After this tutorial you should be able to:

1. Put formulas into prenex normal form.
2. Read and devise proofs in ND for predicate logic.

Problem 1. Prove $\vdash (A \vee \neg A)$.

Hint: Start by assuming the negation with the aim of ending with negation elimination.

Solution 1.

Line	Assumptions	Formula	Justification	References
1	1	$\neg(A \vee \neg A)$	Asmp. I	
2	2	A	Asmp. I	
3	2	$(A \vee \neg A)$	$\vee$ I	2
4	1, 2	$\perp$	$\perp$ I	1, 3
5	1	$\neg A$	$\neg$ I	2, 4
6	1	$(A \vee \neg A)$	$\vee$ I	5
7	1	$\perp$	$\perp$ I	1, 6
8		$(A \vee \neg A)$	$\neg$ E	1, 7

Problem 2. The following are not equivalence laws of predicate logic. Show this by providing counterexamples.

1. $\forall x P(x) \vee \forall x Q(x) \not\equiv \forall x(P(x) \vee Q(x))$
2. $\forall x \exists y R(x, y) \not\equiv \exists y \forall x R(x, y)$

Problem 3. For each of the formulas below, indicate the free and bound occurrences of the variables, and put the formulas into prenex normal form, justifying each step.

1. $(\forall x Q(x) \to \exists z \forall y R(z, y))$
2. $\forall x\big(R(x, y) \wedge \neg \forall y R(x, y)\big)$

Solution 3.

1. All variables are bound. No renaming necessary

$(\forall x Q(x) \rightarrow \exists z \forall y R(z,y))$	
$(\neg \forall x Q(x) \vee \exists z \forall y R(z,y))$	Def Imp
$(\exists x \neg Q(x) \vee \exists z \forall y R(z,y))$	Q Neg
$\exists x(\neg Q(x) \vee \exists z \forall y R(z,y))$	Q Extr
$\exists x \exists z(\neg Q(x) \vee \forall y R(z,y))$	Q Extr
$\exists x \exists z \forall y(\neg Q(x) \vee R(z,y))$	Q Extr

   Note: $\exists z \exists x \forall y, \exists z \forall y \exists x$ are also correct orders, but we could not extract y before z.

2. All occurrences of x are bound, the first y is free, but the second is bound.

$\forall x\big(R(x,y) \wedge \neg \forall y R(x,y)\big)$	
$\forall x\big(R(x,y) \wedge \neg \forall z R(x,z)\big)$	Relabelling
$\forall x\big(R(x,y) \wedge \exists z \neg R(x,z)\big)$	Q Neg
$\forall x \exists z\big(R(x,y) \wedge \neg R(x,z)\big)$	Q Extr

   Note that we had to relabel the *bound* y. We *cannot* relabel the free y.

Problem 4. Consider the following argument: Every dog likes people or hates cats. Rover is a dog. Rover likes cats. Therefore, some dog likes people.

1. Express the English sentences in predicate logic over a suitably chosen vocabulary.
2. The argument itself can be expressed as the consequence $E_1, E_2, E_3 \vdash F$. Prove this consequence in Natural Deduction.

Solution 4.

1. Take the vocabulary with unary predicates D (for the set of dogs), C (for the set of dogs that like cats), P (for the set of dogs that like people), and constant symbol *rover*.
2. Because the Logic Tutor requires constants to be lowercase symbols in the first half of the alphabet, we use a to denote *rover* in the following proof.

Line	Assumptions	Formula	Justification	References
1	1	$\forall x(D(x) \to (P(x) \vee \neg C(x)))$	Asmp. I	
2	2	$D(a)$	Asmp. I	
3	3	$C(a)$	Asmp. I	
4	1	$(D(a) \to (P(a) \vee \neg C(a)))$	($\forall$ E)	1
5	1, 2	$(P(a) \vee \neg C(a))$	$\to$ E	2, 4
6	6	$P(a)$	Asmp. I	
7	7	$\neg C(a)$	Asmp. I	
8	3, 7	$\perp$	$\perp$ I	3, 7
9	3, 7	$P(a)$	$\perp$ E	8
10	1, 2, 3	$P(a)$	$\vee$ E	5, 6, 6, 7, 9
11	1, 2, 3	$(D(a) \wedge P(a))$	$\wedge$ I	2, 10
12	1, 2, 3	$\exists x(D(x) \wedge P(x))$	($\exists$ I)	11

Problem 5. Pick any two of the following and prove them in Natural Deduction:

1. $\exists x P(x) \vdash \exists y P(y)$
2. $\forall y \forall x P(x, y) \ \vdash \ \exists y \exists x P(x, y)$
3. $\forall x P(x), \forall x(P(x) \to Q(x)) \vdash \forall x Q(x)$
4. $\exists x(P(x) \wedge Q(x)) \vdash \exists x P(x) \wedge \exists x Q(x)$
5. $\exists x(P(x) \wedge Q(y)) \vdash \exists x P(x) \wedge Q(y)$

Solution 5.

1.

Line	Assumptions	Formula	Justification	References
1	1	$\exists x P(x)$	Asmp. I	
2	2	$P(c)$	Asmp. I	
3	2	$\exists y P(y)$	$\exists$ I	2
4	1	$\exists y P(y)$	$\exists$ E	1, 2, 3

2.

Line	Assumptions	Formula	Justification	References
1	1	$\forall y \forall x P(x, y)$	Asmp. I	
2	1	$\forall x P(x, b)$	$\forall$ E	1
3	1	$P(a, b)$	$\forall$ E	2
4	1	$\exists x P(x, b)$	$\exists$ I	3
5	1	$\exists y \exists x P(x, y)$	$\exists$ I	4

3.

Line	Assumptions	Formula	Justification	References
1	1	$\forall x P(x)$	Asmp. I	
2	2	$\forall x(P(x) \to Q(x))$	Asmp. I	
3	1	$P(a)$	$\forall$ E	1
4	2	$(P(a) \to Q(a))$	$\forall$ E	2
5	1, 2	$Q(a)$	$\to$ E	3, 4
6	1, 2	$\forall x Q(x)$	$\forall$ I	5

4.

Line	Assumptions	Formula	Justification	References
1	1	$\exists x(P(x) \wedge Q(x))$	Asmp. I	
2	2	$(P(a) \wedge Q(a))$	Asmp. I	
3	2	$P(a)$	$\wedge$ E	2
4	2	$\exists x P(x)$	$\exists$ I	3
5	2	$Q(a)$	$\wedge$ E	2
6	2	$\exists x Q(x)$	$\exists$ I	5
7	2	$(\exists x P(x) \wedge \exists x Q(x))$	$\wedge$ I	4, 6
8	1	$(\exists x P(x) \wedge \exists x Q(x))$	$\exists$ E	1, 2, 7

5.

Line	Assumptions	Formula	Justification	References
1	1	$\exists x(P(x) \wedge Q(y))$	Asmp. I	
2	2	$(P(a) \wedge Q(y))$	Asmp. I	
3	2	$P(a)$	$\wedge$ E	2
4	2	$\exists x P(x)$	$\exists$ I	3
5	2	$Q(y)$	$\wedge$ E	2
6	2	$(\exists x P(x) \wedge Q(y))$	$\wedge$ I	4, 5
7	1	$(\exists x P(x) \wedge Q(y))$	$\exists$ E	1, 2, 6