Name: ________________________________ Period: ________ Date: ________

Unit 6 Exam Review
Fundamental Concepts

Learning Targets

Here are the official learning targets given by OnRamps for Unit 6.

1. Parametric Equations
    a) Examine distance traveled along a path with respect to coordinate positions.
    b) Graph parametric equations.
    c) Convert between rectangular relations and parametric equations.
    d) Create parametric equations to model real-world problems.
    e) Model projectile motion using parametric equations.

2. Vectors
    a) Calculate solutions to problems involving vector addition, subtraction, and scalar multiplication.
    b) Determine the component form of a vector given the magnitude and direction.
    c) Determine the component form and magnitude given a vector on a plane.
    d) Determine the magnitude and direction of a vector given in component form.
    e) Determine parameters of projectile motion such as angle, time, and distance.

3. Polar Coordinate System
    a) Graph and label given points in the polar coordinate system.
    b) Convert between rectangular and polar coordinates.
    c) Graph polar equations.
    d) Write equations of polar functions given a graph.
    e) Convert polar functions into rectangular equations.

Concept Questions

Answer in complete sentences using mathematical vocabulary. Doing so will refine your understanding of the topics in this unit and improve your ability to articulate what you know.

1. What are parametric equations? What do they describe? Based off your experience in class, what parameters have we used and what do they describe?

2. Suppose you are given two functions $x(t)$ and $y(t)$ that describe the horizontal and vertical position of an object with respect to time. What algebraic techniques have we used to eliminate the parameter of time? That is, if we wanted to create a function $y(x)$ that describes the vertical position depending on the object's horizontal position, how do you eliminate t from the equation?
    a) Describe how you eliminate the parameter if $x(t)$ and $y(t)$ are polynomials.
    b) Describe how you eliminate the parameter if $x(t)$ and $y(t)$ describe a conic section such as a circle, ellipse, or hyperbola. *Consider: what *type* of functions would $x(t)$ and $y(t)$ need to be to describe such shapes?

3. Suppose you are given two functions $x(t)$ and $y(t)$ that describe the horizontal and vertical position of an object with respect to time. How could you go about graphing the path of the object without finding the equivalent rectangular-form equation? Moreover, how exactly do you *draw* this curve?

4. Explain how each parametric function describes an objects position (horizontal/vertical) depending on time. That is, identify what each constant/variable represents and explain how the equation works to describe the path of the object. It would benefit you to draw an illustration.
    a) $x(t) = v_0 \cos(\theta) t$
    b) $y(t) = h_0 + v_0 \sin(\theta)\, t - \frac{1}{2} g t^2$

5. How does vector addition work…
   a) Geometrically? Draw a diagram of vector addition and vector subtraction.
   b) Algebraically? Explain how vector addition/subtraction works with component form of vectors.

6. Write down what equation you use to compute the dot product of two vectors $\vec{v}$ and $\vec{w}$ if…
   a) The magnitudes of $\vec{v}$ and $\vec{w}$ are given and the angle between them is given.
   b) The component forms of $\vec{v}$ and $\vec{w}$ are given.
   c) Rearrange your equation from part a) such that you have a formula that will give you the angle between $\vec{v}$ and $\vec{w}$, provided you know the magnitudes of $\vec{v}$ and $\vec{w}$ AND the component form of $\vec{v}$ and $\vec{w}$. If you did it correctly, your formula should *not* include the expression "$\vec{v} \cdot \vec{w}$".

   ***You MUST memorize the two dot product formulas!** $\boldsymbol{\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos\theta = v_1 w_1 + v_2 w_2}$

7. Explain WHY the dot product of two vectors is a scalar quantity and *not* a vector. You might try visualizing this geometrically or justifying it algebraically.

8. Given two vectors with non-zero magnitudes, when will their dot product be zero? Briefly explain why this is true (this is the third question from Exploration 6.4.2).

9. Describe how the polar coordinate system works. What are the parameters of our coordinates? How do you plot these coordinates? Describe how you plot such coordinates if one (or both) of the parameters are negative. Explain why the polar system is not unique (what is meant by "uniqueness?"). It would benefit you to give a few examples and illustrations of such coordinates.

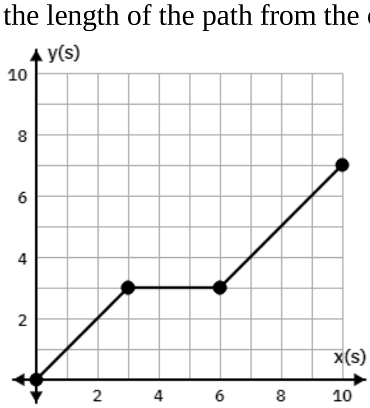
10. List the equations you would use to convert from polar form to rectangular form, and vice versa. It is expected you memorize such conversions use them quickly. It is recommended you practice these conversions for ordered pairs (rectangular to polar coordinates) and expressions (rectangular to polar equations).

11. Know and understand how to graph variations of limaçons and rose curves. You will need to figure out a way that is most memorable to you. It is usually enough to graph r for $\theta = \{0, \pi/2, \pi, 3\pi/2\}$. This gets you four nice points on the graph. Inner loop and convex limaçons are easy to recognize with these points; make sure you know how to discern between a cardiod and dimpled limaçon, however. Do not worry about the equation of lemniscates (but be able to recognize the shape). You can expect these on the multiple choice section; you will probably be asked to 1) write an equation *given* the graph and 2) given a polar equation, *identify* the graph.

Unit 6 Practice Assessment

It is important you practice these questions without the use of calculator. Questions that require the use of a calculator are marked with an asterisk (*).

1. For this question, we are concerned with the movement of an object along a path in the plane. We are assuming that the plane is a coordinate plane and the object starts at the point. As the object moves along the path, each point on that path has two coordinates. The coordinates depend on the distance traveled along the path. Let us call this distance S, the length of the path from the origin to a point P on the path.



   a) What value of s yields the coordinate (3, 3)?
   b) What value of s yields the coordinate (10, 7)?
   c) Evaluate $x(2)$.
   d) Evaluate $y(6)$.

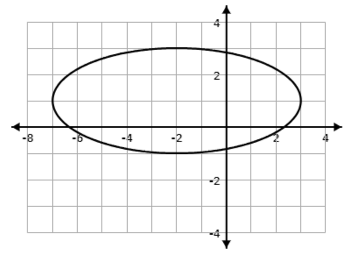
   Instructor's note: $\sqrt{2} \approx 1.4$

2. Answer the following given the parametric function below.

$$\begin{cases} x(t) = t^3 - t \\ y(t) = 2t \end{cases}$$

a) Create a table that gives $x(t)$ and $y(t)$ for values of $t = \{-2, -1, 0, 1, 2\}$.
b) Using the table you wrote, draw the graph generated by this function for $-2 \le t \le 2$ on the xy-plane.

3. Answer the following given the graph to the right.



a) Write the rectangular equation that models the graph.
b) Write a parametric function that models the graph.

4. For each parametric function, eliminate the parameter t and rewrite as a Cartesian equation in $y =$ form.

a) $\begin{cases} x(t) = 3 + t \\ y(t) = 1 - 2t \end{cases}$ b) $\begin{cases} x(t) = e^{-t} - 4 \\ y(t) = e^{-3t} - 1 \end{cases}$ c) $\begin{cases} x(t) = \sin(t) \\ y(t) = 5\cos(t) \end{cases}$

5. You are on a Ferris wheel that has a radius of 32 feet and the bottom of the wheel is 3 feet above the ground. The Ferris wheel starts when you get on at the bottom and rotates counter-clockwise and has a period of 40 seconds. Create a parametric function to model your location on the Ferris wheel at a given time.

6. A train is spotted 8 miles north and 5 miles west of an observer at 11:00 am. At 2:00 pm the train is spotted 4 miles south and 4 miles east of the observer.

a) How far did the train travel in the first hour? At 12:00 pm, where is the train in relation to the observer?
b) Write a parametric function that will give the train's position at any time.
c) Predict the train's location relative to the observer at 5:00 pm. How far away is the train from the observer?

7. Answer the following given $\vec{u} = < -5, 2 >$ and $\vec{v} = < 3, 1 >$.

a) Sketch $\vec{u}$ and $\vec{v}$ on the Cartesian plane.
b) First sketch $\vec{v} - \vec{u}$, then give $\vec{v} - \vec{u}$ in component form.
c) First sketch $2\vec{u} + 5\vec{v}$, then give $2\vec{u} + 5\vec{v}$ in component form.
d) Find $||\vec{u}||$ and $||\vec{v}||$.
e) Find $\vec{u} \cdot \vec{v}$.
f) *Find the direction (in degrees) of $\vec{u}$ and the direction of $\vec{v}$.
g) *Find the angle θ between $\vec{u}$ and $\vec{v}$ by using the equation of the dot product. Express your answer in degrees and round to three decimal places.
h) Look the sketch you drew in part a). Using your answers from part f), find the angle between $\vec{u}$ and $\vec{v}$ using your knowledge of geometry.

Instructor's note: Notice that you needed the information in parts d) and e) in order to answer part g). If I had only asked you to find the angle between the two vectors, you would have had to figure out that information regardless. Unless, of course, you solved it using the simple geometry thinking in part h)!

8. Answer the following on component form and magnitude of vectors.

a) Write the component form of the vector given its magnitude and direction. Give exact values.

i. $\|\vec{a}\| = 2, \theta_{\vec{a}} = 270°$.

ii. $\|\vec{b}\| = 10, \theta_{\vec{b}} = 330°$.

iii. $\|\vec{c}\| = \sqrt{6}, \theta_{\vec{c}} = 135°$.

b) Find the magnitude of $\vec{a} + \vec{b}$.

c) For an extra algebraic headache, find the magnitude of $\vec{a} + \vec{b} + \vec{c}$.

9. Which of the following vectors is perpendicular to $< 6, 9 >$? (This is a multiple choice question). Note: this question would not allow the use of a calculator on the test.

A. $< 2, 3 >$ B. $< 3, -2 >$ C. $< -6, -9 >$ D. $< -1, 0 >$

*10. A football is punted off the ground at an angle of elevation 45° and an initial velocity 72 ft/sec.

a) How long is the ball in the air? Give an exact answer and an approximation rounded to three decimals.

b) What is the maximum height of the ball? Use your exact answer from part a) to find the exact value for the maximum height. Then check your result with a calculator.

c) How far, horizontally, does the ball travel through the air? Again, use an exact value for time. You will find an exact value for distance. You can verify this result with your calculator.

*11. A mortar is launched from ground level and reaches a maximum height of 120 feet in the air before hitting its target on the ground 500 feet away.

a) Determine a quadratic equation that models the path of the mortar, assuming it starts at the origin. Round the coefficient of x^2 to the nearest ten-thousandths place, and round the coefficient of x to the hundredths place.

b) Using your understanding of parametric equations for projectile motion, at what is the angle the mortar is launched? Round your answer to the nearest hundredths place.

c) How long is the mortar in the air? Round your answer to the nearest hundredths place.

d) What is the mortar's speed when it hits the ground? Round your answer to the nearest tenths place.

*12. A daredevil is shot out of a cannon positioned on a platform 30 feet above the ground with an initial velocity of 64 ft/sec and at an angle of elevation of 70°.

Instructor note: this question uses the calculus skills you've learned from Unit 5 and applies them to parametric equations. This is a common test item on AP exams for AB/BC Calculus courses.

a) Write a set of parametric equations that model the football's horizontal and vertical movement.

b) After how many seconds does the daredevil achieve maximum height?

c) Give the location of the daredevil when he reaches maximum height (horizontal and vertical distance).

d) For how many seconds was the daredevil in the air?

e) How far away from the base of the platform did he land?

13. Convert each polar coordinate into a rectangular coordinate.

a) $\left(2, \frac{\pi}{6}\right)$ b) $\left(12, \frac{7\pi}{4}\right)$ c) $\left(-3, \frac{\pi}{2}\right)$ d) $\left(1, -\frac{2\pi}{3}\right)$

14. Convert each rectangular coordinate into polar form. It would be good practice to generate at least two different answers.

a) $(-4, 4)$ b) $(1, \sqrt{3})$ c) $(-7, 0)$ d) $(0, 5)$

15. Convert each rectangular equation into a polar equation.

a) $x = 2$
b) $y = -5$
c) $x^2 + y^2 = 3y$
d) $(x + 4)^2 + y^2 = 16$
e) $y = 2x - 3$

Instructor's note: the test will likely feature questions like a)-c), rather than d) or e). Still, it is good practice!

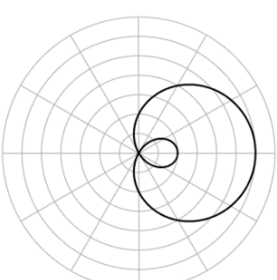
16. Convert each polar equation into a Cartesian equation.

a) $r = 6$
b) $\theta = 330°$
c) $r = 3\csc\theta$
d) $r = -4\sin\theta$

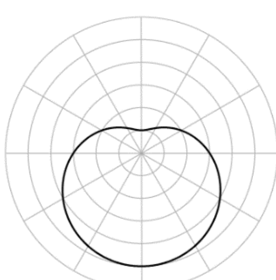
17. For each of the following polar functions, sketch a graph of the function, identify the shape of the graph, and describe its symmetry (either horizontally symmetrical over $\theta = \pi/2$ or vertically symmetrical over $\theta = 0$).

a) $r = 4 + 3\sin\theta$
b) $r = 5\cos(3\theta)$
c) $r = 1 - \sin\theta$
d) $\theta = \frac{5\pi}{6}$
e) $r = 1 - 3\cos\theta$
f) $r = -2\cos\theta$
g) $r = 6$
h) $r = 4\sin(2\theta)$
i) $r = 5 + 2\cos\theta$

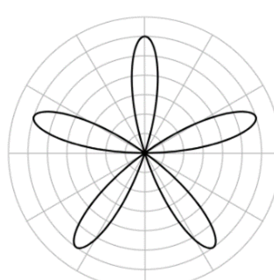
18. Write the equation for each polar graph.



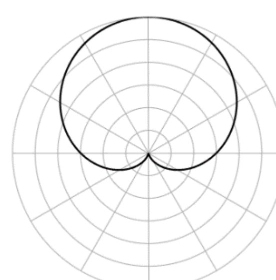
a) ________________



b) ________________



c) ________________



d) ________________

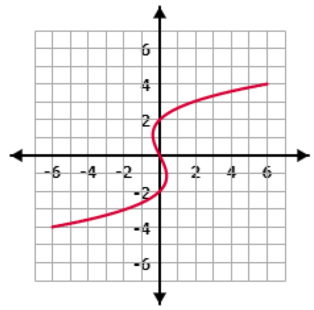
Answer Key

1) Path Graph Using Length as a Parameter

   a) $s = 3\sqrt{2}$
   b) $s = 3\sqrt{2} + 3 + 4\sqrt{2}$
   c) $x(2) = \sqrt{2}$
   d) $y(6) = 3$

2)

t	-2	-1	0	1	2
x	-6	0	0	0	6
y	-4	-2	0	2	4



3) Write the rectangular and parametric equation for an ellipse

   a) $\frac{(x+2)^2}{25} + \frac{(y-1)^2}{4} = 1$

   b) $x(t) = 5\sin t - 2$
      $y(t) = 2\cos t + 1$

*Note: you can interchange sine/cosine in these; same ellipse

4) Write each parametric equation as a Cartesian equation

   a) $y = -2x + 7$
   b) $y = (x + 4)^3 - 1$
   c) $x^2 + \frac{y^2}{25} = 1$

5) Ferris Wheel Problem – assuming you place the *y*-axis through the center of the wheel

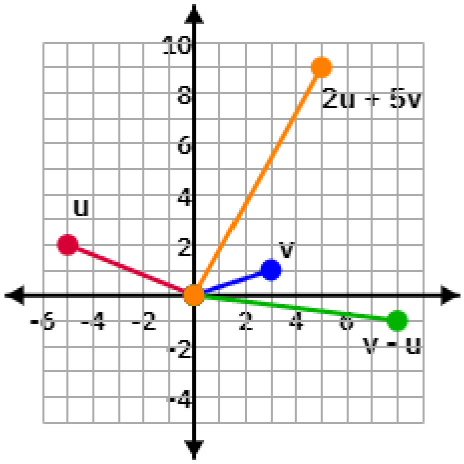
$x(t) = 32\sin(t)$
$y(t) = 35 - 32\cos(t)$

6) Train Problem

   a) Train moved 5 miles in first hour. The train is 2 miles west and 4 miles north of the observer at 12:00 pm.
   b) $x(t) = -5 + 3t$
      $y(t) = 8 - 4t$
   c) At 5:00 pm, train is 13 miles east and 16 miles south of the observer; the train is $5\sqrt{17} \sim 20.6$ miles away from the observer.

7) Vector Math

a), b), c) graphs



b) $\vec{v} - \vec{u} = \langle 8, -1 \rangle$

c) $2\vec{u} + 5\vec{v} = \langle 5, 9 \rangle$

d) $||\vec{u}|| = \sqrt{29}$, $||\vec{v}|| = \sqrt{10}$

e) $\vec{u} \cdot \vec{v} = -13$

f) $\theta_{\vec{u}} \sim 158.199°$, $\theta_{\vec{v}} \sim 18.435°$

g) $\theta = \cos^{-1}\left(\frac{-13}{\sqrt{29} \cdot \sqrt{10}}\right) = 139.764°$

h) $\theta = \theta_{\vec{u}} - \theta_{\vec{v}} = 158.199° - 18.435° = 139.764°$

8) Finding component form and magnitudes.

a) $\vec{a} =< 0, -2 >$, $\vec{b} =< 5\sqrt{3}, -5 >$, $\vec{c} =< -\sqrt{3}, \sqrt{3} >$
b) $\|\vec{a} + \vec{b}\| = 2\sqrt{31}$
c) $\|\vec{a} + \vec{b} + \vec{c}\| = \sqrt{100 - 14\sqrt{3}}$

9) Answer choice B.

10) Projectile motion (football)

a) $t = \frac{9\sqrt{2}}{4} \approx 3.182$ seconds
b) 40.5 feet above ground level
c) 162 feet away

11) Projectile motion (mortar)

a) $y = -0.0019x^2 + 0.96x$
b) $\theta = 43.83°$
c) $t = 5.48$ seconds
d) $v_0 = 126.5$ ft/s

12) Daredevil problem

a) $x(t) = 64\cos(70°)t \quad \sim \quad 21.89t$
$y(t) = 30 + 64\sin(70°)t - 16t^2 \quad \sim \quad 30 + 60.14t - 16t^2$
b) $t \sim 1.88$ seconds
c) Location at max height is $(41.14, 86.51)$
d) $t \sim 4.2$ seconds
e) Lands ~ 92 ft away

13) Converting polar to rectangular

a) $\left(\sqrt{3}, 1\right)$
b) $\left(6\sqrt{2}, -6\sqrt{2}\right)$
c) $(0, -3)$
d) $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

14) Converting rectangular to polar

a) Possible answers: $\left(4\sqrt{2}, \frac{3\pi}{4}\right)$, $\left(-4\sqrt{2}, \frac{7\pi}{4}\right)$, $\left(-4\sqrt{2}, -\frac{\pi}{4}\right)$, $\left(4\sqrt{2}, -\frac{5\pi}{4}\right)$
b) Possible answers: $\left(4, \frac{\pi}{3}\right)$, $(4, -\frac{5\pi}{3})$, $\left(-4, -\frac{2\pi}{3}\right)$, $\left(-4, \frac{4\pi}{3}\right)$
c) Possible answers: $(7, \pi)$, $(-7, 0)$, $(7, -\pi)$
d) Possible answers: $\left(5, \frac{\pi}{2}\right)$, $\left(-5, \frac{3\pi}{2}\right)$, $\left(-5, -\frac{\pi}{2}\right)$, $(5, -\frac{3\pi}{2})$

15) Converting rectangular equation to polar equation

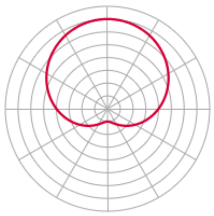
a) $r = 2\sec\theta$
b) $r = -5\csc\theta$
c) $r = 3\sin\theta$
d) $r = -8\cos\theta$

16) Converting polar equations to rectangular equations

a) $x^2 + y^2 = 36$
b) $y = -\frac{\sqrt{3}}{3}x$
c) $y = 3$
d) $x^2 + (y + 2)^2 = 4$

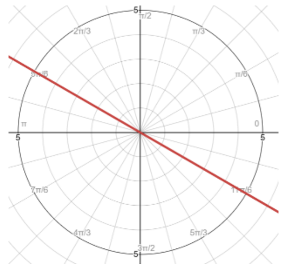
17) Sketch the graph of each function, identify the shape and describe its symmetry.

a)



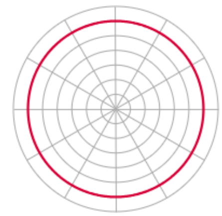
Dimpled Limaçon
Horiz. Symmetry

d)



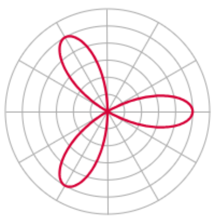
Line
No Symmetry

g)



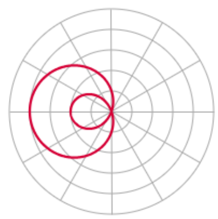
Circle
Horiz. And Vert. Symmetry

b)



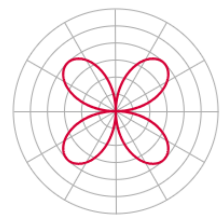
Rose Curve
Vert. Symmetry

e)



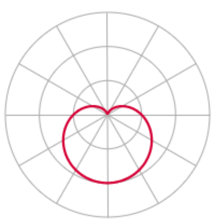
Limaçon with Inner Loop
Vert. Symmetry

h)



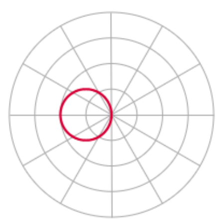
Rose Curve
Horiz. And Vert. Symmetry

c)



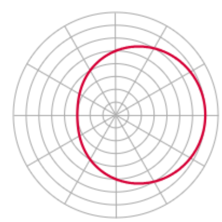
Cardiod
Horiz. Symmetry

f)



Circle
Vert. Symmetry

i)



Convex Limaçon
Vert. Symmetry

18) Write the equation for each polar graph

a) $r = 2 + 4\cos\theta$
b) $r = 3 - 2\sin\theta$
c) $r = 6\sin(5\theta)$
d) $r = 3 + 3\sin\theta$