Semester 1 **Tutorial Exercises for Week 8 — Solutions** 2022

1. Find the inverse of each of the following matrices when it exists:

(i) $\begin{bmatrix} 5 & 2 \\ 3 & -2 \end{bmatrix}$ (ii) $\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (iv) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

(v) $\begin{bmatrix} 0 & -1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ -1 & 1 & 0 \end{bmatrix}$ (vii) $\begin{bmatrix} 2 & 4 & 6 \\ 7 & 11 & 6 \\ -6 & -6 & 12 \end{bmatrix}$ (viii) $\begin{bmatrix} -4 & 3 & 3 \\ 8 & 7 & 3 \\ 4 & 3 & 3 \end{bmatrix}$

Solution:

(i) $\frac{1}{16}\begin{bmatrix} 2 & 2 \\ 3 & -5 \end{bmatrix}$

(ii) The inverse of $\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$ does not exist, because its determinant is $6(1) - 2(3) = 0$.

(iii) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(iv) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

(v) $$\left[\begin{array}{ccc|ccc} 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \end{array}\right].$$

So the inverse is $\begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}$.

(vi) $$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{array}\right].$$

So the inverse is $\begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & 2 \\ -1 & -1 & -1 \end{bmatrix}$.

(vii) $$\left[\begin{array}{ccc|ccc} 2 & 4 & 6 & 1 & 0 & 0 \\ 7 & 11 & 6 & 0 & 1 & 0 \\ -6 & -6 & 12 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1/2 & 0 & 0 \\ 0 & -3 & -15 & -7/2 & 1 & 0 \\ 0 & 6 & 30 & 3 & 0 & 1 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1/2 & 0 & 0 \\ 0 & -3 & -15 & -7/2 & 1 & 0 \\ 0 & 0 & 0 & * & * & * \end{array}\right],$$

so the matrix is not invertible.

(viii) $$\left[\begin{array}{ccc|ccc} -4 & 3 & 3 & 1 & 0 & 0 \\ 8 & 7 & 3 & 0 & 1 & 0 \\ 4 & 3 & 3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} -4 & 3 & 3 & 1 & 0 & 0 \\ 0 & 13 & 9 & 2 & 1 & 0 \\ 0 & 6 & 6 & 1 & 0 & 1 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|ccc} 1 & -3/4 & -3/4 & -1/4 & 0 & 0 \\ 0 & 1 & 1 & 1/6 & 0 & 1/6 \\ 0 & 0 & -4 & -1/6 & 1 & -13/6 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/8 & 0 & 1/8 \\ 0 & 1 & 0 & 1/8 & 1/4 & -3/8 \\ 0 & 0 & 1 & 1/24 & -1/4 & 13/24 \end{array}\right].$$

So the inverse is $\frac{1}{24}\begin{bmatrix} -3 & 0 & 3 \\ 3 & 6 & -9 \\ 1 & -6 & 13 \end{bmatrix}$.

2. Find the inverse of $\begin{bmatrix} 5 & -3 \\ 7 & -4 \end{bmatrix}$ and use it to solve for x, y, z, and w, where

$$\begin{bmatrix} 5 & -3 \\ 7 & -4 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 11 & 4 \\ 15 & 5 \end{bmatrix}$$

Solution: By the formula for 2×2 matrices, the inverse of $\begin{bmatrix} 5 & -3 \\ 7 & -4 \end{bmatrix}$ is $\begin{bmatrix} -4 & 3 \\ -7 & 5 \end{bmatrix}$, so that

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ -7 & 5 \end{bmatrix} \begin{bmatrix} 11 & 4 \\ 15 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & -3 \end{bmatrix}.$$

3. Suppose that A is an invertible matrix. Explain briefly why the matrix equation $AB = AC$ implies $B = C$.

Solution: We have $B = IB = (A^{-1}A)B = A^{-1}(AB) = A^{-1}(AC) = (A^{-1}A)C = IC = C$.

4. Which of the following are true for all invertible matrices A, B, C of the same size:

(i) $(ABC)^{-1} = A^{-1}B^{-1}C^{-1}$ (ii) $(ABA)^{-1} = A^{-1}B^{-1}A^{-1}$

(iii) $(A^{-1})^{-1} = A$ (iv) $-(-A)^{-1} = A^{-1}$

(v) $C^{-1}(ABC^{-1})^{-1}AB = I$ (vi) $(A+B)^{-1} = A^{-1} + B^{-1}$

(vii) $A^{-1}(I+A)A = A+I$ (viii) $(A+I)(A^{-1} - I) = A^{-1} - A$

(ix) $A^2 - 2A + I = 0 \implies A^{-1} = 2I - A$

(x)* $A^2 - 2A + I = 0 \implies A = I$

Find a proof or counterexample in each case.

Solution:

(i) This is false. For example take

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then

$$(ABC)^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

yet

$$A^{-1}B^{-1}C^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

(ii) This is true, since $(ABA)^{-1} = A^{-1}(AB)^{-1} = A^{-1}B^{-1}A^{-1}$.

(iii) This is true. By uniqueness of inverses, since $A^{-1}A = AA^{-1} = I$, we have immediately that $(A^{-1})^{-1} = A$.

(iv) This is true. Observe that

$$(-A)(-A^{-1}) = (-1)(-1)AA^{-1} = AA^{-1} = I$$

and

$$(-A^{-1})(-A) = (-1)(-1)A^{-1}A = A^{-1}A = I,$$

so that, by uniqueness of inverses, $(-A)^{-1} = -A^{-1}$, yielding

$$-(-A)^{-1} = -(-A^{-1}) = A^{-1}.$$

(v) This is true, since $C^{-1}(ABC^{-1})^{-1}AB = C^{-1}(C^{-1})^{-1}B^{-1}A^{-1}AB = I$.

(vi) This is false even for 1×1 matrices, since $(A+B)^{-1}$ may not exist. For example, take $A = 1$ and $B = -1$, so that $A + B = 0$ has no inverse. Even when $(A+B)^{-1}$ exists, the statement is typically false. For example, take $A = B = 1$, so that $(A+B)^{-1} = 1/2 \neq 2 = A^{-1} + B^{-1}$.

(vi) This is true, since $A^{-1}(I+A)A = A^{-1}IA + A^{-1}AA = I + A = A + I$.

(vii) This is true, since $(A+I)(A^{-1} - I) = AA^{-1} - A + A^{-1} - I = A^{-1} - A$.

(viii) This is true, since

$$\begin{aligned} A^2 - 2A + I = 0 \quad &\Longrightarrow \quad 2A - A^2 = I \\ &\Longrightarrow \quad A(2I - A) = (2I - A)A = I \\ &\Longrightarrow \quad A^{-1} = 2I - A\,. \end{aligned}$$

(ix) This is false. For example, take $A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \neq I$, yet

$$A^2 - 2A + I = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix} - \begin{bmatrix} 4 & 2 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0\,.$$

5. Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 3 & 4 & 3 \end{bmatrix}$ and use it to solve for x, y, and z, where

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 2 \\ 2x & + & 2y & + & 3z & = & 0 \\ 3x & + & 4y & + & 3z & = & 1 \end{array}$$

Solution: Observe that

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 3 & 0 & 1 & 0 \\ 3 & 4 & 3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & -1 & -1 \\ 0 & 1 & 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array}\right],$$

so the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 3 & 4 & 3 \end{bmatrix}$ is $\begin{bmatrix} 6 & -1 & -1 \\ -3 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix}$. Observe also that

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 3 & 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix},$$

so that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 & -1 & -1 \\ -3 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ -4 \end{bmatrix}.$$

6. Find the inverse of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ and solve for x, y, z in terms of a, b, c, where

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & a \\ 2x & + & 3y & + & z & = & b \\ 3x & + & y & + & 2z & = & c \end{array}$$

Solution: Observe that

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -5 & -2 & 1 & 0 \\ 0 & -5 & -7 & -3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -7 & -3 & 2 & 0 \\ 0 & 1 & 5 & 2 & -1 & 0 \\ 0 & 0 & 18 & 7 & -5 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5/18 & 1/18 & 7/18 \\ 0 & 1 & 0 & 1/18 & 7/18 & -5/18 \\ 0 & 0 & 1 & 7/18 & -5/18 & 1/18 \end{array}\right]$$

so the inverse of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ is $\frac{1}{18}\begin{bmatrix} -5 & 1 & 7 \\ 1 & 7 & -5 \\ 7 & -5 & 1 \end{bmatrix}$. Observe also that

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix},$$

so that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{18}\begin{bmatrix} -5 & 1 & 7 \\ 1 & 7 & -5 \\ 7 & -5 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{18}\begin{bmatrix} -5a+b+7c \\ a+7b-5c \\ 7a-5b+c \end{bmatrix}.$$

7. Solve the given matrix equations (assume that all matrices are invertible). Simplify your answer as much as possible

(i) $XA^2 = A^{-1}$ (ii) $AXB = (BA)^2$

(iii) $\left(A^{-1}X\right)^{-1} = A\left(B^{-2}A\right)^{-1}$ (iv) $ABXA^{-1}B^{-1} = I + A$.

Solution:

(i) By multiplying both sides with $(A^2)^{-1}$ on the right we have

$$X = A^{-1}(A^2)^{-1} = A^{-1}A^{-2} = A^{-3}.$$

(ii) By multiplying both sides with B^{-1} on the right and A^{-1} on the left, we get

$$X = A^{-1}(BA)^2B^{-1}.$$

This cannot be simplified any further.

(iii) By taking the inverse of both sides, we get

$$A^{-1}X = \left(A\left(B^{-2}A\right)^{-1}\right)^{-1} = \left(\left(B^{-2}A\right)^{-1}\right)^{-1}A^{-1} = B^{-2}AA^{-1} = B^{-2}.$$

Thus $X = AB^{-2}$.

(iv) $ABXA^{-1}B^{-1} = (I + A) \implies ABXA^{-1} = (I + A)B \implies ABX = (I + A)BA \implies BX = A^{-1}(I + A)BA$. Therefore, we obtain

$$X = B^{-1}A^{-1}(I + A)BA = B^{-1}A^{-1}IBA + B^{-1}A^{-1}ABA = B^{-1}A^{-1}BA + A.$$

8. (i) Let A be an invertible matrix, prove that A^T is invertible and $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$

(ii) Prove that if A, B are square matrices and AB is invertible, then both A and B are invertible.

Solution:

(i) Since A is invertible, the inverse of A exists and we have $AA^{-1} = A^{-1}A = I$. Thus, $\left(AA^{-1}\right)^T = \left(A^{-1}A\right)^T = I^T$. This implies $\left(A^{-1}\right)^T A^T = A^T\left(A^{-1}\right)^T = I$.

This means there is a matrix X such that $XA^T = A^TX = I$, namely $X = \left(A^{-1}\right)^T$. Therefore A^T is invertible and its inverse is $X = \left(A^{-1}\right)^T$

(ii) Since AB is invertible, there is matrix X such that $(AB)X = X(AB) = I$.

Using the associative law of matrix multiplication, we have $(AB)X = A(BX)$ and $X(AB) = (XA)B$. Therefore, we obtain $A(BX) = I$ and $(XA)B = I$. Hence, A and B are invertible.

9. When is a diagonal matrix $\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$ invertible, and what is its inverse?

Solution: If any of the diagonal entries is zero, then the matrix has a row of zeros so is not invertible. If all of the diagonal entries are nonzero then

$$\left[\begin{array}{cccc|cccc} d_1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n & 0 & 0 & \cdots & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & d_1^{-1} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & d_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & d_n^{-1} \end{array}\right]$$

so that the inverse exists and is the diagonal matrix with reciprocals down the diagonal.

10. * Let n be a positive integer and J the $n \times n$ matrix for which every entry is 1. Verify that $I - J$ is invertible if and only if $n \geq 2$, in which case

$$(I-J)^{-1} \;=\; I - \frac{1}{n-1} J \,.$$

Solution: If $n = 1$ then $I - J = 1 - 1 = 0$ which is not invertible. Suppose $n \geq 2$. Then $J^2 = nJ$, so that

$$(I-J)\left(I - \frac{1}{n-1}J\right) \;=\; I - \frac{1}{n-1}J - J + \frac{1}{n-1}J^2 \;=\; I - \frac{n}{n-1}J + \frac{n}{n-1}J \;=\; I\,,$$

and similarly $\left(I - \frac{1}{n-1}J\right)(I-J) = I$, so that $(I-J)^{-1} = I - \frac{1}{n-1}J$.

11. * Use row reduction to determine the values of λ for which the matrix $A - \lambda I$ is *not* invertible in each case:

(i) $A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$ (iii) $A = \begin{bmatrix} -3 & 0 & 2 \\ -4 & -1 & 4 \\ -4 & -4 & 7 \end{bmatrix}$

Solution:

(i) Observe that $\begin{bmatrix} 2-\lambda & 0 \\ 0 & -3-\lambda \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ if and only if $\lambda \neq 2$ and $\lambda \neq -3$, so that $A - \lambda I$ is not invertible if and only if $\lambda = 2$ or $\lambda = -3$.

(ii) Observe that

$$\begin{bmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{bmatrix} \sim \begin{bmatrix} 1 & \lambda-4 \\ 1-\lambda & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & \lambda-4 \\ 0 & \lambda^2-5\lambda+6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

if and only if $\lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3) \neq 0$. Hence $A - \lambda I$ is not invertible if and only if $(\lambda-2)(\lambda-3) = 0$, that is, $\lambda = 2$ or $\lambda = 3$.

(iii) Observe that

$$\begin{bmatrix} -3-\lambda & 0 & 2 \\ -4 & -1-\lambda & 4 \\ -4 & -4 & 7-\lambda \end{bmatrix} \sim \begin{bmatrix} -4 & -4 & 7-\lambda \\ 0 & 3-\lambda & \lambda-3 \\ -3-\lambda & 0 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & (\lambda-7)/4 \\ 0 & 3-\lambda & \lambda-3 \\ 0 & \lambda+3 & (\lambda^2-4\lambda-13)/4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & (\lambda-7)/4 \\ 0 & 3-\lambda & \lambda-3 \\ 0 & 6 & (\lambda^2-25)/4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & (\lambda-7)/4 \\ 0 & 1 & (\lambda^2-25)/24 \\ 0 & 3-\lambda & \lambda-3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & (\lambda-7)/4 \\ 0 & 1 & (\lambda^2-25)/24 \\ 0 & 0 & (\lambda-3)(\lambda-1)(\lambda+1)/24 \end{bmatrix},$$

which can be row reduced to the identity matrix if and only if

$$(\lambda-3)(\lambda-1)(\lambda+1) \neq 0\,.$$

Hence $A-\lambda I$ is not invertible if and only if

$$(\lambda-3)(\lambda-1)(\lambda+1) = 0\,,$$

that is, $\lambda = 3\,,\ 1$ or $-1\,.$