MATH1002 Linear Algebra

Topic 3A: Cross product

Just in $\mathbb{R}^3$!

Recall:

2 vectors $u, v \in \mathbb{R}^n$ $\longrightarrow$ $\boxed{\text{Dot product}}$ $\longrightarrow$ 1 scalar $u \cdot v$

Today:

2 vectors $u, v \in \mathbb{R}^3$ $\longrightarrow$ $\boxed{\text{Cross product}}$ $\longrightarrow$ 1 vector $u \times v \in \mathbb{R}^3$

Definition If $u = [u_1, u_2, u_3] \in \mathbb{R}^3$, $v = [v_1, v_2, v_3] \in \mathbb{R}^3$ then the cross product of u and v is the vector

$$u \times v = [u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1]$$

How to remember this?

Array method:

$$\begin{array}{ccccc} u_1 & u_2 & u_3 & u_1 & u_2 \\ \\ v_1 & v_2 & v_3 & v_1 & v_2 \end{array}$$

(1) Compute all products along lines

ie. $u_2 v_3$, $v_2 u_3$, ...

(2) The components of $\underset{\sim}{u} \times \underset{\sim}{v}$ are given by:

Note: correction made here

Red line product minus Green line product

Example $\underset{\sim}{u} = [2, 1, 3]$, $\underset{\sim}{v} = [5, 0, 4]$

1. To find $\underset{\sim}{u} \times \underset{\sim}{v}$:

$$\left.\begin{array}{ccccc} 2 & 1 & 3 & 2 & 1 \end{array}\right\} \; u_1 \; u_2 \; u_3 \; u_1 \; u_2$$

$$\left.\begin{array}{ccccc} 5 & 0 & 4 & 5 & 0 \end{array}\right\} \; v_1 \; v_2 \; v_3 \; v_1 \; v_2$$

$$\underset{\sim}{u} \times \underset{\sim}{v} = [4 \times 1 - 3 \times 0,\ 3 \times 5 - 2 \times 4,\ 2 \times 0 - 1 \times 5]$$

$$= [4,\ 7,\ -5].$$

2. Find $\underset{\sim}{v} \times \underset{\sim}{u}$:

$$\begin{array}{ccccc} 5 & 0 & 4 & 5 & 0 \\ 2 & 1 & 3 & 2 & 1 \end{array}$$

$$\underset{\sim}{v} \times \underset{\sim}{u} = [0\times 3 - 1\times 4,\ 4\times 2 - 3\times 5,\ 5\times 1 - 0\times 2]$$

$$= [-4, -7, 5].$$

3. Compute $\underset{\sim}{u} \cdot (\underset{\sim}{u} \times \underset{\sim}{v})$ and $\underset{\sim}{v} \cdot (\underset{\sim}{u} \times \underset{\sim}{v})$

$$\begin{aligned} \underset{\sim}{u} \cdot (\underset{\sim}{u} \times \underset{\sim}{v}) &= [2, 1, 3] \cdot [4, 7, -5] \\ &= 2\times 4 + 1\times 7 + 3\times(-5) \\ &= 8 + 7 - 15 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \underset{\sim}{v} \cdot (\underset{\sim}{u} \times \underset{\sim}{v}) &= [5, 0, 4] \cdot [4, 7, -5] \\ &= 20 + 0 - 20 \\ &= 0 \end{aligned}$$

The vector $\underset{\sim}{u} \times \underset{\sim}{v}$ is orthogonal to both $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

Theorem (Properties of the cross product)
Let $\underset{\sim}{u}, \underset{\sim}{v} \in \mathbb{R}^3$. Then:

(1) $\underset{\sim}{u} \times \underset{\sim}{v} = -\underset{\sim}{v} \times \underset{\sim}{u}$. (Anticommutativity)

(2) $\underset{\sim}{u} \cdot (\underset{\sim}{u} \times \underset{\sim}{v}) = \underset{\sim}{v} \cdot (\underset{\sim}{u} \times \underset{\sim}{v}) = 0$

($\underset{\sim}{u} \times \underset{\sim}{v}$ is orthogonal to $\underset{\sim}{u}$ and $\underset{\sim}{v}$).

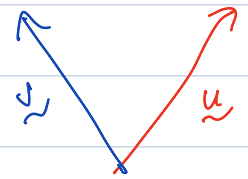
Proof Exercise.

Direction of $\underset{\sim}{u} \times \underset{\sim}{v}$

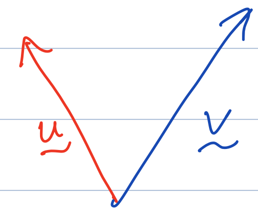
This is given by the right hand rule:

(1) Put fingers of right hand along $\underset{\sim}{u}$

(2) Bend fingers in direction of $\underset{\sim}{v}$
(3) Thumb points in direction of $\underset{\sim}{u} \times \underset{\sim}{v}$



$\underset{\sim}{u} \times \underset{\sim}{v}$ points out of screen



$\underset{\sim}{u} \times \underset{\sim}{v}$ points into screen

Warning: the cross product is not associative ie it is not true that for all $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w} \in \mathbb{R}^3$

$$(\underset{\sim}{u} \times \underset{\sim}{v}) \times \underset{\sim}{w} = \underset{\sim}{u} \times (\underset{\sim}{v} \times \underset{\sim}{w})$$

Example where associativity does not hold:

$$\underset{\sim}{e}_1 = [1,0,0]$$
$$\underset{\sim}{e}_2 = [0,1,0]$$
$$\underset{\sim}{e}_3 = [0,0,1]$$

then

$$\underset{\sim}{e}_1 \times \underset{\sim}{e}_2 = \underset{\sim}{e}_3 \qquad \text{(check!)}$$

$$\underset{\underset{\sim}{u}}{\underset{\sim}{e}_1} \times (\underset{\underset{\sim}{v}}{\underset{\sim}{e}_1} \times \underset{\underset{\sim}{w}}{\underset{\sim}{e}_2}) = \underset{\sim}{e}_1 \times \underset{\sim}{e}_3 = -\underset{\sim}{e}_2 \qquad \text{(check!)}$$

but $(\underset{\underset{\sim}{u}}{\underset{\sim}{e}_1} \times \underset{\underset{\sim}{v}}{\underset{\sim}{e}_1}) \times \underset{\underset{\sim}{w}}{\underset{\sim}{e}_2} = \underset{\sim}{0} \times \underset{\sim}{e}_2 = \underset{\sim}{0}$. (check!)

Theorem (More properties of cross product)
Let $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w} \in \mathbb{R}^3$ and $c \in \mathbb{R}$. Then:

(3) $\underset{\sim}{u} \times \underset{\sim}{u} = \underset{\sim}{0}$

(4) $\underset{\sim}{u} \times (\underset{\sim}{v} + \underset{\sim}{w}) = \underset{\sim}{u} \times \underset{\sim}{v} + \underset{\sim}{u} \times \underset{\sim}{w}$
(Distributive Law)

(5) $c(\underset{\sim}{u} \times \underset{\sim}{v}) = (c\underset{\sim}{u}) \times \underset{\sim}{v} = \underset{\sim}{u} \times (c\underset{\sim}{v})$.

Proof exercises.

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Topic 3B: Cross Product, Length and Area

Proposition If $\underset{\sim}{u}, \underset{\sim}{v} \in \mathbb{R}^3$ then

$$\|\underset{\sim}{u} \times \underset{\sim}{v}\|^2 = \|\underset{\sim}{u}\|^2 \|\underset{\sim}{v}\|^2 - (\underset{\sim}{u} \cdot \underset{\sim}{v})^2$$

Proof exercise.

follows from previous statements

Corollary If $\underset{\sim}{u}, \underset{\sim}{v} \in \mathbb{R}^3$ then

$$\|\underset{\sim}{u} \times \underset{\sim}{v}\| = \|\underset{\sim}{u}\| \|\underset{\sim}{v}\| \sin\theta$$

where θ is the angle between the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$. (So $0 \leq \theta \leq \pi$.)

Proof

Recall $\underset{\sim}{u} \cdot \underset{\sim}{v} = \|\underset{\sim}{u}\| \|\underset{\sim}{v}\| \cos\theta$.

Now by the Prop^n,

$$\begin{aligned}
\|\underset{\sim}{u} \times \underset{\sim}{v}\|^2 &= \|\underset{\sim}{u}\|^2 \|\underset{\sim}{v}\|^2 - (\underset{\sim}{u} \cdot \underset{\sim}{v})^2 \\
&= \|\underset{\sim}{u}\|^2 \|\underset{\sim}{v}\|^2 - \|\underset{\sim}{u}\|^2 \|\underset{\sim}{v}\|^2 \cos^2\theta \\
&= \|\underset{\sim}{u}\|^2 \|\underset{\sim}{v}\|^2 (1 - \cos^2\theta) \\
&= \|\underset{\sim}{u}\|^2 \|\underset{\sim}{v}\|^2 \sin^2\theta
\end{aligned}$$

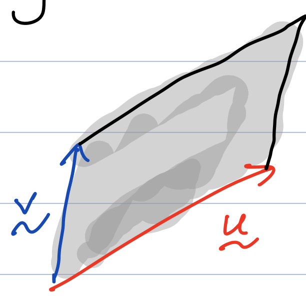
hence

$$\|\underset{\sim}{u} \times \underset{\sim}{v}\| = \|\underset{\sim}{u}\| \|\underset{\sim}{v}\| \sin\theta$$

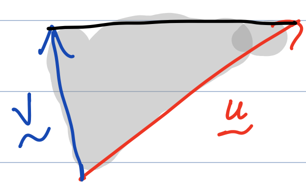
as both sides are non-negative. $\square$

Corollary

(1) $\|\underset{\sim}{u} \times \underset{\sim}{v}\|$ is the area of the parallelogram inscribed by $\underset{\sim}{u}$ and $\underset{\sim}{v}$:



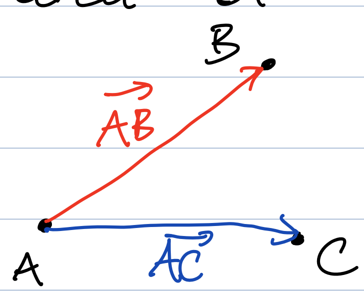
(2) $\frac{1}{2}\|\underset{\sim}{u} \times \underset{\sim}{v}\|$ is the area of the triangle inscribed by $\underset{\sim}{u}$ and $\underset{\sim}{v}$



Proof exercise.

Example $A = (1, 2, -1)$
$B = (2, 4, 1)$
$C = (0, 2, -2)$

Find the area of $\triangle ABC$.



$$\text{Area} = \frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = [2,4,1]-[1,2,-1]$$
$$= [1,2,2]$$
$$\overrightarrow{AC} = [-1,0,-1].$$

$$\overrightarrow{AB}\times\overrightarrow{AC} = [-2,-1,2].$$

$$\begin{array}{ccccc} 1 & 2 & 2 & 1 & 2 \\ -1 & 0 & -1 & -1 & 0 \end{array}$$

$$\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\| = \sqrt{(-2)^2+(-1)^2+2^2}$$
$$= \sqrt{9}$$
$$= 3$$

Area of $\triangle ABC$ is $\frac{3}{2}$.

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Topic 3C: Lines in $\mathbb{R}^2$ – general form and normal form

General form for a line in the plane is

$$ax + by = c$$

where $a, b, c \in \mathbb{R}$, with a and b not both 0.

If $b = 0$, we get $ax = c$, so $x = \frac{c}{a}$
ie. vertical line through $(\frac{c}{a}, 0)$.

If $b \neq 0$, we can divide through by b to get

$$\frac{a}{b}x + y = \frac{c}{b}$$

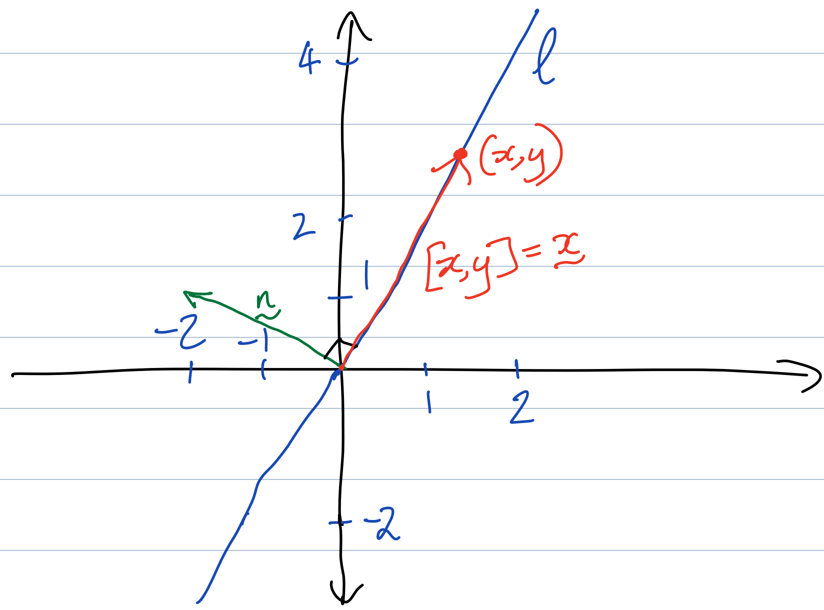
$$\Leftrightarrow \quad y = \left(-\frac{a}{b}\right)x + \frac{c}{b}$$

ie. $\quad y = mx + k$

this is the line of slope m, y-intercept k.

Normal form of a line

Example $\quad y = 2x \quad$ or $\quad -2x + y = 0$.



Notice $\quad -2x + y = [-2, 1] \cdot [x, y] = 0$

Let $\underset{\sim}{n} = [-2, 1]$. Then $\underset{\sim}{n}$ is perpendicular to ℓ.

i.e. $\underset{\sim}{n}$ is orthogonal to any vector parallel to ℓ.

We call $\underset{\sim}{n}$ a <u>normal vector</u> to ℓ.

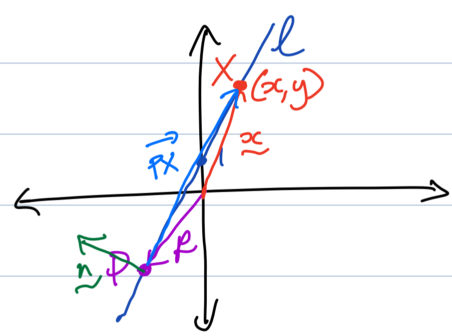
The <u>normal form</u> of this line ℓ is

$$\underset{\sim}{n} \cdot \underset{\sim}{x} = 0$$

where $\underset{\sim}{n} = [-2, 1]$ and $\underset{\sim}{x} = [x, y]$.

<u>Example</u> $\quad y = 2x + 1 \quad$ or $\quad -2x + y = 1.$

$\underset{\sim}{n} = [-2, 1]$



$r = \overrightarrow{OP}$

$\underset{\sim}{x} = [x, y]$

A vector parallel to ℓ is given by $\overrightarrow{PX} = \underset{\sim}{x} - \underset{\sim}{p}$. Then $\underset{\sim}{n}$ is orthogonal to $\overrightarrow{PX}$.

Here:

$$\underset{\sim}{n} \cdot \underset{\sim}{x} = [-2, 1] \cdot [x, y] = -2x + y$$

a point on ℓ

and if $P = (-3, -5)$ then

$$\underset{\sim}{n} \cdot \underset{\sim}{p} = [-2, 1] \cdot [-3, -5] = 1.$$

So

$$\underset{\sim}{n} \cdot \underset{\sim}{x} = \underset{\sim}{n} \cdot \underset{\sim}{p}$$

or

$$\underset{\sim}{n} \cdot (\underset{\sim}{x} - \underset{\sim}{p}) = 0.$$

In general:

Given a line ℓ in the plane

- a normal vector to ℓ is a nonzero vector $\underset{\sim}{n}$ which is orthogonal to any vector parallel to ℓ.
- a direction vector for ℓ is a nonzero vector which is parallel to ℓ.

Def^{n}s The normal form of the equation of a line in $\mathbb{R}^2$ is

$$\underset{\sim}{n} \cdot (\underset{\sim}{x} - \underset{\sim}{p}) = 0$$

where $\underset{\sim}{p}$ is a specific point on the line, n is a normal vector to the line, and $\underset{\sim}{x} = [x, y]$.

The general form of the equation of ℓ can be obtained by expanding this out: if $\underset{\sim}{n} = [a, b]$ then

$\underset{\sim}{p} = [p_1, p_2]$

$$[a, b] \cdot [x - p_1, y - p_2] = 0$$
$$\Rightarrow a(x - p_1) + b(y - p_2) = 0$$
$$\Rightarrow ax + by = c$$

where $c = ap_1 + bp_2 = \underset{\sim}{n} \cdot \underset{\sim}{p}$.

Given general form $ax + by = c$ for ℓ, a normal vector to ℓ is $\underset{\sim}{n} = [a, b]$.

Example $\underset{\sim}{n} = [3, 4]$, $P = (2, -1)$.
Find the normal form and general form for the line through P which is perpendicular to $\underset{\sim}{n}$.

Normal form:

$$\underset{\sim}{n} \cdot (\underset{\sim}{x} - \underset{\sim}{p}) = 0$$
$$[3, 4] \cdot ([x, y] - [2, -1]) = 0$$
$$[3, 4] \cdot [x, y] = [3, 4] \cdot [2, -1]$$
$$3x + 4y = 6 - 4$$
$$3x + 4y = 2$$

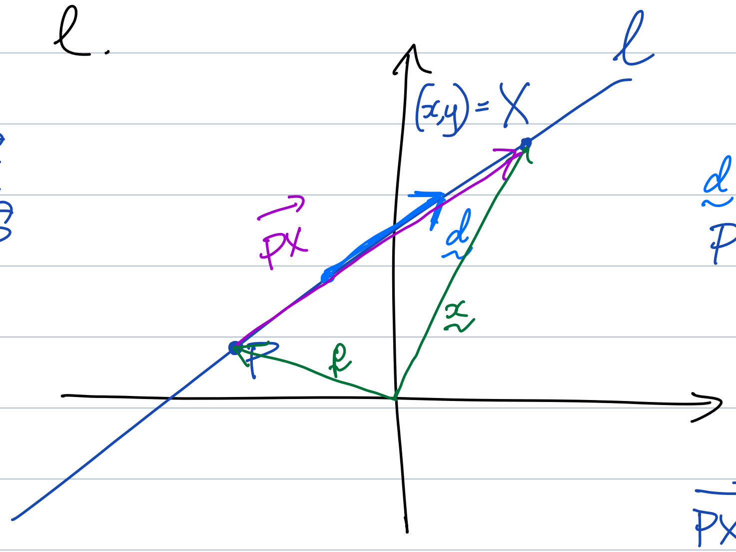
Normal form: $[3, 4] \cdot \underset{\sim}{x} = 2$
General form: $3x + 4y = 2$.

Topic 3D: Lines in $\mathbb{R}^2$ – vector form and parametric equations

A <u>direction vector</u> for a line ℓ is a nonzero vector which is parallel to ℓ.

$\underset{\sim}{x} = \overrightarrow{OX}$

$\underset{\sim}{p} = \overrightarrow{OP}$



$\underset{\sim}{d}$ direction vector

P point on ℓ (fixed)

X arbitrary point on ℓ (varying)

$\overrightarrow{PX} = \overrightarrow{OX} - \overrightarrow{OP}$

$= \underset{\sim}{x} - \underset{\sim}{p}$

$\overrightarrow{PX}$ is parallel to $\underset{\sim}{d}$, so

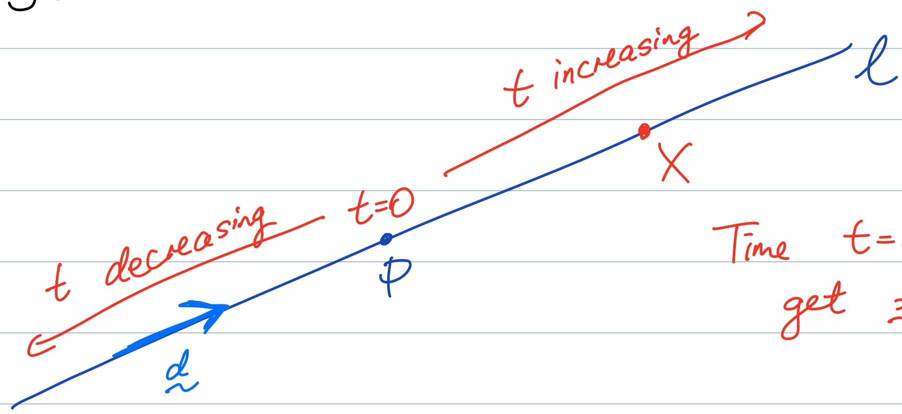
$$\overrightarrow{PX} = t\underset{\sim}{d}$$

for some $t \in \mathbb{R}$.

In other words, $\underset{\sim}{x} - \underset{\sim}{p} = t\underset{\sim}{d}$

$$\Longrightarrow \quad \underset{\sim}{x} = \underset{\sim}{p} + t\underset{\sim}{d}.$$

Defn
The vector form of the equation of a line ℓ is
$$\underset{\sim}{x} = \underset{\sim}{p} + t\underset{\sim}{d}$$
where $\underset{\sim}{p}$ is a specific point on ℓ, $\underset{\sim}{x} = [x, y]$, and $\underset{\sim}{d}$ is a direction vector for ℓ.



Time $t=0$, get $\underset{\sim}{x} = \underset{\sim}{p}$

If $\underset{\sim}{x} = [x, y]$, $\underset{\sim}{p} = [p_1, p_2]$, $\underset{\sim}{d} = [d_1, d_2]$
then from vector form we get
$$[x, y] = [p_1, p_2] + t[d_1, d_2]$$
$$[x, y] = [p_1 + td_1, p_2 + td_2]$$
So we get
$$\left.\begin{aligned} x &= p_1 + td_1 \\ y &= p_2 + td_2 \end{aligned}\right\} t \in \mathbb{R}$$

These are parametric equations for the line ℓ. We call t a parameter of these equations.

Example

Find the vector form and the parametric equations of the line through the points $(1,2)$ and $(5,7)$.

$P = (1,2)$ point on the line

For direction vector $\underset{\sim}{d}$, can take

$$[5,7] - [1,2] = [4,5]$$

ie the vector from $(1,2)$ to $(5,7)$.

Then vector form is

$$\underset{\sim}{x} = [1,2] + t[4,5]. \quad t \in \mathbb{R}$$

Parametric equations:

$$\left.\begin{aligned} x &= 1 + 4t \\ y &= 2 + 5t \end{aligned}\right\} t \in \mathbb{R}.$$

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Topic 3E: Lines in $\mathbb{R}^3$

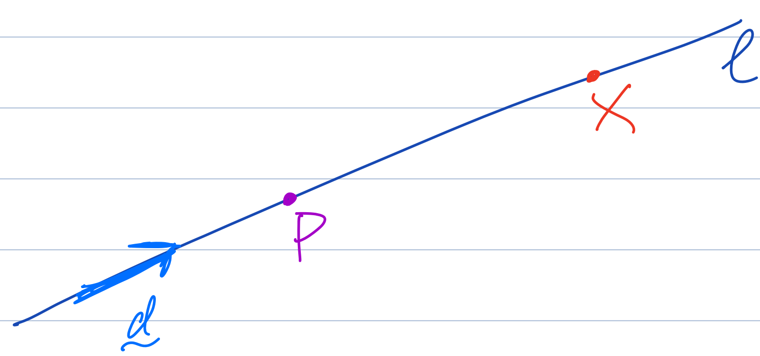
Recall: for lines in $\mathbb{R}^2$, have:

$$\underbrace{\begin{array}{cc}\text{Normal form} & \text{General form} \\ \underset{\sim}{n}\cdot(\underset{\sim}{x}-\underset{\sim}{p})=0 & ax+by=c\end{array}}_{\text{Generalise to planes in } \mathbb{R}^3}
\quad \Bigg| \quad
\underbrace{\begin{array}{cc}\text{Vector form} & \text{parametric equations} \\ \underset{\sim}{x}=\underset{\sim}{p}+t\underset{\sim}{d} & \left.\begin{array}{l} x=p_1+td_1 \\ y=p_2+td_2 \end{array}\right\} \\ & t\in\mathbb{R}\end{array}}_{\text{Generalise to lines in } \mathbb{R}^3}$$

Given a line ℓ in $\mathbb{R}^3$, its vector form is

$$\underset{\sim}{x}=\underset{\sim}{p}+t\underset{\sim}{d}$$

where $\underset{\sim}{p}$ is a point on ℓ, $\underset{\sim}{d}$ is a vector parallel to the line (i.e. a direction vector) and $\underset{\sim}{x}=[x,y,z]$.



The line ℓ has parametric equations

$$\left.\begin{aligned} x &= p_1 + td_1 \\ y &= p_2 + td_2 \\ z &= p_3 + td_3 \end{aligned}\right\} t \in \mathbb{R}$$

where $\underset{\sim}{p} = [p_1, p_2, p_3]$ and $\underset{\sim}{d} = [d_1, d_2, d_3]$.

Examples

1. Find the parametric equations for the line through $P = (1, 2, 0)$ in the direction of the vector $\underset{\sim}{d} = [1, 0, 3]$.

Find vector form first:

$$\underset{\sim}{x} = [x, y, z], \quad \underset{\sim}{p} = [1, 2, 0]$$

$$\underset{\sim}{x} = \underset{\sim}{p} + t\underset{\sim}{d}$$

$$[x, y, z] = [1, 2, 0] + t[1, 0, 3]$$

Parametric equations:

$$\left.\begin{aligned} x &= 1 + t \\ y &= 2 + 0t = 2 \\ z &= 0 + 3t = 3t \end{aligned}\right\} t \in \mathbb{R}$$

2. Find vector form for the line through $P = (1, 1, 3)$ and $Q = (2, 1, 5)$.

Point on line: $\underset{\sim}{p} = [1, 1, 3]$

Direction vector: $\overrightarrow{PQ} = [1, 0, 2]$

Vector form:

$$\underset{\sim}{x} = [1, 1, 3] + t[1, 0, 2] \qquad t \in \mathbb{R}.$$