UNIT 6

Exploring Other Coordinate Systems

Lesson 6.1: A Nonstandard Exploration of the Rate of Change of Functions

In the interest of discovery, we will open this Unit without preliminary explanation by having you complete the next Exploration involving a further and slightly more complicated look at functions and rate of change.

Exploration 6.1.1: Movement in the Plane

For this Exploration, we are concerned with the movement of an object along a path in the plane. We are assuming that the plane is a coordinate plane and the object starts at the point $(0, 0)$.

As the object moves along the path, each point on the path has two coordinates. The coordinates depend on the **distance traveled along the path**. Let us call this distance S. That is, S is the length of the path from the origin $(0, 0)$ to a point P on the path as depicted in Figure 6.1.1-1.

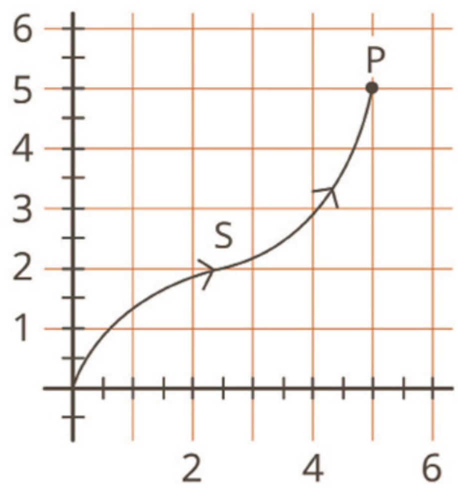
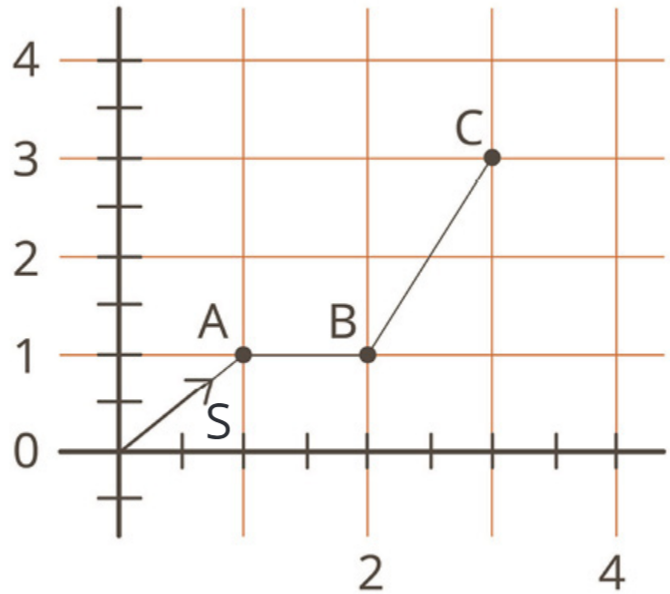


Figure 6.1.1-1

Since the coordinates of P depend on the distance S, we write $(x(s), y(s))$ for the coordinates of P.

We now wish to examine various types of paths and describe the behavior of the functions $x(s)$ and $y(s)$ as S increases.

Part 1: A Polygonal Path



1. Describe the behavior of $x(s)$ between the points O and A, A and B, and B and C.

2. Describe the behavior of $y(s)$ between the points O and A, A and B, and B and C.

3. Can you predict what the graphs of the $x(s)$ and $y(s)$ would look like for the entire polygonal path S between the points O and C?

4. Is it possible to write explicit formulas describing the values of $x(s)$ and $y(s)$?

   (i) If so, use the graph to approximate $x(2)$ and $y(2)$.

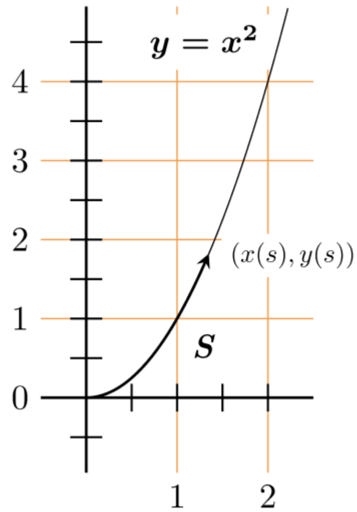
   (ii) What value of s yields the coordinate point $(1.5, 1)$?

   (iii) What is $y(s)$ when $x(s) = 2.5$?

5. Test your prediction from (3) by constructing graphs of $x(s)$ and $y(s)$. What is the domain of each of these functions, based on the supplied information?

Part 2: A Parabolic Path

Movement, in this case, is along a parabola with vertex at the origin. The simplest algebraic of such a parabola is $y = x^2$.



6. Describe the behavior of $x(s)$, $y(s)$.

7. Can we determine $x(2)$ and $y(2)$ from the given information?

8. Can we construct graphs for $x(s)$, $y(s)$ (are they the same graph)? How might you accomplish this task (i.e., what materials are needed for you to do this?).

Consider using a length of string and the scaled length of the provided ruler.

[NOTE: One centimeter on the supplied ruler and one unit on the "Parabolic Path" graph can be taken as the same. Please keep in mind that, due to paper copying irregularities, the provided ruler is likely not an accurate representation

Lesson 6.2: More Information Needed

Based upon your experience with the last section, you might have a better idea as to how to approach Exploration 6.2.1.

Exploration 6.2.1: Another Position- Time Relationship

At 1:00 p.m. a ship is 10 miles due east of port. At 2:00 p.m., it has sailed to a point that is 20 miles east and 50 miles north of the position at 1:00 p.m. Assume that the ship continues to sail in this manner as it did from 1:00 p.m. to 2:00 p.m.

1. Draw a picture or diagram to represent its voyage.
2. Place on your drawing the position of the ship at 3:00 p.m. and at 5:00 p.m.
3. How far north from the port will the ship be at 4:00 p.m.?
4. How far east from the port will the ship be at 4:00 p.m.?
5. Write a function that will give the ship's position at any given time.

As a result of completing Exploration 6.2.1, one realizes that it is not too hard to create a function that describes the ship's *north position* as a function of its *east position*. The challenge occurs when one tries to introduce the additional information parameter of time into the situation. By now, you should have realized that, in the last few sections, we are focusing our attention on the concept of *parametric equations*.

Previous to this Unit we have worked with equations such as $y = 4x^2 + 2x - 5$ where the variables x and y are related in a direct way. However, as a result of the Explorations of this Unit thus far, it should be clear that another way to describe a curve in the plane is to describe the x and y coordinates of a point on a curve in terms of a third parameter or variable. This third *parameter* is usually denoted as t. Thus, a *parallel set* of equations used to describe a curve can be given as:

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

where f and g are functions of the parameter t. This parallel set is called the *parametric equations* of the curve. Note that parametric equations and the concept behind them can be quite useful for modeling the interactions of *vectors*. We will encounter this application of parametrics in future sections.

!! Your instructor may wish to expand upon the examples in this section and further relate some of these equations to trigonometric concepts that you have also studied in previous courses. Your discussion or consideration of these topics might also include techniques for switching between parametric and rectangular representations of curves in the plane.

Exploration 6.2.2: Working with Parametric Equations

For the given parametric equations for problem 1-4:

a) Make a table of the form

t	x	y

and graph the results in the rectangular coordinate system.

b) Eliminate the parameter t in each equation and write each equation in rectangular form. Also state the type of curve represented by the given parametric equations.

1. $\begin{cases} x = t + 1 \\ y = t^2 - 2t \end{cases}$

2. $\begin{cases} x = \cos\ t \\ y = \sin\ t \end{cases}$ [Hint: use the Pythagorean Trigonometric Identity to eliminate t.]

3. $\begin{cases} x = 3 + 2\cos\ t \\ y = 5 + 4\sin\ t \end{cases}$

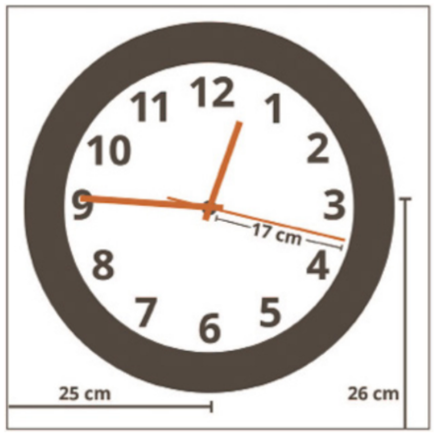
4. $\begin{cases} x = \dfrac{1}{2}t \\ y = t - 2 \end{cases}$

Lesson 6.3: Applications of Parametric Equations

Among other things, the next Exploration will allow you to consolidate what you've learned about sinusoidal equations and parametric equations.

Exploration 6.3.1: Applications of Parametric Equations

Part 1



You have been hired to create a working clock for a game scene with the shown radius (based on the second hand). The tip of the second hand's distances x cm and y cm from the left side of the shown box and the bottom of the box, respectively, depend on the number of seconds, t, since the second hand was pointing straight up. The second hand on the clock must turn with a period of 60 seconds and turn in the clockwise direction.

1. Create parametric equations for x and y in terms of t that satisfy the given conditions that the center of the clock is 25 cm from the left edge of the box and 26 cm above the bottom of the box with a radius of 17 cm.

2. Eliminate the parameter t and write the rectangular equation of the circle that models the shape and location of the clock relative to the box.

Part 2

Parametric equations can be used to model more advanced situations, such as projectile motion. Given

$$h(t) = h_0 + v_0 \sin(\theta)t - \frac{1}{2}gt^2$$

(where $h =$ height; $v_0 =$ initial velocity; $g =$ gravity; $t =$ time; $h_0 =$ initial height) you will use parametric equations in the following Exploration.

3. Using the equation for $h(t)$ on the previous page, notice the initial height h_0 is a vertical component and there is no horizontal component in the formula. Determine a formula for vertical position of an object, $y(t)$.

4. Understanding that both initial height h_0 and gravity are unnecessary to calculate horizontal position, and using the formula for $y(t)$ you just developed, determine a formula for horizontal position of an object, $x(t)$.

5. A basketball is thrown from the roof of the school building as part of a science experiment examining gravity. The basketball is thrown from 25 feet above the ground at an initial velocity of 40 feet per second and at an angle of elevation of $30°$. Write a set of parametric equations that model the basketball's horizontal and vertical position.

6. The basketball reaches its maximum height at $t = 0.625$ seconds. Using your parametric equations from part 5, determine the location of the basketball at its maximum height relative to the starting point.

Lesson 6.4: Vectors

Vectors: A Brief Background

A *vector* is a quantity characterized by possessing both a magnitude and a direction. Velocity and force are good examples of vector quantities. Non-vector quantities represented by numbers such as mass and length are called scalars. Geometrically, a vector can simply be represented of as a directed line segment or arrow such as is pictured in Figure 6.4-1

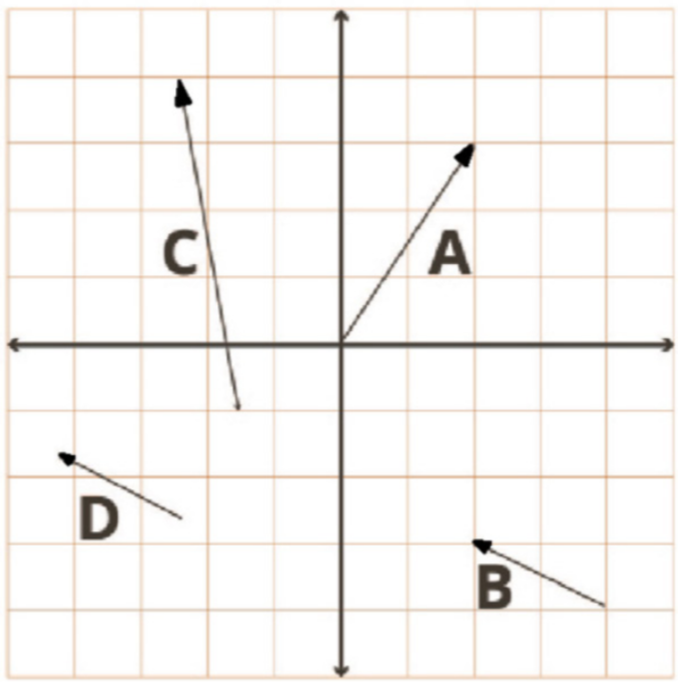


Figure 6.4-1: Vectors in the Plane

Since a vector is characterized by magnitude and direction, it can be moved about in the plane for convenience as long as its magnitude (length) and direction (orientation) is not changed.

One adds vectors, by placing one vector which we call vector a, which may be symbolized a with components $\langle a_1, a_2 \rangle$ at the origin and then placing the vector to be added, b, at the arrow end of a. The components of vector b are $\langle b_1, b_2 \rangle$. The resultant of $a + b$ is the vector drawn from the tail end of a to the arrow end of b. This situation is pictured geometrically in Figure 6.4-2. Algebraically, this situation can be represented by adding the corresponding vector components of a and b to yield the resultant vector in component form

$$\langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle = \langle a_1 + b_1, a_2 + b_2 \rangle$$

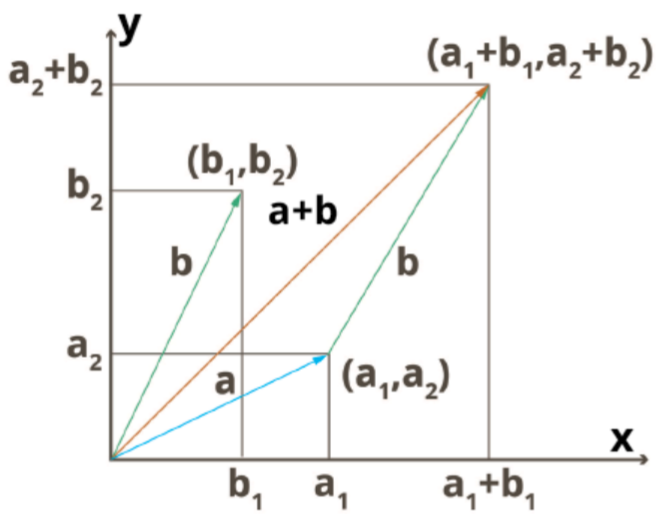


Figure 6.4-2: Vector Addition

One way to subtract vectors is to add the negative of the vector being subtracted from another as pictured in Figure 6.4-3 using vectors $\vec{u}$ and $\vec{v}$ (note that this is yet another way to symbolize vectors). Thus

$$\vec{v} - \vec{u} = \vec{v} + (-\vec{u})$$

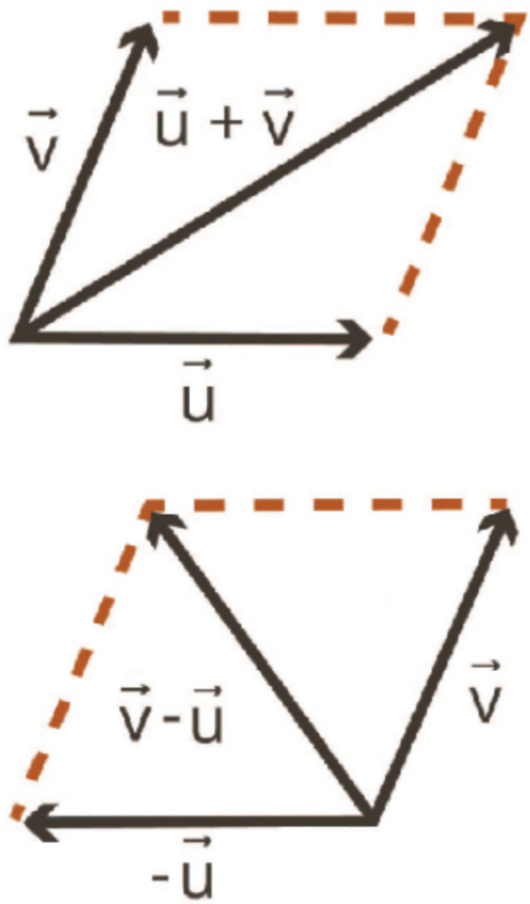


Figure 6.4-3: Vector Subtraction

Two More Properties of Vectors

One property of vectors is that they can be multiplied by scalars such that, if $\vec{a}$ is a vector with components $\langle a_1, a_2 \rangle$, then $2\vec{a} = \langle 2a_1, 2a_2 \rangle$. Another property is that we can define the magnitude or *norm* of a vector in such a way that

The norm of a vector $\vec{v} = \langle v_1, v_2 \rangle$ is a symbolized $||\vec{v}||$ and found by

$$||\vec{v}|| = \sqrt{(v_1)^2 + (v_2)^2}$$

Exploration 6.4.1: Working with Vectors

1. If $\vec{u} = \langle 2, 6 \rangle$ and $\vec{v} = \langle -3, 2 \rangle$,

a. Find the coordinates of $\vec{u} + \vec{v}$ and draw this vector.

b. Find the coordinates of $\vec{u} - \vec{v}$ and draw this vector.

c. Find the coordinates of $2\vec{u} - 4\vec{v}$ and draw this vector.

2. If a directed segment drawn from point $P(2, 3)$ to point $Q(5, 8)$ describes vector PQ, find an equivalent vector, OA, which has its initial point at the origin.

Last, find the norm of this vector.

3. Find the norm $\vec{v} = \langle 4, -3 \rangle$.

4. Write the vector in component form, given $||\vec{a}|| = 10$ and $\theta_{\vec{a}} = 30°$.

5. Determine the resultant vector, $\vec{r} = \vec{c} + \vec{d}$, in component form, if $||\vec{c}|| = 6$ $\theta_{\vec{c}} = 25°,\ ||\vec{d}|| = 4$, and $\theta_{\vec{d}} = 110°$.

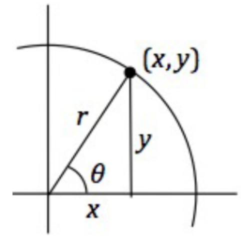
6. Determine the magnitude and direction of the resultant from part 5.

The Dot Product

Recall that we can define two important trigonometric functions as the ratios of the sides of the corresponding triangles:

$$\sin(\theta) = \frac{y}{r} \; (1)$$

$$\cos(\theta) = \frac{x}{r} \; (2)$$



Firgure 6.4-4

From (1) and (2) above, we can make the observations that

$x = r\cos\theta$, and
$y = r\sin\theta$

Now, let's apply this concept to what we've learned about vectors. We've seen that vectors can be written in component form such as

$$\vec{u} = \langle x, y \rangle$$

pictured as

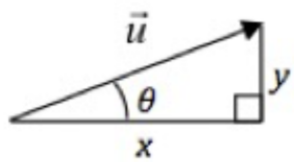


Figure 6.4-5

Thus, we can visualize the magnitude (or norm) of a vector as the length of the hypotenuse of a right triangle. It follows that

$$\vec{u} = \langle x, y \rangle = \langle ||\vec{u}|| \cos\theta, ||\vec{u}|| \sin\theta \rangle$$

We can now define what is meant by the *dot product* between two vectors.

> **Definition.** Let $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$ be two vectors and let θ be the angle between them. The dot product of $\vec{v}$ with $\vec{w}$, is defined as
>
> $$\vec{v} \cdot \vec{w} = ||\vec{v}|| \; ||\vec{w}|| \cos\theta. \;\; (3)$$

The dot product can also be written in component form as

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2. \quad (4)$$

As an exercise, you will be asked to derive the component form (4) of the dot product from angle form (3), in order for you to see why the component form (4) is equivalent (3).

The dot product between two vectors produces a **scalar result**; i.e. the dot product is a real number not a vector. One way to visualize the dot product is to think of this product as being the magnitude of the projection of $\vec{v}$ onto $\vec{w}$ times the magnitude of $\vec{w}$.

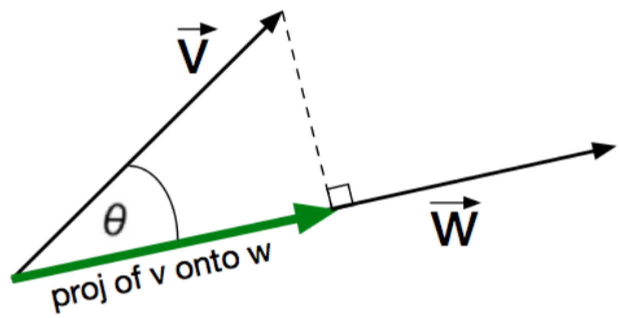


Figure 6.4-6

The dot product has many applications in the sciences and engineering. For example the amount of work done by applying a force to an object at angle in order to move it a certain displacement can be calculated as

$$W = \vec{F} \cdot \vec{d}$$

In an equivalent form

$$W = ||\vec{F}||\ ||\vec{d}|| \cos\theta$$

Dot Product Properties

Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors and let r represent a scalar.

a) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ Commutative Property

b) $r\vec{u} \cdot \vec{v} = \vec{u} \cdot r\vec{v}$ Associative Property

c) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ Distributive Property

d) $\vec{u} \cdot \vec{u} = ||\mathbf{u}||^2$ (Norm Property)

Exploration 6.4.2: The Dot Product

1. Find the dot product for $\langle 3, 4\rangle$ and $\langle 7, 2\rangle$ using the component form.

2. Find the dot product of these two vectors: $\langle 12, -1\rangle,\ \langle 6, 4\rangle$.

3. Given the two vectors with non-zero magnitudes, use this form of the dot product

$$\vec{v} \cdot \vec{w} = ||\vec{v}||\ ||\vec{w}|| \cos\theta$$

to decide when the dot product is equal to zero?

4. Solve

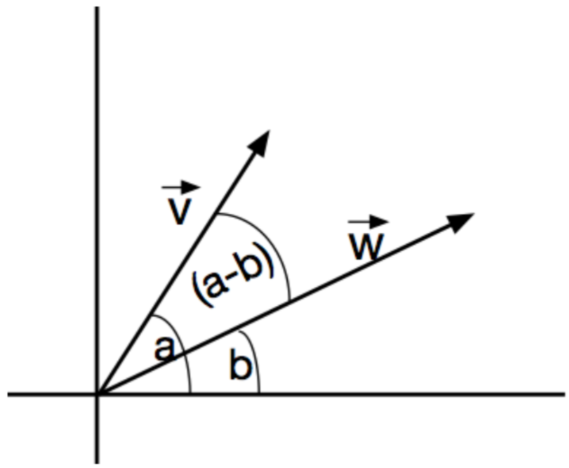
$$\vec{v} \cdot \vec{w} = ||\vec{v}||\ ||\vec{w}|| \cos\theta$$

for $\cos\theta$ and use this form of the equation to find the angle between the vectors represented by $\langle 3, 4\rangle$ and $\langle 7, 2\rangle$.

5. Find the angle between $\langle 8, 6\rangle$ and $\langle -3, 4\rangle$.

6. Determine the work done by force of 30 Newtons applied at an angle of $45°$ with the horizontal that moves an object 12 meters along the floor. (Note: $W = \vec{F} \cdot \vec{d}$ and a Newton is a kg$\frac{\text{meter}}{\text{sec}^2}$).

7. Consider this picture of two vectors in the plane with an angle of (a-b) between them, where $\vec{v}$ has an angle a from the horizontal and $\vec{w}$ is at an angle b from the horizontal axis.



Note that

$$\vec{v}_x = ||\vec{v}|| \cos(a) \quad \vec{v}_y = ||\vec{v}|| \sin(a)$$
$$\vec{w}_x = ||\vec{w}|| \cos(b) \quad \vec{w}_y = ||\vec{w}|| \sin(b)$$

and

$$\vec{v} \cdot \vec{w} = ||\vec{v}||\ ||\vec{w}|| \cos\theta$$

Use the provided figure and a trigonometric identity to show that the angle form and the component form of the dot product are equivalent.

8. Use the fact, $W = ||\vec{F}||\ ||\vec{d}|| \cos\theta$, to show that a satellite orbiting earth does zero work as it travels around the earth.

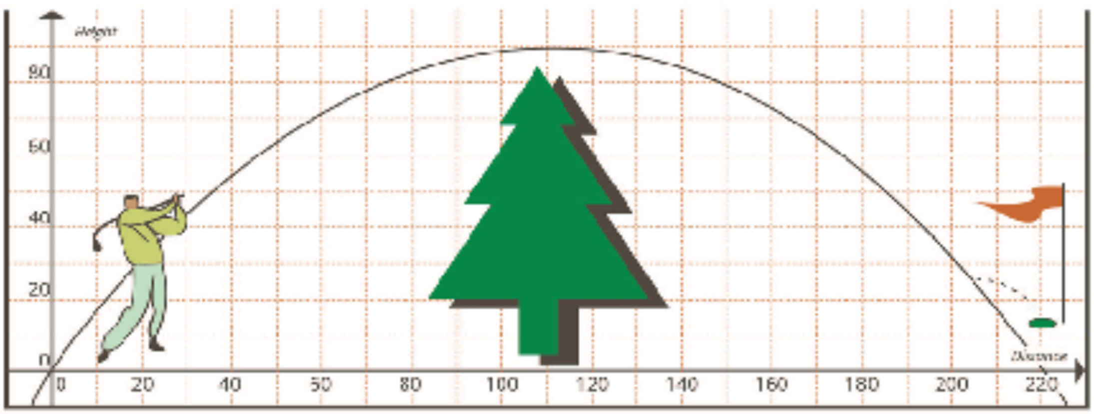
Lesson 6.5: The Golf Shot

"The Golf Shot" activity is another application Exploration that will allow you to use and connect a few different concepts that have been covered thus far in the course. Recall from a previous exploration that projectile motion can be modeled using parametric equations, such as:

$$h(t) = h_0 + v_0 \sin(\theta)t - \frac{1}{2}gt^2$$

where $h =$ height; $v_0 =$ initial velocity; $g =$ gravity; $t =$ time; $h_0 =$ initial height. You will use this equation in the following Exploration.

Exploration 6.5.1: Gunter's Golf Shot



At the Golf Classic Open, the professional golfer, Gunter Money hit a chip shot from the rough that just skimmed the top of a 90 foot pine tree and went right into the hole, 220 feet away, on the green for an eagle on the 12th fairway.

Max. height: 90 feet **Max. distance: 220 feet**

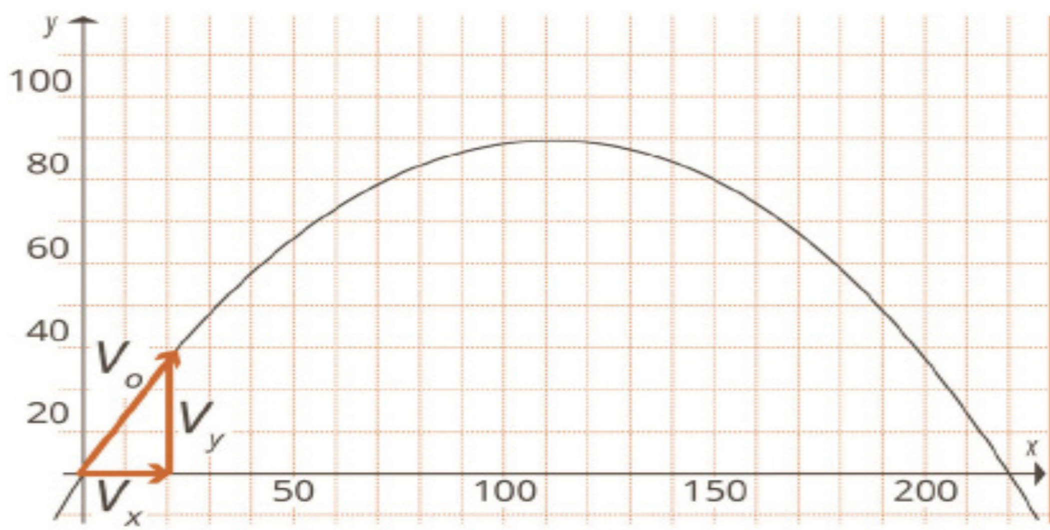
1. Use regression to find an equation for the path of the golf ball.

2. Now use matrices to find the equation for the golf ball's path.

3. Write the equation for the golf ball's path using the vertex form of the equation for a parabola.

4. What is the angle at which the golf ball takes off?

5. In order to introduce the parameter of time, t, into this situation, first find the time it takes the ball to fall to the ground from its maximum height. Then find the ball's speed when it hits the ground.



6. Last, use the diagram above and the information that you have found to write a parametric relation for the ball's path that incorporates time and relates this parameter to the distance and height that the ball travels.

Lesson 6.6: The Polar Coordinate System

The *polar coordinate* system, unlike the rectangular system, is not a *unique* system when it comes to naming points. Since the polar system involves "wrapping" around the origin or *pole*, points can be named many ways due to the periodic nature of the system. In the *polar coordinate system*, a point is defined by the polar coordinates (r, θ) where r is defined as the distance from the pole to a point after sweeping through an angle of θ. In accordance with trigonometric convention, a positive θ sweeps in a counter-clockwise direction about the pole while a negative θ sweeps in a clockwise direction. The value of r is defined as positive, but in a vector sense, we can consider the "negative of r" in that $-r$ will be the reflection of the distance r from the pole in the opposite direction.

!! Your instructor might want to elaborate on the polar system at this point.

Figure 6.6-1 shows a picture of the polar coordinate system with some common radian angle measures noted between 0 and 2π . Each concentric circle represents a further unit measure of r away from the pole. The r values that lie along the 0 radian axis are said to be on the polar axis and those that lie along the $\frac{\pi}{2}$ angle are said to be on the $\frac{\pi}{2}$ axis.

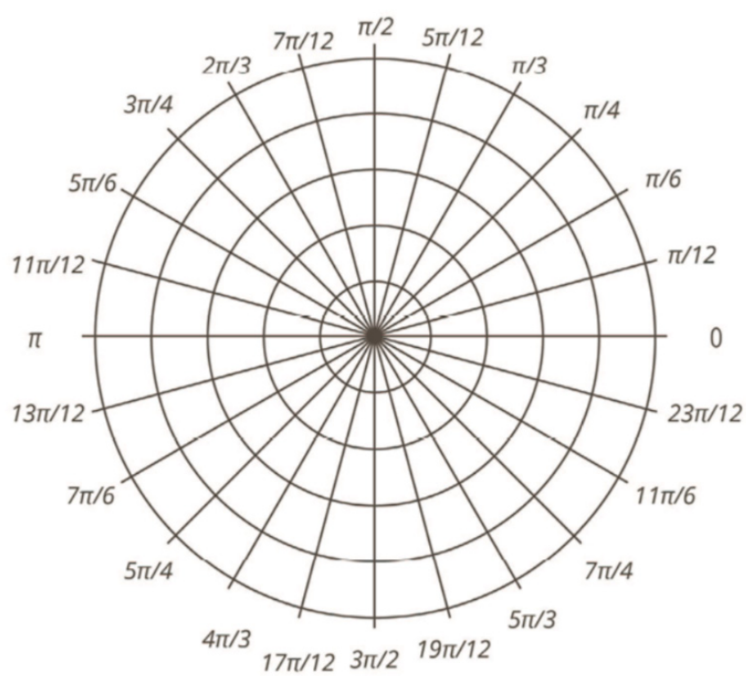
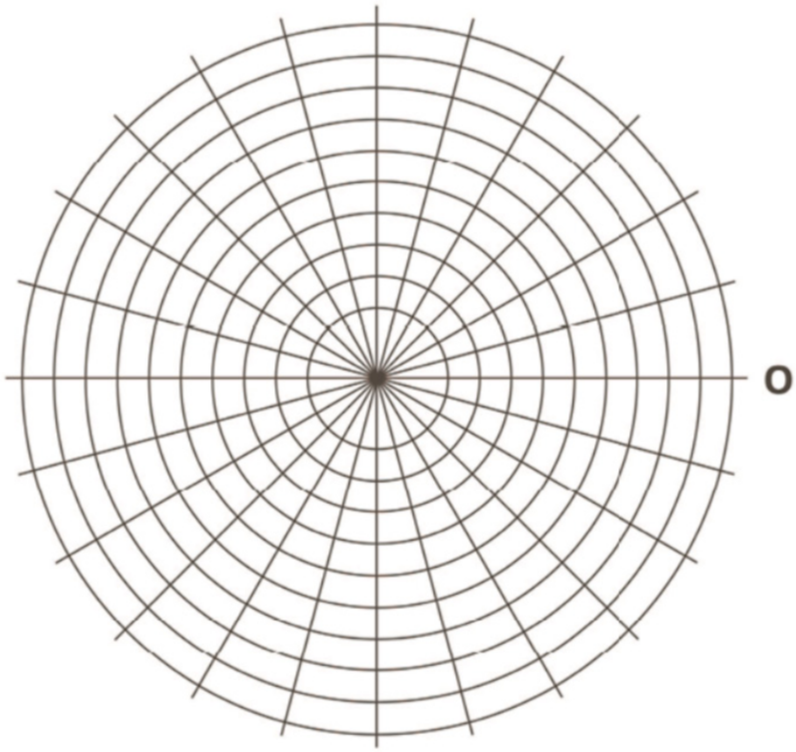


Figure 6.6-1: The Polar Coordinate System

Exploration 6.6.1: Polar Coordinates

Graph and label (w/ letter) the following points given in polar coordinates.

1. $A\left(7, \frac{\pi}{6}\right)$ 2. $B\left(5, -\frac{\pi}{2}\right)$ 3. $C\left(-6, \frac{\pi}{3}\right)$ 4. $D\left(3, -\frac{3\pi}{2}\right)$



5. What would the graph $r = 2$ look like in the polar coordinate system? What about $r = c$, where c is a constant?

6. What would the graph $\theta = 2$ look like in the polar coordinate system? What about $\theta = k$, where k is a constant?

In case you were wondering, if one wanted to convert between the (x, y) rectangular coordinate system and the (r, θ) polar coordinate system or vice versa, the conversions are defined in this way

Coordinate Conversion

$$x = r\cos\theta \qquad \tan\theta = \frac{y}{x}$$

$$y = r\sin\theta \qquad r = \sqrt{x^2 + y^2}$$

Keep in mind that a point in one system should be able to "overlay" onto that point in the other system if one were to conceive of placing one system on top of another. This is good to consider when trying to decide what quadrant a rectangular system point will be in when converting from the polar to the rectangular system.

Exploration 6.6.2: Polar Conversions

1. Fill in the table below.

(x, y)	(r, θ)
	$\left(5, -\frac{\pi}{3}\right)$
$(-3, 0)$	
$(4, 3)$	
$(0, 5)$	

Lesson 6.7: A Nonstandard Exploration of the Polar Coordinate System

It is assumed that you have had previous experience with the polar coordinate system. Your instructor may, however, wish to perform a quick review/refresher of the basics of the system at this point. The standard introduction to the polar system usually involves a detailing of how one locates points in this system and then a jump to looking at classic polar relations involving trigonometric functions such as the *polar roses, cardiods, lemniscates, and circles*. One then explores how to convert from the polar system to the rectangular and back.

The Exploration of this section explores the polar system in a completely different manner that does not depend upon conversions or trigonometric functions. Rather, this Exploration uses comparison as the key tool for understanding the polar system in relation to the rectangular system. This is accomplished by looking at “similar” functions in each system. Estimation and comparison (or commonality) arguments are mostly used to lead to understanding of the objectives of the Exploration.

Keep in mind that the understanding of two basic concepts is needed to conduct the Exploration of this section. The first concept needed is a very basic understanding of vectors only in that one needs only to think of a vector as a *directed line segment* which is a line segment to which a direction has been assigned. The second is an understanding of what a *radian* is. Recall that a radian is defined as an angle such that when its vertex is placed at the center of a circle its sides intersect an arc whose length is equal to the radius. Thus there are 2π radians along the circumference of a given circle.

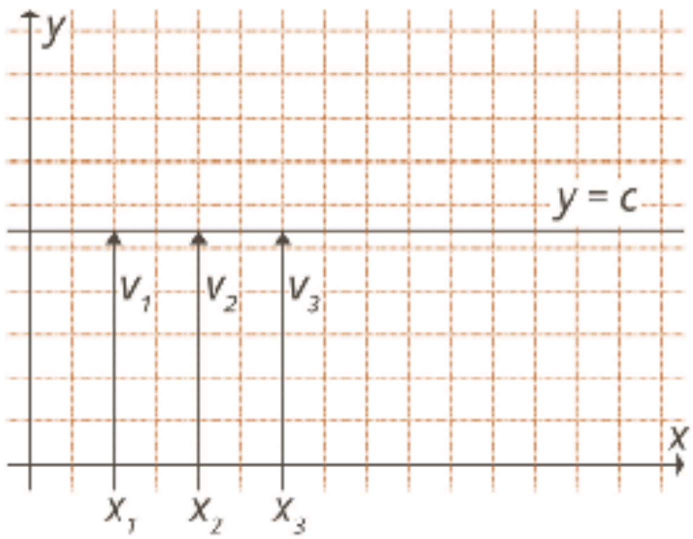
Exploration 6.7.1: Graphing "Cartesian Functions" in Polar Coordinates

Doppler radar used on television to report weather conditions; radar screens used by air traffic controllers to monitor aircraft traffic at an airport; flight plans filed by private aircraft to indicate paths taken as they move from one point to another; sonar positioning techniques employed in submarines; distances and compass bearings for directions used by campers in wilderness areas — these are but a few examples of the occurrence of vectors and polar coordinates in everyday life.

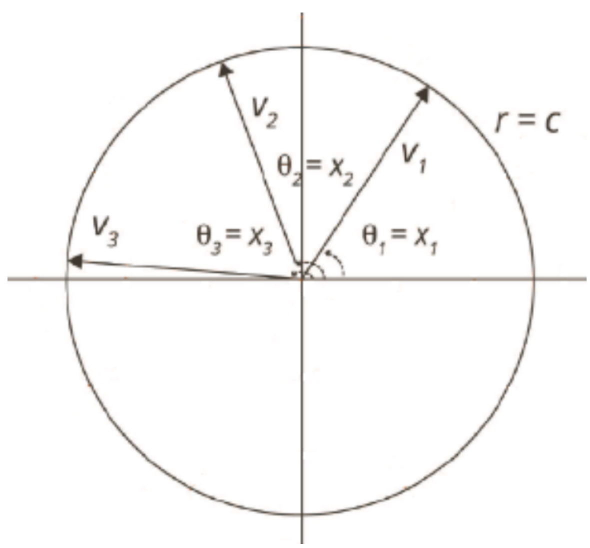
In this Exploration we will use vectors to graph familiar equations in Cartesian coordinates and compare those to the equivalent graph in polar coordinates. We will use “$(\, , \,)$” for Cartesian coordinates and $\langle \, , \, \rangle$ for polar coordinates.

In Cartesian coordinates, a vector will represent the directed line segment from the point $(x, 0)$ to the point $(x, f(x))$ while in polar coordinates, a vector will represent the directed line segment from the pole $\langle 0, 0\rangle$ to the point $\langle f(q), q\rangle$. In the examples and exercises the domain of the functions will be limited to the set of nonnegative real numbers. We will explore both linear and quadratic expressions.

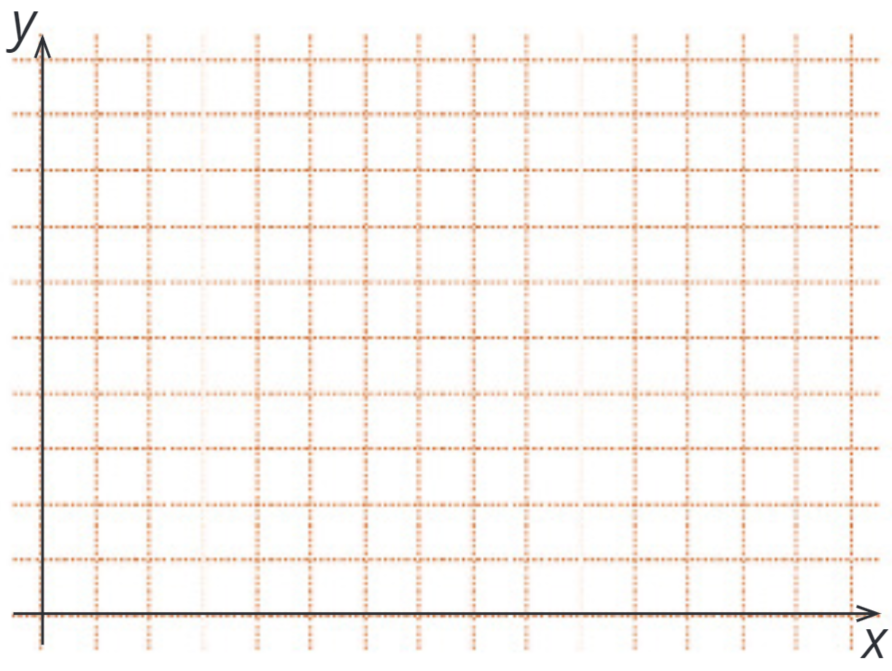
1. (Example for your consideration) We begin with a constant function $y = c$, where $c > 0$.

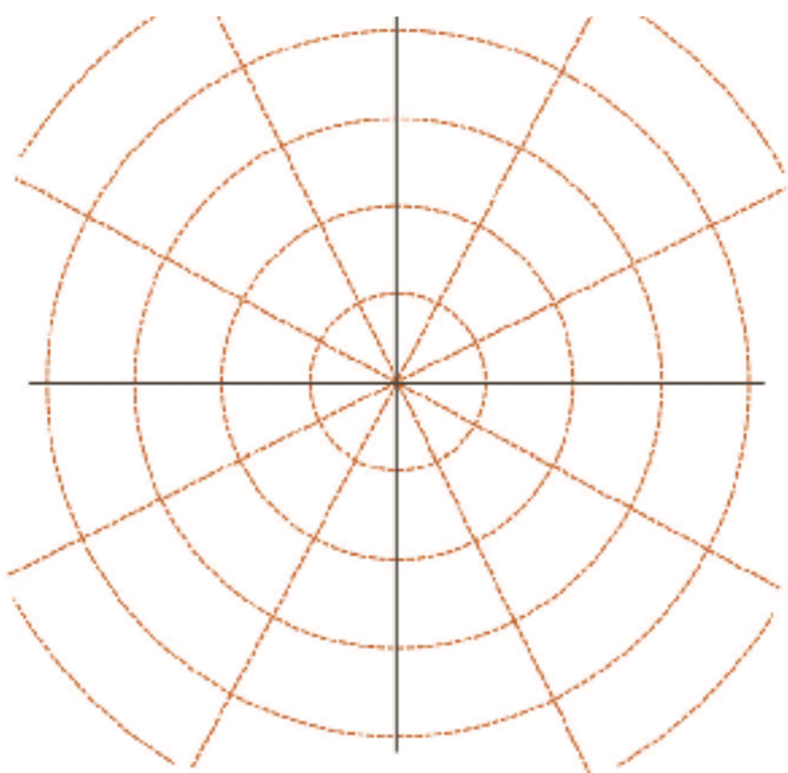


In this example, the vectors in Cartesian coordinates easily translate to vectors of fixed length bound at the origin with the tip of the vectors lying on a circle of radius c.



2. One should have little difficulty in graphing $y = x$, and can use this to interpret what should take place with the graph of $r = \theta$ for $\theta \geq 0$. Use the "vector approach" of the previous example as a guide to do this.

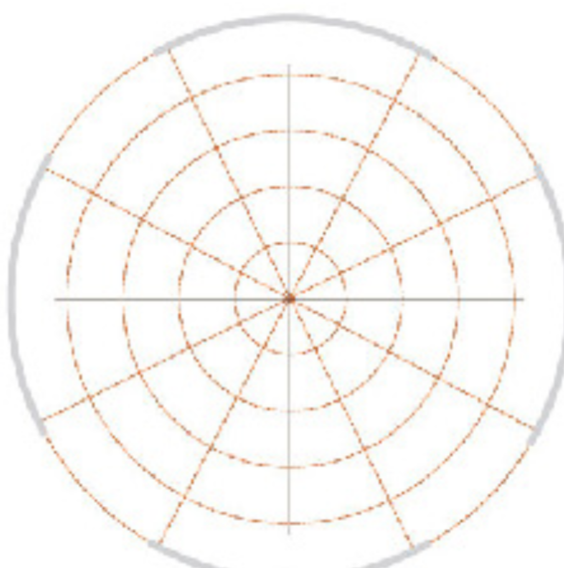




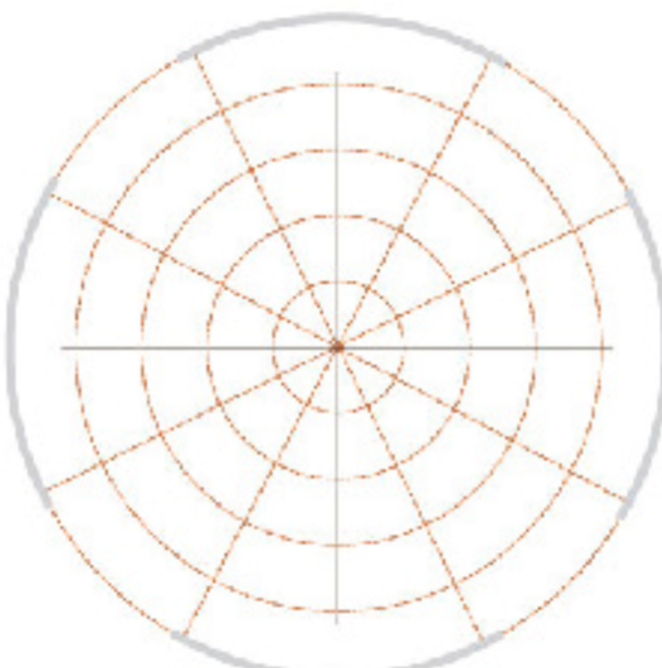
Quadratic Expressions:

We next consider polar quadratic functions of the form: $r = (\theta - a)(\theta - b)$, where $0 < a < b$; $r = (\theta - a)^2$, where $a > 0$; and $r = \theta^2 + a\theta + b$, where $r(\theta) \neq 0$ for all θ.

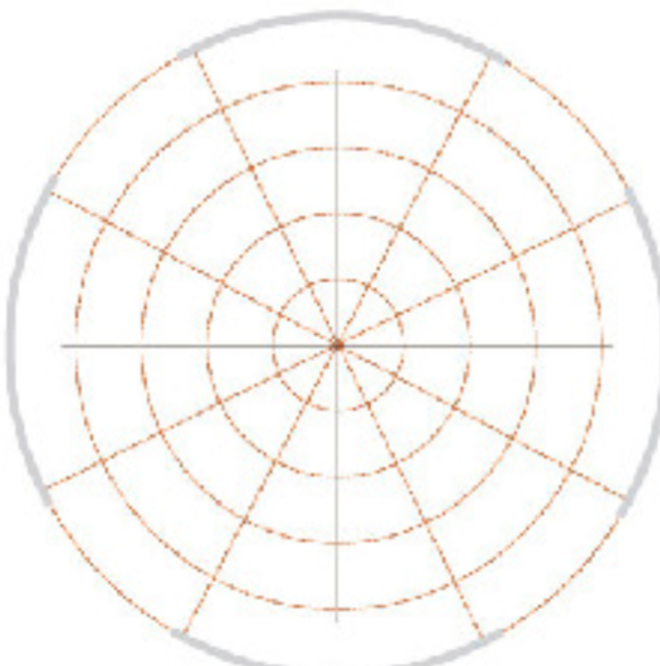
3. Use the same "vector approach" to graph $x^2 - 3x + 2 = (x-1)(x-2)$ and the corresponding polar graph $r = (\theta - 1)(\theta - 2)$ on the grids provided below (you may have to adjust your scale on each axis). The vertex of the parabolic graph is at $\left(\frac{3}{2}, -\frac{1}{4}\right)$ with axis of symmetry at $x = \frac{3}{2}$. For the polar graph, consider three rays corresponding to the values of $\theta = 1$ rad, $\theta = \frac{3}{2}$ rad, and $\theta = 2$ rad.



4. Consider a quadratic that has only one positive real root, $y = x^2 - 4x + 4 = (x-2)^2$ and the corresponding polar curve, $r = \theta^2 - 4\theta + 4 = (\theta - 2)^2$. When constructing the polar graph there is one important ray to consider, the ray $\theta = 2$ rad. Use the same "vector approach" to explore the connection between these graphs in the two systems.

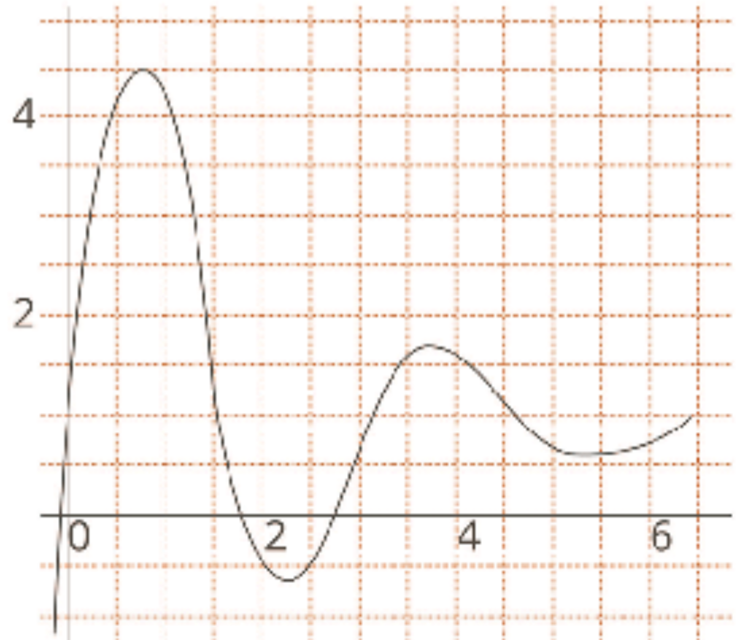


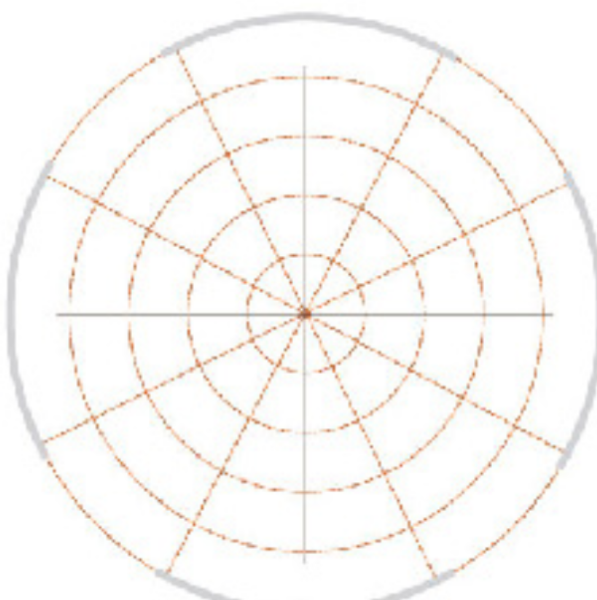
5. Next consider the quadratic, $y = x^2 - 4x + 8$ which has no real roots and is positive for all values of x. Perform the same systems exploration using the Cartesian and polar grids below.



Extension:

Use what you learned previously and the Cartesian graph below to create a "popular version" of the graph for $0 < \theta < 6$.





Historical Note:

While ancient Greek mathematicians such as Archimedes made references to functions of chord length that depended upon angles measured, it was a Persian geographer, Abu Rayhan Biruni (circa 1000) who is credited with developing an early foundation for a polar coordinate system. The polar coordinate system as known and used today, however, is credited as having been developed by Isaac Newton circa 1671, and further refined and used by Jacob Bernoulli circa 1691.

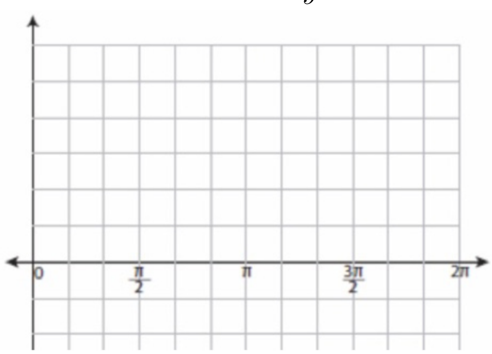
Lesson 6.8: Classic Polar Relations

In this section we will investigate the classic polar relations such as the *polar roses, cardioids, lemniscates, and circles* mentioned in preface to Exploration 6.7.1.

Exploration 6.8.1: Polar Relations and Conversions

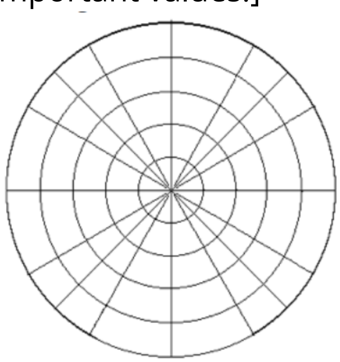
Part 1

1. Consider the Polar relation $r = 2 + 3\cos\theta$. Create the Cartesian graph for the function with $r = y$ and $\theta = x$ so that $y = 2 + 3\cos\ x$ if $0 \le x \le 2\pi$.

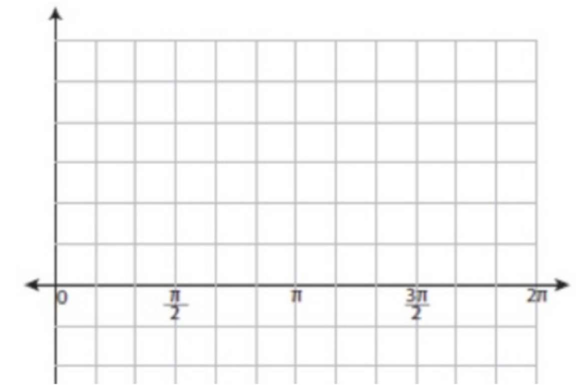


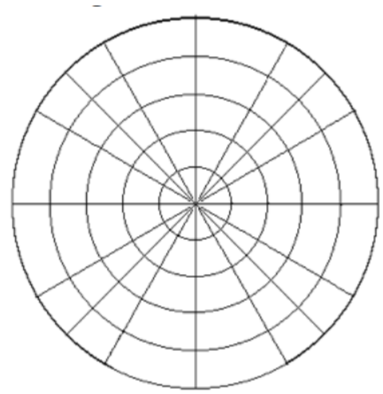
x	$y = 2 + 3\cos\ x$
0	
$\pi/2$	
	0
π	
	0
$3\pi/2$	
2π	

2. Now, plot values on the Polar grid, understanding that r is analogous to y and θ is analogous to x. [**Hint:** it may be helpful to create a table of important values.]

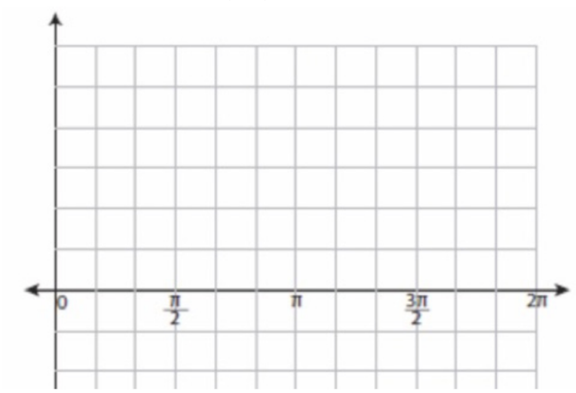


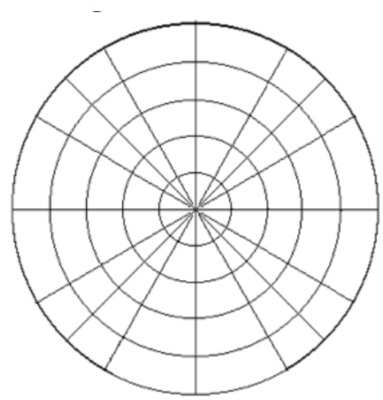
3. Go directly from the Cartesian to the Polar graph now and graph the relation $r = 3 + 2\sin\theta$.



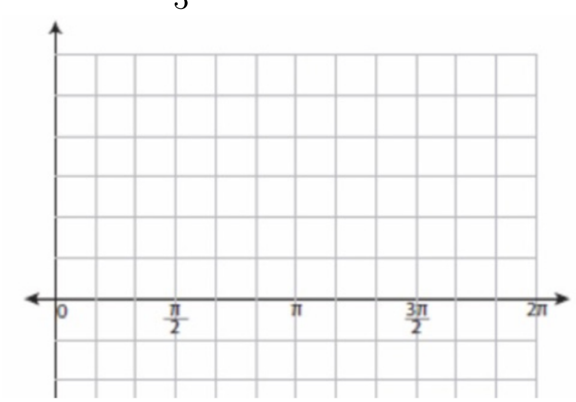


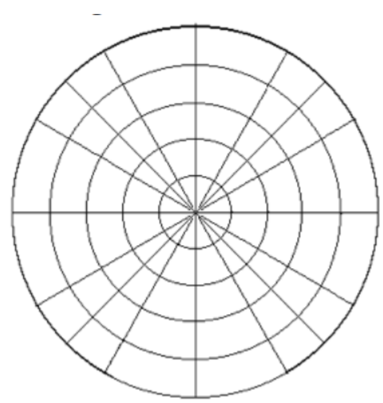
4. Graph $r = 2\sin(2\theta)$.



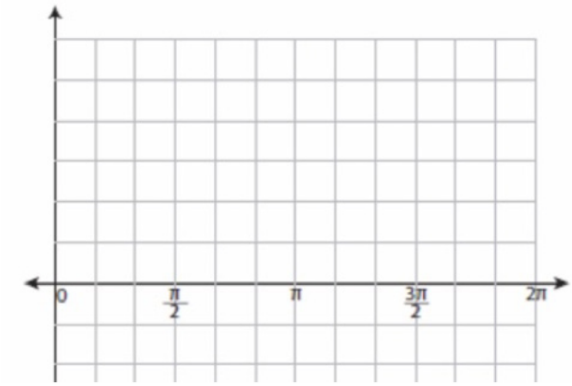


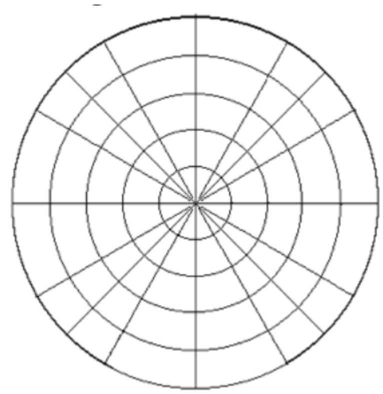
5. Graph $\theta = \dfrac{2\pi}{3}$.





6. Graph $r = 2\cos(3\theta)$.





Part 2

Complete the chart to give the same equations in both polar and rectangular form.

Example:

$x^2 + y^2 - 2x = 0$

$r^2 - 2r\cos\theta = 0$

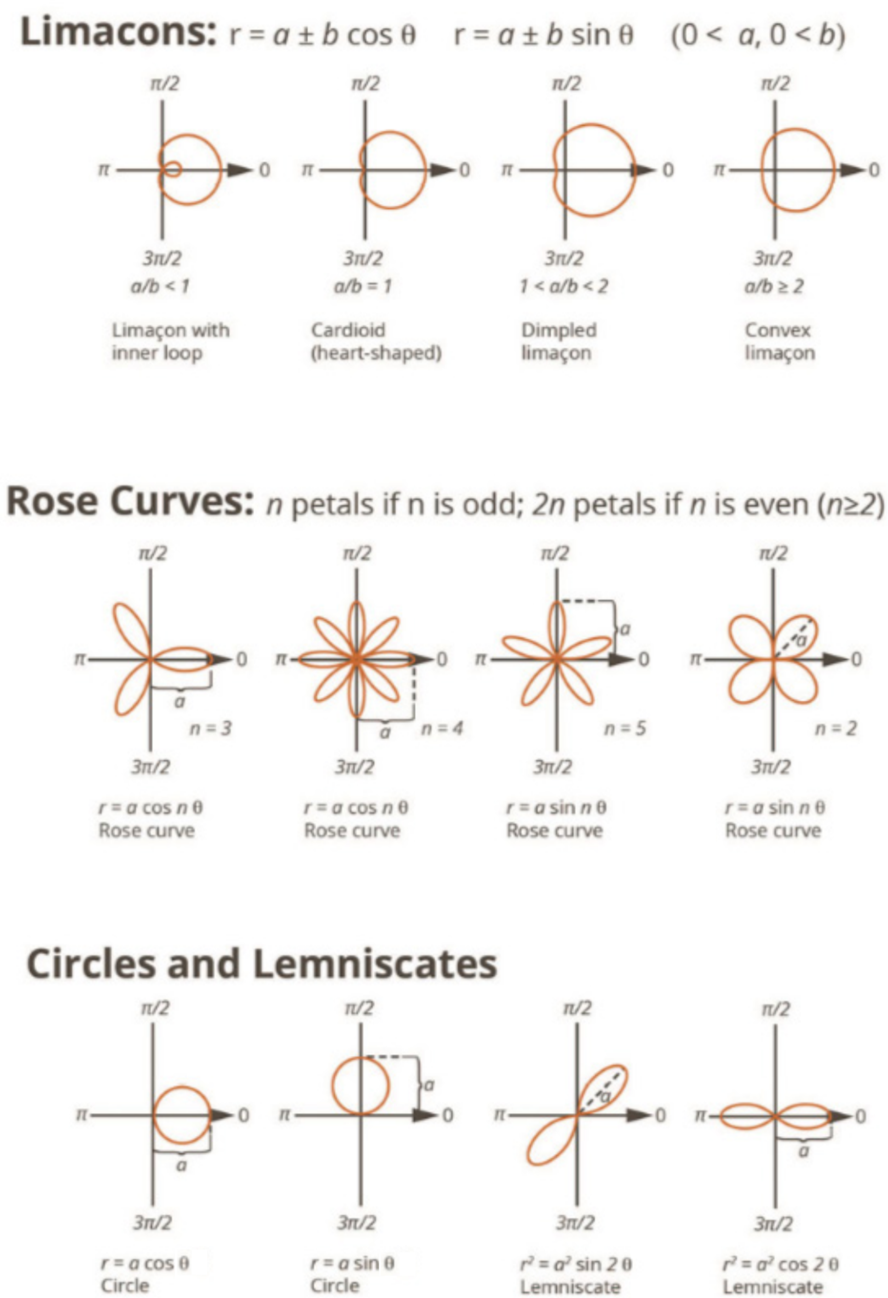
$r(r - 2\cos\theta) = 0$

$r = 2\cos\theta$

Rectangular	Polar
7. $x^2 + y^2 = 4$	
8. $y = 4$	
9. $x = 10$	
10.	$r = 3$
11.	$r\sin\theta = 4$
12.	$r = -2\sec\theta$

Having completed Explorations 6.7.1 and 6.8.1, you should have a good feel for the polar coordinate system. A summary of the classic polar curves and some of their properties is provided below.

The Classic Polar Curves



Lesson 6.9: Complex Numbers

!!As a warm-up for the Exploration of this section, your instructor should first conduct a short review of the field properties of the complex number system and have students try to come up with a way to visualize addition and subtraction of complex numbers in the coordinate plane. The *complex plane* is analogous to the Cartesian or rectangular plane. We can imagine a complex number $z = x + yi$ as analogous to the rectangular point (x, y). Thus the geometry of the point can be represented as in Figure 6.9.1-1

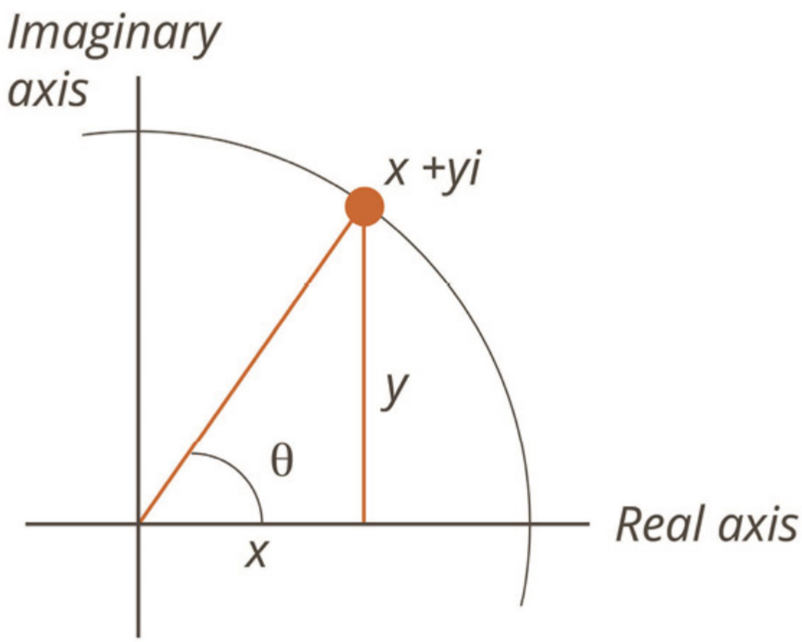


Figure 6.9.1-1: A Geometric Representation of $z = x + yi$

It will also be helpful to understand what is meant by the *modulus* of a complex number. The *modulus* or absolute value of $z = x + yi$ is represented by the symbol $|z|$ and is defined as

$$|z| = \sqrt{x^2 + y^2}$$

which is simply an application of the Pythagorean Theorem.

In the following exploration you will investigate properties of complex numbers, which you investigated previously in Unit 2, in more detail as a precursor to investigating *polar complex numbers and Euler numbers.*

Exploration 6.9.1: The Geometry of Complex Numbers

1. Now that you have reviewed the geometry of complex number addition and subtraction along with the *modulus*, $|a + bi|$, of a complex number; try to provide **geometric evidence** of the triangle inequality for complex numbers:

Theorem A: For any complex numbers w and z, $|w + z| \leq |w| + |z|$.

2. Use **Theorem A**, above, to algebraically prove an extension of the triangle inequality for complex numbers, namely

Theorem B: For any complex numbers w and z, $|w - z| \geq |w| - |z|$.

3. For any $z = a + bi$, we can define $\overline{z} = a - bi$ as the *conjugate* of z.

   a. If w and z are complex numbers , show that $\overline{w + z} = \bar{w} + \bar{z}$

   b. If w and z are complex numbers, show that $\overline{wz} = (\bar{w})(\bar{z})$

   c. Show that $|z| = |-z| = |\bar{z}|$ for all complex numbers z.

   d. Show that $|wz| = (|w|)(|z|)$ for all complex numbers w and z.

   e. Write $(z)(\bar{z})$ in terms of the modules of z.

Definition: If w and z are complex numbers $(z \neq 0)$, then $\dfrac{w}{z}$ is the complex number u such that $w = (z)(u)$.

   f. Find $\dfrac{w}{z}$, [Leave your answer in terms of z and w and verify the result.]

   g. Multiply your above answer above by z. Comment on why this makes sense with regard to the concept of 'division'.

   h. Prove that $\left|\dfrac{w}{z}\right| = \dfrac{|w|}{|z|}$, and that $\left|\overline{\dfrac{w}{z}}\right| = \dfrac{\overline{|w|}}{\overline{|z|}}$

Complex Numbers in Polar Form and Euler Numbers

Thus far in this course, among other topics, you have explored the polar coordinate system and complex number properties. In this section, we will combine concepts learned about each of these topics in order to explore complex numbers in polar form.

A complex number $z = x + yi$ can also be written in *polar form*

$$z = r(\cos\theta + i\sin\theta)$$

where r is equal to $|z|$ and θ is defined as the *argument* of z since

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

The argument θ is found by using the fact that

$$\tan(\theta) = \frac{y}{x}$$

where the quadrant containing z must be considering when finding θ.

Also note that $\cos\theta + i\sin\theta$ can be written as $e^{i\theta}$ by using *Euler's Formula*

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Therefore, we have

$$z = x + yi = re^{i\theta},\ (1)$$

where the latter form in equation (1) is called the *exponential form or Euler Number*. Using the information discussed to this point in the section, one is now ready to attempt Exploration 6.9.2.

Exploration 6.9.2: Complex Numbers in Polar Form

Given two complex numbers in polar form

$$z_1 = r_1(cos\theta_1 + i\sin\theta_1)\ and\ z_2 = r_2(\cos\theta_2 + i\sin\theta_2),$$

1. Show that $z_1z_2 = r_1r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$.

2. Show that $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right]$.

3. Use the result from (1.) to present an argument supporting DeMoivre's Theorem which states that if z is a complex number in polar form, then for any positive integer n

$$z^n = r^n\left[\cos(n\theta) + i\sin(n\theta)\right]$$

Corollary to DeMoivre's Theorem: For any $z = r\cos\theta$ and n any positive integer, the n distinct n^{th} roots of z are given by $\sqrt[n]{r}cis\dfrac{\theta+2\pi k}{n}$ for $k = 0, 1, 2, 3, ..., n-1$

4. Use the corollary above to find the five fifth roots of 1. Then graph these roots n the complex plane.

5. Use the theorem above to factor $p(x) = x^3 - 10$, into linear factors with complex coefficients.

Extension

6. Use the previously discussed fact that the Euler Number $e^{i\theta} = \cos\theta + i\sin\theta$ to derive some common trigonometric identities.

7. Write $e^{i\theta_1} \cdot e^{i\theta_2}$ two different ways in order to derive the *sine* and *cosine* angle addition identities.

8. Now try to derive the *sine* and *cosine* angle subtraction identities by writing $\dfrac{e^{i\theta_1}}{e^{i\theta_2}}$ two different ways.