1. Consider the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix},\ B = \begin{bmatrix} 0 & 1 \\ 1 & -1 \\ -1 & 2 \end{bmatrix},\ C = \begin{bmatrix} 3 & 4 & 2 \end{bmatrix},\ D = \begin{bmatrix} -1 \\ -1 \\ 5 \end{bmatrix}.$$

(i) Find AB, BA, CD, DC and $BA + DC$.

(ii) Explain briefly why A^2, B^2, C^2, D^2, and $AB + CD$ do not exist.

Solution:

(i) $\begin{bmatrix} -1 & 5 \\ 1 & 3 \end{bmatrix}$, $\begin{bmatrix} 3 & 2 & 1 \\ -2 & 0 & 2 \\ 5 & 2 & -1 \end{bmatrix}$, $[\,3\,] = 3$, $\begin{bmatrix} -3 & -4 & -2 \\ -3 & -4 & -2 \\ 15 & 20 & 10 \end{bmatrix}$, $\begin{bmatrix} 0 & -2 & -1 \\ -5 & -4 & 0 \\ 20 & 22 & 9 \end{bmatrix}$

(ii) None of A, B, C, D are square. The matrices AB and CD have different sizes.

2. Consider the following 2×2 matrices:

$$A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix},\quad B = \begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix},\quad C = \begin{bmatrix} 0 & 2 \\ 4 & -3 \end{bmatrix},\quad \text{and} \quad D = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}.$$

Find

(i) $A + B$ (ii) $A - B$ (iii) $B - C$ (iv) $D + C$ (v) $2A$

(vi) $-B$ (vii) $\frac{1}{5}D$ (viii) AB (ix) BA (x) CD

(xi) BC (xii) $A(BC)$ (xiii) $(AB)C$ (xiv) $ABCD$

(xv) A^2 (xvi) B^2 (xvii) $A^2 - B^2$ (xviii) $(A + B)(A - B)$

Solution: (i) $\begin{bmatrix} 4 & 2 \\ -2 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} -2 & -6 \\ 0 & 5 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 & 2 \\ -5 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 10 & 2 \\ 4 & 2 \end{bmatrix}$ (v) $\begin{bmatrix} 2 & -4 \\ -2 & 6 \end{bmatrix}$

(vi) $\begin{bmatrix} -3 & -4 \\ 1 & 2 \end{bmatrix}$ (vii) $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ (viii) $\begin{bmatrix} 5 & 8 \\ -6 & -10 \end{bmatrix}$ (ix) $\begin{bmatrix} -1 & 6 \\ 1 & -4 \end{bmatrix}$ (x) $\begin{bmatrix} 0 & 10 \\ 40 & -15 \end{bmatrix}$

(xi) $\begin{bmatrix} 16 & -6 \\ -8 & 4 \end{bmatrix}$ (xii) $A(BC) = \begin{bmatrix} 32 & -14 \\ -40 & 18 \end{bmatrix}$ (xiii) $(AB)C = \begin{bmatrix} 32 & -14 \\ -40 & 18 \end{bmatrix} = A(BC)$

(xiv) $\begin{bmatrix} 320 & -70 \\ -400 & 90 \end{bmatrix}$ (xv) $A^2 = AA = \begin{bmatrix} 3 & -8 \\ -4 & 11 \end{bmatrix}$ (xvi) $B^2 = BB = \begin{bmatrix} 5 & 4 \\ -1 & 0 \end{bmatrix}$

(xvii) $A^2 - B^2 = \begin{bmatrix} 3 & -8 \\ -4 & 11 \end{bmatrix} - \begin{bmatrix} 5 & 4 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & -12 \\ -3 & 11 \end{bmatrix}$

(xviii) $(A + B)(A - B) = \begin{bmatrix} 4 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -2 & -6 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} -8 & -14 \\ 4 & 17 \end{bmatrix} \neq A^2 - B^2$

3. Find a 2×2 matrix M such that $M^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ but every entry of M is nonzero.

Solution: Suppose $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfies $M^2 = 0$. Then we need $a, b, c, d \in \mathbb{R}$ such that

$$a^2 + bc = ab + bd = ca + db = cb + d^2 = 0.$$

We try $a = c = 1$ and we see that $a = c = 1$, $b = d = -1$ is a solution. So $M = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ satisfies $M^2 = 0$.

4. Consider the following matrices:

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix}, \ B = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 2 & 3 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, \ D = \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix}.$$

Compute the following matrices, if possible.

(i) $C - B^T$ (ii) $(AB)^T - CB$ (iii) BB^T (iv) $B^TC^T - (CB)^T$

Solution: (i) $\begin{bmatrix} -3 & 2 \\ 5 & 2 \\ 4 & 3 \end{bmatrix}$ (ii) Not possible (iii) $\begin{bmatrix} 21 & -1 \\ -1 & 13 \end{bmatrix}$ (iv) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

5. Solve each of the following for x, y and z:

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & -1 & 2 \\ 0 & 6 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 29 \\ -10 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -3 & 3 \\ 4 & 9 & -4 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Solution:

(i)
$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 15 \\ 4 & -1 & 2 & 29 \\ 0 & 6 & -1 & -10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 15 \\ 0 & -9 & -10 & -31 \\ 0 & 6 & -1 & -10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 15 \\ 0 & -18 & -20 & -62 \\ 0 & 18 & -3 & -30 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 15 \\ 0 & -18 & -20 & -62 \\ 0 & 0 & -23 & -92 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 15 \\ 0 & 9 & 10 & 31 \\ 0 & 0 & 1 & 4 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 9 & 0 & -9 \\ 0 & 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{array}\right],$$

so that $x = 5\,,\ y = -1$ and $z = 4\,.$

(ii)
$$\begin{bmatrix} 2 & -3 & 3 \\ 4 & 9 & -4 \\ 2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & 3 \\ 0 & 15 & -10 \\ 0 & 3 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & -2 \\ 0 & 0 & 0 \end{bmatrix},$$

so that $x = -t/2\,,\ y = 2t/3\,,\ z = t\,.$

6. Which of the following do you know to be true or expect to be true for all square matrices A, B, and C of the same size?

(i) $(AB)C = A(BC)$ (ii) $AB = BA$

(iii) $(AB)^2 = A^2B^2$ (iv) $A(B + C) = AB + AC$

(v) $(-A)(-B) = AB$ (vi) $A(B - C) = AB - AC$

(vii) $(A + B)^2 = A^2 + 2AB + B^2$ (viii) $(A + B)(A - B) = A^2 - B^2$

(ix) $(A + I)^2 = A^2 + 2A + I$ (x) $(A + I)(A - I) = A^2 - I$

(xi) $A^TA = AA^T$ (xii) $(AB + AC)^T = (C^T + B^T)A^T$

Find a counterexample to each statement that you believe not to be true in general.

Solution:

(i) This is the familiar associative law of matrix multiplication, which is always true.

(ii) This is false. For example take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, so that

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = AB \neq BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(iii) This is false. For example take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, so that

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = (AB)^2 \neq A^2B^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} .$$

(iv) This is one of the familiar distributive laws, which is always true.

(v) This is always true and follows quickly from properties involving scalars.

(vi) This is always true and follows quickly from the distributive law and properties involving scalars.

(vii) This is false. For example take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, so that

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = (A+B)^2 \neq A^2 + 2AB + B^2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} .$$

(viii) This is false. For example take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, so that

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = (A+B)(A-B) \neq A^2 - B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} .$$

(ix) This is always true because $AI = IA = A$, so that

$$(A+I)^2 = A^2 + AI + IA + I^2 = A^2 + 2A + I .$$

(x) This is always true because $AI = IA = A$, so that

$$(A+I)(A-I) = A^2 - AI + IA - I^2 = A^2 - A + A - I = A^2 - I .$$

(xi) This is false. For example take $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

(xii) This is true. We have

$$(AB + AC)^T = (A(B+C))^T = (B+C)^T A^T = (B^T + C^T)A^T = (C^T + B^T)A^T .$$

7. A square matrix is called *diagonal* if all entries away from the main diagonal are zero. Find a simple rule for multiplying diagonal matrices. More generally, describe in words as simply as you can what happens if you multiply any square matrix (of the same size) (i) on the left by a diagonal matrix, or (ii) on the right by a diagonal matrix.

Solution: Multiply corresponding diagonal elements.

(i) Multiply each row by the corresponding diagonal element.

(ii) Multiply each column by the corresponding diagonal element.

8. Find all x, y, z and w such that the following matrix equation holds:

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

Solution: Multiplying out, the equation becomes

$$\begin{bmatrix} 3x & -2y \\ 3z & -2w \end{bmatrix} = \begin{bmatrix} -x+4z & -y+4w \\ x+2z & y+2w \end{bmatrix}$$

which quickly yields $x = z$ and $y = -4w$, which can be expressed as a parametric solution:

$$x = s\ , \quad y = -4t\ , \quad z = s\ , \quad w = t \quad (s, t \in \mathbb{R})\ .$$

9. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 2 & -1 & -1 \\ 2 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$.

(i) Find AB, BA, CD, and DC.

(ii) Simplify A^2B^2 and $C(DCDCD)^2C$ without any further matrix calculations.

Solution: (i) I_2, I_2, I_3, I_3 (ii) I_2, I_3

10. Let $A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$, $X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $Y = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, $Z = \begin{bmatrix} 5 & -2 \end{bmatrix}$, $W = \begin{bmatrix} 1 & 1 \end{bmatrix}$.

Find AX, AY, ZA, WA, ZX, ZAX, ZY, ZAY, WX, WAX, WY, WAY.

Solution: $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 12 \\ 30 \end{bmatrix}$, $\begin{bmatrix} -5 & 2 \end{bmatrix}$, $\begin{bmatrix} 6 & 6 \end{bmatrix}$, 7, −7, 0, 0, 0, 0, 7, 42

11. * Find XY in each case, given that X is a row matrix and Y is a column matrix, both with the same number of entries:

(i) $YX = \begin{bmatrix} -2 & -3 \\ 2 & 3 \end{bmatrix}$ (ii) $YX = \begin{bmatrix} 3 & -3 & 6 \\ 4 & -4 & 8 \\ -2 & 2 & -4 \end{bmatrix}$

Solution:

(i) Put $X = \begin{bmatrix} a & b \end{bmatrix}$ and $Y = \begin{bmatrix} c \\ d \end{bmatrix}$, so

$$\begin{bmatrix} -2 & -3 \\ 2 & 3 \end{bmatrix} = YX = \begin{bmatrix} ca & cb \\ da & db \end{bmatrix},$$

yielding $XY = [ac + bd] = [-2 + 3] = [1]$.

(ii) Put $X = \begin{bmatrix} a & b & c \end{bmatrix}$ and $Y = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$, so

$$\begin{bmatrix} 3 & -3 & 6 \\ 4 & -4 & 8 \\ -2 & 2 & -4 \end{bmatrix} = YX = \begin{bmatrix} da & db & dc \\ ea & eb & ec \\ fa & fb & fc \end{bmatrix},$$

yielding $XY = [ad + be + cf] = [3 - 4 - 4] = [-5]$.

12. Find necessary and sufficient conditions on a, b, c and d such that the matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ commutes with the matrix A in each of the following cases:

(i) $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 0 & 7 \\ 7 & 0 \end{bmatrix}$ (iii) $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

Solution:

(i) Matrices A and B commute if and only if

$$\begin{bmatrix} a & b \\ -c & -d \end{bmatrix} = AB = BA = \begin{bmatrix} a & -b \\ c & -d \end{bmatrix},$$

which occurs if and only if $b = -b$ and $c = -c$, that is, $b = c = 0$.

(ii) Matrices A and B commute if and only if

$$\begin{bmatrix} 7c & 7d \\ 7a & 7b \end{bmatrix} = AB = BA = \begin{bmatrix} 7b & 7a \\ 7d & 7c \end{bmatrix},$$

which occurs if and only if $a = d$ and $b = c$.

(iii) Matrices A and B commute if and only if

$$\begin{bmatrix} a+2c & b+2d \\ c & d \end{bmatrix} = AB = BA = \begin{bmatrix} a & 2a+b \\ c & 2c+d \end{bmatrix},$$

which occurs if and only if $a = d$ and $c = 0$.

13. Explain briefly why the associative law for matrix multiplication implies that every square matrix commutes with its square.

Solution: $AA^2 = A(AA) = (AA)A = A^2A$

14. A square matrix A is called *skew symmetric* if $A^T = -A$. Which of the following matrices are skew symmetric:

(i) $\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 0 & 3 & -1 \\ -3 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}$ (iv) $\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 5 \\ 2 & 5 & 0 \end{bmatrix}$.

Solution: (ii), (iii)

15. * Consider the matrix

$$M = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}.$$

(i) Verify that $M^2 = 2M - I$.

(ii) Deduce that $M^3 = 3M - 2I$ and guess a general formula for powers of M. (If you know the technique of proof by induction then you can try to prove that your guess is correct.)

(iii) Evaluate M^5, M^{10}, and M^{100}.

Solution:

(i) Observe that

$$\begin{aligned} M^2 - 2M + I &= \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix} - 2\begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 5 & -2 \\ 8 & -3 \end{bmatrix} + \begin{bmatrix} -6 & 2 \\ -8 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

so that $M^2 = 2M - I$.

(ii) By part (i),

$$\begin{aligned} M^3 &= M^2M = (2M-I)M = 2M^2 - IM = 2(2M-I) - M \\ &= 4M - 2I - M = 3M - 2I. \end{aligned}$$

We conjecture that, for any positive integer n,

$$M^n = nM - (n-1)I.$$

Certainly the conjecture is true for $n = 1$, which starts an induction. Suppose the conjecture is true for $n = k$. We verify that it is also true for $n = k+1$:

$$\begin{aligned} M^{k+1} &= M^kM = (kM - (k-1)I)M = kM^2 - (k-1)IM \\ &= k(2M-I) - (k-1)M = (k+1)M - kI. \end{aligned}$$

The conjecture follows for all n by mathematical induction.

(iii) $M^5 = 5M - 4I = \begin{bmatrix} 11 & -5 \\ 20 & -9 \end{bmatrix}$, $M^{10} = 10M - 9I = \begin{bmatrix} 21 & -10 \\ 40 & -19 \end{bmatrix}$,

$M^{100} = 100M - 99I = \begin{bmatrix} 201 & -100 \\ 400 & -199 \end{bmatrix}$.