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# A Note on Elliptic Coordinates

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# Explicit Equations to Transform from Cartesian to Elliptic Coordinates

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**Abstract:** Explicit equations are obtained to convert Cartesian coordinates to elliptic coordinates, based on which a function in elliptic coordinates can be readily mapped in physical space. Application to Kirchhoff vortex shows that its elliptical core induces two counter-rotating irrotational eddies.

**Keywords:** Elliptic Coordinates, Cartesian Coordinates, Kirchhoff Vortex

## 1. Introduction

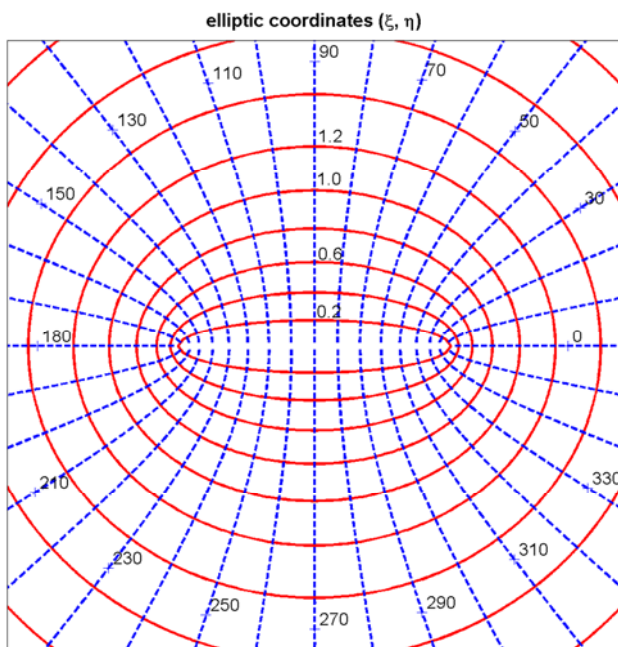


Figure 1. Coordinate lines for an elliptic coordinate system with  $a=1, b=0.5$ .

The elliptical coordinate system  $(\xi, \eta)$  as a two-dimensional orthogonal coordinate system has many dynamical and engineering applications, such as Kirchhoff vortex [1], insect aerodynamics [2], hydrodynamic wave

diffraction [3], and theoretical physics [4]. Its coordinate lines are confocal ellipses and hyperbolae and the transformation from elliptic to Cartesian coordinates is given by

$$\begin{aligned} x &= c \cosh(\xi) \cos(\eta) \\ y &= c \sinh(\xi) \sin(\eta) \\ c^2 &= a^2 - b^2 \\ \xi &\geq 0, \quad 0 \leq \eta < 2\pi \end{aligned} \quad (1)$$

where  $a$  and  $b$  denote the semi-major and semi-minor axes of the ellipse and  $c$  is the elliptical eccentricity (Figure 1).

In the meantime, explicit equations to transform from Cartesian to elliptic coordinates have not been found in the existing literature [5, 6, 7]. Such a conversion relation would be useful in mapping an elliptic-coordinate solution, for example the flow field of Kirchhoff vortex, in physical space.

## 2. Cartesian to Elliptic Coordinates

In order to invert the functional relation (1), we first eliminate  $\xi$  and have

$$\frac{x^2}{\cos^2(\eta)} - \frac{y^2}{\sin^2(\eta)} = c^2$$

which means curves of constant  $\eta$  are hyperbolae. The focus distance is  $c$  and the eccentricity is  $e = \sec(\eta)$ .

Let  $p = \sin^2(\eta)$ , we have

$$\frac{x^2}{1-p} - \frac{y^2}{p} = c^2$$

which becomes

$$c^2 p^2 + (x^2 + y^2 - c^2)p - y^2 = 0 \quad (2)$$

Then eliminating  $\eta$  from (1) we have

$$\frac{x^2}{\cosh^2(\xi)} + \frac{y^2}{\sinh^2(\xi)} = c^2$$

It shows that curves of constant  $\xi$  are ellipses. The focus distance is  $c$  and the eccentricity is  $e = \cosh^{-1}(\xi)$ .

Let  $q = -\sinh^2(\xi)$ , we have

$$\frac{x^2}{1-q} - \frac{y^2}{q} = c^2$$

which leads to

$$c^2 q^2 + (x^2 + y^2 - c^2)q - y^2 = 0 \quad (3)$$

It is essentially the same as (2). Therefore  $(p, q)$  constitute the two roots of a quadratic equation. Since  $0 \leq p \leq 1$ ,  $q \leq 0$ , we have  $p \geq q$ , and the two roots are

$$p = \frac{-B + \sqrt{B^2 + 4c^2 y^2}}{2c^2}, \quad q = \frac{-B - \sqrt{B^2 + 4c^2 y^2}}{2c^2} \quad (4)$$

in which  $B = x^2 + y^2 - c^2$ .

From the definition of  $p$  we obtain

$$\eta_0 = \arcsin(\sqrt{p}) \quad (5)$$

It has four cases depending on which quadrant the Cartesian point  $(x, y)$  is located, i.e.,

$$\begin{aligned} \eta &= \eta_0, & x \geq 0, y \geq 0 \\ \eta &= \pi - \eta_0, & x < 0, y \geq 0 \\ \eta &= \pi + \eta_0, & x \leq 0, y < 0 \\ \eta &= 2\pi - \eta_0, & x > 0, y < 0 \end{aligned} \quad (6)$$

Based on the definition of  $q$ , we can solve  $\xi$  from quadratic equation

$$e^{4\xi} + (4q - 2)e^{2\xi} + 1 = 0,$$

which has two roots

$$e^{2\xi} = 1 - 2q \pm 2\sqrt{q^2 - q}$$

Since  $q \leq 0$ , both roots are real and denoted as  $(\xi_1, \xi_2)$ .

They clearly satisfy  $e^{2\xi_1} \cdot e^{2\xi_2} = 1$ , which leads to  $\xi_2 = -\xi_1 < 0$ . Because in elliptical coordinates only non-negative  $\xi$  value is considered, we obtain

$$\xi = \frac{1}{2} \ln(1 - 2q + 2\sqrt{q^2 - q}) \quad (7)$$

Eqs. (4-7) are explicit equations to derive elliptic coordinates from Cartesian grid. They can easily be realized via computation software such as Matlab.

### 3. Application to Kirchhoff Vortex

Kirchhoff vortex is a rotating elliptical region of uniform vorticity  $\omega$  embedded in an irrotational ideal fluid [8]. It is the simplest example of non-smooth weak solutions to the Euler equations and has wide application in vortex dynamics [9-11]. It has a discontinuity of vorticity across its elliptical boundary

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where  $a$  and  $b$  are semi-major and semi-minor axes, respectively. The boundary corresponds to contour  $\xi = \operatorname{arcsinh}(b/c)$ , and the vortex rotates with constant angular velocity

$$\Omega = \frac{ab}{(a+b)^2} \omega$$

around the origin. From Act. 159 of Lamb [1], the streamfunction outside the core is

$$\psi = \frac{1}{4} \Omega (a+b)^2 e^{-2\xi} \cos(2\eta) + \frac{1}{2} \omega ab \xi \quad (8)$$

The instantaneous streamlines in a unsteady flow are given by the curves  $\psi = \text{const}$ . In a rotating frame with angular velocity  $\Omega$ , Kirchhoff vortex looks steady and its streamfunction outside the elliptic core is related to the inertial-frame streamfunction (8) by

$$\psi_R = \psi - \frac{1}{2} \Omega [x(\xi, \eta)^2 + y(\xi, \eta)^2] \quad (9)$$

From (9) it is straightforward to make a conformal mapping of steady streamfunction on a uniform mesh of elliptic coordinates (Figure 2).

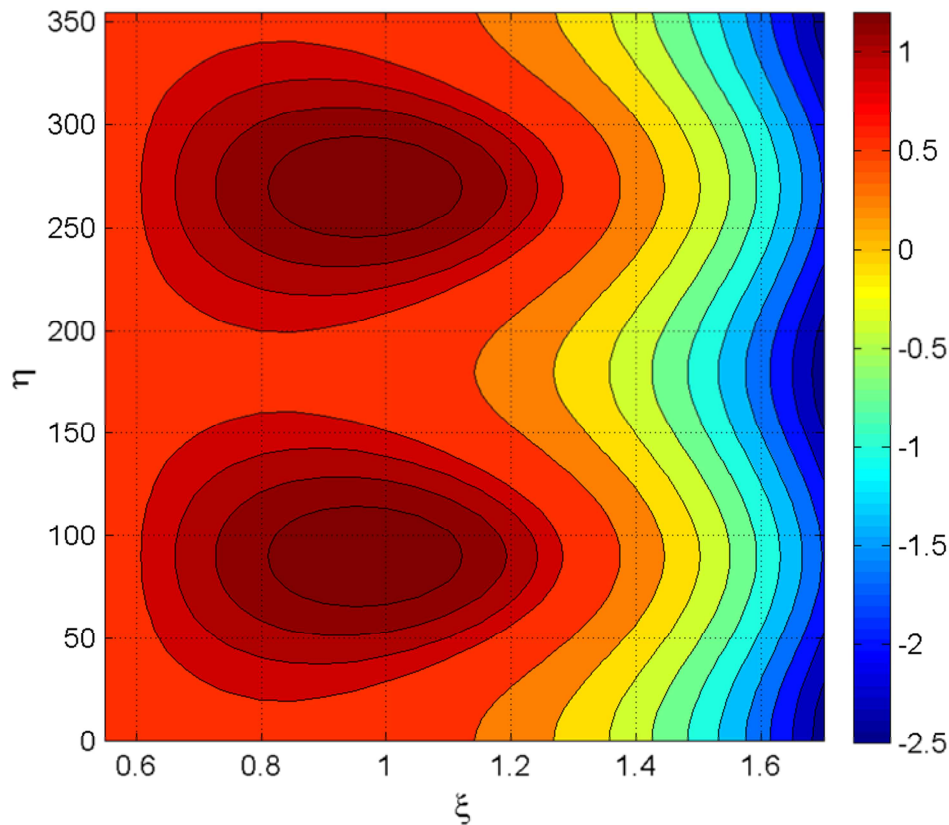


Figure 2. Mapping of streamfunction (9) on a uniform mesh of  $(\xi, \eta)$ , with  $a=1, b=0.5, \omega=10$ .

In order to view the flow field in physical space, we use the conversion equations (4-7) to plot streamfunction (9) on a uniform Cartesian mesh in Figure 3, which shows the elliptical core of Kirchhoff vortex induces two irrotational eddies that rotate in opposite directions.

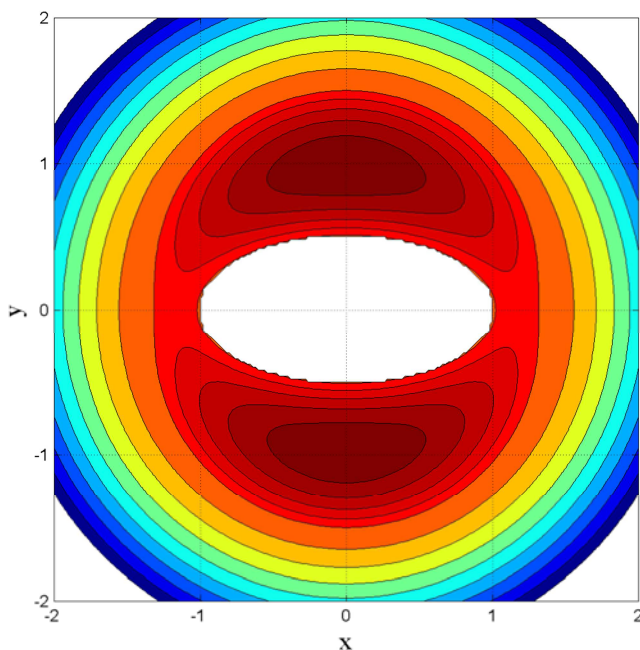


Figure 3. Mapping of streamfunction (9) on a uniform mesh of  $xy$ -plane, with  $a=1, b=0.5, \omega=10$ . The boundary of vortex core corresponds to  $\xi=0.55$ .

## 4. Conclusion

This study obtains explicit equations that convert Cartesian coordinates to elliptic coordinates. The conversion relation can be easily realized with computer software and used to map a known function of elliptic-coordinates, such as the streamfunction of Kirchhoff vortex, on a uniform Cartesian mesh.

## Acknowledgements

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## References

- [1] Lamb, H. (1932) Hydrodynamics. Cambridge Univ. Press, 738pp.
- [2] Wang, Z. (2000), Vortex shedding and frequency selection in flapping flight. *J. Fluid Mech.*, 410, 323-341.
- [3] Chatjigeorgiou, I., & Mavrakos, S. (2012), The analytic form of green's function in elliptic coordinates. *J. Engineering Math.*, 72, 87-105.
- [4] Morse, P., & Feshbach, H. (1953), Methods of theoretical physics, McGraw-Hill, 1978pp.

- [5] Arfken, G. (1970), *Mathematical Methods for Physicists*, 2nd ed., Academic Press, 815pp.
- [6] Hazewinkel, M. (1988), *Encyclopaedia of Mathematics*, Springer, 488pp.
- [7] Korn G., & Korn, T. (2000), *Mathematical handbook for scientists and engineers*, Dover, 1152pp.
- [8] Kirchhoff, G. (1877), *Vorlesungen über mathematische Physik. Mechanik*, Leipzig.
- [9] Miyazaki, T., Imai, T., & Fukumoto, Y. (1995), Three-dimensional instability of Kirchhoff's elliptic vortex. *Phys. Fluids*, 7, 195-202.
- [10] Muller, B., & Yee, H. (2001), High order numerical simulation of sound generated by the Kirchhoff vortex. *Comp. Visual. Sci.*, 4, 197-204.
- [11] Mitchell, T., & Rossi, L. (2008), The evolution of Kirchhoff elliptic vortices. *Phys. Fluids*, 20, 054103.

## Matlab mfile scripts

```
function [zeta,eta]=xy2el(x,y,a,b)
%-----
% Conversion from Cartesian mesh (x,y) to elliptic coordinates (zeta,eta)
% zeta (>=0), eta [0 2pi)
% a -- semi-major axis
% b -- semi-minor axis
% c -- elliptic eccentricity
% (x,y,zeta,eta) same-size matrix
% Author: Che Sun (CAS, 2016)
% Source doi:10.11648/j.mma.20170204.12
% Note 1: An early code error was pointed out by Dr. Vinod Varghese.
% Note 2: A potential round-off error at x=0, as pointed out by Dr. Peter Gangl,
%         is avoided by requiring root p<=1.
%-----
c2=(a^2-b^2);c=2*c2;
x2=x.^2; y2=y.^2;
B=x2+y2-c2;
del2=B.^2+2*c*y2; del=sqrt(del2);
p=(-B+del)/c; iv=find(p>1); p(iv)=1;
p=sqrt(p); et0=asin(p); eta=x;
i=find(x>=0&y>=0); eta(i)=et0(i);
i=find(x<0&y>=0); eta(i)=pi-et0(i);
i=find(x<=0&y<0); eta(i)=pi+et0(i);
i=find(x>0&y<0); eta(i)=2*pi-et0(i);
q=(-B-del)/c;
del2=q.^2-q; del=sqrt(del2);
zeta=log(1-2*q+2*del)/2;% only keep the positive root
```

### **function [x,y]=el2xy(zeta,eta,a,b)**

```
%-----
% Conversion from elliptic coordiantes (zeta,eta) to Cartesian coordinates
% zeta (>=0), eta [0 2pi)
% a -- semi-major axis
% b -- semi-minor axis
% c -- elliptic eccentricity
% (x,y,zeta,eta) same-size matrix
% Author: Che Sun (CAS, 2016)
% Source doi:10.11648/j.mma.20170204.12
%-----
c2=a^2-b^2; c=sqrt(c2);
x=c*cosh(zeta).*cos(eta);
y=c*sinh(zeta).*sin(eta);
```