After this tutorial you should be able to:

1. read and devise natural-deduction proofs for propositional logic.

Problem 1. Prove the following consequents in Natural Deduction:

1. $(A \wedge B) \wedge C \vdash A$
2. $A, B \vdash C \vee (A \wedge B)$
3. $(A \wedge B) \vee (A \wedge C) \vdash A$

Problem 2. The following consequent is called *Modus Tollens*:

$$(F \rightarrow G), \neg G \vdash \neg F$$

Modus Tollens formalises the following reasoning:

1. If it is hot, I wear a hat.
2. I do not wear a hat.
3. So it is not hot.

Prove $(F \rightarrow G), \neg G \vdash \neg F$ in ND.

Solution 2.

Proof strategy: Assume F, get $\perp$, deduce $\neg F$ by ($\neg$ Intr)

Line	Assumptions	Formula	Justification	References
1	1	$F \rightarrow G$	Asmp. Intr	
2	2	$\neg G$	Asmp. Intr	
3	3	F	Asmp. Intr	
4	1,3	G	$\rightarrow$ Elim	1,3
5	1,3,2	$\perp$	$\perp$ Intr	4,2
6	1,2	$\neg F$	$\neg$ Intr	3,5

Problem 3. Prove $(F \vee G), \neg G \vdash F$ in ND.

Solution 3. Proof strategy: Apply reasoning by cases... so we must show F from F (easy) and F from G (we do this using the other assumption $\neg G$ to get $\perp$, which then gets us F, or anything we want, with assumptions $G, \neg G$)).

Line	Assumptions	Formula	Justification	References
1	1	$F \vee G$	Asmp. Intr	
2	2	$\neg G$	Asmp. Intr	
3	3	G	Asmp. Intr	
4	2,3	$\perp$	$\perp$ Intr	2,3
5	2,3	F	$\perp$ Elim	4
6	6	F	Asmp. Intr	
7	1,2	F	$\vee$ E	1,6,6,3,5

References: L1 ($F \vee G$), L6, L6 (F is assumed, F is proven), L3, L5 (G is assumed, F is proven)

Problem 4. $\wedge$ and $\vee$ also have the associativity property. We'll just show one direction of the equivalence, as the proof of the converses of each of these two consequences are almost identical.

1. $(A \wedge (B \wedge C)) \vdash ((A \wedge B) \wedge C)$
2. $(A \vee (B \vee C)) \vdash ((A \vee B) \vee C)$

Solution 4.

1.

Line	Assumptions	Formula	Justification	References
1	1	$(A \wedge (B \wedge C))$	Asmp. I	
2	1	A	$\wedge$ E	1
3	1	$(B \wedge C)$	$\wedge$ E	1
4	1	B	$\wedge$ E	3
5	1	C	$\wedge$ E	3
6	1	$(A \wedge B)$	$\wedge$ I	2, 4
7	1	$((A \wedge B) \wedge C)$	$\wedge$ I	6,5

2.

Line	Assumptions	Formula	Justification	References
1	1	$(A \vee (B \vee C))$	Asmp. I	
2	2	A	Asmp. I	
3	3	$(B \vee C)$	Asmp. I	
4	4	B	Asmp. I	
5	5	C	Asmp. I	
6	2	$(A \vee B)$	$\vee$ I	2
7	2	$((A \vee B) \vee C)$	$\vee$ I	6
8	4	$(A \vee B)$	$\vee$ I	4
9	4	$((A \vee B) \vee C)$	$\vee$ I	8
10	5	$((A \vee B) \vee C)$	$\vee$ I	5
11	3	$((A \vee B) \vee C)$	$\vee$ E	3, 4, 9, 5, 10
12	1	$((A \vee B) \vee C)$	$\vee$ E	1, 2, 7, 3, 11

We ought to prove the converses of these too (e.g. that $((A \wedge B) \wedge C) \vdash (A \wedge (B \wedge C))$ etc.,) if we want to show that these are logical equivalences, but the proofs would be almost identical.

Problem 5. Prove some of de Morgan's Laws:

1. $\neg A \vee \neg B \vdash \neg(A \wedge B)$
   Hint: ($\neg$ I) works well here
2. $\neg(A \vee B) \vdash \neg A \wedge \neg B$
   Hint: assume A, and try to deduce $\neg A$ while cancelling that assumption

Solution 5.

1.

Line	Assumptions	Formula	Justification	References
1	1	$(\neg A \vee \neg B)$	Asmp. I	
2	2	$\neg A$	Asmp. I	
3	3	$\neg B$	Asmp. I	
4	4	$(A \wedge B)$	Asmp. I	
5	4	A	$\wedge$ E	4
6	2, 4	$\perp$	$\perp$ Intr	2, 5
7	2	$\neg(A \wedge B)$	$\neg$ I	4, 6
8	4	B	$\wedge$ E	4
9	3, 4	$\perp$	$\perp$ Intr	3, 8
10	3	$\neg(A \wedge B)$	$\neg$ I	4, 9
11	1	$\neg(A \wedge B)$	$\vee$ E	1, 2, 7, 3, 10

2.

Line	Assumptions	Formula	Justification	References
1	1	$\neg(A \vee B)$	Asmp. I	
2	2	A	Asmp. I	
3	2	$(A \vee B)$	$\vee$ I	2
4	1, 2	$\perp$	$\neg$ E	1, 3
5	1	$\neg A$	$\neg$ I	2, 4
6	6	B	Asmp. I	
7	6	$(A \vee B)$	$\vee$ I	6
8	1, 6	$\perp$	$\perp$ Intr	1, 7
9	1	$\neg B$	$\neg$ I	1, 8
10	1	$(\neg A \wedge \neg B)$	$\wedge$ I	5, 9

Problem 6. Formalise the following in propositional logic and prove it in ND.

A certain Company has Directors.

1. Every Director holds either Bonds or Shares; but no Director holds both.
2. Every Bondholder is a Director.

What can you conclude about this company? Formalise and prove it in ND.

Solution 6. Let D stand for the set of directors, B for the set of bondholders, S for the set of shareholders.

One possible conclusion is "no Bondholder holds Shares".

Then the argument can be formalised in a number of ways, e.g.,

$$D \to (B \vee S), D \to \neg(B \wedge S), B \to D \vdash B \to \neg S$$

Problem 7. Prove the following in ND:

1. $(p \vee (q \vee r)) \vdash (\neg p \to (q \vee r))$
2. $\vdash p \to ((q \to r) \to ((p \to q) \to (p \to r)))$
3. $p \to (q \vee \neg r) \vdash ((q \to \neg p) \wedge r) \to \neg p$

These were 2021 assignment questions.

Solution 7.

1. We want to use $\to I$ to deduce $(\neg p \to (q \vee r))$ from a line corresponding to $\neg p, (p \vee (q \vee r)) \vdash (q \vee r)$. Therefore, we start by assuming $\neg p$. Using that and the original disjunction, we can deduce $(q \vee r)$ using 'reasoning by cases' (lines 3 – 7), then combine these with $\to I$ to finish the proof.

Line	Assumptions	Formula	Justification	References
1	1	$(p \vee (q \vee r))$	Asmp. I	
2	2	$\neg p$	Asmp. I	
3	3	p	Asmp. I	
4	2, 3	$\bot$	$\bot$ I	3, 2
5	2, 3	$(q \vee r)$	$\bot$ E	4
6	6	$(q \vee r)$	Asmp. I	
7	1, 2	$(q \vee r)$	$\vee$ E	1, 3, 5, 6, 6
8	1	$(\neg p \to (q \vee r))$	$\to$ I	2, 7

2. This proof is an exercise in using $\to I$ and $\to E$. The conclusion is a series of nested implications. We will need to use $\to I$ several times, so it's reasonable to assume all the antecedents of each of the subformulas (i.e. p, $q \to r$, and $p \to q$). We deduce the final consequent of the conclusion, $p \to r$ using these, and then repeatedly use $\to I$ to build the answer.

Line	Assumptions	Formula	Justification	References
1	1	p	Asmp. I	
2	2	$(q \to r)$	Asmp. I	
3	3	$(p \to q)$	Asmp. I	
4	1, 3	q	$\to$ E	1, 3
5	1, 2, 3	r	$\to$ E	4, 2
6	2, 3	$(p \to r)$	$\to$ I	1, 5
7	2	$((p \to q) \to (p \to r))$	$\to$ I	3, 6
8		$((q \to r) \to ((p \to q) \to (p \to r)))$	$\to$ I	2, 7
9	1	$(p \wedge ((q \to r) \to ((p \to q) \to (p \to r))))$	$\wedge$ I	1, 8
10	1	$((q \to r) \to ((p \to q) \to (p \to r)))$	$\wedge$ E	9
11		$(p \to ((q \to r) \to ((p \to q) \to (p \to r))))$	$\to$ I	1, 10

The final $\to I$ is tricky, because the definition of $\to I$ requires us to have lines corresponding to $A, S \vdash F$ and $A \vdash A$, but line 8 is of the form $\emptyset \vdash ((q \to r) \to ((p \to q) \to (p \to r)))$ (i.e. it's not dependent on the assumption p!) To introduce the dependency we need, we use $\wedge I$ and $\wedge E$.

A common error is lacking the dependency on p to perform the final $\to I$. Another common error is introducing assumptions of q or of r, which are difficult to discharge in a useful way.

3. We're trying to prove an implication, so we start by assuming the antecedent of that implication, $((q \to \neg p) \wedge r)$. Our goal now, is to somehow deduce $\neg p$ from that. As p is also the antecedent of the starting assumption, a proof by contradiction looks promising, so we start a subproof assuming p. Using p and $\to E$, we deduce $(q \vee \neg r)$, and then do a 'proof by cases', deducing $\perp$ from both sides (with the aim of contradicting p). Now that we have $\perp$ deduced from the first three assumptions, we can use $\neg I$ and then $\to I$ to eliminate the two assumptions we introduced, while constructing the formula we needed.

Line	Assumptions	Formula	Justification	References
1	1	$(p \to (q \vee \neg r))$	Asmp. I	
2	2	$((q \to \neg p) \wedge r)$	Asmp. I	
3	3	p	Asmp. I	
4	1, 3	$(q \vee \neg r)$	$\to$ E	3, 1
5	5	q	Asmp. I	
6	2	$(q \to \neg p)$	$\wedge$ E	2
7	2, 5	$\neg p$	$\to$ E	5, 6
8	2, 3, 5	$\perp$	$\perp$ Intr	3, 7
9	9	$\neg r$	Asmp. I	
10	2	r	$\wedge$ E	2
11	2, 9	$\perp$	$\perp$ I	10, 9
12	1, 2, 3	$\perp$	$\vee$ E	4, 5, 8, 9, 11
13	1, 2	$\neg p$	$\neg$ I	3, 12
14	1	$(((q \to \neg p) \wedge r) \to \neg p)$	$\to$ I	2, 13

A common error is failing to identify the sensible starting assumptions ($((q \to \neg p) \wedge r)$ is a fairly obvious choice, but p is a bit harder to identify). Take care when introducing new assumptions to have a plan about (1) how you will eliminate that assumption, and (2) how it will contribute to deriving the formulas you need.