Semester 1 **Tutorial Exercises for Week 10 — Solutions** 2022

1. Briefly justify the following calculation:

$$\begin{vmatrix} 2 & -3 & -2 \\ -1 & 3 & 4 \\ -7 & -2 & 8 \end{vmatrix} = \begin{vmatrix} 2 & -3 & -2 \\ 3 & -3 & 0 \\ 1 & -14 & 0 \end{vmatrix} = 3\begin{vmatrix} 2 & -3 & -2 \\ 1 & -1 & 0 \\ 1 & -14 & 0 \end{vmatrix} = -6\begin{vmatrix} 1 & -1 \\ 1 & -14 \end{vmatrix} = 78.$$

Solution: Apply $R_2 \to R_2 + 2R_1$, followed by $R_3 \to R_3 + 4R_1$, followed by $R_2 \to \frac{1}{3}R_2$, followed by expansion down the third column and evaluation of 2×2 determinant.

2. Use elementary row and column operations, or otherwise, to find the following:

(i) $\begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 3 \\ 4 & 5 & 1 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix}$ (iii) $\begin{vmatrix} 2 & 3 & 6 & 2 \\ 3 & 1 & 1 & -2 \\ 4 & 0 & 1 & 3 \\ 1 & 1 & 2 & -1 \end{vmatrix}$

Solution:

(i) $$\begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 3 \\ 4 & 5 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 3 & 5 \\ 0 & 1 & -3 \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 1 & -3 \end{vmatrix} = -9 - 5 = -14$$

(ii) $$\begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & 1 \end{vmatrix} = 0$$

(iii) $$\begin{vmatrix} 2 & 3 & 6 & 2 \\ 3 & 1 & 1 & -2 \\ 4 & 0 & 1 & 3 \\ 1 & 1 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 & 4 \\ 0 & -2 & -5 & 1 \\ 0 & -4 & -7 & 7 \\ 1 & 1 & 2 & -1 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 4 \\ -2 & -5 & 1 \\ -4 & -7 & 7 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & 9 \\ 0 & 1 & 23 \end{vmatrix}$$

$$= -\begin{vmatrix} -1 & 9 \\ 1 & 23 \end{vmatrix} = -(-23 - 9) = 32$$

3. We have $\det(A) = \det(A^T)$ for any square matrix A.

Prove this property for 2×2 and 3×3 matrices.

Solution:

- For the 2×2 case, if

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \text{ then } A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}.$$

Using the definition of the determinant for a 2×2 matrix, we have

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}, \text{ and } \det(A^T) = a_{11}a_{22} - a_{21}a_{12}.$$

Therefore, we get $\det(A) = \det(A^T)$.

- For the 3×3 case, if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{ then } A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}.$$

Using the definition of the determinant of a 3×3 matrix, we have

$$\begin{aligned}
\det(A) &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
&= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}, \\
\det(A^T) &= a_{11}\begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{32} \\ a_{13} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \end{vmatrix} \\
&= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{21}a_{12}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13}.
\end{aligned}$$

The two formulas for $\det(A)$ and $\det(A^T)$ are the same, thus $\det(A) = \det(A^T)$.

4. Determine whether the following statements are true for all 2×2 matrices A and B. For any of the statements that are false, find a counterexample.

(i) $\det(AB) = \det(A)\det(B)$ (ii) $\det(A+B) = \det(A) + \det(B)$

(iii) $\det(2A) = 2\det(A)$ (iv) $\det(-A) = \det(A)$

Solution:

(i) This is the usual multiplicative property, which is always true.

(ii) This is false. For example, let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then

$$\det(A+B) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0 = 0 + 0 = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = (\det A) + (\det B)\ .$$

(iii) This is false. In fact, we always have,

$$\det(2A) = \det(2IA) = \det(2I)\det(A) = 4\det(A) \neq 2\det(A)\ ,$$

except when $\det(A) = 0$.

(iv) This is true always since

$$\det(-A) = \det(-IA) = \det(-I)\det A = (-1)(-1)\det A = \det A\ .$$

5. Find $B\mathbf{v}_1$, $B\mathbf{v}_2$, and $B\mathbf{v}_3$, where

$$B = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

By inspection, write down the three eigenvalues of B.

Solution: We have $B\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\mathbf{v}_1$, $B\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \mathbf{v}_2$, and $B\mathbf{v}_3 = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} = 3\mathbf{v}_3$.

So the eigenvalues of B are 0, 1, and 3.

6. Find all eigenvalues and eigenvectors for the following matrices.

(i) $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ (ii) $B = \begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 3 \\ 0 & 0 & 4 \end{bmatrix}$

Solution:

(i) The characteristic polynomial of A is $\begin{vmatrix} 1-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 - 1 + 1 = \lambda^2$. This polynomial has only one root $\lambda = 0$. Thus, A has only one eigenvalue $\lambda = 0$.

Let $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ be an eigenvector corresponding to the eigenvalue $\lambda = 0$. To find all these eigenvectors, we need to find all non-zero vectors $\mathbf{x}$ such that

$$A\mathbf{x} = 0\mathbf{x} = \mathbf{0}.$$

This equation can be written as the following system

$$x - y = 0,$$
$$x - y = 0.$$

Setting $y = t$, where $t \in \mathbb{R}$, we obtain the general solution for this system $x = y = t$, for $t \in \mathbb{R}$. Therefore, the set of all eigenvectors corresponding to the eigenvalue $\lambda = 0$ is

$$\left\{ \mathbf{x} = \begin{bmatrix} t \\ t \end{bmatrix}, t \in \mathbb{R},\ t \neq 0 \right\}.$$

(ii) The characteristic polynomial of B is

$$|B - \lambda I| = \begin{vmatrix} 1-\lambda & -3 & 3 \\ 0 & -5-\lambda & 3 \\ 0 & 0 & 4-\lambda \end{vmatrix} = (1-\lambda)(-5-\lambda)(4-\lambda).$$

Therefore, the eigenvalues of B are $\lambda = 1$, or $\lambda = -5$, or $\lambda = 4$.
Now, to find all eigenvectors corresponding to the eigenvalue $\lambda = 1$, we need to find all non-zero vectors $\mathbf{x}$ such that $B\mathbf{x} = \mathbf{x}$. This is equivalent to find all non-zero solutions of the equation $(B - I)\mathbf{x} = 0$. By writing this equation as a system of linear equations, we have the following augmented matrix

$$\left[\begin{array}{ccc|c} 0 & -3 & 3 & 0 \\ 0 & -6 & 3 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right],$$

which is already in row-echelon form. Using back substitution we get

$$z = 0, y = 0, \text{ and } x = t, \text{ for } t \in \mathbb{R}$$

Thus, the set of all eigenvectors corresponding to the eigenvalue $\lambda = 1$ is

$$\left\{ \mathbf{x} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}, t \in \mathbb{R},\ t \neq 0 \right\}.$$

Similarly, the set of all eigenvectors corresponding to the eigenvalue $\lambda = -5$ is

$$\left\{ \mathbf{x} = \begin{bmatrix} t \\ 2t \\ 0 \end{bmatrix} : t \in \mathbb{R},\ t \neq 0 \right\},$$

and the set of all eigenvectors corresponding the eigenvalue $\lambda = 4$ is

$$\left\{ \mathbf{x} = \begin{bmatrix} 2t \\ t \\ 3t \end{bmatrix} : t \in \mathbb{R},\ t \neq 0 \right\}.$$

7. Prove that every square matrix A has the same eigenvalues as its transpose A^T.

Solution: Since every square matrix has the same determinant as its transpose, we can use properties of the transpose to see that

$$\det(A - \lambda I) = \det\big((A - \lambda I)^T\big) = \det\big(A^T - \lambda I^T\big) = \det\big(A^T - \lambda I\big).$$

Hence A and A^T have identical characteristic polynomials, and therefore they have the same eigenvalues.

8. Let A be a square matrix with eigenvalue λ. Prove the following implications.

(i) $A^2 = 0 \implies \lambda = 0$ (ii) $A^2 = A \implies \lambda = 0$ or $\lambda = 1$ (iii) $A^2 = I \implies \lambda = 1$ or $\lambda = -1$

Solution: Let $\mathbf{v}$ be an eigenvector of A corresponding to the eigenvalue λ.

(i) Suppose that $A^2 = 0$. If $\lambda \neq 0$, then

$$\mathbf{v} \;=\; \lambda^{-2}\lambda^2\mathbf{v} \;=\; \lambda^{-2}A^2\mathbf{v} \;=\; \lambda^{-2}0\mathbf{v} \;=\; \mathbf{0},$$

which contradicts the fact that every eigenvector is nonzero. Hence $\lambda = 0$.

(ii) Suppose that $A^2 = A$ and $\lambda \neq 0$. Then we have

$$\mathbf{v} \;=\; \lambda^{-1}\lambda\mathbf{v} \;=\; \lambda^{-1}A\mathbf{v} \;=\; \lambda^{-1}A^2\mathbf{v} \;=\; \lambda^{-1}A\lambda\mathbf{v} \;=\; \lambda^{-1}\lambda A\mathbf{v} \;=\; \lambda^{-1}\lambda^2\mathbf{v} \;=\; \lambda\mathbf{v},$$

and hence $(1-\lambda)\mathbf{v} = \mathbf{0}$. But $\mathbf{v} \neq \mathbf{0}$, so we must have $1 - \lambda = 0$, and hence $\lambda = 1$.

(iii) Suppose that $A^2 = I$. Then we have

$$\mathbf{v} \;=\; I\mathbf{v} \;=\; A^2\mathbf{v} \;=\; A\lambda\mathbf{v} \;=\; \lambda A\mathbf{v} \;=\; \lambda^2\mathbf{v},$$

and hence $(1-\lambda^2)\mathbf{v} = \mathbf{0}$. But $\mathbf{v} \neq \mathbf{0}$, so we must have $1 - \lambda^2 = 0$, and hence $\lambda = -1$ or $\lambda = 1$.

9. * Prove that if A is invertible and λ is an eigenvalue of A, then $\lambda \neq 0$ and λ^{-1} is an eigenvalue of A^{-1}. What can be said about eigenvalues of A^n, where n is any integer?

Solution: Suppose that $\mathbf{v}$ is an eigenvector of an invertible matrix A corresponding to the eigenvalue λ. If $\lambda = 0$, then

$$\mathbf{v} = A^{-1}A\mathbf{v} = A^{-1}\lambda\mathbf{v} = A^{-1}0\mathbf{v} = \mathbf{0},$$

which contradicts the fact that every eigenvector is nonzero. Hence $\lambda \neq 0$.

From $A\mathbf{v} = \lambda\mathbf{v}$, we deduce that

$$A^{-1}\mathbf{v} = A^{-1}\lambda^{-1}\lambda\mathbf{v} = \lambda^{-1}A^{-1}\lambda\mathbf{v} = \lambda^{-1}A^{-1}A\mathbf{v} = \lambda^{-1}I\mathbf{v} = \lambda^{-1}\mathbf{v}\,,$$

which implies that $\mathbf{v}$ is an eigenvector of A^{-1}, corresponding to the eigenvalue λ^{-1}.

We claim that for each integer n, $\mathbf{v}$ is an eigenvector of A^n corresponding to the eigenvalue λ^n.

This claim is true for $n = 1$, by the definition of $\mathbf{v}$ and λ. We will use mathematical induction on n to show that the claim is true for all integers $n \geq 1$. Suppose that the claim is true for some positive integer $n = k$. Then $\mathbf{v}$ is an eigenvector of A^k corresponding to the eigenvalue λ^k. Hence we have

$$A^{k+1}\mathbf{v} = AA^k\mathbf{v} = A\lambda^k\mathbf{v} = \lambda^k A\mathbf{v} = \lambda^k\lambda\mathbf{v} = \lambda^{k+1}\mathbf{v},$$

and so $\mathbf{v}$ is an eigenvector of A^{k+1} corresponding to the eigenvalue λ^{k+1}. Therefore, by mathematical induction on n, $\mathbf{v}$ is an eigenvector of A^n corresponding to the eigenvalue λ^n, for all integers $n \geq 1$.

Now, since we are assuming that A is invertible, we know that A^n is invertible, with eigenvector $\mathbf{v}$ corresponding to the eigenvalue λ^n. Hence the proof above shows that $\mathbf{v}$ is an eigenvector of $A^{-n} = (A^n)^{-1}$ corresponding to the eigenvalue $\lambda^{-n} = (\lambda^n)^{-1}$.

Finally, the claim is true for $n = 0$, because $\mathbf{v}$ is an eigenvector of $A^0 = I$ corresponding to the eigenvalue $\lambda^0 = 1$, since $I\mathbf{v} = \mathbf{v} = 1\mathbf{v}$.

10. * Suppose that a_1, a_2 and a_3 are distinct real numbers. For any real numbers b_1, b_2 and b_3, prove that there is unique quadratic with equation of the form $y = ax^2 + bx + c$ passing through the points (a_1, b_1), (a_2, b_2) and (a_3, b_3).

Solution: Suppose that three points (a_1, b_1), (a_2, b_2) and (a_3, b_3) lie on the quadratic curve $y = ax^2 + bx + c$. This means we have the three simultaneous equations

$$\begin{aligned} a\;a_1^2 + b\;a_1 + c \quad &= b_1,\\ a\;a_2^2 + b\;a_2 + c \quad &= b_2,\\ a\;a_3^2 + b\;a_3 + c \quad &= b_3. \end{aligned}$$

These equations can be written in the following form with variables a, b and c:

$$\begin{bmatrix} a_1^2 & a_1 & 1 \\ a_2^2 & a_2 & 1 \\ a_3^2 & a_2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad = \quad \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \tag{1}$$

We denote $A = \begin{bmatrix} a_1^2 & a_1 & 1 \\ a_2^2 & a_2 & 1 \\ a_3^2 & a_2 & 1 \end{bmatrix}$.

To prove that there is a unique quadratic curve of the form $y = ax^2 + bx + c$ passing through the three given points, we will prove system (1) has a unique solution or equivalently that $\det(A) \neq 0$.

We use row operations to calculate $\det(A)$:

$$\begin{aligned}
\det(A) = \begin{vmatrix} a_1^2 & a_1 & 1 \\ a_2^2 & a_2 & 1 \\ a_3^2 & a_2 & 1 \end{vmatrix} &= \begin{vmatrix} a_1^2 & a_1 & 1 \\ a_2^2 - a_1^2 & a_2 - a_1 & 0 \\ a_3^2 - a_1^2 & a_3 - a_1 & 0 \end{vmatrix} = \begin{vmatrix} a_2^2 - a_1^2 & a_2 - a_1 \\ a_3^2 - a_1^2 & a_3 - a_1 \end{vmatrix} \\
&= \left(a_2^2 - a_1^2\right)(a_3 - a_1) - (a_2 - a_1)\left(a_3^2 - a_1^2\right) \\
&= (a_2 - a_1)(a_3 - a_1)(a_2 - a_3).
\end{aligned}$$

Since a_1, a_2 and a_3 are distinct numbers, $\det(A) \neq 0$. Thus, we have proven our statement.