

Assignment 4

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Download all latex-tikz codes from

<https://github.com/ImNamitaKumari/Probability-and-Random-Variables/blob/main/Assignment4/Assignment4.tex>

For all options, a_n and b_n are fixed numbers dependent only on n and not a random variable. So, $E(a_n) = a_n$ and $E(b_n) = b_n$.

$$\implies \lim_{n \rightarrow \infty} \frac{E(S_n) - a_n}{b_n} = 0 \quad (2.0.8)$$

$$(2.0.9)$$

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Suppose X_1, X_2, \dots are independent random variables. Assume that X_1, X_3, \dots are identically distributed with mean μ_1 and variance σ_1^2 , while X_2, X_4, \dots are identically distributed with mean μ_2 and variance σ_2^2 . Let $S_n = X_1 + X_2 + \dots + X_n$. Then $\frac{S_n - a_n}{b_n}$ converges in distribution to $N(0, 1)$ if

- 1) $a_n = \frac{n(\mu_1 + \mu_2)}{2}$ and $b_n = \sqrt{n} \sqrt{\frac{\sigma_1^2 + \sigma_2^2}{2}}$
- 2) $a_n = \frac{n(\mu_1 + \mu_2)}{2}$ and $b_n = \frac{n(\sigma_1 + \sigma_2)}{2}$
- 3) $a_n = n(\mu_1 + \mu_2)$ and $b_n = \sqrt{n} \sqrt{\frac{\sigma_1 + \sigma_2}{2}}$
- 4) $a_n = n(\mu_1 + \mu_2)$ and $b_n = \sqrt{n} \sqrt{\frac{\sigma_1^2 + \sigma_2^2}{2}}$

For all given options,

$$\lim_{n \rightarrow \infty} b_n = \infty \quad (2.0.10)$$

So,

$$\lim_{n \rightarrow \infty} E(S_n) - a_n = k, k \in \mathbb{R}, \text{ free of } n \quad (2.0.11)$$

$$\implies \frac{n(\mu_1 + \mu_2)}{2} - a_n = k_1 \quad (2.0.12)$$

$$\text{And, } \frac{n(\mu_1 + \mu_2)}{2} + \frac{\mu_1 - \mu_2}{2} - a_n = k_2 \quad (2.0.13)$$

$$\implies a_n = \frac{n(\mu_1 + \mu_2)}{2} + k_3 \quad (2.0.14)$$

Using equation (2.0.2)

$$\text{Var}(S_n) = \text{Var}(X_1) + \text{Var}(X_2) + \dots + \text{Var}(X_n) \quad (2.0.15)$$

$$\implies \text{Var}(S_n) = \begin{cases} \frac{n(\sigma_1^2 + \sigma_2^2)}{2} & n = \text{even} \\ \frac{n(\sigma_1^2 + \sigma_2^2)}{2} + \frac{\sigma_1^2 - \sigma_2^2}{2} & n = \text{odd} \end{cases} \quad (2.0.16)$$

For X and Y being independent random variables,

$$E(X + Y) = E(X) + E(Y) \quad (2.0.1)$$

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) \quad (2.0.2)$$

$$\text{Var}(aX + b) = a^2 \text{Var}(X) \quad (2.0.3)$$

$$S_n = X_1 + X_2 + \dots + X_n \quad (2.0.4)$$

Using equation (2.0.1),

$$\implies E(S_n) = E(X_1) + E(X_2) + \dots + E(X_n) \quad (2.0.5)$$

$$\implies E(S_n) = \begin{cases} \frac{n(\mu_1 + \mu_2)}{2} & n = \text{even} \\ \frac{n(\mu_1 + \mu_2)}{2} + \frac{\mu_1 - \mu_2}{2} & n = \text{odd} \end{cases} \quad (2.0.6)$$

Given,

$$\lim_{n \rightarrow \infty} E\left(\frac{S_n - a_n}{b_n}\right) = 0 \quad (2.0.7)$$

Given,

$$\lim_{n \rightarrow \infty} \text{Var}\left(\frac{S_n - a_n}{b_n}\right) = 1 \quad (2.0.17)$$

From equation (2.0.3),

$$\implies \lim_{n \rightarrow \infty} \frac{\text{Var}(S_n)}{b_n^2} = 1 \quad (2.0.18)$$

$$\implies \frac{n(\sigma_1^2 + \sigma_2^2)}{2b_n^2} = 1, n = \text{even} \quad (2.0.19)$$

$$\text{And, } \frac{n(\sigma_1^2 + \sigma_2^2)}{2b_n^2} + \frac{\sigma_1^2 - \sigma_2^2}{2b_n^2} = 1, n = \text{odd} \quad (2.0.20)$$

$$\Rightarrow b_n = \sqrt{n} \sqrt{\frac{\sigma_1^2 + \sigma_2^2}{2}}, n = \text{even} \quad (2.0.21)$$

$$\text{And, } b_n = \sqrt{\frac{n(\sigma_1^2 + \sigma_2^2) + (\sigma_1^2 - \sigma_2^2)}{2}}, n = \text{odd} \quad (2.0.22)$$

Hence from equation (2.0.21) and (2.0.14), the correct answer is option 1).

2.1 Proof of Standard Normal Distribution: Central Limit Theorem

The Classical Central Limit Theorem states that:

If X_1, X_2, \dots, X_n are a sequence of independent and identically distributed (i.i.d.) random variables drawn from a distribution with overall mean μ and finite variance σ^2 , and if \bar{X}_n is the sample average, i.e.,

$$\bar{X}_n = \frac{X_1 + X_2 + \dots + X_n}{n}, \quad (2.1.1)$$

then the limiting form of the distribution,

$$Z = \lim_{n \rightarrow \infty} \sqrt{n} \left(\frac{\bar{X}_n - \mu}{\sigma} \right) \quad (2.1.2)$$

is a standard normal distribution.

Let $Z_1 = X_1 + X_2, Z_2 = X_3 + X_4, \dots, Z_{n/2} = X_{n-1} + X_n$. As evident, $Z_1, Z_2, \dots, Z_{n/2}$ are independent and identically distributed random variables with common mean $(\mu_1 + \mu_2)$ and common variance $(\sigma_1^2 + \sigma_2^2)$. By CLT, this implies

$$\lim_{\frac{n}{2} \rightarrow \infty} \sqrt{\frac{n}{2}} \frac{\frac{Z_1 + Z_2 + \dots + Z_{n/2}}{n/2} - (\mu_1 + \mu_2)}{n/2} \quad (2.1.3)$$

is a standard normal distribution.

By replacing $n/2$ by n and by multiplying both numerator and denominator by n , the expression given in (2.1.3) comes out to be same as the expression $\frac{S_n - a_n}{b_n}$. Hence, the given expression converges in distribution to a standard normal distribution.