THE UNIVERSITY OF SYDNEY

MATH1002 LINEAR ALGEBRA

Semester 1 **Tutorial Exercises for Week 4 — Solutions** 2022

1. Given that $\mathbf{a} = [2, -1, 0]$, $\mathbf{b} = [1, 1, 1]$ and $\mathbf{c} = [-2, 0, 1]$ find $\mathbf{a} \times \mathbf{b}$ and $\mathbf{a} \times \mathbf{c}$, and then use these answers and properties of the cross product to find:

(i) $\mathbf{b} \times \mathbf{a}$

(ii) $\mathbf{a} \times \big(\mathbf{a} + \mathbf{c}\big)$

(iii) $\big(\mathbf{a} \times \mathbf{a}\big) \times \mathbf{c}$

(iv) $\mathbf{a} \times \big(\mathbf{b} - 2\mathbf{c}\big)$

(v) the sine of the angle between $\mathbf{a}$ and $\mathbf{b}$

(vi) the area of the parallelogram inscribed by $\mathbf{a}$ and $\mathbf{c}$

Solution: We have $\mathbf{a} \times \mathbf{b} = [-1, -2, 3]$ and $\mathbf{a} \times \mathbf{c} = [-1, -2, -2]$. Now

(i) $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b} = [1, 2, -3]$.

(ii) $\mathbf{a} \times \big(\mathbf{a} + \mathbf{c}\big) = \mathbf{a} \times \mathbf{a} + \mathbf{a} \times \mathbf{c} = \mathbf{0} + \mathbf{a} \times \mathbf{c} = [-1, -2, -2]$.

(iii) $\big(\mathbf{a} \times \mathbf{a}\big) \times \mathbf{c} = \mathbf{0} \times \mathbf{c} = \mathbf{0}$.

(iv) $\mathbf{a} \times \big(\mathbf{b} - 2\mathbf{c}\big) = \mathbf{a} \times \mathbf{b} - 2(\mathbf{a} \times \mathbf{c}) = [1, 2, 7]$.

(v) $\sin\theta = \frac{\|\mathbf{a}\times\mathbf{b}\|}{\|\mathbf{a}\|\|\mathbf{b}\|} = \frac{\sqrt{14}}{\sqrt{5}\sqrt{3}} = \sqrt{\frac{14}{15}}$.

(vi) $\|\mathbf{a} \times \mathbf{c}\| = 3$.

2. Find two unit vectors orthogonal to both $\mathbf{v}$ and $\mathbf{w}$, where $\mathbf{v} = [1, 2, -7]$ and $\mathbf{w} = [5, 1, 1]$.

Solution: Observe that

$$\mathbf{v} \times \mathbf{w} = [9, -36, -9] = 9[1, -4, -1]$$

which has length $9\sqrt{1 + 16 + 1} = 9\sqrt{18} = 27\sqrt{2}$. Hence two unit vectors orthogonal to both $\mathbf{v}$ and $\mathbf{w}$ are

$$\pm\frac{\sqrt{2}}{6}[1, -4, -1].$$

3. Given that $P = (8, 4, -1)$, $Q = (6, 3, -4)$, and $R = (7, 5, -5)$, find $\overrightarrow{QP} \times \overrightarrow{QR}$ and the area of $\triangle PQR$.

Solution: We have

$$\overrightarrow{QP} \times \overrightarrow{QR} = [-7, 5, 3].$$

Hence the area of the triangle $\triangle PQR$ is

$$\frac{1}{2}\|[-7, 5, 3]\| = \frac{\sqrt{83}}{2}.$$

4. Let $\mathbf{a}$ and $\mathbf{b}$ be vectors in $\mathbb{R}^3$. Calculate $\|\mathbf{a} \times \mathbf{b}\|$ given that $\|\mathbf{a}\| = 7$, $\|\mathbf{b}\| = 4$ and $\mathbf{a} \cdot \mathbf{b} = -21$.

Solution: Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$. Then

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \frac{-21}{28} = -\frac{3}{4},$$

so $\sin\theta = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}$. Hence

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\|\sin\theta = 7\sqrt{7}.$$

5. Find a vector form and parametric equations for the line passing through P in the direction of $\mathbf{v}$ in each of the following cases:

(i) $P = (-6, 5), \quad \mathbf{v} = [2, 1]$

(ii) $P = (1, 0), \quad \mathbf{v} = [2, 2]$

(iii) $P = (0, 1, -1), \quad \mathbf{v} = [1, 2, 0]$

(iv) $P = (2, 3, -3), \quad \mathbf{v} = [-1, 0, 0]$

Solution:

(i) A vector form is: $\mathbf{x} = [-6, 5] + t[2, 1]$ where $t \in \mathbb{R}$.
Parametric equations:
$$x = -6 + 2t, \quad y = 5 + t, \quad \text{where } t \in \mathbb{R}.$$

(ii) A vector form is: $\mathbf{x} = [1, 0] + t[2, 2]$ where $t \in \mathbb{R}$.
Parametric equations:
$$x = 1 + 2t, \quad y = 2t, \quad \text{where } t \in \mathbb{R}.$$

(iii) A vector form is: $\mathbf{x} = [0, 1, -1] + t[1, 2, 0]$ where $t \in \mathbb{R}$.
Parametric equations:
$$x = t, \quad y = 1 + 2t, \quad z = -1, \quad \text{where } t \in \mathbb{R}.$$

(iv) A vector form is: $\mathbf{x} = [2, 3, -3] + t[-1, 0, 0]$ where $t \in \mathbb{R}$.
Parametric equations:
$$x = 2 - t, \quad y = 3, \quad z = -3, \quad \text{where } t \in \mathbb{R}.$$

6. Find a vector form and parametric equations for the line passing through P and Q in each of the following cases:

(i) $P = (3, 1), \quad Q = (5, 4)$.
Also find a normal form and a general equation for the line passing through P and Q for this case.

(ii) $P = (-4, 3, 5), \quad Q = (-2, 4, -1)$

Solution:

(i) We find $\overrightarrow{PQ} = [2, 3]$. Then a vector form is $\mathbf{x} = [3, 1] + t[2, 3]$ where $t \in \mathbb{R}$.
Parametric equations:
$$x = 3 + 2t, \quad y = 1 + 3t, \quad \text{where } t \in \mathbb{R}.$$
Let $\mathbf{n} = [a, b]$ be a normal vector for the line passing through P and Q. Then $\mathbf{n}$ is orthogonal to $\overrightarrow{PQ}$, i.e.
$$\mathbf{n}.\overrightarrow{PQ} = 0.$$
This implies $2a + 3b = 0$. If we choose $a = 3$ then $b = -2$, we obtain a normal vector $\mathbf{n} = [3, -2]$ for the line. So, a normal form is
$$[3, -2].\,([x, y] - [3, 1]) = 0.$$
Hence, a general equation is $3x - 2y = 7$.

(ii) We find $\overrightarrow{PQ} = [2, 1, -6]$. Then a vector form is $\mathbf{x} = [-4, 3, 5] + t[2, 1, -6]$ where $t \in \mathbb{R}$.
Parametric equations:
$$x = -4 + 2t, \quad y = 3 + t, \quad z = 5 - 6t, \quad \text{where } t \in \mathbb{R}.$$

7. * Two lines in $\mathbb{R}^3$ are *skew* if they are not parallel and do not intersect. Show that the following lines are not skew.

$$\ell_1 : \quad \mathbf{x} = [1,1,1] + t[3,-1,4] \quad \text{and} \quad \ell_2 : \quad \mathbf{x} = [6,-6,1] + s[-7,5,-6]$$

Solution: These lines are not parallel since their direction vectors $[3,-1,4]$ and $[-7,5,-6]$ are not parallel. We now determine whether these lines intersect. The parametric equations for ℓ_1 are

$$x = 1+3t, \quad y = 1-t, \quad z = 1+4t, \quad \text{where } t \in \mathbb{R}$$

and the parametric equations for ℓ_2 are

$$x = 6-7s, \quad y = -6+5s, \quad z = 1-6s, \quad \text{where } s \in \mathbb{R}.$$

So we need to see whether there is a solution to the system

$$\begin{aligned} 1+3t &= 6-7s \\ 1-t &= -6+5s \\ 1+4t &= 1-6s. \end{aligned}$$

Solving the first two equations simultaneously gives $t = -3$ and $s = 2$. This is also a solution to the third equation, so these two lines do intersect (at the point $(-8,4,-11)$, found by subbing $t = -3$ into the parametric equations for ℓ_1 or subbing $s = 2$ into the parametric equations for ℓ_2). Hence these lines are not skew.

8. * Prove the following distributive law: for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^3$, we have

$$(\mathbf{u}+\mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}.$$

Solution: Let $\mathbf{u} = [u_1,u_2,u_3]$, $\mathbf{v} = [v_1,v_2,v_3]$, and $\mathbf{w} = [w_1,w_2,w_3]$. Then

$$\begin{aligned} (\mathbf{u}+\mathbf{v}) \times \mathbf{w} &= ([u_1,u_2,u_3] + [v_1,v_2,v_3]) \times [w_1,w_2,w_3] \\ &= [u_1+v_1, u_2+v_2, u_3+v_3] \times [w_1,w_2,w_3] \\ &= [(u_2+v_2)w_3 - (u_3+v_3)w_2, (u_3+v_3)w_1 - (u_1+v_1)w_3, (u_1+v_1)w_2 - (u_2+v_2)w_1] \\ &= [u_2w_3 - u_3w_2 + v_2w_3 - v_3w_2, u_3w_1 - u_1w_3 + v_3w_1 - v_1w_3, u_1w_2 - u_2w_1 + v_1w_2 - v_2w_1] \\ &= [u_2w_3 - u_3w_2, u_3w_1 - u_1w_3, u_1w_2 - u_2w_1] + [v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1] \\ &= \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w} \end{aligned}$$

as required.

9. * Verify that, for any vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ in $\mathbb{R}^3$

$$\big(\mathbf{a} \times \mathbf{b}\big) \times \mathbf{c} = \big(\mathbf{a} \cdot \mathbf{c}\big)\mathbf{b} - \big(\mathbf{b} \cdot \mathbf{c}\big)\mathbf{a}.$$

Solution: Put $\mathbf{a} = [a_1,a_2,a_3]$, $\mathbf{b} = [b_1,b_2,b_3]$, and $\mathbf{c} = [c_1,c_2,c_3]$. Then

$$\begin{aligned} \big(\mathbf{a} \times \mathbf{b}\big) \times \mathbf{c} &= \big[a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\big] \times \mathbf{c} \\ &= \big[(a_3b_1 - a_1b_3)c_3 - (a_1b_2 - a_2b_1)c_2, (a_1b_2 - a_2b_1)c_1 - (a_2b_3 - a_3b_2)c_3, \\ &\quad (a_2b_3 - a_3b_2)c_2 - (a_3b_1 - a_1b_3)c_1\big] \\ &= \big[(a_2c_2 + a_3c_3)b_1, (a_1c_1 + a_3c_3)b_2, (a_1c_1 + a_2c_2)b_3\big] - \\ &\quad \big[(b_2c_2 + b_3c_3)a_1, (b_1c_1 + b_3c_3)a_2, (b_1c_1 + b_2c_2)a_3\big] \\ &= \big[(a_1c_1 + a_2c_2 + a_3c_3)b_1, (a_1c_1 + a_2c_2 + a_3c_3)b_2, (a_1c_1 + a_2c_2 + a_3c_3)b_3\big] - \\ &\quad \big[(b_1c_1 + b_2c_2 + b_3c_3)a_1, (b_1c_1 + b_2c_2 + b_3c_3)a_2, (b_1c_1 + b_2c_2 + b_3c_3)a_3\big] \\ &= \big(\mathbf{a} \cdot \mathbf{c}\big)\mathbf{b} - \big(\mathbf{b} \cdot \mathbf{c}\big)\mathbf{a}. \end{aligned}$$