The University of Sydney

MATH1002 Linear Algebra

Semester 1 **Tutorial Exercises for Week 12 — Solutions** 2022

**1.** Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$.

(i) Find the eigenvalues and corresponding eigenspaces of $A$.

(ii) Find an invertible matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1}$.

(iii) Evaluate $A^n = PD^nP^{-1}$ for any positive integer $n$.

(iv) Find $A^3$ and $A^5$.

***Solution:***

(i) The characteristic polynomial of $A$ is

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) + 2 = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3).$$

Setting $\det(A - \lambda I) = 0$, we see that the eigenvalues of $A$ are $\lambda = 2$ and $\lambda = 3$.
For $\lambda = 2$, we solve the system

$$\left[A - 2I \mid \mathbf{0}\right] = \left[\begin{array}{cc|c} -1 & 2 & 0 \\ -1 & 2 & 0 \end{array}\right] \xrightarrow{R_2 \mapsto R_2 - R_1} \left[\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

and we get $y = t$ and $x = 2t$, for $t \in \mathbb{R}$. Hence the 2-eigenspace of $A$ is $E_2 = \left\{ t\begin{bmatrix} 2 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$.
For $\lambda = 3$, we solve the system

$$\left[A - 3I \mid \mathbf{0}\right] = \left[\begin{array}{cc|c} -2 & 2 & 0 \\ -1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 \mapsto R_2 - \frac{1}{2}R_1} \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

and we get $y = t$ and $x = t$, for $t \in \mathbb{R}$. Hence the 3-eigenspace of $A$ is $E_3 = \left\{ t\begin{bmatrix} 1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$.

(ii) Since $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a 2-eigenvector of $A$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a 3-eigenvector of $A$, we take

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

(iii) We have

$$\begin{aligned} A^n &= \left(PDP^{-1}\right)^n \\ &= PD^nP^{-1} \\ &= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2^{n+1} & 3^n \\ 2^n & 3^n \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2^{n+1} - 3^n & -2^{n+1} + 2(3)^n \\ 2^n - 3^n & -2^n + 2(3)^n \end{bmatrix}. \end{aligned}$$

(iv) We have

$$A^3 = \begin{bmatrix} 2^{3+1} - 3^3 & -2^{3+1} + 2(3)^3 \\ 2^3 - 3^3 & -2^3 + 2(3)^3 \end{bmatrix} = \begin{bmatrix} -11 & 38 \\ -19 & 46 \end{bmatrix},$$

and

$$A^5 = \begin{bmatrix} 2^{5+1} - 3^5 & -2^{5+1} + 2(3)^5 \\ 2^5 - 3^5 & -2^5 + 2(3)^5 \end{bmatrix} = \begin{bmatrix} -179 & 422 \\ -211 & 454 \end{bmatrix}.$$

**2.** The matrix $B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ has eigenvalues 2 and 4, with corresponding eigenvectors $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(i) Write down an invertible matrix $P$ and a diagonal matrix $D$ such that $B = PDP^{-1}$.

(ii) Find a formula for $B^n$ (for integers $n \geq 0$), and use it to find $B^3$ and $B^4$.

***Solution:***

(i) We may take $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$.

(ii) We have

$$\begin{aligned} B^n &= PD^nP^{-1} \\ &= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & 4^n \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & 2^{2n} \end{bmatrix} \begin{bmatrix} -2^{-1} & 2^{-1} \\ 2^{-1} & 2^{-1} \end{bmatrix} \\ &= \begin{bmatrix} -2^n & 2^{2n} \\ 2^n & 2^{2n} \end{bmatrix} \begin{bmatrix} -2^{-1} & 2^{-1} \\ 2^{-1} & 2^{-1} \end{bmatrix} \\ &= \begin{bmatrix} 2^{n-1} + 2^{2n-1} & -2^{n-1} + 2^{2n-1} \\ -2^{n-1} + 2^{2n-1} & 2^{n-1} + 2^{2n-1} \end{bmatrix} \\ &= 2^{n-1} \begin{bmatrix} 1 + 2^n & -1 + 2^n \\ -1 + 2^n & 1 + 2^n \end{bmatrix} \end{aligned}$$

In particular, the above formula implies that

$$B^3 = 2^{3-1} \begin{bmatrix} 1 + 2^3 & -1 + 2^3 \\ -1 + 2^3 & 1 + 2^3 \end{bmatrix} = \begin{bmatrix} 36 & 28 \\ 28 & 36 \end{bmatrix},$$

and

$$B^4 = 2^{4-1} \begin{bmatrix} 1 + 2^4 & -1 + 2^4 \\ -1 + 2^4 & 1 + 2^4 \end{bmatrix} = \begin{bmatrix} 136 & 120 \\ 120 & 136 \end{bmatrix}.$$

**3.** The matrix

$$C = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

has eigenvalues 0, 1, and 3, with corresponding eigenvectors $[1, -1, 1]$, $[1, -1, 0]$, and $[1, 2, 1]$.

(i) Write down an invertible matrix $P$ and a diagonal matrix $D$ such that $C = PDP^{-1}$.

(ii) Find a formula for $C^n$ (for integers $n \geq 0$), and use it to find $C^4$.

***Solution:***

(i) We may take $P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

(ii) Using the row operations $R_2 \mapsto R_2 + R_1$, $R_3 \mapsto R_3 - R_1$, $R_2 \leftrightarrow R_3$, $R_2 \mapsto -R_2$, $R_3 \mapsto \frac{1}{3}R_3$, $R_1 \mapsto R_1 - R_3$, and $R_1 \mapsto R_1 - R_2$ (in that order), we see that

$$\begin{bmatrix} P \mid I_3 \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & 0 \end{array}\right] = \begin{bmatrix} I_3 \mid P^{-1} \end{bmatrix},$$

and hence

$$P^{-1} = \frac{1}{3} \begin{bmatrix} -1 & -1 & 3 \\ 3 & 0 & -3 \\ 1 & 1 & 0 \end{bmatrix}.$$

Hence we have

$$\begin{aligned}
C^n &= PD^nP^{-1} \\
&= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3^n \end{bmatrix}\begin{bmatrix} -1 & -1 & 3 \\ 3 & 0 & -3 \\ 1 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3^{-1} & 0 \\ 0 & 0 & 3^{n-1} \end{bmatrix}\begin{bmatrix} -1 & -1 & 3 \\ 3 & 0 & -3 \\ 1 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 3^{-1} & 3^{n-1} \\ 0 & -3^{-1} & 2(3)^{n-1} \\ 0 & 0 & 3^{n-1} \end{bmatrix}\begin{bmatrix} -1 & -1 & 3 \\ 3 & 0 & -3 \\ 1 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1+3^{n-1} & 3^{n-1} & -1 \\ -1+2(3)^{n-1} & 2(3)^{n-1} & 1 \\ 3^{n-1} & 3^{n-1} & 0 \end{bmatrix}.
\end{aligned}$$

In particular, the above formula implies that

$$C^4 = \begin{bmatrix} 1+3^{4-1} & 3^{4-1} & -1 \\ -1+2(3)^{4-1} & 2(3)^{4-1} & 1 \\ 3^{4-1} & 3^{4-1} & 0 \end{bmatrix} = \begin{bmatrix} 28 & 27 & -1 \\ 53 & 54 & 1 \\ 27 & 27 & 0 \end{bmatrix}.$$

**4.** Let

$$A = \begin{bmatrix} 3 & 2 & 1 \\ -2 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

(i) Find all of the eigenvalues of $A$.

(ii) For each eigenvalue, find the corresponding eigenspace.

(iii) Write down an invertible matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1}$.

(iv) Evaluate $A^n = PD^nP^{-1}$ for any positive integer $n$, and hence find $A^4$.

***Solution:***

(i) By expanding along the top row and then factorising, the determinant of the matrix $A - \lambda I$ is

$$\begin{aligned}
\det(A-\lambda I) &= \begin{vmatrix} 3-\lambda & 2 & 1 \\ -2 & -1-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} -1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - 2\begin{vmatrix} -2 & 1 \\ 1 & -\lambda \end{vmatrix} + \begin{vmatrix} -2 & -1-\lambda \\ 1 & 1 \end{vmatrix} \\
&= (3-\lambda)(\lambda^2+\lambda-1) - 2(2\lambda-1) + (-2+1+\lambda) \\
&= -\lambda^3 + 2\lambda + \lambda - 2 \\
&= -(2-\lambda)(1-\lambda)(1+\lambda).
\end{aligned}$$

Hence $A$ has eigenvalues $\lambda = -1$, $\lambda = 1$, and $\lambda = 2$.

(ii) To find the $(-1)$-eigenspace of $A$, we solve

$$\left[A+I \mid \mathbf{0}\right] = \left[\begin{array}{ccc|c} 4 & 2 & 1 & 0 \\ -2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow[R_3 \mapsto R_3 - \frac{1}{4}R_1]{R_2 \mapsto R_2 + \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 4 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & \frac{1}{2} & \frac{3}{4} & 0 \end{array}\right] \xrightarrow{R_3 \mapsto R_3 - \frac{1}{2}R_2} \left[\begin{array}{ccc|c} 4 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

and we get $z = 2t$, $y = -3t$, and $x = t$, for $t \in \mathbb{R}$. So the $(-1)$-eigenspace of $A$ is

$$E_{-1} = \left\{ t\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} : t \in \mathbb{R} \right\}.$$

To find the 1-eigenspace of $A$, we solve

$$\begin{bmatrix} A - I \mid \mathbf{0} \end{bmatrix} = \left[\begin{array}{ccc|c} 2 & 2 & 1 & 0 \\ -2 & -2 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right] \xrightarrow[R_3 \mapsto R_3 - \frac{1}{2}R_1]{R_2 \mapsto R_2 + R_1} \left[\begin{array}{ccc|c} 2 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -\frac{3}{2} & 0 \end{array}\right] \xrightarrow{R_3 \mapsto R_3 + \frac{3}{4}R_2} \left[\begin{array}{ccc|c} 2 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

and we get $z = 0$, $y = t$, and $x = -t$, for $t \in \mathbb{R}$. So the 1-eigenspace of $A$ is

$$E_1 = \left\{ t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\}.$$

To find the 2-eigenspace of $A$, we solve

$$\begin{bmatrix} A - 2I \mid \mathbf{0} \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \xrightarrow[R_3 \mapsto R_3 - R_1]{R_2 \mapsto R_2 + 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & -1 & -3 & 0 \end{array}\right] \xrightarrow{R_3 \mapsto R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

and we get $z = t$, $y = -3t$, and $x = 5t$, for $t \in \mathbb{R}$. So the 2-eigenspace of $A$ is

$$E_2 = \left\{ t \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}.$$

(iii) Since $A$ has 3 distinct eigenvalues, we deduce that $A$ is diagonalisable. We may take

$$P = \begin{bmatrix} -1 & 5 & 1 \\ 1 & -3 & -3 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

(iv) By row reducing the augmented matrix $\begin{bmatrix} P \mid I \end{bmatrix}$ until it is transformed into the augmented matrix $\begin{bmatrix} I \mid P^{-1} \end{bmatrix}$, we see that

$$P^{-1} = \frac{1}{6} \begin{bmatrix} 3 & 9 & 12 \\ 2 & 2 & 2 \\ -1 & -1 & 2 \end{bmatrix}.$$

Therefore, for any positive integer $n$, we have

$$\begin{aligned} A^n &= PD^nP^{-1} = \frac{1}{6} \begin{bmatrix} -1 & 5 & 1 \\ 1 & -3 & -3 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & (-1)^n \end{bmatrix} \begin{bmatrix} 3 & 9 & 12 \\ 2 & 2 & 2 \\ -1 & -1 & 2 \end{bmatrix} \\ &= \frac{1}{6} \begin{bmatrix} -3 + 5(2)^{n+1} - (-1)^n & -9 + 5(2)^{n+1} - (-1)^n & -12 + 5(2)^{n+1} + 2(-1)^n \\ 3 - 6(2)^n + 3(-1)^n & 9 - 6(2)^n + 3(-1)^n & 12 - 6(2^n) - 6(-1)^n \\ 2^{n+1} - 2(-1)^n & 2^{n+1} - 2(-1)^n & 2^{n+1} + 4(-1)^n \end{bmatrix}; \end{aligned}$$

and so, in particular, we have

$$A^4 = \begin{bmatrix} 26 & 25 & 25 \\ -15 & -14 & -15 \\ 5 & 5 & 6 \end{bmatrix}.$$

**5.** Diagonalise $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$, and hence find $A^n$ for any positive integer $n$.

***Solution:*** The matrix $A$ is triangular, so its eigenvalues are just its diagonal entries. By solving the system $(A - 2I)\mathbf{x} = \mathbf{0}$, we see that $[1, 0]$ is a 2-eigenvector; and by solving the system $(A - I)\mathbf{x} = \mathbf{0}$, we see that $[-1, 1]$ is a 1-eigenvector. Since $A$ has 2 linearly independent eigenvectors, it is diagonalisable. So we have $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, for any positive integer $n$, we have

$$A^n = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2^n & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{bmatrix}.$$

**6.** Diagonalise $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$, and hence find $A^n$ for any positive integer $n$.

***Solution:*** The matrix $A$ is triangular, so its eigenvalues are just its diagonal entries. By solving the system $(A - I)\mathbf{x} = \mathbf{0}$, we see that $[1, 0, 0]$ is a 1-eigenvector; by solving the system $(A - 2I)\mathbf{x} = \mathbf{0}$, we see that $[1, 1, 0]$ is a 2-eigenvector; and by solving the system $(A - 3I)\mathbf{x} = \mathbf{0}$, we see that $[0, 1, -1]$ is a 3-eigenvector. Since $A$ has 3 linearly independent eigenvectors, it is diagonalisable. So we have $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

By row reducing the augmented matrix $\left[P \mid I\right]$ until it is transformed into the augmented matrix $\left[I \mid P^{-1}\right]$, we see that

$$P^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

Therefore, for any positive integer $n$, we have

$$\begin{aligned} A^n &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 2^n & 0 \\ 0 & 2^n & 3^n \\ 0 & 0 & -3^n \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2^n - 1 & 2^n - 1 \\ 0 & 2^n & 2^n - 3^n \\ 0 & 0 & 3^n \end{bmatrix}. \end{aligned}$$

**7.** A population with three age groups has birth parameters $b_1 = 0$, $b_2 = 1$, $b_3 = 2$; and survival probabilities $s_1 = 0.2$, $s_2 = 0.5$. If the initial population vector is $\mathbf{x}_0 = \begin{bmatrix} 10 \\ 4 \\ 3 \end{bmatrix}$, use the Leslie population model to compute $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$.

***Solution:*** The Leslie matrix is

$$L = \begin{bmatrix} 0 & 1 & 2 \\ 0.2 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}.$$

We have $\mathbf{x}_1 = L\mathbf{x}_0 = \begin{bmatrix} 0 & 1 & 2 \\ 0.2 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 10 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \\ 2 \end{bmatrix}$,

$\mathbf{x}_2 = L\mathbf{x}_1 = \begin{bmatrix} 0 & 1 & 2 \\ 0.2 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 10 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$

$\mathbf{x}_3 = L\mathbf{x}_2 = \begin{bmatrix} 0 & 1 & 2 \\ 0.2 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1.2 \\ 1 \end{bmatrix}$

**8.** * Suppose that the Leslie matrix for a population of female Nifflers is

$$L = \begin{bmatrix} 0 & 0 & 20 \\ 0.1 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}.$$

Starting with an arbitrary $\mathbf{x}_0$, determine the behaviour of this population.

***Solution:*** To understand the long time behaviour for this population growth model, we have to estimate the entries of $L^n$ as $n \to \infty$. Notice that

$$L^2 = \begin{bmatrix} 0 & 10 & 0 \\ 0 & 0 & 2 \\ 0.05 & 0 & 0 \end{bmatrix},$$

and so

$$L^4 = (L^2)^2 = \begin{bmatrix} 0 & 0 & 20 \\ 0.1 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} = L.$$

Hence we have

$$\mathbf{x}_4 = L^4 \mathbf{x}_0 = L\mathbf{x}_0 = \mathbf{x}_1,$$

and we see that in fact

$$\mathbf{x}_{3n+1} = \mathbf{x}_1,$$

for every positive integer $n$. Therefore, the population distribution of female Nifflers has a periodic behaviour. In particular, the population has the same distribution every three years.

**9.** * Prove that the matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is not diagonalisable.

***Solution:*** We argue by contradiction. Suppose that $A$ is diagonalisable. Then there exists an invertible $2 \times 2$ matrix

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

such that $P^{-1}AP$ is diagonal. But

$$P^{-1}AP = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} 2(ad - bc) + cd & d^2 \\ -c^2 & 2(ad - bc) - cd \end{bmatrix};$$

so if $P^{-1}AP$ is diagonal, then $c = d = 0$. But this implies that $\det(P) = ad - bc = 0$, which contradicts the fact that $P$ is invertible.

Note also that $\lambda = 2$ is an eigenvalue of $A$ with algebraic multiplicity 2 and geometric multiplicity $\dim(E_2) = 1$, and this difference in multiplicities implies that $A$ is not diagonalisable.

**10.** * The sequence of *Fibonacci numbers* is obtained by writing down 1 twice, and obtaining each successive number by adding the previous two numbers together:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots.$$

If we let $x_n$ denote the $n$th Fibonacci number, then

$$x_1 = x_2 = 1, \quad \text{and} \quad x_n = x_{n-1} + x_{n-2}, \quad \text{for } n \geq 3,$$

and hence

$$\begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{n-1} \\ x_{n-2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-2} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Diagonalise $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, and hence find a general formula for the $n$th Fibonacci number.

***Solution:*** The characteristic polynomial of $A$ is

$$\det(A - \lambda I_2) = \begin{vmatrix} 1 - \lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1,$$

and hence the eigenvalues of $A$ are

$$\lambda_1 = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{1 - \sqrt{5}}{2}.$$

Note that for each $i \in \{1,2\}$, we have

$$1-(1-\lambda_i)(-\lambda_i) = 1+\lambda_i-\lambda_i^2 = 0,$$

because $\lambda = \lambda_i$ is a solution to the characteristic equation $\lambda^2-\lambda-1=0$.

Hence, for each $i \in \{1,2\}$, we have

$$\left[A-\lambda_i I \mid \mathbf{0}\right] = \left[\begin{array}{cc|c} 1-\lambda_i & 1 & 0 \\ 1 & -\lambda_i & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & -\lambda_i & 0 \\ 1-\lambda_i & 1 & 0 \end{array}\right] \xrightarrow{R_2 \mapsto R_2-(1-\lambda_i)R_1} \left[\begin{array}{cc|c} 1 & -\lambda_i & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Hence $\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$ is a $\lambda_1$-eigenvector of $A$, and $\begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$ is a $\lambda_2$-eigenvector of $A$.

Therefore, for each positive integer $n$, we have

$$\begin{aligned} A^n &= \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}^{-1} \\ &= \frac{1}{\lambda_1-\lambda_2} \begin{bmatrix} \lambda_1^{n+1} & \lambda_2^{n+1} \\ \lambda_1^n & \lambda_2^n \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^{n+1}-\lambda_2^{n+1} & \lambda_1\lambda_2^{n+1}-\lambda_2\lambda_1^{n+1} \\ \lambda_1^n-\lambda_2^n & \lambda_1\lambda_2^n-\lambda_2\lambda_1^n \end{bmatrix}. \end{aligned}$$

Hence we have

$$\begin{aligned} \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} = A^{n-2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^{n-1}-\lambda_2^{n-1} & \lambda_1\lambda_2^{n-1}-\lambda_2\lambda_1^{n-1} \\ \lambda_1^{n-2}-\lambda_2^{n-2} & \lambda_1\lambda_2^{n-2}-\lambda_2\lambda_1^{n-2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^{n-1}(1-\lambda_2)-\lambda_2^{n-1}(1-\lambda_1) \\ \lambda_1^{n-2}(1-\lambda_2)-\lambda_2^{n-2}(1-\lambda_1) \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^n-\lambda_2^n \\ \lambda_1^{n-1}-\lambda_2^{n-1} \end{bmatrix}, \end{aligned}$$

using the fact that $\lambda_1+\lambda_2=1$ to get the final equality.

Therefore, we have the following formula for the $n$th Fibonacci number:

$$x_n = \frac{1}{\sqrt{5}}\left(\lambda_1^n-\lambda_2^n\right) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$