MATH1002 Linear Algebra

Topic 8A: Elementary matrices

Def^n An *elementary matrix* is any matrix which is obtained from an identity matrix by performing one elementary row operation.

Examples

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$ is obtained from I_3 by performing $R_3 \mapsto 4R_3$

$$E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ is obtained from I_3 by performing $R_1 \leftrightarrow R_2$

$$E_3 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ is obtained from I_3 by performing $R_1 \mapsto R_1 + 3R_3$

Now let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad a,b,c,d,e,f,g,h,i \in \mathbb{R}$$

Then

$$E_1A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}}_{\substack{\text{matrix obtained} \\ \text{from } I_3 \text{ by} \\ R_3 \mapsto 4R_3}} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \underbrace{\begin{bmatrix} a & b & c \\ d & e & f \\ 4g & 4h & 4i \end{bmatrix}}_{\substack{\text{matrix obtained} \\ \text{from } A \text{ by } R_3 \mapsto 4R_3}}$$

$$E_2A = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{matrix obtained} \\ \text{from } I_3 \text{ by} \\ R_1 \leftrightarrow R_2}} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \underbrace{\begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}}_{\substack{\text{matrix obtained} \\ \text{from } A \text{ by} \\ R_1 \leftrightarrow R_2}}$$

$$E_3A = \underbrace{\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{matrix obtained} \\ \text{from } I_3 \text{ by} \\ R_1 \mapsto R_1 + 3R_3}} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \underbrace{\begin{bmatrix} a+3g & b+3h & c+3i \\ d & e & f \\ g & h & i \end{bmatrix}}_{\substack{\text{matrix obtained} \\ \text{from } A \text{ by} \\ R_1 \mapsto R_1 + 3R_3}}$$

Theorem Let E be the elementary matrix obtained by performing the row operation R on I. Then EA is the matrix obtained by performing R on A.

Proof Generalise the examples above. $\square$

Theorem Every elementary matrix is invertible, and its inverse is an elementary matrix of the same type.

Examples

For E_1, E_2, E_3 as above

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$$\begin{tikzcd} I \arrow[r, "R_3 \mapsto 4R_3"] & E_1 \arrow[l, bend left, "R_3 \mapsto \frac{1}{4}R_3"] \end{tikzcd}$$

correction here

ie. E_1^{-1} is obtained from I by applying the row operation which undoes $R_3 \mapsto 4R_3$.

$$E_2^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_2$$

$$\begin{tikzcd} I \arrow[r, bend left, "R_1 \leftrightarrow R_2"] & E_2 \arrow[l, bend left, "R_1 \leftrightarrow R_2"] \end{tikzcd}$$

$R_1 \leftrightarrow R_2$ undoes itself

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{tikzcd} I \arrow[r, bend left, "R_1 \mapsto R_1 + 3R_3"] & E_3 \arrow[l, bend left, "R_1 \mapsto R_1 - 3R_3"] \end{tikzcd}$$

Proof of previous theorem: Generalise these examples.

Theorem

Let A be an $n \times n$ matrix.
The following are equivalent:

(1) A is invertible.
(2) $A\underset{\sim}{x} = \underset{\sim}{b}$ has a unique solution for all $\underset{\sim}{b} \in \mathbb{R}^n$.
(3) $A\underset{\sim}{x} = \underset{\sim}{0}$ has unique solution $\underset{\sim}{x} = \underset{\sim}{0}$.
(4) the reduced row echelon form of A is I_n.
(5) A is a product of elementary matrices.

Proof We'll show (4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (1)

(4) $\Rightarrow$ (5):
$$[A \mid I_n] \xrightarrow[\text{row ops}]{\text{elem}} [I_n \mid A^{-1}]$$

Performing elementary row operations on A is the same as multiplying A on the left by elementary matrices.
Suppose row ops used to get A to I_n are R_1 then R_2 then --- then R_k.
Let $E_1, E_2, \ldots, E_k$ be the corresponding elementary matrices ie. $I_n \xrightarrow{R_i} E_i$.
Then
$$E_k \cdots E_3 E_2 E_1 A = I_n.$$
Now each E_i is invertible, and its inverse is also an elementary matrix.

So
$$E_k \cdots E_2 E_1 A = I_n$$
$$\Rightarrow \quad E_k^{-1} E_k E_{k-1} \cdots E_2 E_1 A = E_k^{-1} I_n$$
$$\Rightarrow \quad I E_{k-1} \cdots E_2 E_1 A = E_k^{-1}$$
$$\Rightarrow \quad E_{k-1} \cdots E_2 E_1 A = E_k^{-1}$$
and carrying on we eventually get
$$A = E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1}.$$
So A is a product of elementary matrices i.e. (5) holds.

(5) $\Rightarrow$ (1): Elementary matrices are invertible, and products of invertible matrices are invertible. So if
$$A = E_1 E_2 \cdots E_k$$
where the E_i are elementary matrices, A is invertible. $\square$

Example Express $A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$ as a product of elementary matrices.

Row reduce A:

$$\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \xrightarrow{R_2 \mapsto R_2 - 3R_1} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 \mapsto R_1 - 2R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Let $E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ obtained from I by $R_2 \mapsto R_2 - 3R_1$

$$E_2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}.$$ obtained from I by $R_1 \mapsto R_1 - 2R_2$

Then $$E_2 E_1 A = I$$

so $$A = E_1^{-1} E_2^{-1}$$
$$= \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$ (Check!)

//

MATH1002 Linear Algebra

Topic 8B: Determinants

Recall: if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

then

$$\det(A) = a_{11}a_{22} - a_{12}a_{21} = |A|.$$

We'll also sometimes write $|A|$ for the determinant of A.

We'll now extend this definition to $n \times n$ matrices:

$n=1$ $\quad A = [a_{11}]$ and $|A| = a_{11}$.

$n=3$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det(A) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

For any matrix A, we write A_{ij} for the submatrix of A obtained by deleting row i and column j

So for A a 3×3 matrix

$$\det(A) = a_{11}|A_{11}| - a_{12}|A_{12}| + a_{13}|A_{13}|$$
$$= \sum_{j=1}^{3} (-1)^{1+j} a_{1j} |A_{1j}|$$

Σ sum

Example Find $\det(A)$ for

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 0 \\ 7 & 3 & 9 \end{bmatrix}$$

$$\det(A) = 1\begin{vmatrix} 1 & 0 \\ 3 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 0 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 1 \\ 7 & 3 \end{vmatrix}$$
$$= 1(9) - 2(36) + 3(12-7)$$
$$= -48.$$

Def^n Let $A = (a_{ij})$ be an $n\times n$ matrix $(n \geq 2)$.

Then the determinant of A, denoted by $\det(A)$ or $|A|$, is the scalar

$$\det(A) = |A| = a_{11}|A_{11}| - a_{12}|A_{12}| + \cdots + (-1)^{1+n} a_{1n} |A_{1n}|$$

$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} |A_{1j}|.$$

Theorem To compute $\det(A)$, we can expand along any row or column. More precisely, let

$$c_{ij} = (-1)^{i+j} |A_{ij}|$$

Then

$$\det(A) = \sum_{j=1}^{n} a_{ij} c_{ij} \quad \text{for any } i$$

(can expand along any row)

$$= \sum_{i=1}^{n} a_{ij} c_{ij} \quad \text{for any } j$$

(can expand down any column)

Proof Definitely not.

Examples

1. Find $\det(A)$ for $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 0 \\ 7 & 3 & 9 \end{bmatrix}$

using column 3, since it has a zero.

Pattern of $+$ and $-$ signs in determinant formula:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Checkerboard pattern
$+$ in top LH corner

$$\det(A) = 3\begin{vmatrix} 4 & 1 \\ 7 & 3 \end{vmatrix} - 0\begin{vmatrix} 1 & 2 \\ 7 & 3 \end{vmatrix} + 9\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix}$$

4of5

$$= 3(12-7) + 9(1-8)$$
$$= 15 - 63$$
$$= -48.$$

2. Find $\det(B)$ for

$$B = \begin{bmatrix} 5 & 1 & 0 & 1 \\ 7 & 0 & 3 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 2 & 4 & 9 \end{bmatrix}$$

We'll use row 3 since it has the most 0s.

$$\det(B) = 2\det\begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 2 & 4 & 9 \end{bmatrix}$$
$$= 2\left(3\begin{vmatrix} 1 & 1 \\ 2 & 9 \end{vmatrix}\right)$$
$$= 2(3(9-2))$$
$$= 42$$

3. Find $\det(C)$ where

$$C = \underbrace{\begin{bmatrix} 5 & 1 & 0 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}}_{\text{upper-triangular}}$$

correction here

Using column 1:

$$\det(C) = 5\begin{vmatrix} 2 & 3 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{vmatrix} = 5\left(2\begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix}\right) = 5 \times 2 \times 1 \times 2$$
$$= 20.$$

Defn An $n\times n$ matrix is upper-triangular if all its entries below the diagonal are 0s. An $n\times n$ matrix is lower-triangular if all its entries above the diagonal are 0s. A matrix is triangular if it's either upper- or lower-triangular

Example $D = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 3 & 2 \end{bmatrix}$ is lower-triangular

Theorem If $A = (a_{ij})$ is triangular then

$$\det(A) = a_{11}\, a_{22} \cdots a_{nn}$$

i.e. the product of its diagonal entries.

Theorem A matrix A is invertible $\iff \det(A) \neq 0$.

Proof Not an exercise. Uses earlier results and elementary matrices. $\square$

MATH1002 Linear Algebra

Topic 8C: Determinant method for cross product

Recall: For $\underset{\sim}{u} = [u_1, u_2, u_3]$ and $\underset{\sim}{v} = [v_1, v_2, v_3]$

$$\underset{\sim}{u} \times \underset{\sim}{v} = [u_2 u_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1].$$

Recall: $\underset{\sim}{e}_1 = [1, 0, 0]$, $\underset{\sim}{e}_2 = [0, 1, 0]$, $\underset{\sim}{e}_3 = [0, 0, 1]$.

Consider the "determinant" of the 3×3 object:

$$\begin{bmatrix} \underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \underset{\sim}{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

We get, expanding along row 1:

$$\underset{\sim}{e}_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \underset{\sim}{e}_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \underset{\sim}{e}_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$\begin{aligned}
&= (u_2 v_3 - u_3 v_2)\underset{\sim}{e}_1 - (u_1 v_3 - u_3 v_1)\underset{\sim}{e}_2 + (u_1 v_2 - u_2 v_1)\underset{\sim}{e}_3 \\
&= (u_2 v_3 - u_3 v_2)[1, 0, 0] - (u_1 v_3 - u_3 v_1)[0, 1, 0] \\
&\qquad + (u_1 v_2 - u_2 v_1)[0, 0, 1] \\
&= [u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1] \\
&= \underset{\sim}{u} \times \underset{\sim}{v}.
\end{aligned}$$

In short:

$$\underset{\sim}{u} \times \underset{\sim}{v} = \begin{vmatrix} \underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \underset{\sim}{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Examples

Find $\underset{\sim}{u} \times \underset{\sim}{v}$ for $\underset{\sim}{u} = [2, 0, 1]$ and $\underset{\sim}{v} = [4, 1, 3]$.

$$\begin{vmatrix} \underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \underset{\sim}{e}_3 \\ 2 & 0 & 1 \\ 4 & 1 & 3 \end{vmatrix} = \underset{\sim}{e}_1 \begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix} - \underset{\sim}{e}_2 \begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} + \underset{\sim}{e}_3 \begin{vmatrix} 2 & 0 \\ 4 & 1 \end{vmatrix}$$

$$= -\underset{\sim}{e}_1 - 2\underset{\sim}{e}_2 + 2\underset{\sim}{e}_3$$

$$= [-1, -2, 2].$$

Find $\underset{\sim}{v} \times \underset{\sim}{u}$:

$$\begin{vmatrix} \underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \underset{\sim}{e}_3 \\ 4 & 1 & 3 \\ 2 & 0 & 1 \end{vmatrix} = \underset{\sim}{e}_1 + 2\underset{\sim}{e}_2 - 2\underset{\sim}{e}_3$$

$$= [1, 2, -2].$$

//