The University of Sydney

MATH1002 Linear Algebra

Semester 1 **Tutorial Exercises for Week 9 — Solutions** 2022

1. Explain briefly why the inverse of an elementary matrix is elementary. [Hint: think about inverting elementary row operations.]

Solution: Suppose that E is an elementary matrix corresponding to the elementary row operation ρ. Let the inverse operation of ρ be called σ. In all possible cases, σ is itself an elementary row operation:

(i) if $\rho : R_i \leftrightarrow R_j$ then $\rho = \sigma$;

(ii) if $\rho : R_i \to \lambda R_i$ where $\lambda \neq 0$ then $\sigma : R_i \to \frac{1}{\lambda} R_i$;

(iii) if $\rho : R_j \to R_j + \lambda R_i$ then $\sigma : R_j \to R_j + (-\lambda)R_i$.

Denote by F the elementary matrix corresponding to σ. But E is the effect of ρ on I, so the effect of σ on E must be I, so $FE = I$. Hence $E^{-1} = F$ is elementary.

2. Find the inverse for each of the following elementary matrices.

(i) $E_1 = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}$ (ii) $E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (iii) $E_3 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

(iv) $E_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (v) $E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{bmatrix}$, $c \neq 0$ (vi) $E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$, $c \neq 0$

Solution: From question 1, we know that the inverse of a given elementary matrix is the elementary matrix corresponding to the row operation that reduces the given matrix to the identity matrix. Therefore, we have:

(i) $E_1^{-1} = \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$ (ii) $E_2^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (iii) $E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

(iv) $E_4^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $E_5^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$, $c \neq 0$ (v) $E_6^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$, $c \neq 0$

3. For the following matrices A, find elementary matrices $E_1, \dots, E_k$ such that $E_k \cdots E_1 A = I_2$. Use these matrices to write A and A^{-1} as products of elementary matrices.

(i) $A = \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$

Solution:

(i) We use elementary row operations $R_2 \to R_2 + R_1$ and $R_2 \to -\frac{1}{2}R_2$ to get

$$\begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The corresponding elementary matrices are $E_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$, and we have $E_2 E_1 A = I_2$. The inverse row operations are $R_2 \to R_2 - R_1$ and $R_2 \to -2R_2$, and so $E_1^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ and $E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$. Hence

$$A = E_1^{-1} E_2^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \quad \text{and} \quad A^{-1} = E_2 E_1 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

(ii) We use elementary row operations $R_1 \leftrightarrow R_2$, $R_2 \to R_2 - 3R_1$, $R_2 \to -\frac{1}{5}R_2$ and $R_1 \to R_1 - 2R_2$ to get

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The corresponding elementary matrices are

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ E_2 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix},\ E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{5} \end{bmatrix},\ E_4 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}.$$

and we have $E_4E_3E_2E_1A = I_2$. The inverse row operations are $R_1 \leftrightarrow R_2$, $R_2 \to R_2 + 3R_1$, $R_2 \to -5R_2$ and $R_1 \to R_1 + 2R_2$, and so

$$E_1^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix},\ E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix},\ E_4^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Hence

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

and

$$A^{-1} = E_4E_3E_2E_1 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{5} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

4. Let $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & -2 & 3 \\ 0 & 4 & 2 \end{bmatrix}$. Find elementary matrices E_1, E_2 and E_3, and a matrix B such that $A = E_1E_2E_3B$.

Solution: There are many different answers. We use the row operations $R_2 \to R_2 + R_1$, $R_2 \leftrightarrow R_3$, $R_1 \to R_1 + R_2$ to get

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & -2 & 3 \\ 0 & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & -2 & 5 \\ 0 & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 4 & 2 \\ 0 & -2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 4 \\ 0 & 4 & 2 \\ 0 & -2 & 5 \end{bmatrix}$$

These correspond to elementary matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ -1 & -2 & 3 \\ 0 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 4 \\ 0 & 4 & 2 \\ 0 & -2 & 5 \end{bmatrix},$$

and hence

$$\begin{aligned} A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & -2 & 3 \\ 0 & 4 & 2 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 4 & 4 \\ 0 & 4 & 2 \\ 0 & -2 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 & 4 \\ 0 & 4 & 2 \\ 0 & -2 & 5 \end{bmatrix}. \end{aligned}$$

So we take

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},\ E_3 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 4 & 4 \\ 0 & 4 & 2 \\ 0 & -2 & 5 \end{bmatrix}.$$

5. Express each of $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ as products of elementary matrices.

Solution: Observe that

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

using elementary row operations ρ_1 and ρ_2, in that order, that correspond to elementary matrices

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad E_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

respectively. We have

$$E_2 E_1 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so that

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = E_1^{-1} E_2^{-1} = E_1 E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Observe that

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

using elementary row operations ρ_1, ρ_2, ρ_3, ρ_4, in that order, that correspond to elementary matrices

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

respectively. We have

$$E_4 E_3 E_2 E_1 \begin{bmatrix} 0 & -1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so that

$$\begin{aligned} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} &= E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} \\ &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}. \end{aligned}$$

6. Find the determinants of the following matrices and hence determine whether or not they are invertible.

(i) $\begin{vmatrix} 5 & 2 \\ 3 & -2 \end{vmatrix}$ (ii) $\begin{vmatrix} 6 & 2 \\ 3 & 1 \end{vmatrix}$ (iii) $\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$ (iv) $\begin{vmatrix} 0 & -1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{vmatrix}$

(v) $\begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ -1 & 1 & 0 \end{vmatrix}$ (vi) $\begin{vmatrix} 5 & 2 & 2 \\ -1 & 1 & 2 \\ 3 & 0 & 0 \end{vmatrix}$ (vii) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

Solution:

(i) $\begin{vmatrix} 5 & 2 \\ 3 & -2 \end{vmatrix} = 5(-2) - 2(3) = -16$ (ii) $\begin{vmatrix} 6 & 2 \\ 3 & 1 \end{vmatrix} = 6(1) - 2(3) = 0$

(iii) $\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = 0 - (-1) = 1$ (iv) $\begin{vmatrix} 0 & -1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} = -1$

(v) $\begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ -1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} -1 & -2 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix} = 0 - (-2) + 0 - 1 = 1$

(vi) $\begin{vmatrix} 5 & 2 & 2 \\ -1 & 1 & 2 \\ 3 & 0 & 0 \end{vmatrix} = 3 \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = 3(4-2) = 6$

(vii) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = (45-48) - 2(36-42) + 3(32-35) = 0$

All the matrices given in this question are invertible except for the ones given in part (ii) and part (vii).

7. Write down quickly the determinants of the following matrices:

(i) $\begin{bmatrix} 5 & 0 & 0 \\ 3 & -2 & 0 \\ 1 & -5 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 3 & 8 \\ 0 & -6 & -7 \\ 0 & 0 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} -4 & -5 & 11 \\ 0 & 0 & 0 \\ 2 & -1 & 2 \end{bmatrix}$

(iv) $\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 1 & 0 & 0 \end{bmatrix}$ (v) $\begin{bmatrix} 0 & 0 & 5 \\ 6 & 0 & 0 \\ 0 & -3 & 0 \end{bmatrix}$ (vi) $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 3 & -2 & 0 & 0 \\ 1 & -5 & 2 & 0 \\ -6 & -3 & -7 & -1 \end{bmatrix}$

Solution: (i) $\begin{vmatrix} 5 & 0 & 0 \\ 3 & -2 & 0 \\ 1 & -5 & -1 \end{vmatrix} = 5(-2)(-1) = 10$ (ii) $\begin{vmatrix} 3 & 3 & 8 \\ 0 & -6 & -7 \\ 0 & 0 & 2 \end{vmatrix} = 3(-6)(2) = -36$

(iii) $\begin{vmatrix} -4 & -5 & 11 \\ 0 & 0 & 0 \\ 2 & -1 & 2 \end{vmatrix} = -0 + 0 - 0 = 0$ (iv) $\begin{vmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 1 & 0 & 0 \end{vmatrix} = 1(-1)(-2) = 2$

(v) $\begin{vmatrix} 0 & 0 & 5 \\ 6 & 0 & 0 \\ 0 & -3 & 0 \end{vmatrix} = 5(6)(-3) = -90$ (vi) $\begin{vmatrix} 4 & 0 & 0 & 0 \\ 3 & -2 & 0 & 0 \\ 1 & -5 & 2 & 0 \\ -6 & -3 & -7 & -1 \end{vmatrix} = 4(-2)(2)(-1) = 16$

8. We can use a "determinant" to calculate the cross product of vectors. (We use inverted commas in "determinant" because we are forming a matrix whose entries are both numbers and vectors.) For $\mathbf{v} = [v_1, v_2, v_3]$ and $\mathbf{w} = [w_1, w_2, w_3]$, verify the cross product formula by calculating the "determinant" (by expanding across the first row)

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Use this method to calculate $\mathbf{v} \times \mathbf{w}$ when

(i) $\mathbf{v} = [1, 2, 3]$ and $\mathbf{w} = [4, 5, 6]$ (ii) $\mathbf{v} = [2, -1, 6]$ and $\mathbf{w} = [-1, 1, -3]$

Solution:

(i) $\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}\mathbf{e}_1 - \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}\mathbf{e}_2 + \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}\mathbf{e}_3 = -3\mathbf{e}_1 + 6\mathbf{e}_2 - 3\mathbf{e}_3 = [-3, 6, -3]$

(ii) $\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 2 & -1 & 6 \\ -1 & 1 & -3 \end{vmatrix} = \begin{vmatrix} -1 & 6 \\ 1 & -3 \end{vmatrix}\mathbf{e}_1 - \begin{vmatrix} 2 & 6 \\ -1 & -3 \end{vmatrix}\mathbf{e}_2 + \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix}\mathbf{e}_3 = -3\mathbf{e}_1 + \mathbf{e}_3 = [-3, 0, 1]$

9. (i) * Let A, B be $n \times n$ upper triangular matrices. Prove that $\det(AB) = \det(A)\det(B)$.

(ii) Use (i) to prove that if A and B are $n \times n$ matrices in row echelon form, then $\det(AB) = \det(A)\det(B)$.

Solution:

(i) Since A, B are $n \times n$ upper triangular matrices, they have the following form

$$A = [a_{ij}] = \begin{bmatrix} * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & * & * \\ 0 & 0 & \cdots & 0 & * \end{bmatrix} \quad \text{and} \quad B = [b_{ij}] = \begin{bmatrix} * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & * & * \\ 0 & 0 & \cdots & 0 & * \end{bmatrix},$$

where $a_{ij} = 0$ and $b_{ij} = 0$ for $i > j$.

We will prove that $C = AB$ is also an upper triangular matrix whose entries on the diagonal are $c_{ii} = a_{ii}b_{ii}$ for $1 \le i \le n$. For $1 \le j < i \le n$, we have

$$\begin{aligned} c_{ij} &= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} \\ &= \sum_{k=1}^{n} a_{ik}b_{kj} \\ &= \left(\sum_{k=1}^{i-1} a_{ik}b_{kj}\right) + \left(\sum_{k=i}^{n} a_{ik}b_{kj}\right) \\ &= 0, \end{aligned}$$

where we have used $a_{ik} = 0$ for $k \le i-1$ and $b_{kj} = 0$ for $k \ge i > j$. Similarly, we obtain

$$\begin{aligned} c_{ii} &= a_{i1}b_{1i} + a_{i2}b_{2i} + \cdots + a_{in}b_{ni} \\ &= \sum_{k=1}^{n} a_{ik}b_{ki} \\ &= \left(\sum_{k=1}^{i-1} a_{ik}b_{ki}\right) + a_{ii}b_{ii} + \left(\sum_{k=i+1}^{n} a_{ik}b_{ki}\right) \\ &= a_{ii}b_{ii}, \end{aligned}$$

where we have used $a_{ik} = 0$ for $k \le i-1$ and $b_{ki} = 0$ for $k \ge i+1$.

Since A, B, C are triangular, we have $\det(A) = a_{11}a_{22}\cdots a_{nn}$, $\det(B) = b_{11}b_{22}\cdots b_{nn}$ and

$$\det(C) = c_{11}c_{22}\cdots c_{nn} = (a_{11}b_{11})\ (a_{22}b_{22})\ \cdots\ (a_{nn}b_{nn}).$$

Therefore, we obtain $\det(AB) = \det(C) = \det(A)\det(B)$.

(ii) Since A and B are in row echelon form, A and B are upper triangular. Using the previous part, we have $\det(AB) = \det(A)\det(B)$.