MATH1002 Linear Algebra

Topic 6A: Matrices

Def^n A matrix is a rectangular array of numbers, called the entries or elements of the matrix.

A matrix has size $\underset{\text{no. of rows}}{m} \times \underset{\text{no. of columns}}{n}$ if it has m rows and n columns.

A $1 \times n$ matrix is a row matrix or row vector.

An $m \times 1$ matrix is a column matrix or column vector.

Examples

$A = \begin{bmatrix} 2 & 0 & 5 \\ 1 & 4 & -1 \end{bmatrix}$ is a 2×3 matrix

$B = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}$ is a 1×3 matrix, i.e. a row matrix, a row vector

$C = \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}$ is a 3×1 matrix i.e. a column matrix, a column vector.

$$D = \begin{bmatrix} 1 & 2 \\ 4 & 0 \\ 6 & -1 \end{bmatrix}$$ is a 3×2 matrix.

The entry of a matrix A in row i and column j will be referred to as a_{ij}.

Example

$$A = \begin{bmatrix} 2 & 0 & 5 \\ 1 & 4 & -1 \end{bmatrix}$$ has

$a_{11} = 2$
$a_{12} = 0$
$a_{13} = 5$
$a_{21} = 1$
$a_{22} = 4$
$a_{23} = -1$.

If A is an $m \times n$ matrix, we can write

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

or for short

$$A = [a_{ij}].$$

If the columns of A are the column vectors $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_n$ we sometimes write

$$A = \begin{bmatrix} \underset{\sim}{a}_1 & \underset{\sim}{a}_2 & \cdots & \underset{\sim}{a}_n \end{bmatrix}$$

and if rows of A are row vectors $\underset{\sim}{A}_1, \underset{\sim}{A}_2, \ldots, \underset{\sim}{A}_m$ we sometimes write

$$A = \begin{bmatrix} \underset{\sim}{A}_1 \\ \underset{\sim}{A}_2 \\ \vdots \\ \underset{\sim}{A}_m \end{bmatrix}.$$

If $m = n$ we say A is a square matrix, and that its diagonal entries are $a_{11}, a_{22}, \ldots, a_{nn}$.
(read from top left to bottom right)

Examples

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}$$ is square, with

diagonal entries $a_{11} = 1$, $a_{22} = 4$.

$$B = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 7 \end{bmatrix}$$ is not square.

A square matrix is diagonal if all its off-diagonal entries are 0.

Examples

$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ is diagonal.

$B = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ is diagonal.

$C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is not diagonal.

The $n \times n$ identity matrix I_n is the diagonal matrix with all diagonal entries equal to 1.

Examples

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, ...

Examples

$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 4 \\ 0 & 0 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

A, B, C are all square
A has diagonal entries 1, 4, 8, but
is not a diagonal matrix.
B, C are both diagonal matrices.
$C = I_3$.

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Topic 6B: Matrix addition and scalar multiplication

Addition

If A and B are $m \times n$ matrices then $A+B$ is the $m \times n$ matrix with (i,j)-entry given by

$$a_{ij} + b_{ij}.$$

Added later: ie. add together the corresponding entries of A and B

Examples

1. $A = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$

then

$$A+B = \begin{bmatrix} 1+2 & 0+(-1) \\ 2+3 & 4+(-2) \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 5 & 2 \end{bmatrix}$$

2. $A = \begin{bmatrix} -1 & 2 & 0 \\ 3 & -1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 5 & 2 \\ -1 & 4 & 0 \end{bmatrix}$

$$A+B = \begin{bmatrix} (-1)+0 & 2+5 & 0+2 \\ 3+(-1) & (-1)+4 & 5+0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 7 & 2 \\ 2 & 3 & 5 \end{bmatrix}$$

Note We can only add together matrices which are the same size i.e. same number of rows and same number of columns.

Scalar multiplication

If A is a matrix and c is a scalar then cA is the matrix where each entry of A is multiplied by c.

Examples

1. $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$ $\quad (-1)A = \begin{bmatrix} -2 & -1 \\ 1 & -3 \end{bmatrix} = -A$

$2A = \begin{bmatrix} 4 & 2 \\ -2 & 6 \end{bmatrix}$, $\quad -5A = \begin{bmatrix} -10 & -5 \\ 5 & -15 \end{bmatrix}$.

We write $-A$ for the matrix $(-1)A$. Then

$$A - B = A + (-B)$$

for $m \times n$ matrices A and B.

A matrix with all entries 0 is called a zero matrix.

Theorem (Properties of matrix addition and scalar multiplication)
Let A, B and C be $m \times n$ matrices. Let c and d be scalars. Then:

1. $A + B = B + A$ (commutative law)
2. $(A+B)+C = A + (B+C)$ (associative law)
3. $A + O_{m\times n} = A$
   where $O_{m\times n}$ is the $m \times n$ zero matrix
4. $A + (-A) = O_{m\times n}$
5. $c(A+B) = cA + cB$
6. $(c+d)A = cA + dA$

   5–6: distributive laws
7. $c(dA) = (cd)A$
8. $1A = A$.

Proof Exercise. All these properties hold for addition and multiplication in $\mathbb{R}$, and matrix addition and scalar multiplication are defined on the entries of the matrix, which are real numbers.

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Topic 6C: Matrix multiplication

number of columns of A = number of rows of B

If A is an $m \times n$ matrix and B is $n \times r$ then AB is the $m \times r$ matrix with (i,j)-entry given by taking the dot product of row i of A with column j of B.

Examples

1. If $A = \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix}}_{2\times 2}$ and $B = \underbrace{\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}}_{2\times 2}$

$m = n = r = 2$

$$AB = \begin{bmatrix} (\text{row } 1 \text{ of } A)\cdot(\text{col } 1 \text{ of } B) & (\text{row } 1 \text{ of } A)\cdot(\text{col } 2 \text{ of } B) \\ (\text{row } 2 \text{ of } A)\cdot(\text{col } 1 \text{ of } B) & (\text{row } 2 \text{ of } A)\cdot(\text{col } 2 \text{ of } B) \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 2 + 0\times(-1) & 1\times 1 + 0\times 4 \\ 2\times 2 + (-3)\times(-1) & 2\times 1 + (-3)\times 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 7 & -10 \end{bmatrix}$$

2. $A = \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 & 4 \\ 1 & -1 & 0 \end{bmatrix}$

2×2 $\qquad$ 2×3

$$AC = \begin{bmatrix} R1\cdot C1 & R1\cdot C2 & R1\cdot C3 \\ R2\cdot C1 & R2\cdot C2 & R2\cdot C3 \end{bmatrix}$$

2×3

$$= \begin{bmatrix} 1 & 2 & 4 \\ -1 & 7 & 8 \end{bmatrix}$$

3. $D = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 4 & 2 \end{bmatrix}$ $\quad E = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & 2 & 4 \\ 0 & 1 & 0 & 3 \end{bmatrix}$

2×3 $\quad 3\times 4$

corrections here

$$DE = \begin{bmatrix} 1-4+0 & -2+1 & -1-4 & 1-8+3 \\ 0+8+0 & 6 & 8 & 16+6 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -1 & -5 & -4 \\ 8 & 6 & 8 & 22 \end{bmatrix}.$$

Note If A is $m\times n$ then we can only do the multiplication AB for B an $n\times r$ matrix
i.e. if no. of columns of A is not equal to no. of rows of B, the product AB is not defined.

Examples For $\overset{2\times2}{A}, \overset{2\times2}{B}, \overset{2\times3}{C}, \overset{2\times3}{D}, \overset{3\times4}{E}$ as above
We can do
$AB, BA, AC, BC, AD, BD, CE, DE$

but no other products of pairs of these.

Warning! Even when AB and BA are both defined, they are not in general equal. Matrix multiplication is not commutative.

Example If $A = \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$

then $AB = \begin{bmatrix} 2 & 1 \\ 7 & -10 \end{bmatrix}$.

correction

Now $BA = \begin{bmatrix} 4 & -3 \\ 7 & -12 \end{bmatrix} \neq AB$.

Note If A is $m \times n$ and B is $p \times q$ then AB is defined $\iff n = p$ and BA is defined $\iff q = m$.

So AB and BA are both defined $\iff$ A is $m \times n$ and B is $n \times m$.

Warning! You can have $AA = O$ the zero matrix, even if A is not the zero matrix.

Example $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ then

$$AA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Matrix Powers

If A is a square matrix ie. A is $n \times n$ then define

$$A^2 = AA$$

and for $k \geqslant 1$ an integer, define

$$A^k = \underbrace{AA \cdots A}_{k \text{ times}}.$$

So $A^1 = A$. Also define $A^0 = I_n$.

Example If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

then

$$A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

Can prove by induction that for $k \geqslant 1$

$$A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}.$$

Theorem (Properties of matrix multiplication)
Let A, B and C be matrices, and let $k \in \mathbb{R}$.
Then when the following products are defined:

1. $A(BC) = (AB)C$ (associativity)
2. $A(B+C) = AB + AC$ (distributivity)
3. $(A+B)C = AC + BC$ (")
4. $k(AB) = (kA)B = A(kB)$
5. $I A = A I_n = A$ if A is $m \times n$.

Proof exercise.

Example for 5.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 3 \end{bmatrix} \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad 2\times 2$$

2×3

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad 3\times 3$$

$$\underset{2\times 3}{A I_3} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 3 \end{bmatrix} = A$$

$$I_2 A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 3 \end{bmatrix} = A.$$

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Topic 6D: Transpose of a matrix

Def^n The *transpose* of an $m \times n$ matrix A is the $n \times m$ matrix A^T obtained by interchanging the rows and columns of A ie.

row i of A = column i of A^T
column j of A = row j of A^T

Examples

If $A = \underbrace{\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}}_{2\times 2}$ then $A^T = \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}_{2\times 2}$.

If $B = \underbrace{\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}}_{2\times 3}$ then $B^T = \underbrace{\begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 1 & 4 \end{bmatrix}}_{3\times 2}$.

In terms of matrix entries, the (i,j)-entry of A is the (j,i)-entry of A^T.

ie. if $A = (a_{ij})$ then $A^T = (a_{ji})$

Def^n A square matrix A is *symmetric* if $A = A^T$ ie. A equals its own transpose.

Examples

$A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$ is not symmetric as

$$A^T = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \neq A.$$

$B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 7 \\ 1 & 7 & 5 \end{bmatrix}$ is symmetric.

In terms of matrix entries, A is symmetric $\Longleftrightarrow$ $a_{ij} = a_{ji}$ for all i, j

Theorem (Properties of the transpose).
Let A and B be matrices, let $k \in \mathbb{R}$.
When the following operations can be performed:

1. $(A^T)^T = A$.
2. $(A+B)^T = A^T + B^T$
3. $(kA)^T = kA^T$
4. $(AB)^T = B^T A^T$
5. $(A^m)^T = (A^T)^m$ for all integers $m \geq 0$.

Proof of 1, 2, 3, 5 exercise.

Proof of 4:

Suppose A is $m \times n$, B is $n \times r$.
Then AB is $m \times r$. So $(AB)^T$ is $r \times m$.
Also B^T is $r \times n$ and A^T is $n \times m$,
so $B^T A^T$ is $r \times m$ matrix.

To see that $(AB)^T$ has same entries as B^TA^T:

The (i,j)-entry of $(AB)^T$
$= \text{the } (j,i)\text{-entry of } AB$
$= (\text{row } j \text{ of } A) \cdot (\text{col } i \text{ of } B)$
$= (\text{col } j \text{ of } A^T) \cdot (\text{row } i \text{ of } B^T)$
$= (\text{row } i \text{ of } B^T) \cdot (\text{col } j \text{ of } A^T)$
$= (i,j)\text{-entry of } B^TA^T.$

Thus $(AB)^T = B^TA^T.$ $\square$