \quad \eta > 1.\end{aligned} \tag{15}$$

The above is a sufficient condition for stability of the "cruise" mode in $[\theta, \dot{\theta}]^{\mathrm{T}}$ subspace. The equation shows that the energy maintenance component of the acceleration should be the dominating component.

It is trivial to note that the "cruise" mode exists only for a maximum of a single swing if the condition in Eq. (15) is satisfied. After this the pendulum is caught by the stabilizing controller (the third mode), since $\mathscr{E}_{\text{rp}}$ is maintained in the second mode close to the value corresponding to the upright (static) condition.

Local stability around this operating point is ensured by design of the LQ regulator.

7. Conclusions and further work

In this paper, a new technique has been proposed for the swing-up and upright stabilization of an inverted pendulum on a cart with a restricted track length. This is achieved by generalizing the simple energy control law with the introduction of a cart potential well to penalize motions close to and beyond the boundary. The energy control law is modified to inject energy at a fixed rate. This, however, requires a "cruise" mode of operation to ensure that energy is maintained at the desired value after a sufficient value is acquired that can take it to the upright configuration. Finally, when the pendulum comes close to an upright configuration, an LQ regulator designed for this operating point stabilizes it.

The proposed control law has been found to work well in simulation as well as experimentally. It has been proved that the control laws for swing-up and "cruise" mode ensures stability under fairly reasonable conditions. These conditions have been verified through simulations.

A major problem with the proposed scheme was to design suitable potential wells and coefficients in the various control laws which have been obtained using intuition and iterations, rather than from an analytical perspective. It would be interesting to consider what class of functions would satisfy the desired specifications and how to optimize the associated parameters. The robustness of the control scheme with respect to knowledge of the pendulum parameters also needs to be studied. Some of these issues are currently being investigated by the authors.

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