**1.** For the points $A = (2,1)$, $B = (3,-1)$, $C = (0,5)$, and $D = (-2,-2)$, calculate $\overrightarrow{AB} \cdot \overrightarrow{CD}$.

***Solution:*** We know that if $P = (p_1, p_2)$ and $Q = (q_1, q_2)$, then $\overrightarrow{PQ} = [q_1 - p_1, q_2 - p_2]$. So

$$\overrightarrow{AB} = [3-2, -1-1] = [1,-2] \quad \text{and} \quad \overrightarrow{CD} = [-2-0, -2-5] = [-2,-7],$$

and hence $\overrightarrow{AB} \cdot \overrightarrow{CD} = [1,-2] \cdot [-2,-7] = -2 + 14 = 12$.

**2.** Given that $\mathbf{a} = [3,1]$, $\mathbf{b} = [\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}]$, and $\mathbf{c} = [-1,2]$, find

(i) $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a} \cdot \mathbf{c}$
(ii) the lengths of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$
(iii) the unit vectors in the directions of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$
(iv) the projection of $\mathbf{a}$ onto $\mathbf{b}$
(v) the projection of $\mathbf{c}$ onto $\mathbf{a}$

***Solution:***

(i) $\mathbf{a} \cdot \mathbf{b} = \sqrt{2}$ and $\mathbf{a} \cdot \mathbf{c} = -1$
(ii) $\|\mathbf{a}\| = \sqrt{10}$, $\|\mathbf{b}\| = 1$ and $\|\mathbf{c}\| = \sqrt{5}$
(iii) the unit vector in the direction of $\mathbf{a}$ is $[\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}]$, the unit vector in the direction of $\mathbf{b}$ is $\mathbf{b}$ and the unit vector in the direction of $\mathbf{c}$ is $[\frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}]$
(iv) $\text{proj}_{\mathbf{b}}(\mathbf{a}) = [1,-1]$
(v) $\text{proj}_{\mathbf{a}}(\mathbf{c}) = [-\frac{3}{10}, -\frac{1}{10}]$

**3.** Given that $\mathbf{u} = [1,2,2]$ and $\mathbf{v} = [-4,4,1]$, find

(i) $\mathbf{u} \cdot \mathbf{v}$
(ii) the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$
(iii) the projection of $\mathbf{u}$ onto $\mathbf{v}$

***Solution:***

(i) 6
(ii) $\frac{2}{\sqrt{33}}$
(iii) $\text{proj}_{\mathbf{v}}\, \mathbf{u} = [-\frac{24}{33}, \frac{24}{33}, \frac{6}{33}]$

**4.** Let $\mathbf{v} = [2,-6,9,0]$ and $\mathbf{w} = [4,0,2,-4]$. Find

(i) $\mathbf{v} \cdot \mathbf{w}$
(ii) the unit vectors in the directions of $\mathbf{v}$ and $\mathbf{w}$
(iii) $\|\mathbf{v} + \mathbf{w}\|$

***Solution:***

(i) 26
(ii) the unit vector in the direction of $\mathbf{v}$ is $[\frac{2}{11}, \frac{-6}{11}, \frac{9}{11}, 0]$ and the unit vector in the direction of $\mathbf{w}$ is $[\frac{2}{3}, 0, \frac{1}{3}, -\frac{2}{3}]$.
(iii) $\sqrt{209}$. (Note this is not equal to $\|\mathbf{v}\| + \|\mathbf{w}\|$.)

**5.** Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^n$ with $n \geq 2$, and let $c$ be a scalar. Explain why the following expressions make no sense: $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$, $c \cdot (\mathbf{u} + \mathbf{v})$, $\mathbf{u} \cdot \mathbf{v} + \mathbf{w}$, $\mathbf{vw}$.

***Solution:*** First, $\mathbf{u}$ is a vector but $\mathbf{v} \cdot \mathbf{w}$ is scalar, and we cannot take the dot product of a vector with a scalar. Second, $c$ is a scalar but $\mathbf{u} + \mathbf{v}$ is a vector, and we cannot take the dot product of a scalar with a vector. Third, $\mathbf{u} \cdot \mathbf{v}$ is a scalar and $\mathbf{w}$ is a vector, and we can't add a scalar to a vector. Fourth, $\mathbf{vw}$ is not defined: you must always specify which operation on vectors you intend (addition, dot product, cross product, etc).

**6.** * Prove that if $\mathbf{a}$ and $\mathbf{b}$ are vectors in $\mathbb{R}^n$ with $\|\mathbf{a}\| = \|\mathbf{b}\|$, then $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are orthogonal.

***Solution:*** Using properties of the dot product, we have

$$\begin{aligned}
(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) &= \mathbf{a} \cdot (\mathbf{a} - \mathbf{b}) + \mathbf{b} \cdot (\mathbf{a} - \mathbf{b}) \\
&= \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{b} \\
&= \|\mathbf{a}\|^2 - \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b} - \|\mathbf{b}\|^2 \\
&= \|\mathbf{a}\|^2 - \|\mathbf{b}\|^2 \\
&= 0
\end{aligned}$$

since $\|\mathbf{a}\| = \|\mathbf{b}\|$. Hence $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are orthogonal.

**7.** * Let $\mathbf{u} = [3, 1]$ and $\mathbf{v} = [-1, 1]$. Show that the vector $\mathbf{w} = [-7, -1]$ can be expressed as a linear combination of $\mathbf{u}$ and $\mathbf{v}$, and draw a picture to illustrate this geometrically.

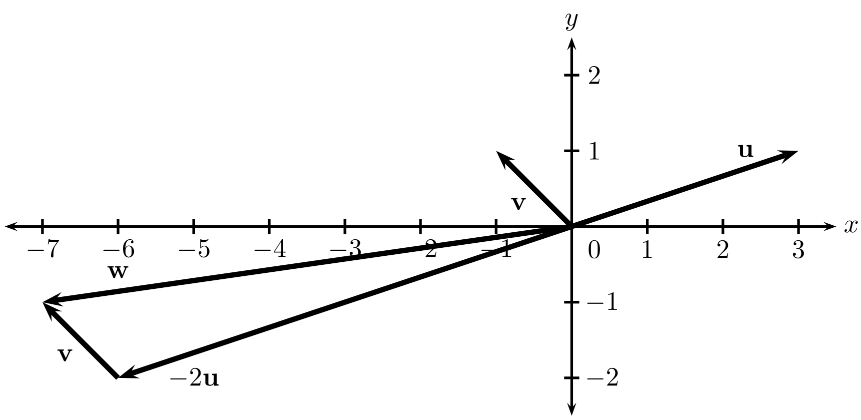
***Solution:*** We want to find scalars $c_1$ and $c_2$ so that $\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v}$. Now

$$c_1\mathbf{u} + c_2\mathbf{v} = [3c_1, c_1] + [-c_2, c_2] = [3c_1 - c_2, c_1 + c_2]$$

and $\mathbf{w} = [-7, -1]$. So by comparing components, we need to solve the simultaneous equations

$$3c_1 - c_2 = -7 \quad \text{and} \quad c_1 + c_2 = -1.$$

These have (unique) solution $c_1 = -2$ and $c_2 = 1$, so $\mathbf{w} = -2\mathbf{u} + \mathbf{v}$. (You should check this answer.)




**8.** Let $\mathbf{v} = \overrightarrow{PQ}$ where $P = (-3, 2, 0)$ and $Q = (4, -2, 3)$. Find the vector $\mathbf{v}$, the length of $\mathbf{v}$ and the angles $\mathbf{v}$ makes (to the nearest degree) with each of the positive $x$, $y$ and $z$-axes. (These will be the angles between $\mathbf{v}$ and the vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$.)

***Solution:*** We have $\mathbf{v} = [7, -4, 3]$ and $\|\mathbf{v}\| = \sqrt{74}$. So the cosines of the angles made with the positive $x$, $y$ and $z$-axes are

$$\frac{7}{\sqrt{74}}, \quad -\frac{4}{\sqrt{74}}, \quad \frac{3}{\sqrt{74}},$$

yielding angles of approximately 36°, 118° and 70° respectively.

**9.** Consider the points $A(1, 2, -3)$, $B(-2, 1, 1)$, and $C(0, 2, 1)$.

(i) Find the point $D$ such that $ABCD$ is a parallelogram.

(ii) Let $P$ be the midpoint of $AC$. Find the vector $\overrightarrow{OP}$.

(iii) Find the vectors $\overrightarrow{BP}$ and $\overrightarrow{PD}$, and deduce that the diagonals $AC$ and $BD$ bisect each other.

(iv) Find the lengths of $\overrightarrow{AC}$ and $\overrightarrow{BD}$. Is the parallelogram $ABCD$ a rectangle?

***Solution:***

(i) We want $D(x, y, z)$ such that $\overrightarrow{AB} = \overrightarrow{DC}$, so that

$$[-3, -1, 4] = [-x, 2 - y, 1 - z],$$

yielding $x = 3$, $y = 3$, $z = -3$. Hence $D$ is the point $(3, 3, -3)$.

(ii) The coordinates of $P$ are the averages of the respective coordinates of $A$ and $C$, so $P = (\frac{1}{2}, 2, -1)$ and $\overrightarrow{OP} = [\frac{1}{2}, 2, -1]$.

(iii) We have $\overrightarrow{BP} = \overrightarrow{PD} = [\frac{5}{2}, 1, -2]$, so that $P$ must be the midpoint of the line segment joining $B$ and $D$. Thus the diagonals $AC$ and $BD$ bisect each other.

(iv) We have

$$\|\overrightarrow{AC}\| = \|[-1, 0, 4]\| = \sqrt{17} \quad \text{and} \quad \|\overrightarrow{BD}\| = \|[5, 2, -4]\| = 3\sqrt{5}.$$

Since these lengths are different, the parallelogram $ABCD$ a not a rectangle.

**10.** Find all values of the scalars $\alpha$ and $\beta$ such that

(i) $[2, 3]$ is orthogonal to $[\alpha + 1, \alpha - 1]$

(ii) $[3, \beta, 3\beta]$ has the same length as $[12, 0, -5]$

***Solution:***

(i) We want to find $\alpha$ such that $[2, 3] \cdot [\alpha + 1, \alpha - 1] = 0$. Now

$$[2, 3] \cdot [\alpha + 1, \alpha - 1] = 2(\alpha + 1) + 3(\alpha - 1) = 5\alpha - 1$$

and so $\alpha = \frac{1}{5}$.

(ii) The vector $[3, \beta, 3\beta]$ has length $\sqrt{3^2 + \beta^2 + (3\beta)^2} = \sqrt{9 + 10\beta^2}$. The vector $[12, 0, -5]$ has length 13. Squaring both sides, we get $9 + 10\beta^2 = 169$ hence $\beta^2 = 16$ and so $\beta = \pm 4$.

**11.** * Prove the following distributive law: for all vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ in $\mathbb{R}^n$, we have

$$(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}.$$

***Solution:*** Let $\mathbf{a} = [a_1, a_2, \ldots, a_n]$, $\mathbf{b} = [b_1, b_2, \ldots, b_n]$, and $\mathbf{c} = [c_1, c_2, \ldots, c_n]$. Then

$$\begin{aligned}
(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} &= ([a_1, a_2, \ldots, a_n] + [b_1, b_2, \ldots, b_n]) \cdot [c_1, c_2, \ldots, c_n] \\
&= [a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n] \cdot [c_1, c_2, \ldots, c_n] \\
&= (a_1 + b_1)c_1 + (a_2 + b_2)c_2 + \cdots + (a_n + b_n)c_n \\
&= a_1c_1 + b_1c_1 + a_2c_2 + b_2c_2 + \cdots + a_nc_n + b_nc_n \\
&= a_1c_1 + a_2c_2 + \cdots + a_nc_n + b_1c_1 + b_2c_2 + \ldots b_nc_n \\
&= [a_1, a_2, \ldots, a_n] \cdot [c_1, c_2, \ldots, c_n] + [b_1, b_2, \ldots, b_n] \cdot [c_1, c_2, \ldots, c_n] \\
&= \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}
\end{aligned}$$

as required.

**12.** * Prove that if $\mathbf{a}$ and $\mathbf{b}$ are orthogonal vectors in $\mathbb{R}^n$ then

$$\|\mathbf{a} + \mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2.$$

Interpret this result in terms of a well-known fact about triangles.

***Solution:*** If $\mathbf{a}$ and $\mathbf{b}$ are orthogonal then $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} = 0$, so that

$$\begin{aligned}
\|\mathbf{a}+\mathbf{b}\|^2 &= \big(\mathbf{a}+\mathbf{b}\big) \cdot \big(\mathbf{a}+\mathbf{b}\big) \\
&= \mathbf{a}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b} \\
&= \|\mathbf{a}\|^2 + 0 + 0 + \|\mathbf{b}\|^2 \\
&= \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2.
\end{aligned}$$

This is just the usual Theorem of Pythagoras where $\mathbf{a}$ and $\mathbf{b}$ label directed edges of a right-angled triangle.

**13.** * Suppose we have $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, where $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $\mathbb{R}^n$ and $\mathbf{u} \neq \mathbf{0}$. Does it follow that $\mathbf{v} = \mathbf{w}$? Either give a proof, or give a counterexample, that is, give specific vectors $\mathbf{u} \neq \mathbf{0}$, $\mathbf{v}$ and $\mathbf{w}$ such that $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ but $\mathbf{v} \neq \mathbf{w}$.

***Solution:*** No, this does not follow. For example in $\mathbb{R}^2$ let $\mathbf{u} = [1, 0]$, $\mathbf{v} = [0, 1]$ and $\mathbf{w} = [0, 2]$. Then $\mathbf{u} \cdot \mathbf{v} = 0 = \mathbf{u} \cdot \mathbf{w}$, but $\mathbf{v} \neq \mathbf{w}$. There are many other counterexamples.