MATH1002 Linear Algebra

Topic 5A: Matrices and echelon form

Recall:

A system of linear equations

$$\left.\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right\} \text{m linear equations}$$

$a_{ij} \in \mathbb{R}$ coefficients

$b_i \in \mathbb{R}$ constant terms

x_i are variables

n variables

has augmented matrix

$$[A \mid \underset{\sim}{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

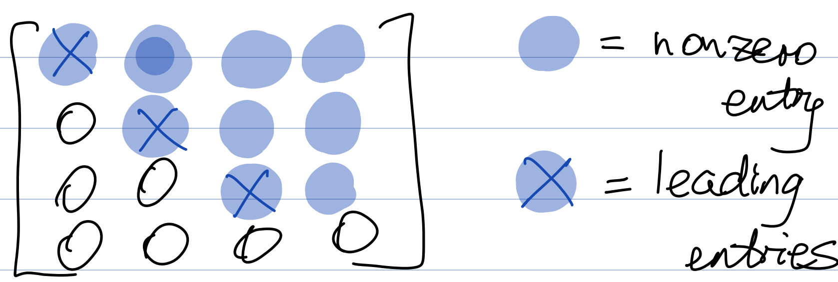
and has either

$\underbrace{\text{no solution}}_{\text{inconsistent}}$, $\underbrace{\text{unique solution or infinitely many solutions}}_{\text{consistent}}$

Def^n A matrix is in <u>row echelon form</u> (staircase) 12 of 5
if

1. Any rows which consist entirely of 0s are at the bottom.
2. In each row which isn't all 0s, the first nonzero entry in that row (called the <u>leading entry</u>) is in a column to the left of any leading entries in rows further down the matrix.

i.e. a matrix in row-echelon form has a staircase of nonzero entries



Examples

The following matrices are in row echelon form:

$$\left[\begin{array}{cc|c} 2 & 4 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 4 \end{array}\right], \quad \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

= leading entries

The following matrices are not in row echelon form:

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

If an augmented matrix is in row echelon form, then it's easy to solve the associated system of linear equations.

Examples

1. $\left[\begin{array}{cc|c} 2 & 4 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{array}\right]$ has system of linear equations

$$\begin{aligned} 2x + 4y &= 1 \\ 0x - y &= 2 \\ 0x + 0y &= 0 \end{aligned}$$

holds for all values of x, y

ie. $$\begin{aligned} 2x + 4y &= 1 \\ -y &= 2 \end{aligned}$$

To solve this system: back-substitution

Start with the last equation, then substitute back into the previous equation, and continue on.

Start with $-y=2 \Rightarrow y=-2$.
Sub into

$$2x+4y=1$$
$$\Rightarrow 2x+4(-2)=1$$
$$\Rightarrow 2x-8=1$$
$$\Rightarrow 2x=9$$
$$\Rightarrow x=\frac{9}{2}.$$

2. $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 4 \end{array}\right]$

$$1x+0y=1$$
$$0x+1y=5$$
$$0x+0y=4$$

$$\left.\begin{array}{rl} x & =1 \\ y & =5 \\ 0 & =4 \end{array}\right]$$

This system has <u>no</u> solution, as $0 \neq 4$.

3. $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$$\begin{aligned} x \quad + y \quad + 2z &= 1 \\ z &= 3 \end{aligned}$$

Use a parameter $t \in \mathbb{R}$ for y ie. put $y = t$, then solve for x.

$$\begin{aligned} x \quad + t \quad + 2(3) &= 1 \\ x &= -5 - t. \end{aligned}$$

correction here

Solution is $\begin{bmatrix} -5 - t \\ t \\ 3 \end{bmatrix}$, $t \in \mathbb{R}$

There are infinitely many solutions, one for each $t \in \mathbb{R}$.

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Topic 5B: Elementary row operations

System of linear equations

$\downarrow$

Augmented matrix

row reduction $\Big\downarrow$ elementary row operations

Augmented matrix in row echelon form

$\downarrow$

Solution to original system

Recall: A matrix is in row echelon form if

1. any all-0 rows are at the bottom
2. the first nonzero entry in each row, called the leading entry, is to the left of the leading entries lower down.

Elementary row operations

1. Swap 2 rows $\quad R_i \leftrightarrow R_j$
   (swap 2 equations) $\quad$ swap row i and row j

2. Multiply a row by a nonzero scalar
(multiply an equation by a nonzero scalar)

$\mapsto$ means "maps to"

$R_i \mapsto cR_i$ multiply row i by c

3. Add a nonzero multiple of one row to another

$R_i \mapsto R_i + cR_j$ replace row i by row i plus c times row j

(add a nonzero multiple of one equation to another)

Each of these corresponds to an operation on the original system of linear equations which doesn't change the solution. (see blue)

Notation see green

The process of using elementary row operations to get the matrix into row echelon form is called row reduction.

Examples

1. Use elementary row operations to obtain a row echelon form of

$$\left[\begin{array}{ccc|c} 2 & 4 & 0 & 6 \\ -1 & 0 & -2 & -5 \\ -3 & -5 & 1 & -4 \end{array}\right]$$

want to get a 1 here

$$\xrightarrow{R_1 \mapsto \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ -1 & 0 & -2 & -5 \\ -3 & -5 & 1 & -4 \end{array}\right]$$

want to get 0s here

$$\xrightarrow[R_3 \mapsto R_3 + 3R_1]{R_2 \mapsto R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & \boxed{2} & -2 & -2 \\ 0 & 1 & 1 & 5 \end{array}\right]$$

want a 1 here

$$\xrightarrow{R_2 \mapsto \frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & \boxed{1} & 1 & 5 \end{array}\right]$$

want a 0 here

$$\xrightarrow{R_3 \mapsto R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & 6 \end{array}\right]$$

This is in row echelon form.

A matrix has lots of different row echelon forms.

2. want 1 here

$$\left[\begin{array}{cc|c} \boxed{2} & 3 & 0 \\ 2 & -1 & 0 \\ 1 & -2 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 2 & -1 & 0 \\ 2 & 3 & 0 \end{array}\right]$$

want 0s here

$$\xrightarrow[R_3 \mapsto R_3 - 2R_1]{R_2 \mapsto R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & \boxed{3} & 0 \\ 0 & 7 & 0 \end{array}\right] \xrightarrow{R_2 \mapsto \frac{1}{3}R_2} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & \boxed{7} & 0 \end{array}\right]$$

want 1 here

want 0 here

$$\xrightarrow{R_3 \mapsto R_3 - 7R_2} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

This is in row echelon form.

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Topic 5C: Gaussian Elimination

Gaussian Elimination is the following process:

system of linear equations

$\downarrow$

augmented matrix

$\downarrow$ elementary row operations

augmented matrix in row echelon form

$\downarrow$ back-substitution

solve the system

Examples

1.

$$\begin{aligned} 2x + 4y &= 6 \\ -x - 2z &= -5 \\ -3x - 5y + z &= -4 \end{aligned}$$

correction here

$$\left[\begin{array}{ccc|c} 2 & 4 & 0 & 6 \\ -1 & 0 & -2 & -5 \\ -3 & -5 & 1 & -4 \end{array}\right] \xrightarrow{\text{see 5B}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & -4 & -12 \end{array}\right]$$

System to solve is:

$$\begin{aligned} x + 2y &= 3 \\ y + z &= 5 \\ -4z &= -12 \end{aligned}$$

Now $z = 3$

$y + z = 5$ so $y = 2$

$x + 2y = 3$ so $x = -1$.

ALWAYS check your solution in the <u>original</u> system of equations.

Check:
$$2(-1) + 4(2) = 6 \checkmark$$
$$-(-1) - 2(3) = -5 \checkmark$$
$$-3(-1) - 5(2) + 3 = -4. \checkmark$$

2. $$\begin{aligned} 2x + 3y &= 0 \\ 2x - y &= 0 \\ x - 2y &= 0 \end{aligned}$$ homogeneous system

$$\left[\begin{array}{cc|c} 2 & 3 & 0 \\ 2 & -1 & 0 \\ 1 & -2 & 0 \end{array}\right] \xrightarrow{\text{see 5B}} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

System to solve: $$\begin{aligned} x - 2y &= 0 \\ y &= 0 \end{aligned}$$

Unique solution is $x = 0$, $y = 0$.

3. $$\begin{array}{c} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 3 \\ 0 & 1 & 3 & -2 & -2 \end{array}\right] \\ \begin{array}{cccc} \underbrace{\quad}_{x_1 \text{ column}} & \underbrace{\quad}_{x_2} & \underbrace{\quad}_{x_3} & \underbrace{\quad}_{x_4} \end{array} \end{array}$$ in row echelon form

● = leading entries

4 variables x_1, x_2, x_3, x_4

Introduce a parameter for each variable which does not have a leading entry in its column.

Put $x_3 = s$, $x_4 = t$. $(s, t \in \mathbb{R})$

$$x_2 + 3x_3 - 2x_4 = -2$$
$$x_2 = -2 - 3s + 2t$$

$$x_1 + x_2 + x_3 - x_4 = 3$$
$$x_1 = 3 - (-2 - 3s + 2t) - s + t$$
$$= 5 + 2s - t$$

Infinitely many solutions

$$\left.\begin{aligned} x_1 &= 5 + 2s - t \\ x_2 &= -2 - 3s + 2t \\ x_3 &= s \\ x_4 &= t \end{aligned}\right\} s, t \in \mathbb{R}.$$

4.
$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

4 variables
x_1, x_2, x_3, x_4

6 equations

System:

$$\begin{aligned} x_1 + x_2 + 2x_3 \quad\quad &= 1 \\ x_2 \quad\quad + x_4 &= 0 \\ x_3 - x_4 &= 1 \\ x_4 &= 0. \end{aligned}$$

Get $x_3 = 1$, $x_2 = 0$, $x_1 = -1$.

Unique solution

$$x_1 = -1,\ x_2 = 0,\ x_3 = 1,\ x_4 = 0.$$

5. $$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 5 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Variables x_1, x_2, x_3, x_4

Columns 2 and 4 have no leading entries. So we set

$$x_2 = s, \quad x_4 = t \qquad s, t \in \mathbb{R}.$$

Then $x_3 - t = 2$ (from row 2)

$$\Rightarrow x_3 = 2 + t$$

$$x_1 + s - (2+t) + 2t = 5$$

$$\Rightarrow x_1 = 7 - s - t.$$

Infinitely many solutions $\left.\begin{aligned} x_1 &= 7-s-t \\ x_2 &= s \\ x_3 &= 2+t \\ x_4 &= t \end{aligned}\right\} s, t \in \mathbb{R}.$

6. $\left[\begin{array}{ccc|c} 0 & 1 & 5 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ = leading entries

Last row gives equation

$$0x + 0y + 0z = 1$$

This has no solution.

Thus the system has no solution.

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Topic 5D: Gauss-Jordan Elimination

Gaussian Elimination:	Gauss-Jordan Elimination:
system of linear equations	system of linear equations
↓	↓
augmented matrix	augmented matrix
↓ elementary row ops	↓ elementary row ops
augmented matrix in row echelon form	augmented matrix in reduced row echelon form
↓	↓
solve the system	solve the system

Def^n A matrix is in reduced row echelon form if

1. it is in row echelon form
2. the leading entry in each nonzero row is a 1 (called a leading 1)
3. Each column containing a leading 1 has zeros everywhere else.

Examples

These matrices are in reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

These matrices are in row echelon but not reduced row echelon form:

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Examples

Find reduced row echelon form for:

1. $$\left[\begin{array}{ccc|c} 2 & 4 & 0 & 6 \\ -1 & 0 & -2 & -5 \\ -3 & -5 & 1 & -4 \end{array}\right] \xrightarrow{5B} \left[\begin{array}{ccc|c} 1 & \boxed{2} & 0 & 3 \\ 0 & 1 & \boxed{1} & 5 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

want 0 here

row-echelon form

Use leading 1s to clear entries above in the same column (entries below are already all 0s from row echelon form).

$$\xrightarrow{R_1 \mapsto R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -7 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow[R_2 \mapsto R_2 - R_3]{R_1 \mapsto R_1 + 2R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

reduced row echelon form

Solve the system:

$$x = -1$$
$$y = 2$$
$$z = 3$$

want 0s here

want 0 here

2. $$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_2 \mapsto R_2 - R_4]{R_3 \mapsto R_3 + R_4} \left[\begin{array}{cccc|c} 1 & 1 & \boxed{2} & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

row echelon form

want 0 here

$$\xrightarrow{R_1 \mapsto R_1 - 2R_3} \left[\begin{array}{cccc|c} 1 & \boxed{1} & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \mapsto R_1 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$x_1 = -1$, $x_2 = 0$, $x_3 = 1$
$x_4 = 0$.

want 0 here

3. $$\left[\begin{array}{cccc|c} 1 & \boxed{1} & 1 & -1 & 3 \\ 0 & 1 & 3 & -2 & -2 \end{array}\right]$$

correction here

$$\xrightarrow{R_1 \mapsto R_1 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 5 \\ 0 & 1 & 3 & -2 & -2 \end{array}\right]$$

Put $x_3 = s$, $x_4 = t$

(columns with no leading 1 get a parameter)

$$x_2 + 3s - 2t = -2$$
$$x_1 - 2s + t = 5$$

Infinitely many solutions

$$\left.\begin{aligned} x_1 &= 5+2s-t \\ x_2 &= -2-3s+2t \\ x_3 &= s \\ x_4 &= t \end{aligned}\right\} s,t\in\mathbb{R}$$

<u>Theorem</u> The reduced row echelon form of a matrix is unique.

<u>Proof</u> have a look online (not just an exercise).

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