The University of Sydney

MATH1002 Linear Algebra

Semester 1 **Tutorial Exercises for Week 11 — Solutions** 2022

**1.** Write down the eigenvalues immediately for the following triangular matrices, and then find all of the corresponding eigenspaces. Write down the algebraic and geometric multiplicities of each eigenvalue.

(i) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix}$ (iii) $A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{bmatrix}$

***Solution:***

(i) The only eigenvalue of $A$ is $\lambda = 1$, and it has algebraic multiplicity 2.
To find the 1-eigenspace of $A$, we solve

$$\big[A - I \mid \mathbf{0}\big] = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

and we get $y = 0$ and $x = t$, for $t \in \mathbb{R}$. So the 1-eigenspace of $A$ is

$$E_1 = \left\{ t\begin{bmatrix} 1 \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\} = \operatorname{span}\left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right).$$

The geometric multiplicity of the eigenvalue 1 is 1 as the number of free parameters in $E_1$ is 1.

(ii) The eigenvalues of $A$ are $\lambda = -1$ and $\lambda = 2$, which each have algebraic multiplicity 1.
To find the $(-1)$-eigenspace of $A$, we solve

$$\big[A + I \mid \mathbf{0}\big] = \left[\begin{array}{cc|c} 3 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \mapsto \frac{1}{3}R_1} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \mapsto R_2 + R_1} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

and we get $x = 0$ and $y = t$, for $t \in \mathbb{R}$. So the $(-1)$-eigenspace of $A$ is

$$E_{-1} = \left\{ t\begin{bmatrix} 0 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\} = \operatorname{span}\left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right),$$

and the geometric multiplicity of the eigenvalue $-1$ is 1.
To find the 2-eigenspace of $A$, we solve

$$\big[A - 2I \mid \mathbf{0}\big] = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ -1 & -3 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} -1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \mapsto -R_1} \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

and we get $y = t$ and $x = -3t$, for $t \in \mathbb{R}$. So the 2-eigenspace of $A$ is

$$E_2 = \left\{ t\begin{bmatrix} -3 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\} = \operatorname{span}\left( \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right)$$

and the geometric multiplicity of the eigenvalue 2 is 1.

(iii) The eigenvalues of $A$ are $\lambda = 3$ and $\lambda = 5$. The algebraic multiplicity of $\lambda = 3$ is 2, and the algebraic multiplicity of $\lambda = 5$ is 1.
To find the 3-eigenspace of $A$, we solve

$$\big[A - 3I \mid \mathbf{0}\big] = \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \xrightarrow{R_3 \mapsto R_3 - 2R_2} \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

and we get $z = 0$, $y = 0$, and $x = t$, for $t \in \mathbb{R}$. So the 3-eigenspace of $A$ is

$$E_3 = \left\{ t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\} = \operatorname{span}\left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right),$$

and the geometric multiplicity of the eigenvalue 3 is 1.

To find the 5-eigenspace of $A$, we solve

$$[A - 5I \mid \mathbf{0}] = \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

and we get $z = 4t$, $y = 2t$, and $x = 3t$, for $t \in \mathbb{R}$. So the 5-eigenspace of $A$ is

$$E_5 = \left\{ t \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} : t \in \mathbb{R} \right\} = \operatorname{span}\left( \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} \right),$$

and the geometric multiplicity of the eigenvalue 5 is 1.

**2.** Find all of the eigenvalues and eigenspaces for the following matrices. Write down the algebraic and geometric multiplicities of each eigenvalue.

(i) $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ (ii) * $B = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$ (iii) * $C = \begin{bmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{bmatrix}$

[*Hint for (ii) and (iii):* After subtracting $\lambda I$ from the matrix, perform the row operation $R_1 \mapsto R_1 - R_2$ before expanding the determinant.]

***Solution:***

(i) $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) + 1 = \lambda^2$, so the only eigenvalue is $\lambda = 0$, and it has algebraic multiplicity 2.

To find the 0-eigenspace of $A$, we solve

$$[A - 0I \mid \mathbf{0}] = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right] \xrightarrow{R_2 \mapsto R_2 - R_1} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

and we get $y = t$ and $x = t$, for $t \in \mathbb{R}$. So the 0-eigenspace of $A$ is

$$E_0 = \left\{ t \begin{bmatrix} 1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\} = \operatorname{span}\left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right),$$

and the geometric multiplicity of the eigenvalue 0 is 1.

(ii) By performing the row operation $R_1 \mapsto R_1 - R_2$ to the matrix $B - \lambda I$ (which does not change the determinant) and then expanding along the first row, we see that

$$\det(B - \lambda I) = \begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix} = \begin{vmatrix} -2-\lambda & 2+\lambda & 0 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix}$$

$$= (-2-\lambda) \begin{vmatrix} -5-\lambda & 3 \\ -6 & 4-\lambda \end{vmatrix} - (2+\lambda) \begin{vmatrix} 3 & 3 \\ 6 & 4-\lambda \end{vmatrix} = (-2-\lambda)(\lambda^2 - 2\lambda - 8) = (-2-\lambda)^2(4-\lambda).$$

Hence $B$ has eigenvalues $\lambda = -2$ and $\lambda = 4$. The algebraic multiplicity of $\lambda = -2$ is 2, and the algebraic multiplicity of $\lambda = 4$ is 1.

To find the $(-2)$-eigenspace of $B$, we solve

$$[B + 2I \mid \mathbf{0}] = \left[\begin{array}{ccc|c} 3 & -3 & 3 & 0 \\ 3 & -3 & 3 & 0 \\ 6 & -6 & 6 & 0 \end{array}\right] \xrightarrow[R_3 \mapsto R_3 - 2R_1]{R_2 \mapsto R_2 - R_1} \left[\begin{array}{ccc|c} 3 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

and we get $z = t$, $y = s$, and $x = s - t$, for $s, t \in \mathbb{R}$. So the $(-2)$-eigenspace of $B$ is

$$E_{-2} = \left\{ s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\} = \operatorname{span}\left( \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right),$$

and the geometric multiplicity of the eigenvalue $-2$ is 2.

To find the 4-eigenspace of $B$, we solve

$$\left[B-4I \mid \mathbf{0}\right]=\left[\begin{array}{ccc|c}-3 & -3 & 3 & 0\\ 3 & -9 & 3 & 0\\ 6 & -6 & 0 & 0\end{array}\right]\xrightarrow[R_3\mapsto R_3+2R_1]{R_2\mapsto R_2+R_1}\left[\begin{array}{ccc|c}-3 & -3 & 3 & 0\\ 0 & -12 & 6 & 0\\ 0 & -12 & 6 & 0\end{array}\right]$$

$$\xrightarrow{R_3\mapsto R_3-R_2}\left[\begin{array}{ccc|c}-3 & -3 & 3 & 0\\ 0 & -12 & 6 & 0\\ 0 & 0 & 0 & 0\end{array}\right],$$

and we get $z=2t$, $y=t$, and $x=t$, for $t\in\mathbb{R}$. So the 4-eigenspace of $B$ is

$$E_4=\left\{t\begin{bmatrix}1\\1\\2\end{bmatrix} : t\in\mathbb{R}\right\}=\operatorname{span}\left(\begin{bmatrix}1\\1\\2\end{bmatrix}\right),$$

and the geometric multiplicity of the eigenvalue 4 is 1.

(iii) By performing the row operation $R_1\mapsto R_1-R_2$ to the matrix $C-\lambda I$ (which does not change the determinant) and then expanding along the first row, we see that

$$\det(C-\lambda I)=\begin{vmatrix}-3-\lambda & 1 & -1\\ -7 & 5-\lambda & -1\\ -6 & 6 & -2-\lambda\end{vmatrix}=\begin{vmatrix}4-\lambda & -4+\lambda & 0\\ -7 & 5-\lambda & -1\\ -6 & 6 & -2-\lambda\end{vmatrix}$$

$$=(4-\lambda)\begin{vmatrix}5-\lambda & -1\\ 6 & -2-\lambda\end{vmatrix}-(-4+\lambda)\begin{vmatrix}-7 & -1\\ -6 & -2-\lambda\end{vmatrix}=(4-\lambda)(\lambda^2+4\lambda+4)=(4-\lambda)(-2-\lambda)^2.$$

Hence $C$ has eigenvalues $\lambda=-2$ and $\lambda=4$. The algebraic multiplicity of $\lambda=-2$ is 2, and the algebraic multiplicity of $\lambda=4$ is 1.

To find the $(-2)$-eigenspace of $C$, we solve

$$\left[C+2I \mid \mathbf{0}\right]=\left[\begin{array}{ccc|c}-1 & 1 & -1 & 0\\ -7 & 7 & -1 & 0\\ -6 & 6 & 0 & 0\end{array}\right]\xrightarrow[R_3\mapsto R_3-7R_1]{R_2\mapsto R_2-6R_1}\left[\begin{array}{ccc|c}-1 & 1 & -1 & 0\\ 0 & 0 & 6 & 0\\ 0 & 0 & 6 & 0\end{array}\right]\xrightarrow{R_3\mapsto R_3-R_2}\left[\begin{array}{ccc|c}-1 & 1 & -1 & 0\\ 0 & 0 & 6 & 0\\ 0 & 0 & 0 & 0\end{array}\right],$$

and we get $z=0$, $y=t$, and $x=t$, for $t\in\mathbb{R}$. So the $(-2)$-eigenspace of $C$ is

$$E_{-2}=\left\{t\begin{bmatrix}1\\1\\0\end{bmatrix} : t\in\mathbb{R}\right\}=\operatorname{span}\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right),$$

and the geometric multiplicity of the eigenvalue $-2$ is 1.

To find the 4-eigenspace of $C$, we solve

$$\left[C-4I \mid \mathbf{0}\right]=\left[\begin{array}{ccc|c}-7 & 1 & -1 & 0\\ -7 & 1 & -1 & 0\\ -6 & 6 & -6 & 0\end{array}\right]\sim\left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0\end{array}\right],$$

where the final augmented matrix is obtained by using the row operations $R_2\mapsto R_2-R_1$, $R_3\mapsto R_3-\frac{6}{7}R_1$, $R_3\mapsto\frac{7}{36}R_3$, $R_1\mapsto R_1-R_3$, $R_1\mapsto-\frac{1}{7}R_1$, and $R_2\leftrightarrow R_3$ (in that order). The solution to this system is $z=t$, $y=t$, and $x=0$, for $t\in\mathbb{R}$. So the 4-eigenspace of $C$ is

$$E_4=\left\{t\begin{bmatrix}0\\1\\1\end{bmatrix} : t\in\mathbb{R}\right\}=\operatorname{span}\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right).$$

The geometric multiplicity of the eigenvalue 4 is 1.

**3.** For each matrix in question **2.**, calculate its trace (using the definition) and its determinant. Then compare the trace with the sum of all the eigenvalues (repetition included) and compare the determinant with the product of all the eigenvalues (repetition included) of this matrix.

***Solution:***

(i) We have $\operatorname{tr}(A) = 1 + (-1) = 0$ and $\det(A) = 0$. The matrix $A$ has eigenvalues $\lambda_1 = \lambda_2 = 0$. It is easy to see that $\operatorname{tr}(A) = \lambda_1 + \lambda_2$ and $\det(A) = \lambda_1\lambda_2$.

(ii) We have $\operatorname{tr}(B) = 1 + (-5) + 4 = 0$ and

$$\det(B) = \begin{vmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 3 & 4 & -6 \\ 6 & 12 & -14 \end{vmatrix} = \begin{vmatrix} 4 & -6 \\ 12 & -14 \end{vmatrix} = 4 \times (-14) - (-6) \times 12 = 16.$$

The matrix $B$ has eigenvalues $\lambda_1 = \lambda_2 = -2$ and $\lambda_3 = 4$. It is easy to see that $\operatorname{tr}(B) = \lambda_1 + \lambda_2 + \lambda_3$ and $\det(B) = \lambda_1\lambda_2\lambda_3$.

(iii) We have $\operatorname{tr}(C) = -3 + 5 + (-2) = 0$ and

$$\det(C) = \begin{vmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{vmatrix} = \begin{vmatrix} -3 & 1 & -1 \\ -4 & 4 & 0 \\ 0 & 4 & 0 \end{vmatrix} = (-1) \begin{vmatrix} -4 & 4 \\ 0 & 4 \end{vmatrix} = -(-16 - 0) = 16.$$

The matrix $C$ has eigenvalues $\lambda_1 = \lambda_2 = -2$ and $\lambda_3 = 4$. It is easy to see that $\operatorname{tr}(C) = \lambda_1 + \lambda_2 + \lambda_3$ and $\det(C) = \lambda_1\lambda_2\lambda_3$.

**4.** * Use the multiplicative property of the determinant to prove that if $A$ and $B$ are square matrices of the same size, and $B$ is invertible, then $A$ and $B^{-1}AB$ have the same eigenvalues.

***Solution:*** By the multiplicative property of the determinant and distributivity, we have

$$\begin{aligned}
\det(B^{-1}AB - \lambda I) &= \det(B^{-1}AB - \lambda B^{-1}B) \\
&= \det(B^{-1}AB - B^{-1}\lambda IB) \\
&= \det(B^{-1}(A - \lambda I)B) \\
&= \det(B^{-1}) \det(A - \lambda I) \det(B) \\
&= \frac{1}{\det(B)} \det(B) \det(A - \lambda I) \\
&= \det(A - \lambda I).
\end{aligned}$$

Since their characteristic polynomials are identical, the matrices $A$ and $B^{-1}AB$ have the same eigenvalues.

**5.** Suppose that $0 \le \theta \le \pi$. Prove that $A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ has real eigenvalues if and only if $\theta = 0$ or $\theta = \pi$.

***Solution:*** Observe that

$$\begin{aligned}
\det(A - \lambda I_2) &= \begin{vmatrix} \cos(\theta) - \lambda & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) - \lambda \end{vmatrix} \\
&= (\cos(\theta) - \lambda)^2 + \sin^2(\theta) \\
&= \cos^2(\theta) - 2\lambda\cos(\theta) + \lambda^2 + \sin^2(\theta) \\
&= \lambda^2 - 2\lambda\cos(\theta) + 1
\end{aligned}$$

which has roots

$$\lambda = \frac{2\cos(\theta) \pm \sqrt{4\cos^2(\theta) - 4}}{2} = \cos(\theta) \pm i\sqrt{-\sin^2(\theta)} = \cos(\theta) \pm i\sin(\theta).$$

Therefore, the eigenvalues of $A$ are real if and only if $\sin(\theta) = 0$, which occurs precisely when $\theta = 0$ or $\theta = \pi$.

**6.** Suppose that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of a matrix $A$ corresponding to distinct eigenvalues $\lambda_1$ and $\lambda_2$, respectively. Prove that $\mathbf{v}_1$ cannot be a scalar multiple of $\mathbf{v}_2$.

***Solution:*** We argue by contradiction. Suppose that $\mathbf{v}_1 = \alpha\mathbf{v}_2$, for some $\alpha \in \mathbb{R}$. Then

$$\lambda_1\mathbf{v}_1 = A\mathbf{v}_1 = A\alpha\mathbf{v}_2 = \alpha A\mathbf{v}_2 = \alpha\lambda_2\mathbf{v}_2 = \lambda_2\alpha\mathbf{v}_2 = \lambda_2\mathbf{v}_1,$$

and hence $(\lambda_1 - \lambda_2)\mathbf{v}_1 = \mathbf{0}$. Since $\mathbf{v}_1$ is an eigenvector and hence is nonzero, we must have $\lambda_1 - \lambda_2 = 0$, which contradicts the fact that $\lambda_1$ and $\lambda_2$ are distinct. Thus $\mathbf{v}_1$ cannot be a scalar multiple of $\mathbf{v}_2$.

**7.** * Recall that three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are *linearly independent* if

$$\alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \gamma\mathbf{v}_3 = \mathbf{0} \implies \alpha = \beta = \gamma = 0,$$

where $\alpha, \beta, \gamma \in \mathbb{R}$. Explain why three eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ corresponding to three distinct eigenvalues $\lambda_1$, $\lambda_2$, and $\lambda_3$ of a matrix $A$ must be linearly independent.

***Solution:*** Suppose that $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \gamma\mathbf{v}_3 \;=\; \mathbf{0}. \tag{1}$$

Multiplying equation (1) through by $\lambda_1$ gives

$$\lambda_1\alpha\mathbf{v}_1 + \lambda_1\beta\mathbf{v}_2 + \lambda_1\gamma\mathbf{v}_3 \;=\; \mathbf{0}. \tag{2}$$

Multiplying equation (1) through by $A$, and using the fact that $A\mathbf{v}_i = \lambda_i\mathbf{v}_i$ for all $i \in \{1, 2, 3\}$, gives

$$\alpha\lambda_1\mathbf{v}_1 + \beta\lambda_2\mathbf{v}_2 + \gamma\lambda_3\mathbf{v}_3 \;=\; \mathbf{0}. \tag{3}$$

Subtracting equation (3) from equation 2 gives

$$(\lambda_1 - \lambda_2)\beta\mathbf{v}_2 + (\lambda_1 - \lambda_3)\gamma\mathbf{v}_3 \;=\; \mathbf{0}. \tag{4}$$

Multiplying equation (4) through by $\lambda_2$ gives

$$(\lambda_1 - \lambda_2)\lambda_2\beta\mathbf{v}_2 + (\lambda_1 - \lambda_3)\lambda_2\gamma\mathbf{v}_3 \;=\; \mathbf{0}. \tag{5}$$

Multiplying equation (4) through by $A$, and using the fact that $A\mathbf{v}_2 = \lambda_2\mathbf{v}_2$ and $A\mathbf{v}_3 = \lambda_3\mathbf{v}_3$ gives

$$(\lambda_1 - \lambda_2)\lambda_2\beta\mathbf{v}_2 + (\lambda_1 - \lambda_3)\lambda_3\gamma\mathbf{v}_3 \;=\; \mathbf{0}. \tag{6}$$

Subtracting equation (6) from equation (5) gives

$$(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)\gamma\mathbf{v}_3 \;=\; \mathbf{0}. \tag{7}$$

But since $\lambda_1$, $\lambda_2$, and $\lambda_3$ are all distinct, we have $\lambda_1 - \lambda_3 \neq 0$ and $\lambda_2 - \lambda_3 \neq 0$. And we also have $\mathbf{v}_3 \neq \mathbf{0}$, since $\mathbf{v}_3$ is an eigenvector. Therefore, equation (7) implies that $\gamma = 0$. So by equation (1), we have

$$\alpha\mathbf{v}_1 + \beta\mathbf{v}_2 \;=\; \mathbf{0}.$$

If $\alpha \neq 0$ or $\beta \neq 0$, then one of $\mathbf{v}_1$ or $\mathbf{v}_2$ is a scalar multiple of the other, but this contradicts the result from Question **6.**. Hence we have

$$\alpha \;=\; \beta \;=\; \gamma \;=\; 0,$$

which proves that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent.

**8.** Verify that the characteristic polynomial of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$\det(A - \lambda I_2) = \lambda^2 - (a + d)\lambda + ad - bc.$$

Now also verify that

$$A^2 - (a + d)A + (ad - bc)I_2 = 0_{2\times 2}.$$

This result says that, in matrix algebra, $A$ is a root of its own characteristic polynomial. This is the $2 \times 2$ case of the more general *Cayley–Hamilton theorem.*

***Solution:*** Observe that

$$\det(A - \lambda I_2) \;=\; \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} \;=\; (a - \lambda)(d - \lambda) - bc \;=\; \lambda^2 - (a + d)\lambda + ad - bc,$$

and that

$$\begin{aligned}
A^2-(a+d)A+(ad-bc)I_2 &= \begin{bmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{bmatrix} - \begin{bmatrix} a^2+da & ab+db \\ ac+dc & ad+d^2 \end{bmatrix} + \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} \\
&= \begin{bmatrix} a^2+bc-a^2-da+ad-bc & ab+bd-ab-db \\ ca+dc-ac-dc & cb+d^2-ad-d^2+ad-bc \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
&= 0_{2\times 2},
\end{aligned}$$

which implies that $A$ is a root of its characteristic polynomial.

**9.** * Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be a complete set of eigenvalues (repetition included) of the $n \times n$ matrix $A$. Prove that

$$\det(A) = \lambda_1 \lambda_2 \ldots \lambda_n.$$

[*Hint:* Write the characteristic polynomial of $A$ as $\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$.]

***Solution:*** Since characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ of the $n \times n$ matrix $A$ is a degree-$n$ polynomial in $\lambda$ with the leading coefficient $(-1)^n$ and since $\lambda_1, \lambda_2, \ldots, \lambda_n$ are all the roots of the characteristic polynomial of $A$, we have

$$\begin{aligned}
p(\lambda) = \det(A - \lambda I) &= (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n) \\
&= (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda).
\end{aligned}$$

By substituting $\lambda = 0$ in the above formula we get $p(0) = \lambda_1 \lambda_2 \cdots \lambda_n$. On the other hand, we have $p(0) = \det(A - 0I) = \det(A)$. Thus, $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$.