Assignment 4

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Download all latex-tikz codes from

https://github.com/ImNamitaKumari/Probability−and−Random−Variables/blob/main/Assignment4/Assignment4.tex

1 CSIR UGC NET - June 2015 Q. 105

Suppose X_1, X_2, ... are independent random variables. Assume that X_1, X_3, ... are identically distributed with mean μ_1 and variance σ_1^2, while X_2, X_4, ... are identically distributed with mean μ_2 and variance σ_2^2. Let $S_n = X_1 + X_2 + ... + X_n$. Then $\frac{S_n - a_n}{b_n}$ converges in distribution to $N(0, 1)$ if

1) $a_n = \frac{n(\mu_1+\mu_2)}{2}$ and $b_n = \sqrt{n}\sqrt{\frac{\sigma_1^2+\sigma_2^2}{2}}$
2) $a_n = \frac{n(\mu_1+\mu_2)}{2}$ and $b_n = \frac{n(\sigma_1+\sigma_2)}{2}$
3) $a_n = n(\mu_1 + \mu_2)$ and $b_n = \sqrt{n}\frac{(\sigma_1+\sigma_2)}{2}$
4) $a_n = n(\mu_1 + \mu_2)$ and $b_n = \sqrt{n}\sqrt{\frac{\sigma_1^2+\sigma_2^2}{2}}$

2 Solution

Lemma 2.1. *For X and Y being independent random variables,*

$$E(X + Y) = E(X) + E(Y) \tag{2.0.1}$$
$$Var(X + Y) = Var(X) + Var(Y) \tag{2.0.2}$$
$$Var(aX + b) = a^2 Var(X) \tag{2.0.3}$$

Corollary 2.1.

$$E(S_n) = \begin{cases} \frac{n(\mu_1+\mu_2)}{2} & n = \text{even} \\ \frac{n(\mu_1+\mu_2)}{2} + \frac{\mu_1-\mu_2}{2} & n = \text{odd} \end{cases} \tag{2.0.4}$$

Proof.

$$S_n = X_1 + X_2 + ... + X_n \tag{2.0.5}$$

Using lemma (2.1),

$$\implies E(S_n) = E(X_1) + E(X_2) + ... + E(X_n) \tag{2.0.6}$$
$$\implies E(S_n) = \begin{cases} \frac{n(\mu_1+\mu_2)}{2} & n = \text{even} \\ \frac{n(\mu_1+\mu_2)}{2} + \frac{\mu_1-\mu_2}{2} & n = \text{odd} \end{cases} \tag{2.0.7}$$

□

Corollary 2.2.

$$Var(S_n) = \begin{cases} \frac{n(\sigma_1^2+\sigma_2^2)}{2} & n = \text{even} \\ \frac{n(\sigma_1^2+\sigma_2^2)}{2} + \frac{\sigma_1^2-\sigma_2^2}{2} & n = \text{odd} \end{cases} \tag{2.0.8}$$

Proof.

$$S_n = X_1 + X_2 + ... + X_n \tag{2.0.9}$$

Using lemma (2.1)

$$Var(S_n) = Var(X_1) + Var(X_2) + ... + Var(X_n) \tag{2.0.10}$$

$$\implies Var(S_n) = \begin{cases} \frac{n(\sigma_1^2+\sigma_2^2)}{2} & n = \text{even} \\ \frac{n(\sigma_1^2+\sigma_2^2)}{2} + \frac{\sigma_1^2-\sigma_2^2}{2} & n = \text{odd} \end{cases} \tag{2.0.11}$$

□

Given,

$$\lim_{n\to\infty} E\left(\frac{S_n - a_n}{b_n}\right) = 0 \tag{2.0.12}$$

For all options, a_n and b_n are fixed numbers dependent only on n and not a random variable. So, $E(a_n) = a_n$ and $E(b_n) = b_n$.

$$\implies \lim_{n\to\infty} \frac{E(S_n) - a_n}{b_n} = 0 \tag{2.0.13}$$

$$\tag{2.0.14}$$

For all given options,

$$\lim_{n\to\infty} b_n = \infty \tag{2.0.15}$$

So,

$$\lim_{n\to\infty} E(S_n) - a_n = k, k \in \mathbb{R}, \text{free of } n \tag{2.0.16}$$

Using corollary (2.1),

$$\implies \frac{n(\mu_1+\mu_2)}{2} - a_n = k_1 \tag{2.0.17}$$

$$\text{And, } \frac{n(\mu_1+\mu_2)}{2} + \frac{\mu_1-\mu_2}{2} - a_n = k_2 \tag{2.0.18}$$

$$\boxed{\implies a_n = \frac{n(\mu_1+\mu_2)}{2} + k_3} \tag{2.0.19}$$

Given,

$$\lim_{n\to\infty} Var\left(\frac{S_n - a_n}{b_n}\right) = 1 \tag{2.0.20}$$

Using lemma (2.1),

$$\implies \lim_{n\to\infty} \frac{Var(S_n)}{b_n^2} = 1 \tag{2.0.21}$$

$$\implies \frac{n(\sigma_1^2+\sigma_2^2)}{2b_n^2} = 1, n = \text{even} \tag{2.0.22}$$

$$\text{And, } \frac{n(\sigma_1^2+\sigma_2^2)}{2b_n^2} + \frac{\sigma_1^2-\sigma_2^2}{2b_n^2} = 1, n = \text{odd} \tag{2.0.23}$$

$$\boxed{\implies b_n = \sqrt{n}\sqrt{\frac{\sigma_1^2+\sigma_2^2}{2}}, n = \text{even}} \tag{2.0.24}$$

$$\text{And, } b_n = \sqrt{\frac{n(\sigma_1^2+\sigma_2^2)+(\sigma_1^2-\sigma_2^2)}{2}}, n = \text{odd} \tag{2.0.25}$$

Hence from equation (2.0.24) and (2.0.19), the correct answer is option 1).

2.1 Proof of Standard Normal Distribution: Classical Central Limit Theorem

Theorem 2.3. *If $X_1, X_2, \ldots, X_n$ are a sequence of independent and identically distributed (i.i.d.) random variables drawn from a distribution with overall mean μ and finite variance σ^2, and if $\bar{X}_n$ is the sample average, i.e.,*

$$\bar{X}_n = \frac{X_1 + X_2 + ... + X_n}{n}, \tag{2.1.1}$$

then the limiting form of the distribution,

$$Z = \lim_{n\to\infty} \sqrt{n}\left(\frac{\bar{X}_n - \mu}{\sigma}\right) \tag{2.1.2}$$

is a standard normal distribution.

Corollary 2.4. $\frac{S_n - a_n}{b_n}$ *converges to a standard normal distribution.*

Proof. Let $Z_1 = X_1 + X_2$, $Z_2 = X_3 + X_4$, ... $Z_{n/2} = X_{n-1} + X_n$. As evident, $Z_1, Z_2, ..., Z_{n/2}$ are independent and identically distributed random variables with common mean $(\mu_1 + \mu_2)$ and common variance $(\sigma_1^2 + \sigma_2^2)$. By CLT, this implies

$$\lim_{\frac{n}{2}\to\infty} \sqrt{\frac{n}{2}} \frac{\frac{Z_1+Z_2+...+Z_{n/2}}{n/2} - (\mu_1+\mu_2)}{n/2} \tag{2.1.3}$$

is a standard normal distribution.

By replacing $n/2$ by n and by multiplying both numerator and denominator by n, the expression given in (2.1.3) comes out to be same as the expression $\frac{S_n - a_n}{b_n}$. Hence, the given expression converges in distribution to a standard normal distribution. □