Big data: Mathematical modelling
Eigenvalues and Eigenvectors

The eigenvalue problem concerns the determination of all vectors $\mathbf{x}$ such that $\mathbf{x}$ and $A\mathbf{x}$, where $A \in \mathbb{R}^{n\times n}$, are scalar multiples of one another. This scalar multiple is called the *eigenvalue* corresponding with the *eigenvector* $\mathbf{x}$. We use it here as part of our data analytic toolbox.

A formal statement of the eigenvalue problem is that a nonzero vector $\mathbf{x}$ is called an eigenvector of square matrix $A \in \mathbb{R}^{n\times n}$ if a scalar λ (eigenvalue) can be found such that $A\mathbf{x} = \lambda\mathbf{x}$. In this case, $\mathbf{x}$ is said to be an eigenvector of A corresponding to λ. We call $(\lambda, \mathbf{x})$ an *eigenpair*.

Clearly $\mathbf{x} = \mathbf{0}$ trivially satisfies $A\mathbf{x} = \lambda\mathbf{x}$, however it provides us with no useful information. For this reason we restrict our search only for nonzero $\mathbf{x}$. Note, however, that it is possible to have $\lambda = 0$ as an eigenvalue.

The eigenvalues of $A \in \mathbb{R}^{n\times n}$ are determined using the following strategy:

1. Write $A\mathbf{x} = \lambda\mathbf{x}$ as the homogeneous system of equations

$$(\lambda I_n - A)\,\mathbf{x} = \mathbf{0}.$$

2. This system admits a nontrivial solution when

$$\det\left(\lambda I_n - A\right) = 0.$$

   This equation is called the *characteristic equation* of A.

3. Expand the characteristic equation to obtain the *characteristic polynomial*:

$$p_A(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + a_3\lambda^{n-3} + \cdots + a_n = 0,$$

   which is a monic polynomial (ie, its leading coefficient is 1).

4. Determine the roots of the characteristic polynomial $p_A(\lambda)$. These roots correspond with the eigenvalues of $\mathbf{A}$.

The set of distinct eigenvalues, ie, the set of all roots of $p_A(\lambda)$, is called the spectrum of A and is denoted $\sigma(A)$. Note that every matrix $A \in \mathbb{R}^{n\times n}$ has n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$.

Remark: Some texts write the characteristic equation as $\tilde{p}_A(\lambda) = \det(A - \lambda I_n)$. However, by using the properties of the determinant, it can be shown that $\tilde{p}_A(\lambda) = (-1)^n p_A(\lambda)$ and that these two polynomials have the same roots. Our definition is preferred, as the coefficient of λ^n is always 1.

The algebraic multiplicity of λ_j, denoted $\operatorname{alg\,mult}_A(\lambda_j)$, is the multiplicity of λ_j as a root of $p_A(\lambda)$, that is $\operatorname{alg\,mult}_A(\lambda_j)$ is the smallest $k \in \mathbb{N}$ such that $p_A(\lambda) = (\lambda - \lambda_j)^k s(\lambda)$, where $s(\lambda_j) \neq 0$. If A has m distinct eigenvalues and $\operatorname{alg\,mult}_A(\lambda_i) = a_i$ for each $i = 1, \ldots, m$, then $p_A(\lambda) = \prod_{i=1}^{m}(\lambda - \lambda_i)^{a_i}$ is the characteristic polynomial of A. An eigenvalue λ is said to be *simple* if $\operatorname{alg\,mult}_A(\lambda) = 1$, otherwise it is said to be *repeated (multiple)*.

Associated with each eigenvalue λ there corresponds an *eigenspace* $E(\lambda)$ defined as:

$$E(\lambda) = \{\mathbf{x} \mid (\lambda I_n - A)\,\mathbf{x} = \mathbf{0}\} \subseteq \mathbb{R}^n.$$

The eigenspace $E(\lambda)$ is a vector subspace of $\mathbb{R}^n$ consisting of the eigenvectors for A together with the zero vector $\mathbf{0}$. Thus, once the eigenvalues have been determined, the set of all eigenvectors associated with λ is $E(\lambda)\backslash\{\mathbf{0}\}$.

The geometric multiplicity of λ_j, denoted $\operatorname{geo\,mult}_A(\lambda_j)$, is the number of independent eigenvectors of A associated with λ_j. For any eigenvalue λ_j of $A \in \mathbb{R}^{n\times n}$ it is true that $\operatorname{geo\,mult}_A(\lambda_j) \leq \operatorname{alg\,mult}_A(\lambda_j)$. A matrix is said to be *semisimple* if it has n linearly independent eigenvectors. If the matrix has an eigenvalue whose geometric multiplicity is less than its algebraic multiplicity, it is said to be *defective*. Two other interesting facts are that the trace $\operatorname{tr}(A)$ is the sum of the eigenvalues and the determinant $\det(A)$ is the product of the eigenvalues.

To illustrate these concepts further, we now use MATLAB to compute the eigenvalues of the following matrices. We also state their algebraic and geometric multiplicities, find their eigenspaces and determine if the matrix is either semisimple or defective. We'll show you the first one as an example, then you give the others a try and discuss your answers with your peers:

$$(a)\begin{pmatrix}1 & -3 & 3\\ 3 & -5 & 3\\ 6 & -6 & 4\end{pmatrix},\quad (b)\begin{pmatrix}5 & 0 & 1\\ 1 & 1 & 0\\ -7 & 1 & 0\end{pmatrix},\quad (c)\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 2\\ 0 & 0 & 1\end{pmatrix},\quad (d)\begin{pmatrix}-5 & 2 & 3\\ 2 & -4 & 2\\ 3 & 2 & -5\end{pmatrix}$$

Let's start by entering each of these matrices into MATLAB.

```
% Enter the matrices
A = [1 -3 3; 3 -5 3; 6 -6 4];
B = [5 0 1; 1 1 0; -7 1 0];
C = [0 0 1; 0 1 2; 0 0 1; ];
D = [-5 2 3; 2 -4 2; 3 2 -5];
```

Now compute the eigenvalues and eigenvectors.

```
% Compute the eigenvalues and eigenvectors for each matrix
[evecA, evalA] = eig(A);
[evecB, evalB] = eig(B);
[evecC, evalC] = eig(C);
[evecD, evalD] = eig(D);
```

If we look at the eigenvalues of matrix A, we see that the eigenvalues are

$$\lambda_{1,2} = -2, \quad (\text{repeated twice}),$$

$$\lambda_3 = 4, \quad (\text{simple}),$$

giving us three eigenvalues in total. These eigenvalues determine the characteristic polynomial for A as $p_A(\lambda){=}(\lambda+2)^2(\lambda-4)$, where it can be deduced that the algebraic multiplicity of $\lambda_{1,2} = -2$ is 2, and for $\lambda_3 = 4$ is 1. Looking at the eigenvectors (the columns of `evecA`), we have

$$\begin{pmatrix}-0.8103\\ -0.3185\\ 0.4918\end{pmatrix}, \begin{pmatrix}0.1933\\ -0.5904\\ -0.7836\end{pmatrix}, \begin{pmatrix}-0.4082\\ -0.4082\\ -0.8165\end{pmatrix},$$

corresponding to λ_1 , λ_2 , and λ_3 respectively. We can see that we have two linearly independent eigenvectors (not a scalar multiple of each other) corresponding to $\lambda_{1,2}$ and one eigenvector corresponding to λ_3 .

The set of these first two eigenvectors,

$$\left\{\begin{pmatrix}-0.8103\\-0.3185\\0.4918\end{pmatrix},\begin{pmatrix}0.1933\\-0.5904\\-0.7836\end{pmatrix}\right\}$$

form a basis for the eigenspace $E(-2)$. The dimensions of this space, which is the geometric multiplicity, is 2.

Similarly, the set containing the eigenvector

$$\left\{\begin{pmatrix}-0.4082\\-0.4082\\-0.8165\end{pmatrix}\right\}$$

forms a basis for the eigenspace $E(4)$ with dimension = 1, and therefore the geometric multiplicity also equals 1. So in all cases $\operatorname{alg\,mult}_A(\lambda_j) = \operatorname{geo\,mult}_A(\lambda_j)$ and we have three linearly independent eigenvectors. Hence, matrix A is *semisimple*. We can also see that $\operatorname{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3$ by entering the commands:

```
% Compare trace(A) with sum of eigenvalues
trace(A)
evalA(1,1) + evalA(2,2) + evalA(3,3)
```

giving zero as the result. Similarly, we can show that $\det(A) = \lambda_1\lambda_2\lambda_3$ by

```
% Compare det(A) with product of eigenvalues
det(A)
evalA(1,1) * evalA(2,2) * evalA(3,3)
```

giving the result 16.

An interesting question to pose at this point is whether all of the corresponding bases for the eigenspaces of a matrix $A \in \mathbb{R}^{n\times n}$ can be assembled together to form a basis for $\mathbb{R}^n$. If this is possible, then as we will see, the types of problems we wish to tackle in this course can be dealt with very concisely. However, before we answer this question, we need to provide some more background that eventually leads us to the *diagonalisation* of a matrix.

Two matrices $A, B \in \mathbb{R}^{n\times n}$ are said to be similar if there exists an invertible matrix S with $B = S^{-1}AS$. An important result is that similar matrices have the same eigenvalues.

A matrix $A \in \mathbb{R}^{n\times n}$ is said to be *diagonalisable* when there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. In this case the matrix P is said to *diagonalise* A.

The question that arises now is under what conditions is a matrix $A \in \mathbb{R}^{n \times n}$ diagonalisable? Three key observations follow:

1. A can be diagonalised if it is semisimple.
2. A can be diagonalised if it has n distinct eigenvalues.
3. A can be diagonalised if $\operatorname{geo\,mult}_A(\lambda) = \operatorname{alg\,mult}_A(\lambda)$ for each eigenvalue of A.

Determine which of the matrices from above are diagonalisable and if so, find the matrix P that diagonalises A and hence determine $P^{-1}AP$. We'll show you a couple of examples first, then leave the remaining matrices to you.

So let's consider matrix A from before. We saw that it is semisimple, so that means we can diagonalise it. But how do we find the matrix P? Well, we already have it but just don't know it yet! It's just the matrix with columns corresponding to the linearly independent eigenvectors. In MATLAB, this is just the matrix `evecA` from before.

```
% Diagonalise A
Adiag = inv(evecA) * A * evecA;
```

What does `Adiag` look like? Compare it to the matrix of eigenvalues, `evalA`, and what do you see? They should be the same (or very close to it due to the numerical approximations used). So, this tells us that we can represent A using the matrix of eigenvectors, P, and the diagonal matrix of corresponding eigenvalues. To see that this is true, enter the following commands into MATLAB:

```
% Reconstruct A
evecA * Adiag * inv(evecA)
```

You should end up with the original matrix A.

Let's now have a look at what happens when we can't diagonalise a matrix. We'll use matrix C from above. We've already computed the eigenvalues and eigenvectors, let's have a look at them:

```
% Show eigenvalues
evalC

% Show eigenvectors
evecC
```

Output

```
ans =

     0


ans =

   8.8818e-16


ans =

    16


ans =

   16.0000


ans =

   1.0000 - 0.0000i  -3.0000 + 0.0000i   3.0000 - 0.0000i
   3.0000 - 0.0000i  -5.0000 + 0.0000i   3.0000 - 0.0000i
   6.0000 + 0.0000i  -6.0000 + 0.0000i   4.0000 + 0.0000i


evalC =

     0     0     0
     0     1     0
     0     0     1


evecC =

    1.0000         0    0.0000
         0    1.0000   -1.0000
         0         0    0.0000
```

We can see that the eigenvalues are $\lambda_1 = 0$ and $\lambda_{2,3} = 1$ (repeated twice). If we look at the eigenvectors, we have one vector corresponding to λ_1, but MATLAB has returned us two eigenvectors corresponding to $\lambda_{2,3}$. Note that these two vectors are **not** linearly independent - they're a multiple (-1) of each other. If we obtained the eigenvector by hand, we would have only found one. This tells us that C is not semisimple, it is in fact defective and cannot be diagonalised!

Give the remaining matrices a go, and discuss your results with your peers. Remember, make sure that the matrix can be diagonalised before actually trying to do it!