MATH1002 Linear Algebra

Topic 9A: Determinants and elementary row operations

Theorem Let A be an $n \times n$ matrix.

1. If B is obtained from A by swapping two rows then
$$\det(B) = -\det(A).$$
2. If B is obtained from A by multiplying one row by a scalar $c \in \mathbb{R}$, then
$$\det(B) = c \det(A).$$
3. If B is obtained from A by adding a multiple of one row of A to another row of A, then
$$\det(B) = \det(A).$$

Proof exercise. $\square$

We can use this result to compute determinants more efficiently.

Goal: get a row or column with lots of 0s, then compute determinant by expanding along this row or column.

Warning: you must keep track of the elementary row operations and their effect (if any) on the determinant

Example

Find $\det(A)$ where

$$A = \begin{bmatrix} 2 & 5 & 7 \\ 1 & 4 & 5 \\ 3 & 6 & 12 \end{bmatrix}.$$

$$\begin{bmatrix} 2 & 5 & 7 \\ 1 & 4 & 5 \\ 3 & 6 & 12 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 4 & 5 \\ 2 & 5 & 7 \\ 3 & 6 & 12 \end{bmatrix}$$

$$\xrightarrow[R_3 \mapsto R_3 - 3R_1]{R_2 \mapsto R_2 - 2R_1} \begin{bmatrix} 1 & 4 & 5 \\ 0 & -3 & -3 \\ 0 & -6 & -3 \end{bmatrix} \xrightarrow{R_2 \mapsto -\frac{1}{3}R_2} \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & -6 & -3 \end{bmatrix}$$

could stop here and compute using column 1 that

$$\det(A) = 1 \begin{vmatrix} -3 & -3 \\ -6 & -3 \end{vmatrix}$$

$$\xrightarrow{R_3 \mapsto R_3 + 6R_2} \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix} = B$$

Now $\det(B) = 1 \times 1 \times 3$
$= 3$

product of diagonal entries since B is triangular

Now

$$\det(B) = \underbrace{\left(-\frac{1}{3}\right)}_{\text{since we did } R_2 \mapsto -\frac{1}{3}R_2}\left(-\det(A)\right)$$

since we did $R_1 \leftrightarrow R_2$

so

$$3 = \frac{1}{3}\det(A)$$

so

$$\det(A) = 9.$$

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Topic 9B: Properties of determinants

Recall

$$\begin{array}{l} n\times n \text{ matrix } A \xrightarrow{R_i \leftrightarrow R_j} B \text{ with } \det(B) = -\det(A) \\ n\times n \text{ matrix } A \xrightarrow{R_i \mapsto cR_i} B \text{ with } \det(B) = c\det(A) \\ n\times n \text{ matrix } A \xrightarrow{R_i \mapsto R_i + cR_j} B \text{ with } \det(B) = \det(A) \end{array}$$

In each case above, $B = EA$ for E an elementary matrix.

$$\begin{array}{l} A \xrightarrow{R_i \leftrightarrow R_j} EA \text{ where } I \xrightarrow{R_i \leftrightarrow R_j} E \\ A \xrightarrow{R_i \mapsto cR_i} EA \text{ where } I \xrightarrow{R_i \mapsto cR_i} E \\ A \xrightarrow{R_i \mapsto R_i + cR_j} EA \text{ where } I \xrightarrow{R_i \mapsto R_i + cR_j} E \end{array}$$

From the above, we get:

Theorem Let E be an elementary matrix. Then:

(1) If E is obtained from I by swapping two rows then
$$\det(E) = -1.$$
(2) If E is obtained from I by multiplying a row by a scalar c, then
$$\det(E) = c.$$
(3) If E is obtained from I by adding a multiple of one row to another row, then
$$\det(E) = 1.$$

Proof Put $A = I$ above, then
$$\begin{aligned}
\det(E) &= \det(EI) \\
&= \det(EA) \\
&= \begin{cases} -\det(A) & \text{if } A \xrightarrow{R_i \leftrightarrow R_j} EA \\ c\det(A) & \text{if } A \xrightarrow{R_i \mapsto cR_i} EA \\ \det(A) & \text{if } A \xrightarrow{R_i \mapsto R_i + cR_j} EA \end{cases} \\
&= \begin{cases} -1 & \text{if (1) holds} \\ c & \text{if (2) holds} \\ 1 & \text{if (3) holds.} \end{cases} \quad \square
\end{aligned}$$

Corollary 1 If A is an $n \times n$ matrix and E is an $n \times n$ elementary matrix then
$$\underbrace{\det(EA)}_{\text{matrix multiplication}} = \underbrace{\det(E)\det(A)}_{\text{multiplication in } \mathbb{R}}.$$

Corollary 2 An $n \times n$ matrix A is invertible $\iff$ $\det(A) \neq 0$.

Proof Recall $A \xrightarrow[\text{ops}]{\text{elem. row}} R$

for R the reduced row echelon form of A.

Also A is invertible $\iff$ $R = I_n$.

We have

$$R = E_k \cdots E_2 E_1 A$$

where $E_1, \dots, E_k$ are elementary matrices.

So

$$\begin{aligned} \det(R) &= \det(E_k \cdots E_2 E_1 A) \\ &= \det(E_k) \det(E_{k-1} \cdots E_1 A) \\ &\qquad \text{by Corollary 1} \\ &= \det(E_k) \det(E_{k-1}) \cdots \det(E_1) \det(A) \\ &\qquad \text{by Corollary 1 applied } k \text{ times} \end{aligned}$$

By the theorem above,

$$\det(E_i) \neq 0$$

for $1 \leq i \leq k$. Hence

$$\det(E_k)\det(E_{k-1}) \cdots \det(E_1) \neq 0.$$

So

$$\det(R) = 0 \iff \det(A) = 0.$$

Equivalently,

$$\det(R) \neq 0 \iff \det(A) \neq 0.$$

Suppose A is invertible. Then $R = I_n$ so $\det(R) = 1$ so $\det(A) \neq 0$.

Suppose $\det(A) \neq 0$. Then $\det(R) \neq 0$ so the $n \times n$ matrix R cannot have an all-0 row. Thus $R = I_n$ so A is invertible. $\square$

Theorem (Further properties of the determinant)

Let A and B be $n \times n$ matrices and let $c \in \mathbb{R}$. Then:

(1) $\det(cA) = c^n \det(A)$.

(2) $\det(\underbrace{AB}_{\text{matrix multiplication}}) = \underbrace{\det(A)\det(B)}_{\text{multiplication in } \mathbb{R}}$.

(3) If A is invertible, then
$$\det(\underbrace{A^{-1}}_{\text{inverse of the matrix } A}) = \frac{1}{\det(A)} \Big\} \text{ scalar, nonzero}$$

(4) $\det(A) = \det(A^T)$.

Proof

(1) $A \longrightarrow cA$ involves n applications of row operation $R_i \mapsto cR_i$.

(2) If A is invertible, then

$$A = E_1 E_2 \cdots E_k$$

where $E_1, E_2, \ldots, E_k$ are elementary matrices.
Then

$$\begin{aligned} \det(AB) &= \det(E_1 E_2 \cdots E_k B) \\ &= \det(E_1 E_2 \cdots E_k)\det(B) \quad \text{by Corollary 1} \\ &= \det(A)\det(B). \end{aligned}$$

If A is not invertible, then AB is not invertible (why?).
So $\det(A) = 0$ and $\det(AB) = 0$
so

$$0 = \det(AB) \quad \text{and} \quad 0 = \det(A)\det(B)$$

hence

$$\det(AB) = \det(A)\det(B).$$

(3) Since A is invertible, $AA^{-1} = I$.
Thus

$$\det(AA^{-1}) = \det(I)$$

$$\det(A)\det(A^{-1}) = 1$$

$$\det(A^{-1}) = \frac{1}{\det(A)} \quad \text{since } \det(A) \neq 0$$

(4) $\det(A)$ can be found by expanding along any row of A; this is the same as expanding along any column of A^T. $\square$

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Topic 9C: Introduction to eigenvalues and eigenvectors

Def^n Let A be an $n \times n$ matrix. A scalar λ is called an eigenvalue of A if there is a nonzero vector $\underset{\sim}{x}$ so that

$$A\underset{\sim}{x} = \lambda \underset{\sim}{x}.$$

Such a vector $\underset{\sim}{x}$ is called an eigenvector of A corresponding to λ.

eigen = own

Examples

1. Show that $\underset{\sim}{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, and find the corresponding eigenvalue.

$$A\underset{\sim}{x} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3\underset{\sim}{x}.$$

So $\underset{\sim}{x}$ is an eigenvector, with eigenvalue $\lambda = 3$.

2. Find all eigenvectors of $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = 3$.

We are looking for all $\underset{\sim}{x} \in \mathbb{R}^2$ with $\underset{\sim}{x} \neq \underset{\sim}{0}$ and $A\underset{\sim}{x} = 3\underset{\sim}{x}$.

Now

$$
\begin{aligned}
& A\underset{\sim}{x} = 3\underset{\sim}{x} \\
\iff\ & A\underset{\sim}{x} - 3\underset{\sim}{x} = \underset{\sim}{0} \\
\iff\ & A\underset{\sim}{x} - 3I\underset{\sim}{x} = \underset{\sim}{0} \\
\iff\ & (A - 3I)\underset{\sim}{x} = \underset{\sim}{0}.
\end{aligned}
$$

Now

$$
\begin{aligned}
A - 3I &= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}
\end{aligned}
$$

so we consider augmented matrix

$$
\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \xrightarrow{R_2 \mapsto R_2 + R_1} \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]
$$

so we have $-x + y = 0$ $\left(\text{for } \underset{\sim}{x} = \begin{bmatrix} x \\ y \end{bmatrix}\right)$

so we let $y = t$, then $x = t$ and then solutions to this system are

$$
\left\{ \begin{bmatrix} t \\ t \end{bmatrix} : t \in \mathbb{R} \right\}
$$

Thus the solutions to $(A-3I)\underset{\sim}{x} = \underset{\sim}{0}$ are

$$\left\{ t\begin{bmatrix}1\\1\end{bmatrix} : t \in \mathbb{R} \right\}$$

$$= \operatorname{span}\left(\begin{bmatrix}1\\1\end{bmatrix}\right).$$

Remember that eigenvectors are non-zero. Then the eigenvectors of $A = \begin{bmatrix}2&1\\1&2\end{bmatrix}$

for eigenvalue $\lambda = 3$ are

$$\left\{ t\begin{bmatrix}1\\1\end{bmatrix} : t \in \mathbb{R} \text{ and } t \neq 0 \right\}.$$

<u>Def^n</u> Let λ be an eigenvalue of the $n \times n$ matrix A. The collection of all eigenvectors corresponding to λ, together with $\underset{\sim}{0}$, is the <u>λ-eigenspace</u> of A,

$$E_\lambda = \{ \underset{\sim}{x} \in \mathbb{R}^n : A\underset{\sim}{x} = \lambda \underset{\sim}{x} \}.$$

<u>Example</u> For $A = \begin{bmatrix}2&1\\1&2\end{bmatrix}$, the 3-eigenspace is

$$E_3 = \left\{ t\begin{bmatrix}1\\1\end{bmatrix} : t \in \mathbb{R} \right\} = \operatorname{span}\left(\begin{bmatrix}1\\1\end{bmatrix}\right)$$

<u>Note</u> $E_\lambda = \{ \underset{\sim}{x} \in \mathbb{R}^n : (A-\lambda I)\underset{\sim}{x} = \underset{\sim}{0} \}$

since

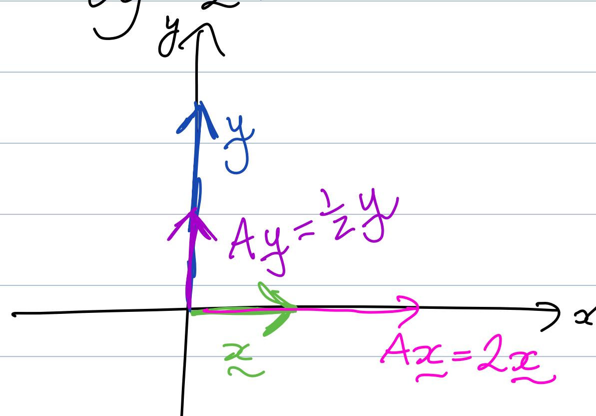
$$\begin{aligned} & A\underset{\sim}{x} = \lambda \underset{\sim}{x} \\ \iff\quad & A\underset{\sim}{x} - \lambda\underset{\sim}{x} = \underset{\sim}{0} \\ \iff\quad & A\underset{\sim}{x} - \lambda I\underset{\sim}{x} = \underset{\sim}{0} \\ \iff\quad & (A - \lambda I)\underset{\sim}{x} = \underset{\sim}{0}. \end{aligned}$$

Example Let $A = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$.

Then for $\underset{\sim}{x} = \begin{bmatrix} x \\ y \end{bmatrix}$,

$$A\underset{\sim}{x} = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ \frac{1}{2}y \end{bmatrix}$$

So A stretches vectors in the x-axis by 2, and compresses vectors in the y-axis by $\frac{1}{2}$!



The vectors $\begin{bmatrix} x \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ y \end{bmatrix}$ are the only ones which are sent to scalar multiples of themselves by A.

For any $x \neq 0$

$$A\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 2x \\ 0 \end{bmatrix} = 2\begin{bmatrix} x \\ 0 \end{bmatrix}$$

For any $y \neq 0$

$$A\begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2}y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ y \end{bmatrix}$$

So the eigenvalues of A are

$$\lambda_1 = 2 \quad , \quad \lambda_2 = \frac{1}{2}$$

with corresponding eigenspaces

$$E_{2} = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} : x \in \mathbb{R} \right\} = \operatorname{span}(\underset{\sim}{e}_1)$$

correction here

$$E_{\frac{1}{2}} = \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} : y \in \mathbb{R} \right\} = \operatorname{span}(\underset{\sim}{e}_2)$$

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Topic 9D: Characteristic polynomials

Recall: Let A be an $n \times n$ matrix.

An eigenvalue of A is a scalar $\lambda \in \mathbb{R}$ so that $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$ for some $\underset{\sim}{x} \neq \underset{\sim}{0}$.

An eigenvector of A is a vector $\underset{\sim}{x} \neq \underset{\sim}{0}$ so that $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$ for some $\lambda \in \mathbb{R}$.

Now $A\underset{\sim}{x} = \lambda \underset{\sim}{x} \iff (A - \lambda I)\underset{\sim}{x} = \underset{\sim}{0}$.

We proved earlier that a matrix B is invertible $\iff$ $B\underset{\sim}{x} = \underset{\sim}{0}$ has unique solution $\underset{\sim}{x} = \underset{\sim}{0}$

So:

$(A - \lambda I)$ is invertible $\iff$ $(A - \lambda I)\underset{\sim}{x} = \underset{\sim}{0}$ has unique solution $\underset{\sim}{x} = \underset{\sim}{0}$

or equivalently

$(A - \lambda I)$ is not invertible $\iff$ $(A - \lambda I)\underset{\sim}{x} = \underset{\sim}{0}$ has a solution $\underset{\sim}{x} \neq \underset{\sim}{0}$.

$$\Updownarrow$$

$\det(A - \lambda I) = 0$.

i.e. $\underset{\sim}{x}$ is an eigenvector for the eigenvalue λ

Theorem The eigenvalues of an $n\times n$ matrix A are the solutions λ to the equation

$$\det(A-\lambda I)=0.$$

Def^n The polynomial $\det(A-\lambda I)$ is called the characteristic polynomial of A, and the equation $\det(A-\lambda I)=0$ is called the characteristic equation.

Examples

Find all eigenvalues of $A=\begin{bmatrix}2&1\\1&2\end{bmatrix}$.

We must find all λ so that

$$\det(A-\lambda I)=0.$$

Now

$$A-\lambda I=\begin{bmatrix}2&1\\1&2\end{bmatrix}-\lambda\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$=\begin{bmatrix}2-\lambda&1\\1&2-\lambda\end{bmatrix}$$

so

$$\begin{aligned}\det(A-\lambda I)&=(2-\lambda)^2-1\\&=\lambda^2-4\lambda+4-1\\&=\lambda^2-4\lambda+3\\&=(\lambda-3)(\lambda-1).\end{aligned}$$

So the eigenvalues of A are $\lambda_1=3,\lambda_2=1$.

Recall: If λ is an eigenvalue for A, then the corresponding eigenspace is

$$E_\lambda = \{ \underset{\sim}{x} \in \mathbb{R}^n \mid A\underset{\sim}{x} = \lambda \underset{\sim}{x} \}.$$

Now $A\underset{\sim}{x} = \lambda \underset{\sim}{x} \iff (A - \lambda I)\underset{\sim}{x} = \underset{\sim}{0}$

To find an eigenspace, you solve system with augmented matrix

$$[A - \lambda I \mid \underset{\sim}{0}].$$

Example Find the 1-eigenspace for $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

$$[A - 1I \mid \underset{\sim}{0}] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \mapsto R_2 - R_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

$$x + y = 0$$

Put $y = t$ then $x = -t$.

$$E_1 = \left\{ \begin{bmatrix} -t \\ t \end{bmatrix} : t \in \mathbb{R} \right\}.$$

Procedure for finding eigenvalues and eigenvectors

1. Compute the characteristic polynomial $\det(A-\lambda I)$.
2. Find eigenvalues by solving characteristic equation $\det(A-\lambda I)=0$.
3. For each eigenvalue λ, solve $[A-\lambda I|\underline{0}]$. This gives the eigenspace E_λ.

Example

Find the eigenvalues and corresponding eigenspaces for

$$A=\begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The characteristic polynomial is

$$\det(A-\lambda I)=\begin{vmatrix} 1-\lambda & 2 & 0 \\ -1 & -1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix}$$

$$=(1-\lambda)\begin{vmatrix} -1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - (-1)\begin{vmatrix} 2 & 0 \\ 1 & 1-\lambda \end{vmatrix}$$

$$=(1-\lambda)\left[(-1-\lambda)(1-\lambda)-1\right] + 2(1-\lambda)$$

$$= (1-\lambda)\left[(-1-\lambda)(1-\lambda) - 1 + 2\right]$$
$$= (1-\lambda)\left[\lambda^2 - 1 - 1 + 2\right]$$
$$= \lambda^2(1-\lambda).$$

The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 1$.

To find eigenspace for $\lambda_1 = 0$:

$$[A + 0I \mid \underset{\sim}{0}] = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \mapsto R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \mapsto R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Put $z = t$, then $y + t = 0$ so $y = -t$.
Then $x + 2y = 0 \Rightarrow x - 2t = 0$
$\Rightarrow x = 2t.$

So

$$E_0 = \left\{ \begin{bmatrix} 2t \\ -t \\ t \end{bmatrix} : t \in \mathbb{R} \right\}.$$

To find eigenspace for $\lambda_2 = 1$:

$$[A - 1I \mid \underset{\sim}{0}] = \left[\begin{array}{ccc|c} 0 & 2 & 0 & 0 \\ -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{ccc|c} -1 & -2 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \mapsto R_3 - \frac{1}{2}R_2} \left[\begin{array}{ccc|c} -1 & -2 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Put $z = t$. Then $2y = 0$ so $y = 0$.

And $-x - 2y + z = 0$

so $-x + t = 0$

so $x = t$.

Thus

$$E_1 = \left\{ \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} : t \in \mathbb{R} \right\}.$$

<u>Theorem</u> The characteristic polynomial of an $n \times n$ matrix A is a degree n polynomial.

<u>Corollary</u> An $n \times n$ matrix has at most n distinct eigenvalues.

Proof A degree n polynomial has at most n distinct roots. $\square$

Theorem An $n \times n$ matrix A is invertible $\iff$ 0 is <u>not</u> an eigenvalue of A.

Proof A is invertible $\iff \det(A) \neq 0$

$\iff \det(A - 0I) \neq 0$

$\iff 0$ is <u>not</u> a solution to $\det(A - \lambda I) = 0$

$\iff 0$ is <u>not</u> an eigenvalue. $\square$