After this tutorial you should be able to:

1. Translate between English and Predicate Logic.
2. Understand syntax and semantics of Predicate Logic.
3. Understand normal forms and general facts about Predicate Logic.

Warmup

Problem 1.

1. Write the following as predicate logic formulas:
   (a) P is a reflexive binary relation.
   (b) P is a transitive binary relation.
2. Consider the formula $\forall x\, E(x)$ where E is a unary predicate-symbol.
   (a) Provide a domain and interpretation of the predicate symbol E in which this formula is true.
   (b) Provide a domain and interpretation of the predicate symbol E in which this formula is false.
3. Find a sentence which is true about the integers but not about strings. You should say what the predicate symbols mean for each domain.

Solution 1.

1. (a) $\forall x P(x,x)$
   (b) $\forall x \forall y \forall z((P(x,y) \wedge P(y,z)) \rightarrow P(x,z))$
2. (a) domain $\mathbb{N}$, where $E(x)$ says $x \geq 0$.
   (b) domain $\mathbb{Z}$, where $E(x)$ says $x \geq 0$.
3. Let $T(x,y,z)$ be a ternary predicate symbol. Over the integers we interpret it to mean that $x + y = z$, and over strings we interpret it to mean that $xy = z$ (i.e., x concatenated with y equals z).

   Then the formula
   $$\forall x \forall y \forall z(T(x,y,z) \leftrightarrow T(y,x,z))$$
   is true in the domain of integers (it simply states the commutativity of addition), but not in the domain of strings since, e.g., we can take an assignment α that maps x to be "hello", y to be "world", and z to be "helloworld". Then $T(\alpha(x), \alpha(y), \alpha(z))$ is true, but $T(\alpha(y), \alpha(x), \alpha(z))$ is not.

Translations

Problem 2. Let $\texttt{child}(x, y)$ be a binary relation expressing that x is a child of y. Write predicate logic formulas expressing the following:

1. the sibling relation,
2. the cousin relation,
3. the second-cousin relation (i.e., their parents are cousins),
4. no pair of siblings has any children,
5. no pair of first cousins has any children,
6. there are two second cousins (i.e., their grandparents are siblings) each of which have children.

Solution 2.

1. the sibling relation $\texttt{siblings}(x_1, x_2)$ can be expressed by
$$\exists y(\texttt{child}(x_1, y) \wedge \texttt{child}(x_2, y))$$
2. the cousin relation $\texttt{cousins}(x_1, x_2)$ can be expressed by
$$\exists y_1 \exists y_2(\texttt{siblings}(y_1, y_2) \wedge \texttt{child}(x_1, y_1) \wedge \texttt{child}(x_2, y_2))$$
3. the second-cousin relation can be expressed by
$$\exists z_1 \exists z_2(\texttt{cousins}(z_1, z_2) \wedge \texttt{child}(x_1, z_1) \wedge \texttt{child}(x_2, z_2))$$

Problem 3. Consider the domain $\mathbb{H}$ of humans, and the predicates $\texttt{man}(x)$, $\texttt{woman}(x)$, and $\texttt{parent}(x, y)$ which states that x is the parent of y.

The formula $(\texttt{parentof(x,y)} \wedge \texttt{man(x)})$ means that x is the father of y. Write formulas that express that x is the sister of y, that x is the uncle of y, and other family relations.

Problem 4. Suppose the domain $\mathbb{D}$ consists of books and children. Using the following predicates:

- $\texttt{child}(x)$ which says x is a child
- $\texttt{book}(x)$ which says x is a book
- $\texttt{likes}(x, y)$ which says x likes y
- $\texttt{ed}(x)$ which says x is educational

Express the following sentences in predicate logic:

1. All children like all books
2. Some child likes every single book
3. Not all children like all books
4. Some child does not like any book
5. There is a book that all children like
6. Books are always educational
7. All educational books are liked by all children
8. Books are not always educational
9. There is a child who likes all books that are not educational

Solution 4.

We use C for `child`, etc.

1. All children like all books

$$\forall x \forall y((C(x) \land B(y)) \rightarrow L(x,y)))$$

2. Some child likes every single book

$$\exists x \Big(C(x) \land \forall y \big(B(y) \rightarrow L(x,y)\big)\Big)$$

3. Not all children like all books

$$\neg \forall x \forall y \Big(\big(C(x) \land B(y)\big) \rightarrow L(x,y)\Big)$$

4. Some child does not like any book

$$\exists x \forall y \Big(C(x) \land \big(B(y) \rightarrow \neg L(x,y)\big)\Big)$$

5. There is a book that all children like

$$\exists y \forall x \Big(B(y) \land \big(C(x) \rightarrow L(x,y)\big)\Big)$$

6. Books are always educational

$$\forall x (B(x) \rightarrow E(x))$$

7. All educational books are liked by all children

$$\forall x \forall y \Big(\big((C(x) \land B(y)) \land E(y)\big) \rightarrow L(x,y)\Big)$$

8. Books are not always educational

$$\neg\forall x(B(x) \to E(x))$$

9. There is a child who likes all books that are not educational

$$\exists x \forall y \Big(C(x) \land \big((B(y) \land \neg E(y)) \to L(x,y) \big) \Big)$$

Syntax and Semantics

Problem 5. If F is a formula and F occurs as part of the formula G then F is called a *subformula* of G.

Give a recursive procedure for determinining if F is a subformula of G.

Apply it to the formula $\forall x \forall y (P(x,y) \land \neg P(y,x))$.

Solution 5. Define $Sub(F)$ as follows:

1. If F is an atomic formula then its subformulas are itself, F, and its arguments, e.g., $Sub(P(x,c)) = \{P(x,c), x, c\}$.
2. If $F = \neg G$ then its subformulas are $Sub(G) \cup \{F\}$.
3. If $F = (G \land H)$ or $F = (G \lor H)$, then its subformulas are $Sub(G) \cup Sub(H) \cup \{F\}$.
4. If $F = \exists x.G$ or $F = \forall x.G$ then its subformuluas are $Sub(G) \cup \{F\}$.

The subformulas of $\forall x \forall y (P(x,y) \land \neg P(y,x))$ are $\forall x \forall y (P(x,y) \land \neg P(y,x))$, $\forall y (P(x,y) \land \neg P(y,x))$, $(P(x,y) \land \neg P(y,x))$, $P(x,y)$, $\neg P(y,x))$, and $P(y,x)$.

Problem 6. For each of the following expressions, indicate if it is a formula of predicate logic. If it is, then indicate the free occurrences of variables, the bound occurrences of variables, and the predicate symbols. The domain is the integers.

1. $(\forall x P(x) \land P(4))$
2. $\forall x (P(x) \land P(y))$
3. $\forall x (P(x) \land Q(y,x))$
4. $\forall x (P(x,3) \land (\exists y))$
5. $(\forall x \exists y P(y,x) \land P(x,y))$
6. $(\forall x P(y,x) \land \exists y P(x,y))$

Solution 6.

P and Q are the predicate symbols.

1. Correct syntax, x is bound,
2. Correct syntax, x is bound, y is free
3. Correct syntax, the x's are bound, y is free
4. Incorrect syntax, the $\exists y$ quantifier does not introduce a formula
5. Correct syntax, the first x and y are bound, but the second ones are free.
6. Correct syntax, the first x is bound but the second is free, the first y is free but the second is bound.

Problem 7. Let $\mathrm{Free}(F)$ be the set of all variables that occur free in F. Define $\mathrm{Free}(F)$ by a recursive procedure.

Solution 7. The set $\mathrm{Free}(F)$ of free variables of F is defined by the following recursive procedure:

1. for an atomic formula F, $\mathrm{Free}(F)$ is the set of variables occuring in F.
2. $\mathrm{Free}(\neg F) = \mathrm{Free}(F)$
3. $\mathrm{Free}((F \vee G)) = \mathrm{Free}((F \wedge G)) = \mathrm{Free}(F) \cup \mathrm{Free}(G)$
4. $\mathrm{Free}(\exists x F) = \mathrm{Free}(\forall x F) = \mathrm{Free}(F) \setminus \{x\}$.