MATH1002 Linear Algebra

Topic 11A: Similar matrices

Def^n Let A and B be $n \times n$ matrices. We say that A is similar to B if there is an invertible matrix P so that

$$P^{-1}AP = B.$$

If A is similar to B, we write $A \sim B$.

Remarks

1. For P invertible,
$$\begin{aligned} P^{-1}AP = B &\iff PP^{-1}AP = PB \\ &\iff IAP = PB \\ &\iff AP = PB. \end{aligned}$$
2. For $n \times n$ matrices A, P and B, we could have $AP = PB$ even when P is not invertible.

Example

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Notice $\det(P) = -1 \neq 0$ so P is invertible.

Now

$$AP = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$PB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Since P is invertible, and $AP = PB$, we have $P^{-1}AP = B$ so A is similar to B.

Theorem Let A, B and C be $n \times n$ matrices. Then

1. $A \sim A$
2. if $A \sim B$ then $B \sim A$
3. if $A \sim B$ and $B \sim C$ then $A \sim C$.

Proof

1. $A = I^{-1} A I$
2. Suppose $P^{-1}AP = B$. Let $Q = P^{-1}$. Then Q is invertible and
$$Q^{-1} = (P^{-1})^{-1} = P.$$
So
$$\begin{aligned} A &= PBP^{-1} \quad \text{(check!)} \\ &= Q^{-1}BQ \end{aligned}$$
so $B \sim A$.
3. Suppose $B = P^{-1}AP$ and $C = Q^{-1}BQ$. Then
$$\begin{aligned} C &= Q^{-1}BQ \\ &= Q^{-1}P^{-1}APQ \\ &= (PQ)^{-1}A(PQ) \end{aligned}$$
so $A \sim C$.

$\square$

Theorem Let A and B be $n\times n$ matrices. If A is similar to B then:

(1) $\det(A) = \det(B)$

(2) $\det(A-\lambda I) = \det(B-\lambda I)$.

Proof

(1) Suppose $B = P^{-1}AP$. Then

$$\begin{aligned}\det(B) &= \det(P^{-1}AP) \\ &= \det(P^{-1})\det(A)\det(P) \\ &= \det(A)\det(P^{-1})\det(P) \quad \text{since multiplication in } \mathbb{R} \text{ is commutative} \\ &= \det(A)\frac{1}{\cancel{\det(P)}}\cancel{\det}(P) \\ &= \det(A).\end{aligned}$$

(2) exercise. $\square$

Corollary Let A and B be $n\times n$ matrices. If A is similar to B then:

(1) A is invertible $\Leftrightarrow$ B is invertible.

(2) A and B have the same eigenvalues.

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Topic 11B: Diagonal matrices

Def^n An $n\times n$ matrix D is diagonal if all its off-diagonal entries are 0s.

Examples

1. The matrices

$$A=\begin{bmatrix}2&0\\0&\frac{1}{2}\end{bmatrix},\quad B=\begin{bmatrix}2&0\\0&2\end{bmatrix},\quad C=\begin{bmatrix}1&0\\0&1\end{bmatrix},$$

$$D=\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

are all diagonal.

2. I_n (identity matrix) is diagonal.

3. The matrices

$$E_1=\begin{bmatrix}1&0&3\\0&1&0\\0&0&1\end{bmatrix} \quad \text{or} \quad E_2=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$$

are not diagonal.

Theorem If D is diagonal, then its eigenvalues are its diagonal entries $d_{11}, d_{22}, \dots, d_{nn}$.

Proof If D is diagonal then D is triangular (in fact, D is both upper-triangular and lower-triangular), and triangular matrices have eigenvalues given by their diagonal entries. $\square$

Examples

$A = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ has eigenvalues 2 and $\frac{1}{2}$.

$B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ has eigenvalues 2 and 2 (including multiplicities)

Products of diagonal matrices

Theorem If D and E are diagonal $n \times n$ matrices, then DE is the diagonal matrix with diagonal entries

$$d_{11}e_{11}, \; d_{22}e_{22}, \; \ldots, \; d_{nn}e_{nn}$$

ie. the product of the diagonal entries of D and E.

Example If $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $E = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

then

$$DE = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 2 & 0 & 0 \\ 0 & 4\times 0 & 0 \\ 0 & 0 & 5\times -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -5 \end{bmatrix}.$$

Corollary If D is diagonal then for any $k \geq 1$:

(1) D^k is diagonal, with diagonal entries

$$d_{11}^k, d_{22}^k, \ldots, d_{nn}^k$$

(2) D^k has eigenvalues

$$d_{11}^k, d_{22}^k, \ldots, d_{nn}^k.$$

Example

If $A = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ then

$$A^k = \begin{bmatrix} 2^k & 0 \\ 0 & \left(\frac{1}{2}\right)^k \end{bmatrix} = \begin{bmatrix} 2^k & 0 \\ 0 & \frac{1}{2^k} \end{bmatrix}$$

and the eigenvalues of A^k are 2^k and $\frac{1}{2^k}$.

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Topic 11C: Diagonalisation

Let A be an $n \times n$ matrix, with
- eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (including multiplicity)
- corresponding eigenvectors $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_n$

ie. $A\underset{\sim}{x}_i = \lambda_i \underset{\sim}{x}_i$ for $1 \leq i \leq n$

Let P be the matrix with columns $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_n$ ie. $P = \begin{bmatrix} | & | & & | \\ \underset{\sim}{x}_1 & \underset{\sim}{x}_2 & \cdots & \underset{\sim}{x}_n \\ | & | & & | \end{bmatrix}$

Let D be the diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$

ie. $D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$

Theorem For A, P and D as above,
$$AP = PD.$$

Proof $AP = A[\underset{\sim}{x}_1 \; \underset{\sim}{x}_2 \cdots \underset{\sim}{x}_n]$

$= [A\underset{\sim}{x}_1 \; A\underset{\sim}{x}_2 \cdots A\underset{\sim}{x}_n]$

$= [\lambda_1\underset{\sim}{x}_1 \; \lambda_2\underset{\sim}{x}_2 \cdots \lambda_n\underset{\sim}{x}_n]$.

Now $PD = [\underset{\sim}{x}_1 \; \underset{\sim}{x}_2 \cdots \underset{\sim}{x}_n]\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$

$$= \begin{bmatrix} \lambda_1 \underset{\sim}{x}_1 & \lambda_2 \underset{\sim}{x}_2 & \cdots & \lambda_n \underset{\sim}{x}_n \end{bmatrix}$$
$$= AP. \qquad \square$$

Note If P is invertible, then
$$AP = PD \iff D = P^{-1}AP.$$

Def^n An $n \times n$ matrix A is diagonalisable if there is a diagonal matrix D and an invertible matrix P so that
$$D = P^{-1}AP.$$
ie. A is diagonalisable if it is similar to a diagonal matrix.

Diagonalisation Theorem

Let A be an $n \times n$ matrix.
The following are equivalent:

(1) A is diagonalisable

(2) A has n linearly independent eigenvectors

(3) each eigenvalue of A has algebraic multiplicity equal to its geometric multiplicity.

Moreover: there is an invertible matrix P and a diagonal matrix D so that

$$P^{-1}AP = D$$

if and only if:

- the columns of P are n linearly independent eigenvectors of A
- the diagonal entries of D are the corresponding eigenvalues.

Proof Suppose (2) holds ie A has n linearly independent eigenvectors $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \dots, \underset{\sim}{x}_n$ with corresp. eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$.

Let

$$P = [\underset{\sim}{x}_1 \; \underset{\sim}{x}_2 \cdots \underset{\sim}{x}_n] \text{ and } D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

Then we showed above that $AP = PD$.

Fact P is invertible $\Longleftrightarrow$ its columns are lin. indep.

So as the $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \dots, \underset{\sim}{x}_n$ are lin. indep., P is invertible, so

$$P^{-1}AP = D$$

ie A is diagonalisable. ie $(2) \Rightarrow (1)$.

For $(1) \Rightarrow (2)$: exercise.

Equivalence of (1) and (2) with (3): not an exercise.

Rest of proof: exercise. $\square$ 14wk6

Examples

1\. Is
$$A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$
diagonalisable?

We found that A has eigenvalues

$\lambda_1 = 1$ with algebraic multiplicity 1

$\lambda_2 = 0$ " " " 2.

We found eigenspaces

$$E_1 = \left\{ t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

$$E_0 = \left\{ t\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

so $\lambda_1 = 1$ has geometric multiplicity 1

and $\lambda_2 = 0$ " " " 1.

Since A has an eigenvalue $\lambda_2 = 0$ which has algebraic multiplicity different to its geometric multiplicity, A is <u>not</u> diagonalisable.

Alternatively:

Every eigenvector of A is a scalar multiple of either $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

Thus A does not have 3 linearly independent eigenvectors.

So A is not diagonalisable.

2. If possible, find a matrix P so that $P^{-1}AP$ is diagonal, for
$$A=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}.$$

We found that A has eigenvalues $\lambda_1=3$, $\lambda_2=1$
and eigenspaces
$$E_3=\left\{t\begin{bmatrix}1\\1\end{bmatrix} : t\in\mathbb{R}\right\}$$
$$E_1=\left\{t\begin{bmatrix}-1\\1\end{bmatrix} : t\in\mathbb{R}\right\}.$$

So both eigenvalues have algebraic multiplicity 1 and geometric multiplicity 1, so A is diagonalisable.

Let
$$P=\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}, \quad D=\begin{bmatrix}\boxed{3} & 0\\ 0 & \boxed{1}\end{bmatrix}$$

eigenvector for $\lambda_1=3$ eigenvector for $\lambda_2=1$

Then $P^{-1}AP=D$. (Check: $AP=PD$)

Alternatively: A has 2 linearly independent eigenvectors $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}-1\\1\end{bmatrix}$.

So A is diagonalisable and

$$P^{-1}AP = D$$

for P and D as above.

Theorem If an $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalisable.

Proof Follows from:

Fact If A has n distinct eigenvalues, then the corresponding n eigenvectors are linearly independent.

and the Diagonalisation Theorem. $\square$

Note A matrix can be diagonalisable even if its eigenvalues are not all distinct.

Example $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is diagonalisable

(since it is diagonal; $I^{-1}AI = A$; $A \underset{\text{similar}}{\sim} A$)

and A has eigenvalues 2 and 2. //

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Topic 11D: Powers of diagonalisable matrices

Recall: If $D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$ is diagonal,

then for every $k \geq 1$

$$D^k = \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n^k \end{bmatrix}$$

An $n \times n$ matrix A is diagonalisable if

$$P^{-1}AP = D$$

for an invertible matrix P and a diagonal matrix D.

Suppose A is diagonalisable, with $P^{-1}AP = D$. Then

$$
\begin{aligned}
& P^{-1}AP = D \\
\iff\ & PP^{-1}AP = PD \\
\iff\ & IAP = PD \\
\iff\ & AP = PD \\
\iff\ & APP^{-1} = PDP^{-1} \\
\iff\ & A = PDP^{-1}.
\end{aligned}
$$

Now

$$A^2 = PDP^{-1}PDP^{-1}$$
$$= PDIDP^{-1}$$
$$= PDDP^{-1}$$
$$= PD^2P^{-1}$$
$$A^3 = A^2A$$
$$= PD^2\cancel{P^{-1}}\cancel{P}DP^{-1}$$
$$= PD^3P^{-1}$$

Theorem If A is diagonalisable, with $P^{-1}AP = D$, then for all $k \geq 1$,

$$A^k = PD^kP^{-1}.$$

Proof exercise.

Examples

1. Let $A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$. Find a formula for A^k and compute A^{1272}.

We need to diagonalise A. (This can be done if you're being asked this kind of question.)

Find eigenvalues and eigenspaces.

Check:

A has eigenvalues 2 and 3

A has eigenspaces

$$E_2 = \left\{ t \begin{bmatrix} 1 \\ 2 \end{bmatrix} : t \in \mathbb{R} \right\}$$

$$E_3 = \left\{ t \begin{bmatrix} 1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}.$$

Then

$$P = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

satisfy

$$A = PDP^{-1}.$$

Find P^{-1}:

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \mapsto R_2 - 2R_1} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \mapsto -R_2} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right] \xrightarrow{R_1 \mapsto R_1 - R_2} \left[\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

So $$P^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$ (check!)

Thus

$$A^k = PD^kP^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2^k & 0 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

$2 \times 2^k = 2^{k+1}$

$$= \begin{bmatrix} 2^k & 3^k \\ 2 \times 2^k & 3^k \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2^k + 2.3^k & 2^k - 3^k \\ -2^{k+1} + 2.3^k & 2^{k+1} - 3^k \end{bmatrix}.$$

$$A^{1272} = \begin{bmatrix} -2^{1272} + 2.3^{1272} & 2^{1272} - 3^{1272} \\ -2^{1273} + 2.3^{1272} & 2^{1273} - 3^{1272} \end{bmatrix}.$$

2. $A = \begin{bmatrix} -1 & 4 & -4 \\ 0 & 7 & -8 \\ 0 & 4 & -5 \end{bmatrix}$

Find a formula for A^k.

Check:

Eigenvalues: $\begin{cases} \lambda_1 = -1 & \text{alg. mult. } 2 \\ \lambda_2 = 3 & \text{" \quad " \quad } 1. \end{cases}$

Eigenspaces:

Check

$$E_{-1} = \left\{ s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} : s,t \in \mathbb{R} \right\}$$

$$E_3 = \left\{ t\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

So $\lambda_1 = -1$ has geom. mult. $2 =$ its alg. mult.

$\lambda_2 = 3$ " " " $1 =$ " " " .

So $A = PDP^{-1}$ where

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Find P^{-1}:

$$P^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & -1 \end{bmatrix} \qquad \text{(check!)}$$

Now

$$\begin{aligned}
A^k &= PD^kP^{-1} \\
&= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} (-1)^k & 0 & 0 \\ 0 & (-1)^k & 0 \\ 0 & 0 & 3^k \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} (-1)^k & 0 & 3^k \\ 0 & (-1)^k & 2.3^k \\ 0 & (-1)^k & 3^k \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} (-1)^k & (-1)^{k+1} + 3^k & (-1)^k - 3^k \\ 0 & (-1)^{k+1} + 2.3^k & 2(-1)^k - 2.3^k \\ 0 & (-1)^{k+1} + 3^k & 2(-1)^k - 3^k \end{bmatrix}
\end{aligned}$$

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Topic 11E: Leslie model of population growth

Example We model the female population of koalas in NSW. We use 4 age groups, each of the same duration:

0 – 4 years : baby koalas
5 – 9 " : young adults
10 – 14 " : adults
15 – 19 " : seniors

Population vector at time period n lots of 5 years:

$$\underset{\sim}{x}_n = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \begin{array}{l} \to \text{no. of baby koalas after } 5n \text{ years} \\ \to \text{" " young adult " " " "} \\ \to \text{" " adult " " " "} \\ \to \text{" " senior " " " "} \end{array}$$

For example, if initially (ie. when $n=0$) we have 10 babies, 20 young adults, 50 adults and 15 seniors, then

$$\underset{\sim}{x}_0 = \begin{bmatrix} 10 \\ 20 \\ 50 \\ 15 \end{bmatrix}$$

$\underset{\sim}{x}_1$ describes population after 5 yrs
$\underset{\sim}{x}_2$ " " " 10 yrs
⋮

Birthrates:

Every 5 years, each

- baby has, on average, b_1 babies
- young adult has, on average, b_2 babies
- adult " " " b_3 "
- senior " " " b_4 "

Note $b_i \geqslant 0$.

Survival rates:

In each 5 year period:

- baby koalas survive with probability s_1
- young adults " " " s_2
- adults " " " s_3
- seniors " " " s_4

Note $0 \leq s_i \leq 1$.

Suppose the koala population after n time periods (i.e. after 5n years) is

$$\underset{\sim}{x}_n = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

Then

$$\underset{\sim}{x}_{n+1} = \begin{bmatrix} \overbrace{b_1 x_1}^{\text{babies of babies}} + \overbrace{b_2 x_2}^{\text{babies of young adults}} + b_3 x_3 + b_4 x_4 \\ \underbrace{s_1 x_1}_{\text{babies who survive}} \\ \underbrace{s_2 x_2}_{\text{young adults who survive}} \\ \underbrace{s_3 x_3}_{\text{adults who survive}} \end{bmatrix}$$

$$= \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ s_1 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 \\ 0 & 0 & s_3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

So

$$\underset{\sim}{x}_{n+1} = \underbrace{\begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ s_1 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 \\ 0 & 0 & s_3 & 0 \end{bmatrix}}_{\text{Leslie matrix}} \underset{\sim}{x}_n$$

General set-up

We model the female population of some species, divided into k age groups of equal duration.

Define

$$\underset{\sim}{x}_n = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}$$

where x_i is the population in the i^{th} age group in time period n.

Let b_i be the birthrates of group i:
average number of new members of group 1 that each member of group i produces. $(b_i \geq 0)$

Let s_i be the survival probability of group i:
proportion of group i that survives to join group $(i+1)$ in the next time period.
$(0 \leq s_i \leq 1)$

Then in the next time period:

group 1 will have population $b_1 x_1 + b_2 x_2 + \cdots + b_k x_k$

group 2 " " " $s_1 x_1$

group 3 " " " $s_2 x_2$

⋮

group k " " " $s_{k-1} x_{k-1}$.

So if $\underset{\sim}{x}_n = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ then

$$\underset{\sim}{x}_{n+1} = \begin{bmatrix} b_1 x_1 + b_2 x_2 + \cdots + b_k x_k \\ s_1 x_1 \\ s_2 x_2 \\ \vdots \\ s_{k-1} x_{k-1} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} b_1 & b_2 & \cdots & & b_k \\ s_1 & 0 & \cdots & & 0 \\ 0 & s_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & s_{k-1} & 0 \end{bmatrix}}_{} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}$$

This is called the <u>Leslie matrix</u>. It is $k \times k$.

If we put L for the Leslie matrix then

$$\underset{\sim}{x}_{n+1} = L \underset{\sim}{x}_n$$

So

$$\underset{\sim}{x}_1 = L \underset{\sim}{x}_0$$

$$\underset{\sim}{x}_2 = L \underset{\sim}{x}_1 = L(L\underset{\sim}{x}_0) = L^2 \underset{\sim}{x}_0$$

$$\underset{\sim}{x}_3 = L \underset{\sim}{x}_2 = L(L^2\underset{\sim}{x}_0) = L^3 \underset{\sim}{x}_0$$

Thus

$$\underset{\sim}{x}_n = L^n \underset{\sim}{x}_0.$$

To compute L^n, we diagonalise and write

$$L = PDP^{-1}.$$

Then $L^n = PD^nP^{-1}.$

Example

A population of female mice has 2 age groups:

junior : ≤ 1 year old.

senior : $1 \leq y \leq 2$ y year olds

Initially, there are 1000 junior mice and 0 senior mice.

So

$$\underset{\sim}{x}_0 = \begin{bmatrix} 1000 \\ 0 \end{bmatrix}.$$

Birthrates:

a junior mouse has, on average, 1.5 babies per year $b_1 = 1.5$

a senior " " " "

2 babies a year. $b_2 = 2$.

Survival probabilities:

Half of the junior mice survive each year. $s_1 = 0.5$.

So

$$L = \begin{bmatrix} 1.5 & 2 \\ 0.5 & 0 \end{bmatrix}$$

is the Leslie matrix.

Find the population of junior and senior mice after n years.

$$\underset{\sim}{x}_n = L^n \underset{\sim}{x}_0 .$$

We need to diagonalise L.
It turns out (check!)

$$L = PDP^{-1}$$

where

$$P = \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}, \quad P^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix}$$

So

$$L^n = PD^nP^{-1}$$
$$= \frac{1}{5}\begin{bmatrix} 4.2^n + \left(-\frac{1}{2}\right)^n & 4.2^n - 4\left(-\frac{1}{2}\right)^n \\ 2^n - \left(-\frac{1}{2}\right)^n & 2^n + 4\left(-\frac{1}{2}\right)^n \end{bmatrix}$$

Thus

$$\underset{\sim}{x}_n = L^n \underset{\sim}{x}_0$$
$$= L^n \begin{bmatrix} 1000 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 200\left(4.2^n + \left(-\frac{1}{2}\right)^n\right) \\ 200\left(2^n - \left(-\frac{1}{2}\right)^n\right) \end{bmatrix}.$$

So after n years, there are approximately
$200\left(4.2^n + \left(-\frac{1}{2}\right)^n\right)$ juniors
$200\left(2^n - \left(-\frac{1}{2}\right)^n\right)$ seniors.

What happens to the population in the long term? i.e. as $n \to \infty$?

Now $\left(-\frac{1}{2}\right)^n \to 0$ as $n \to \infty$.

So as $n \to \infty$, we get approximately
$800 \cdot 2^n$ juniors
$200 \cdot 2^n$ seniors

There are 4 times as many juniors as seniors, in the long term.