1. Let $\mathbf{a} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

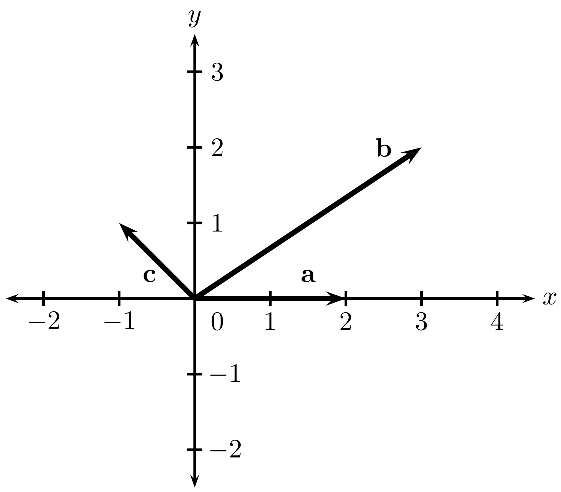
(i) Draw these vectors in standard position in $\mathbb{R}^2$.

(ii) Compute the vectors $\mathbf{a}+\mathbf{b}$, $\mathbf{b}+\mathbf{c}$ and $\mathbf{a}-\mathbf{c}$. How can these results be obtained geometrically?

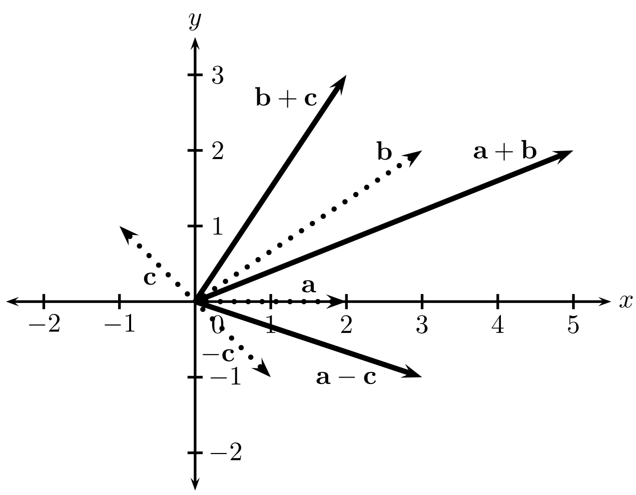
(iii) Draw the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ with their tails at the point $(2, -1)$.

Solution:

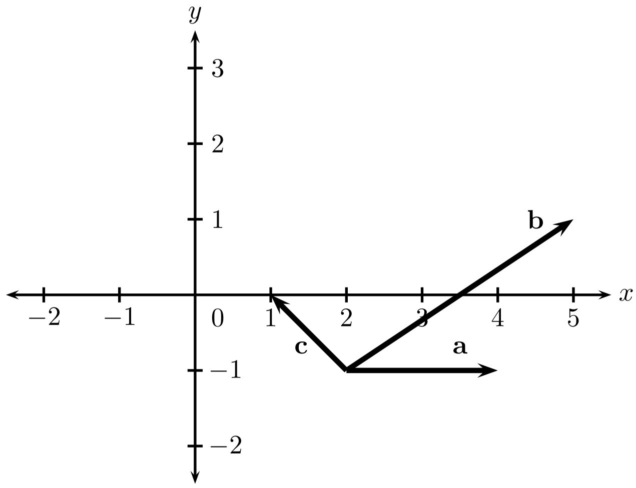
(i)



(ii) $\mathbf{a}+\mathbf{b} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, $\mathbf{b}+\mathbf{c} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{a}-\mathbf{c} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. You can obtain these results geometrically by drawing $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $-\mathbf{c}$ in standard position, and then using the head-to-tail law or parallelogram law for vector addition. (Note that $\mathbf{a}-\mathbf{c} = \mathbf{a}+(-\mathbf{c})$.)



(iii)



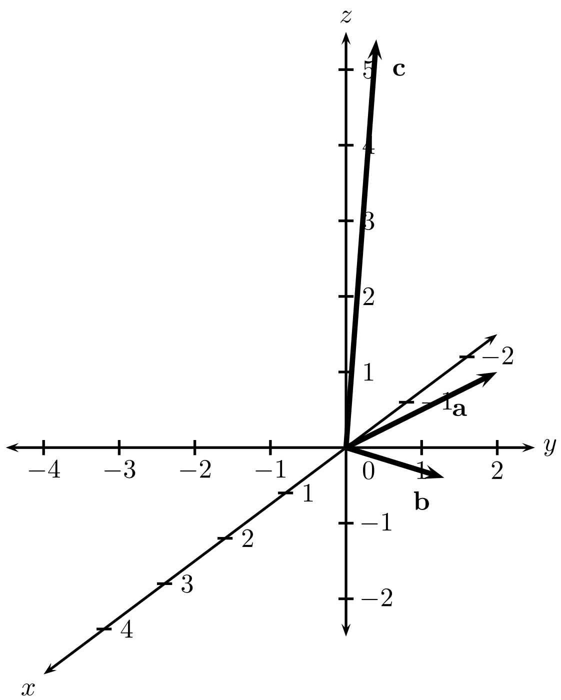
2. Let $\mathbf{a} = [0, 2, 1]$, $\mathbf{b} = [1, 2, \frac{1}{3}]$ and $\mathbf{c} = [-1, -\frac{1}{2}, 5]$.

(i) Draw these vectors in standard position in $\mathbb{R}^3$.

(ii) Compute the vectors $2\mathbf{a} + 3\mathbf{b}$ and $-\mathbf{a} + 4\mathbf{b} - \mathbf{c}$.

Solution:

(i) The difficulty of drawing vectors accurately even in $\mathbb{R}^3$ is an important reason for working algebraically when studying vectors in $\mathbb{R}^n$.



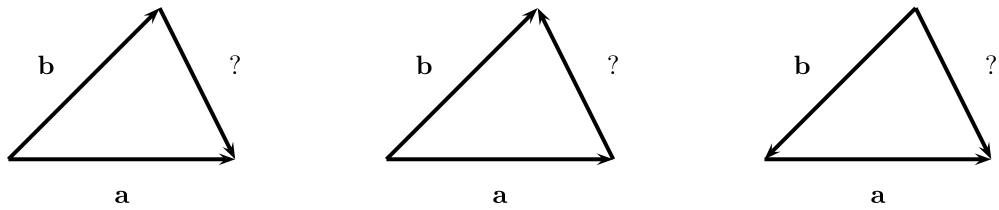
(ii) $2\mathbf{a} + 3\mathbf{b} = [3, 10, 3]$ and $-\mathbf{a} + 4\mathbf{b} - \mathbf{c} = [5, \frac{13}{2}, -\frac{14}{3}]$.

3. If the vector $\mathbf{v}$ has length 2, find the length of the vector $\mathbf{u}$ in each of the following cases.

(i) $\mathbf{u} = 3\mathbf{v}$ (ii) $\mathbf{u} = \frac{1}{2}\mathbf{v}$ (iii) $\mathbf{u} = -3\mathbf{v}$ (iv) $\mathbf{v} = 3\mathbf{u}$

Solution: (i) 6 (ii) 1 (iii) 6 (iv) 2/3

4. In each diagram below, find the unknown vector in terms of $\mathbf{a}$ and $\mathbf{b}$.



Solution: From left to right, $\mathbf{a} - \mathbf{b}$, $\mathbf{b} - \mathbf{a}$ and $\mathbf{a} + \mathbf{b}$.

5. Solve for $\mathbf{x}$ in terms of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in each case.

(i) $\mathbf{v} + \mathbf{x} = \mathbf{u} - \mathbf{w}$

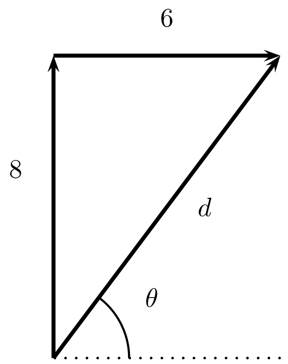
(ii) $\mathbf{v} - \mathbf{x} = \mathbf{w} - \mathbf{u}$

(iii) $2\mathbf{v} + \mathbf{x} = 2\mathbf{w} - 2\mathbf{u} - \mathbf{x}$

Solution: (i) $\mathbf{x} = \mathbf{u} - \mathbf{v} - \mathbf{w}$ (ii) $\mathbf{x} = \mathbf{u} + \mathbf{v} - \mathbf{w}$ (iii) $\mathbf{x} = -\mathbf{u} - \mathbf{v} + \mathbf{w}$

6. A balloon experiences two forces, a buoyancy force of 8 newtons vertically upwards and a wind force of 6 newtons acting horizontally to the right. Calculate the magnitude and direction of the resultant force.

Solution:



By Pythagoras $d = \sqrt{8^2 + 6^2} = 10$. If θ is the angle to the horizontal then $\cos\theta = 6/10$, yielding an angle $\theta \approx 53^\circ$. Thus the resultant force is 10 newtons in a direction 53° to the horizontal, towards the right.

7. * Prove the associative law for vector addition: for all vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in $\mathbb{R}^n$,

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}).$$

Solution: Let $\mathbf{a} = [a_1, a_2, \dots, a_n]$, $\mathbf{b} = [b_1, b_2, \dots, b_n]$ and $\mathbf{c} = [c_1, c_2, \dots, c_n]$. Then

$$\begin{aligned}
(\mathbf{a} + \mathbf{b}) + \mathbf{c} &= ([a_1, a_2, \dots, a_n] + [b_1, b_2, \dots, b_n]) + [c_1, c_2, \dots, c_n] \\
&= [a_1 + b_1, a_2 + b_2, \dots, a_n + b_n] + [c_1, c_2, \dots, c_n] \\
&= [(a_1 + b_1) + c_1, (a_2 + b_2) + c_2, \dots, (a_n + b_n) + c_n] \\
&= [a_1 + (b_1 + c_1), a_2 + (b_2 + c_2), \dots, a_n + (b_n + c_n)] \text{ since addition of scalars is associative} \\
&= [a_1, a_2, \dots, a_n] + [b_1 + c_1, b_2 + c_2, \dots, b_n + c_n] \\
&= [a_1, a_2, \dots, a_n] + ([b_1, b_2, \dots, b_n] + [c_1, c_2, \dots, c_n]) \\
&= \mathbf{a} + (\mathbf{b} + \mathbf{c})
\end{aligned}$$

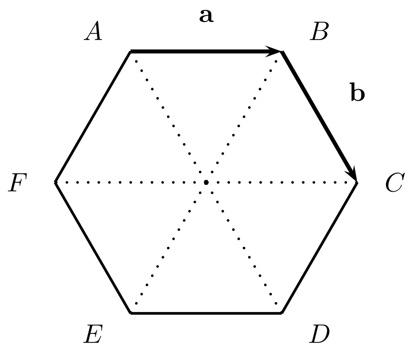
as required.

8. Express $2\mathbf{a} - 3\mathbf{b}$ in terms of $\mathbf{u}$ and $\mathbf{v}$, and simplify, when $\mathbf{a} = \mathbf{u} + \mathbf{v}$ and $\mathbf{b} = 3\mathbf{u} - 2\mathbf{v}$.

Solution: $2\mathbf{a} - 3\mathbf{b} = 2(\mathbf{u} + \mathbf{v}) - 3(3\mathbf{u} - 2\mathbf{v}) = 2\mathbf{u} + 2\mathbf{v} - 9\mathbf{u} + 6\mathbf{v} = -7\mathbf{u} + 8\mathbf{v}$.

9. Let $ABCDEF$ be a regular hexagon and put $\mathbf{a} = \overrightarrow{AB}$ and $\mathbf{b} = \overrightarrow{BC}$. Find vector expressions in terms of $\mathbf{a}$ and $\mathbf{b}$ for the displacements $\overrightarrow{CD}$, $\overrightarrow{DE}$, $\overrightarrow{EF}$ and $\overrightarrow{FA}$.

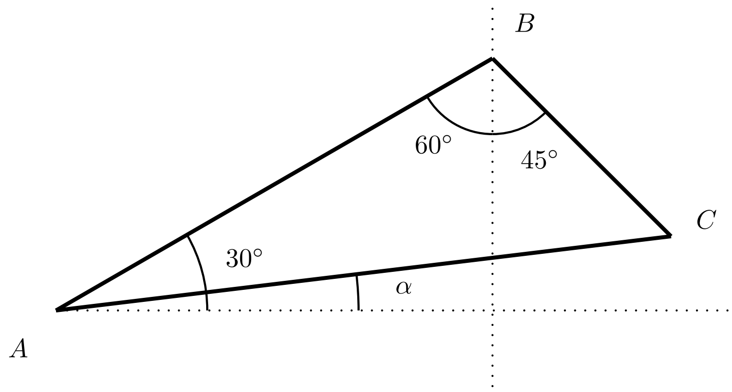
Solution:



We get $\overrightarrow{CD} = \mathbf{b} - \mathbf{a}$, $\overrightarrow{DE} = -\mathbf{a}$, $\overrightarrow{EF} = -\mathbf{b}$ and $\overrightarrow{FA} = \mathbf{a} - \mathbf{b}$.

10. A plane travels 20km in the direction 30° north of east and then 10 km southeast. Use trigonometry and your calculator to find the final distance and direction of the aircraft from the starting position.

Solution:



We have $|\overrightarrow{AB}| = 20$ and $|\overrightarrow{BC}| = 10$. By the Cosine Rule,

$$|\overrightarrow{AC}| = \sqrt{20^2 + 10^2 - 2(10)(20)\cos 105^\circ} \approx 25.$$

By the Sine Rule,

$$\sin(30^\circ - \alpha) = \frac{10 \sin 105^\circ}{|\overrightarrow{AC}|},$$

from which it follows that

$$30^\circ - \alpha \approx 23^\circ,$$

so that $\alpha \approx 7^\circ$. Hence the final distance and direction of the aircraft from the starting point are approximately 25 km and 7° north of east respectively.

11. * Prove the following distributive laws:

(i) For all scalars c and all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.

(ii) For all scalars c and d and all vectors $\mathbf{u}$ in $\mathbb{R}^n$, $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.

Solution:

(i) Let $\mathbf{u} = [u_1, u_2, \ldots, u_n]$ and $\mathbf{v} = [v_1, v_2, \ldots, v_n]$. Then

$$\begin{aligned} c(\mathbf{u} + \mathbf{v}) &= c([u_1, u_2, \ldots, u_n] + [v_1, v_2, \ldots, v_n]) \\ &= c[u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n] \\ &= [c(u_1 + v_1), c(u_2 + v_2), \ldots, c(u_n + v_n)]. \end{aligned}$$

On the other hand

$$\begin{aligned}
c\mathbf{u} + c\mathbf{v} &= c[u_1, u_2, \ldots, u_n] + c[v_1, v_2, \ldots, v_n] \\
&= [cu_1, cu_2, \ldots, cu_n] + [cv_1, cv_2, \ldots, cv_n] \\
&= [cu_1 + cv_1, cu_2 + cv_2, \ldots, cu_n + cv_n] \\
&= [c(u_1 + v_1), c(u_2 + v_2), \ldots, c(u_n + v_n)] \quad \text{using the distributive law for scalars.}
\end{aligned}$$

Since both $c(\mathbf{u} + \mathbf{v})$ and $c\mathbf{u} + c\mathbf{v}$ are equal to $[c(u_1 + v_1), c(u_2 + v_2), \ldots, c(u_n + v_n)]$, we have that $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ as required. General comment: sometimes it is easier to prove an equality by getting two expressions to equal the same thing, rather than trying to work on just one of them until you obtain the other.

(ii) Let $\mathbf{u} = [u_1, u_2, \ldots, u_n]$. Then

$$\begin{aligned}
(c + d)\mathbf{u} &= (c + d)[u_1, u_2, \ldots, u_n] \\
&= [(c + d)u_1, (c + d)u_2, \ldots, (c + d)u_n] \\
&= [cu_1 + du_1, cu_2 + du_2, \ldots, cu_n + du_n] \quad \text{using the distributive law for scalars} \\
&= [cu_1, cu_2, \ldots, cu_n] + [du_1, du_2, \ldots, du_n] \\
&= c[u_1, u_2, \ldots, u_n] + d[u_1, u_2, \ldots, u_n] \\
&= c\mathbf{u} + d\mathbf{u}
\end{aligned}$$

as required.