THE UNIVERSITY OF SYDNEY

MATH1002 LINEAR ALGEBRA

Semester 1 **Tutorial Exercises for Week 13 — Solutions** 2022

1. Let

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{3} & 0 \end{bmatrix},$$

be the transition matrix for a Markov chain with three states. Let

$$\mathbf{x}_0 = \begin{bmatrix} 120 \\ 180 \\ 90 \end{bmatrix}$$

be the initial state vector for the population.

(i) Compute $\mathbf{x}_1$ and $\mathbf{x}_2$,

(ii) What proportion of the state 1 population will be in state 1 after two steps?

(iii) What proportion of the state 2 population will be in state 3 after two steps?

(iv) Find the steady-state probability vector, the steady-state vector, and discuss the asymptotic behaviour of the population.

Solution:

(i)

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} 60+60+30 \\ 60+60 \\ 60+60 \end{bmatrix} = \begin{bmatrix} 150 \\ 120 \\ 120 \end{bmatrix},$$

$$\mathbf{x}_2 = P^2\mathbf{x}_0 = P\mathbf{x}_1 = \begin{bmatrix} 75+40+40 \\ 40+80 \\ 75+40 \end{bmatrix} = \begin{bmatrix} 155 \\ 120 \\ 115 \end{bmatrix}.$$

(ii) We are required to find the $(1,1)$-entry of P^2; that is, the dot product of the first row of P and the first column of P:

$$\left(\frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{3} \times 0\right) + \left(\frac{1}{3} \times \frac{1}{2}\right) = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}.$$

So proportion of the state 1 population that will be in state 1 after two steps is $\frac{5}{12}$.

(iii) We are required to find the $(3,2)$-entry of P^2. So we must find the dot product of the third row of P and the second column of P. The answer is $\frac{1}{6} + \frac{1}{9} = \frac{5}{18}$.

(iv) We are asked to find the steady-state probability vector $\mathbf{x}$, which is the unique vector $\mathbf{x}$ satisfying the equation $P\mathbf{x} = \mathbf{x}$, with nonnegative entries summing to 1. We solve the corresponding system of linear equations $(P - I)\mathbf{x} = \mathbf{0}$:

$$P - I = \begin{bmatrix} -\frac{1}{2} & \frac{1}{3} & \frac{1}{3} \\ 0 & -\frac{2}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{3} & -1 \end{bmatrix} \xrightarrow{R_3 \mapsto R_3 + R_1} \begin{bmatrix} -\frac{1}{2} & \frac{1}{3} & \frac{1}{3} \\ 0 & -\frac{2}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

$$\xrightarrow{R_3 \mapsto R_3 + R_2} \begin{bmatrix} -\frac{1}{2} & \frac{1}{3} & \frac{1}{3} \\ 0 & -\frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \mapsto -\frac{3}{2} R_2} \begin{bmatrix} -\frac{1}{2} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, the solution is $z = t$, $y = z = t$, and $-\frac{x}{2} = -\frac{y+z}{3} = -\frac{2t}{3}$, and hence $x = \frac{4t}{3}$. To find the probability vector, we solve the equation

$$1 = x + y + z = \frac{4t}{3} + t + t$$

for t, and obtain $t = \frac{3}{10}$. Therefore, the steady-state probability vector is

$$\mathbf{x} = \begin{bmatrix} \frac{4}{10} \\ \frac{3}{10} \\ \frac{3}{10} \end{bmatrix}.$$

To get the steady-state vector, we just multiply the steady-state probability vector by the total population (which is $120+180+90 = 390$) to get $[156, 117, 117]$. In the long term, the populations will tend towards the steady-state vector $[156, 117, 117]$ (so the state 1 population will tend towards 156, and the state 2 and state 3 populations will both tend towards 117).

2. (i) Show that the square of a 2×2 stochastic matrix is also a stochastic matrix.

(ii) Show that if a 2×2 stochastic matrix P is invertible, then the entries in each column of P^{-1} sum to 1, but that P^{-1} is in general *not* stochastic.

(iii) * Prove that the square of an $n \times n$ stochastic matrix is also a stochastic matrix.

Solution:

(i) We are given a matrix $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, such that $a, b, c, d, \geq 0$, and $a + c = b + d = 1$. So

$$P^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ca + dc & cb + d^2 \end{bmatrix}.$$

Since $a, b, c, d \geq 0$, all the entries of P^2 are nonnegative. To see that the sum of the entries in every column is 1, we perform the following calculations:

$$(a^2 + bc) + (ca + dc) = a(a + c) + c(b + d) = a \times 1 + c \times 1 = a + c = 1,$$

and

$$(ab + bd) + (cb + d^2) = b(a + c) + d(b + d) = b \times 1 + d \times 1 = b + d = 1.$$

(ii) Since P is stochastic, we have

$$P = \begin{bmatrix} p & q \\ 1-p & 1-q \end{bmatrix}, \quad \text{for some } 0 \leq p, q \leq 1.$$

If P is invertible, then

$$P^{-1} = \frac{1}{p(1-q) - q(1-p)} \begin{bmatrix} 1-q & -q \\ p-1 & p \end{bmatrix} = \frac{1}{p-q} \begin{bmatrix} 1-q & -q \\ p-1 & p \end{bmatrix}$$

The following calculations show that the entries in the columns of P^{-1} add to 1:

$$\frac{1}{p-q}\left((1-q) + (p-1)\right) = \frac{p-q}{p-q} = 1 \quad \text{and} \quad \frac{1}{p-q}(-q+p) = \frac{p-q}{p-q} = 1.$$

To see that P^{-1} is in general *not* stochastic, we provide a counterexample: for

$$P = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix}, \quad \text{we have} \quad P^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix},$$

and P^{-1} is clearly not a stochastic matrix, because it contains some negative entries.

(iii) If $P = (p_{i,j})$ is an $n \times n$ stochastic matrix, then every entry of P is nonnegative, and for every $1 \le j \le n$, we have $\sum_{i=1}^{n} p_{ij} = 1$. Then the (i,j) entry of P^2 is equal to

$$\sum_{k=1}^{n} p_{ik}\, p_{kj},$$

and this value must be nonnegative since the entries of P are all nonnegative. Moreover, the sum of all of the entries in column j of P^2 is equal to

$$\sum_{i=1}^{n}\sum_{k=1}^{n} p_{ik}\, p_{kj},$$

for all $1 \le j \le n$. Now, by rearranging the summation and using the fact that the entries in each column of a stochastic matrix sum to 1, we see that

$$\sum_{i=1}^{n}\sum_{k=1}^{n} p_{ik}\, p_{kj} = \sum_{k=1}^{n} p_{kj}\left(\sum_{i=1}^{n} p_{ik}\right) = \left(\sum_{k=1}^{n} p_{kj}\right) \times 1 = \sum_{k=1}^{n} p_{kj} = 1.$$

3. Find the long-range transition matrix L of the given regular transition matrices P of a Markov chain.

(i) $P = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{2}{3} & \frac{5}{6} \end{bmatrix}$ (ii) $P = \begin{bmatrix} 0.2 & 0.3 & 0.4 \\ 0.6 & 0.1 & 0.4 \\ 0.2 & 0.6 & 0.2 \end{bmatrix}$

Solution:

(i) We first find the steady-state probability vector $\mathbf{x}$:

$$[P - I_2 \,|\, \mathbf{0}] = \left[\begin{array}{cc|c} -\frac{2}{3} & \frac{1}{6} & 0 \\ \frac{2}{3} & -\frac{1}{6} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Therefore, the steady-state probability vector is $\mathbf{x} = \begin{bmatrix} t \\ 4t \end{bmatrix}$, where $t + 4t = 1$. Thus $t = \frac{1}{5}$, and the steady-state probability vector is $\mathbf{x} = \begin{bmatrix} \frac{1}{5} \\ \frac{4}{5} \end{bmatrix}$. Hence the long-range transition matrix is $L = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{4}{5} & \frac{4}{5} \end{bmatrix}$.

(ii) We first find the steady-state probability vector $\mathbf{x}$:

$$[P - I_3 \,|\, \mathbf{0}] = \left[\begin{array}{ccc|c} -0.8 & 0.3 & 0.4 & 0 \\ 0.6 & -0.9 & 0.4 & 0 \\ 0.2 & 0.6 & -0.8 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} -0.8 & 0.3 & 0.4 & 0 \\ 0.6 & -0.9 & 0.4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -\frac{3}{8} & -\frac{1}{2} & 0 \\ \frac{3}{5} & -\frac{9}{10} & \frac{2}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -\frac{3}{8} & -\frac{1}{2} & 0 \\ 0 & -\frac{9}{10} + \frac{9}{40} & \frac{2}{5} + \frac{3}{10} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -\frac{3}{8} & -\frac{1}{2} & 0 \\ 0 & -\frac{27}{40} & \frac{7}{10} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -\frac{3}{8} & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{28}{27} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} - \frac{3}{8} \times \frac{28}{27} & 0 \\ 0 & 1 & -\frac{28}{27} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{8}{9} & 0 \\ 0 & 1 & -\frac{28}{27} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Therefore, the steady-state probability vector is

$$\mathbf{x} = t \begin{bmatrix} \frac{8}{9} \\ \frac{28}{27} \\ 1 \end{bmatrix},$$

where $\frac{8}{9}t + \frac{28}{27}t + t = 1$. Thus $t = \frac{27}{79}$, and the steady-state probability vector is

$$\mathbf{x} = \frac{27}{79}\begin{bmatrix} \frac{8}{9} \\ \frac{28}{27} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{24}{79} \\ \frac{28}{79} \\ \frac{27}{79} \end{bmatrix}.$$

Hence the long-range transition matrix is

$$L = \begin{bmatrix} \frac{24}{79} & \frac{24}{79} & \frac{24}{79} \\ \frac{28}{79} & \frac{28}{79} & \frac{28}{79} \\ \frac{27}{79} & \frac{27}{79} & \frac{27}{79} \end{bmatrix}.$$

4. * It is a theorem about 2×2 regular stochastic matrices P that

$$\lim_{n\to\infty} P^n = \begin{bmatrix} \mathbf{v} & \mathbf{v} \end{bmatrix},$$

where $\mathbf{v}$ is the unique steady-state probability vector of P (that is, $\mathbf{v}$ is the unique eigenvector of P corresponding to the eigenvalue 1, whose entries add to 1). Consider the regular stochastic matrix

$$P = \begin{bmatrix} \frac{1}{2} & \frac{2}{5} \\ \frac{1}{2} & \frac{3}{5} \end{bmatrix}.$$

Diagonalise P and verify this limiting behaviour in this particular example.

Solution: $\begin{vmatrix} \frac{1}{2} - \lambda & \frac{2}{5} \\ \frac{1}{2} & \frac{3}{5} - \lambda \end{vmatrix} = (\lambda - \frac{1}{2})(\lambda - \frac{3}{5}) - \frac{1}{5} = \lambda^2 - \frac{11}{10}\lambda + \frac{1}{10} = (\lambda - 1)(\lambda - \frac{1}{10})$, which has roots 1 and $\frac{1}{10}$. But

$$[P - I_2 \,|\, \mathbf{0}] = \left[\begin{array}{cc|c} -\frac{1}{2} & \frac{2}{5} & 0 \\ \frac{1}{2} & -\frac{2}{5} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -\frac{4}{5} & 0 \\ 0 & 0 & 0 \end{array}\right],$$

so the eigenspace corresponding to the eigenvalue 1 is $\left\{ \begin{bmatrix} 4t \\ 5t \end{bmatrix} : t \in \mathbb{R} \right\}$. Thus one eigenvector corresponding to the eigenvalue 1 is $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$. And the steady-state probability vector of P is the unique element of this eigenspace whose entries add to 1, so $\mathbf{v} = \begin{bmatrix} \frac{4}{9} \\ \frac{5}{9} \end{bmatrix}$. Also,

$$\left[P - \tfrac{1}{10}I_2 \,|\, \mathbf{0}\right] = \left[\begin{array}{cc|c} \frac{2}{5} & \frac{2}{5} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

so one eigenvector corresponding to the eigenvalue $\frac{1}{10}$ is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Therefore, we have

$$\begin{aligned} P^n &= \begin{bmatrix} 4 & -1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{10} \end{bmatrix}^n \begin{bmatrix} 4 & -1 \\ 5 & 1 \end{bmatrix}^{-1} \\ &= \frac{1}{9}\begin{bmatrix} 4 & -(\frac{1}{10})^n \\ 5 & (\frac{1}{10})^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -5 & 4 \end{bmatrix} \\ &= \frac{1}{9}\begin{bmatrix} 4 + 5(\frac{1}{10})^n & 4 - 4(\frac{1}{10})^n \\ 5 - 5(\frac{1}{10})^n & 5 + 4(\frac{1}{10})^n \end{bmatrix}. \end{aligned}$$

But $(\frac{1}{10})^n \to 0$ as $n \to \infty$, so

$$\lim_{n\to\infty} P^n = \frac{1}{9}\begin{bmatrix} 4 & 4 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} \frac{4}{9} & \frac{4}{9} \\ \frac{5}{9} & \frac{5}{9} \end{bmatrix} = \begin{bmatrix} \mathbf{v} & \mathbf{v} \end{bmatrix},$$

as the general theory predicted.