Assignment 5

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Download all latex-tikz codes from

https://github.com/ImNamitaKumari/Probability-and-Random-Variables/blob/main/Assignment5/Assignment5.tex

1 CSIR UGC NET - June 2013 Q. 75

Let X be a non-negative integer valued random variable with probabilty mass function $f(x)$ satisfying $(x+1)f(x+1) = (\alpha+\beta x)f(x)$, x=0,1,2, ...; $\beta \neq 1$. You may assume that $E(X)$ and $Var(X)$ exist. Then which of the following statements are true?

1) $E(X) = \frac{\alpha}{1-\beta}$
2) $E(X) = \frac{\alpha^2}{(1-\beta)(1+\alpha)}$
3) $Var(X) = \frac{\alpha^2}{(1-\beta)^2}$
4) $Var(X) = \frac{\alpha}{(1-\beta)^2}$

2 Solution

Definition 1.

$$A(n) = \sum_{x=0}^{n} f(x) \tag{2.0.1}$$

$$B(n) = \sum_{x=0}^{n} xf(x) \tag{2.0.2}$$

$$C(n) = \sum_{x=0}^{n} x^2 f(x) \tag{2.0.3}$$

Definition 2. *Sum of probabilities is equal to 1.*

$$\lim_{n\to\infty} A(n) = 1 \tag{2.0.4}$$

$$\lim_{n\to\infty} B(n) = E(X) \tag{2.0.5}$$

$$\lim_{n\to\infty} C(n) - (E(X))^2 = Var(X) \tag{2.0.6}$$

Lemma 2.1.

$$\lim_{n\to\infty} nf(n) = 0 = \lim_{n\to\infty} n^2 f(n) \tag{2.0.7}$$

Proof. As given in question, $E(X)$ and $Var(X)$ are finite.

$$E(X) = \sum xf(x) \tag{2.0.8}$$

$$Var(X) = \sum x^2 f(x) - (E(X))^2 \tag{2.0.9}$$

Hence at infinity, the contribution of $nf(n)$ and $n^2 f(n)$ has to be zero for mean and variance to be finite. □

Lemma 2.2.

$$B(n) = \frac{\alpha A(n) - (n+1)f(n+1)}{1-\beta} \tag{2.0.10}$$

Proof.

$$(x+1)f(x+1) = (\alpha + \beta x)f(x) \tag{2.0.11}$$

$$\implies \alpha x = (x+1)f(x+1) - \beta x f(x) \tag{2.0.12}$$

$$\implies \alpha f(0) = f(1) - 0 \tag{2.0.13}$$

$$\alpha f(1) = 2f(2) - \beta f(1) \tag{2.0.14}$$

$$\alpha f(2) = 3f(3) - 2\beta f(2) \tag{2.0.15}$$

$$\vdots \tag{2.0.16}$$

$$\alpha f(n) = (n+1)f(n+1) - n\beta f(n) \tag{2.0.17}$$

Adding equations from (2.0.13) to (2.0.17),

$$\alpha \sum_{x=0}^{n} f(x) = (n+1)f(n+1) + (1-\beta)\sum_{x=0}^{n} xf(x) \tag{2.0.18}$$

Using definition (1),

$$\implies B(n) = \frac{\alpha A(n) - (n+1)f(n+1)}{1-\beta} \tag{2.0.19}$$

□

Lemma 2.3.

$$C(n) = \frac{(\alpha+\beta)B(n) + \alpha A(n) - (n+1)^2 f(n+1)}{1-\beta} \tag{2.0.20}$$

Proof. Multiplying equation (2.0.11) with $(x+1)$ on both sides and rearranging,

$$(x+1)^2 f(x+1) - \beta x^2 f(x) = (\alpha+\beta)xf(x) + \alpha f(x) \tag{2.0.21}$$

$$\implies 1^2 f(1) - 0 = 0 + \alpha f(0) \tag{2.0.22}$$
$$2^2 f(2) - 1^2 \beta f(1) = (\alpha + \beta) f(1) + \alpha f(1) \tag{2.0.23}$$
$$3^2 f(3) - 2^2 \beta f(2) = 2(\alpha + \beta) f(2) + \alpha f(2) \tag{2.0.24}$$
$$\vdots \tag{2.0.25}$$
$$(n+1)^2 f(n+1) - n^2 \beta f(n) = n(\alpha + \beta) f(n) + \alpha f(n) \tag{2.0.26}$$

Adding equations from (2.0.22) to (2.0.26),

$$(n+1)^2 f(n+1) + (1-\beta) \sum_{x=0}^{n} x^2 f(x) = \alpha \sum_{x=0}^{n} f(x) + (\alpha + \beta) \sum_{x=0}^{n} x f(x) \tag{2.0.27}$$

Using definition (1),

$$\implies C(n) = \frac{(\alpha + \beta) B(n) + \alpha A(n) - (n+1)^2 f(n+1)}{1 - \beta} \tag{2.0.28}$$

□

Using definition (2) and lemma (2.2),

$$\lim_{n\to\infty} B(n) = \lim_{n\to\infty} \frac{\alpha A(n) - (n+1) f(n+1)}{1-\beta} \tag{2.0.29}$$
$$= \lim_{n\to\infty} \frac{\alpha - (n+1) f(n+1)}{1-\beta} \tag{2.0.30}$$

Using lemma (2.1),

$$\boxed{\implies E(X) = \frac{\alpha}{1-\beta}} \tag{2.0.31}$$

Using definition (2) and lemma (2.3),

$$\lim_{n\to\infty} C(n) = \frac{(\alpha + \beta) E(X) + \alpha - (n+1)^2 f(n+1)}{1-\beta} \tag{2.0.32}$$

Using lemma (2.1),

$$\lim_{n\to\infty} C(n) = \frac{(\alpha + \beta) E(X) + \alpha}{1-\beta} \tag{2.0.33}$$
$$\implies Var(X) = \frac{(\alpha + \beta) E(X) + \alpha}{1-\beta} - (E(X))^2 \tag{2.0.34}$$

Using equation (2.0.31),

$$Var(X) = \frac{\frac{\alpha(\alpha+\beta)}{1-\beta} + \alpha}{1-\beta} - \frac{\alpha^2}{(1-\beta)^2} \tag{2.0.35}$$

$$\boxed{\implies Var(X) = \frac{\alpha}{(1-\beta)^2}} \tag{2.0.36}$$

Hence, the correct options are 1) and 4).