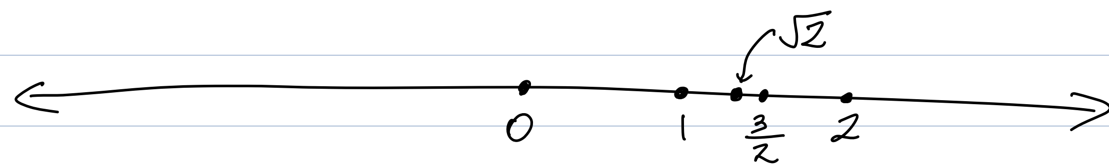
MATH1002 Linear Algebra

Topic 1A: Notation

$\mathbb{R}$ the set of real numbers



We call a real number a scalar.

The symbol $\in$ means "is an element of"

e.g. $0 \in \mathbb{R}$, $\sqrt{2} \in \mathbb{R}$

and $\notin$ means "is not an element of"

complex number $i = \sqrt{-1}$

e.g. $i \notin \mathbb{R}$

Let P and Q be two statements which are either true or false

$P \Rightarrow Q$ means "P implies Q"
or "if P then Q"
or "P only if Q"

Examples 1. P is statement "$x \in \mathbb{R}$"

Q is statement "$x^2 \geqslant 0$"

then $P \Rightarrow Q$

2. P is statement "$x+1=3$"

Q " " "$x=2$"

then $P \Rightarrow Q$.

Now $P \Leftarrow Q$ means "P follows from Q"

or "if Q then P"

or "Q implies P"

Example If $\underbrace{x=2}_{Q}$ then $\underbrace{x+1=3}_{P}$

$Q \Rightarrow P$.

Also $P \Leftrightarrow Q$ means

"P if and only if Q"

"P is equivalent to Q"

Example $\underbrace{x+1=3}_{P} \Leftrightarrow \underbrace{x=2.}_{Q}$

$P \Leftrightarrow Q$

Topic 1B: Vectors in the plane

The set of all ordered pairs of real numbers is denoted by $\mathbb{R}^2$ i.e.

such that are elements of

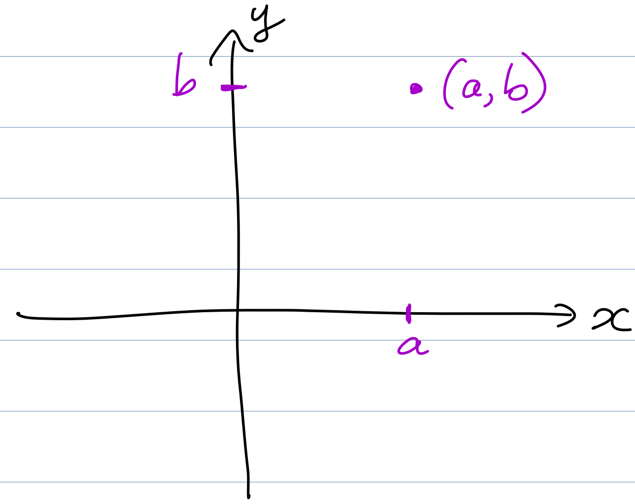
$$\mathbb{R}^2 = \{ (x,y) : x,y \in \mathbb{R} \}$$

curly brackets for sets

set of real numbers

use round brackets for elements of $\mathbb{R}^2$

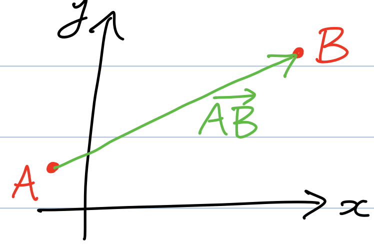
We identify $\mathbb{R}^2$ with the Cartesian plane:



A vector is a directed line segment

A is initial point or tail

B is terminal point or head



Vector from A to B

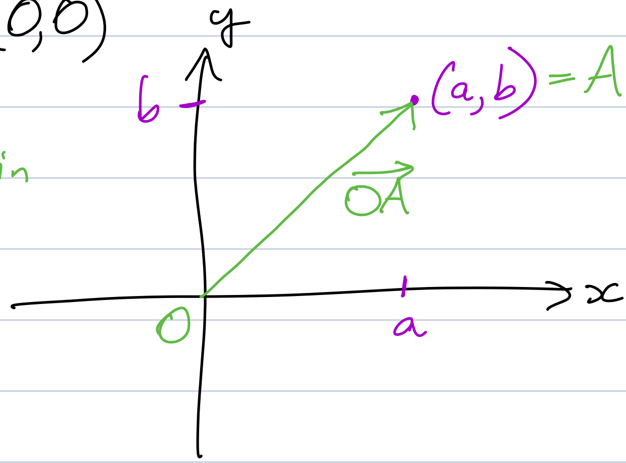
Important Notation

We will also write vectors

- in handwriting, with a tilde underneath $\underset{\sim}{u}$, $\underset{\sim}{v}$, $\underset{\sim}{w}$
- in typeset maths, in bold $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$

We can identify all points in the plane i.e. all of $\mathbb{R}^2$ with the set of vectors with tails at the origin $(0,0)$

Letter O for origin
$A = (a, b)$



A vector with tail at the origin is in <u>standard position</u>.

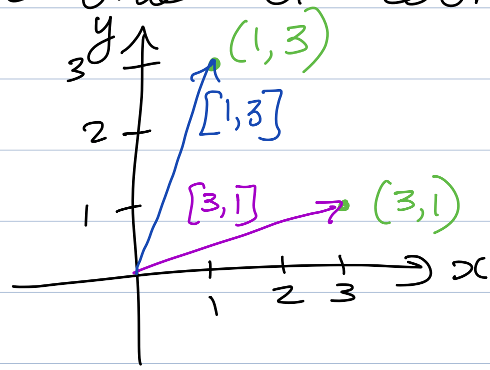
round brackets for points

If $A = (a_1, a_2)$ then the vector $\underset{\sim}{a} = \overrightarrow{OA}$ has <u>coordinates</u> $[a_1, a_2]$.

square brackets for vectors

Write $\underset{\sim}{a} = [a_1, a_2]$,
we say a_1 is the first coordinate,
a_2 " " second " .

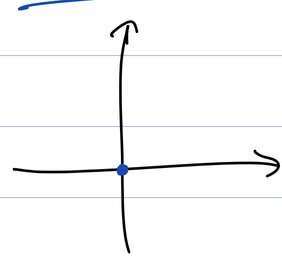
Warning! The order of coordinates matters



$[a_1, a_2]$ is a row vector

We'll also use column vectors $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

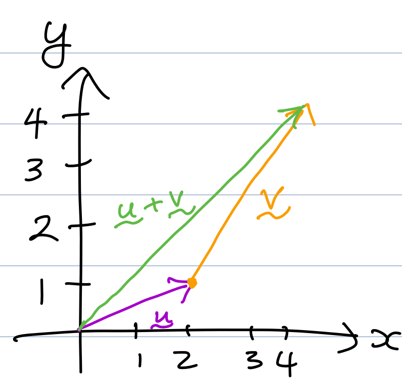
The zero vector is $\underset{\sim}{0} = [0, 0]$.
(hard to draw)



$\underset{\sim}{0} = \vec{00}$

Vector addition

Example



$\underset{\sim}{u} = [2, 1]$

$\underset{\sim}{v} = [2, 3]$

$\underset{\sim}{u} + \underset{\sim}{v} = [4, 4]$

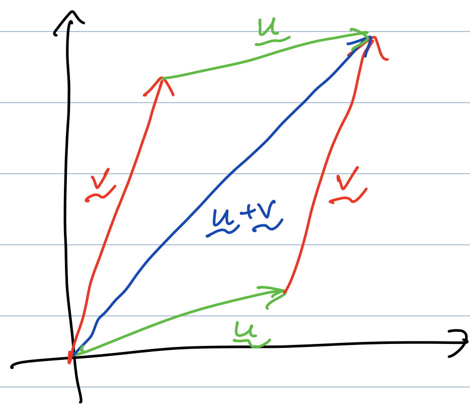
To get $\underset{\sim}{u}+\underset{\sim}{v}$:
Put tail of $\underset{\sim}{v}$ at head of $\underset{\sim}{u}$

Algebraically: $\underset{\sim}{u} + \underset{\sim}{v} = [2, 1] + [2, 3]$
$= [2+2, 1+3]$
$= [4, 4]$

In general: if $\underset{\sim}{u} = [u_1, u_2]$ and $\underset{\sim}{v} = [v_1, v_2]$ then

$$\underset{\sim}{u} + \underset{\sim}{v} = [u_1 + v_1, u_2 + v_2]$$

Geometrically, we have the parallelogram rule:



$\underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{v} + \underset{\sim}{u}$, and is the diagonal of the parallelogram formed by $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

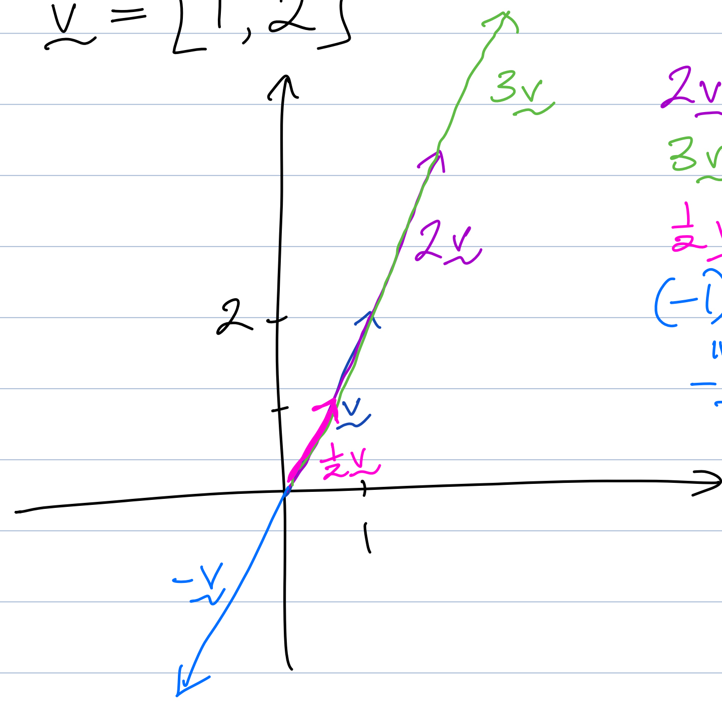
Greek letter lambda

Scalar multiplication

Given a vector $\underset{\sim}{v} = [v_1, v_2]$ and a scalar λ the scalar multiple $\lambda \underset{\sim}{v}$ is the vector

$$\lambda \underset{\sim}{v} = [\lambda v_1, \lambda v_2].$$

Example $\underset{\sim}{v} = [1, 2]$



$2\underset{\sim}{v} = [2, 4]$
$3\underset{\sim}{v} = [3, 6]$
$\frac{1}{2}\underset{\sim}{v} = [\frac{1}{2}, 1]$
$(-1)\underset{\sim}{v} = [-1, -2]$
$\quad \| $
$-\underset{\sim}{v}$

If $\lambda > 0$, $\lambda \underset{\sim}{v}$ has same direction as $\underset{\sim}{v}$

If $\lambda < 0$, $\lambda \underset{\sim}{v}$ has opposite direction to $\underset{\sim}{v}$

The length of $\lambda \underset{\sim}{v}$ is $|\lambda|$ times the length of $\underset{\sim}{v}$.

absolute value

The <u>negative</u> of a vector $\underset{\sim}{v}$ is $-\underset{\sim}{v} = (-1)\underset{\sim}{v}$

and <u>vector subtraction</u> is given by

$$\underset{\sim}{u} - \underset{\sim}{v} = \underset{\sim}{u} + (-1)\underset{\sim}{v} = \underset{\sim}{u} + (-\underset{\sim}{v})$$

i.e. $[u_1, u_2] - [v_1, v_2] = [u_1 - v_1, u_2 - v_2]$.

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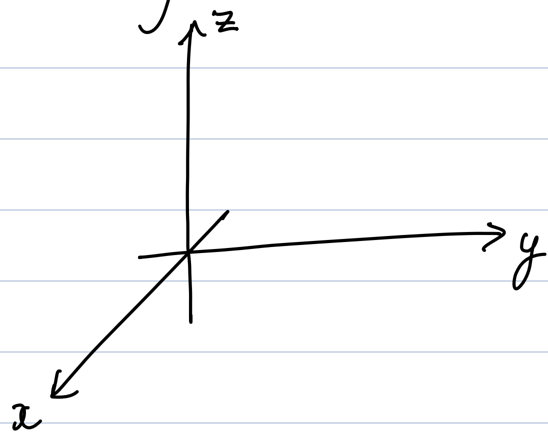
Topic 1C: Vectors in $\mathbb{R}^3$

$\mathbb{R}^3$ is the set of all ordered triples of real numbers
ie $\mathbb{R}^3 = \{ (x, y, z) : x, y, z \in \mathbb{R} \}$

$\mathbb{R}^3$ can be thought of as 3-dim space



Vectors in $\mathbb{R}^3$ have the same operations as we discussed for vectors in $\mathbb{R}^2$:
if $\underset{\sim}{a} = [a_1, a_2, a_3]$ and $\underset{\sim}{b} = [b_1, b_2, b_3]$
then
$$\underset{\sim}{a} + \underset{\sim}{b} = [a_1 + b_1, a_2 + b_2, a_3 + b_3]$$
(vector addition)

and if $\lambda \in \mathbb{R}$ then
$$\lambda \underset{\sim}{a} = [\lambda a_1, \lambda a_2, \lambda a_3]$$
(scalar multiplication)

Check out Geogebra (under Modules > Online Resources).

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Topic 1D: Vectors in $\mathbb{R}^n$ $(n \geq 1)$

$\mathbb{R}^n$ is the set of all ordered n-tuples of real numbers

ie $\mathbb{R}^n = \{ (x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \in \mathbb{R} \}$

A vector in $\mathbb{R}^n$ can be written as a row vector

$$\underset{\sim}{v} = [v_1, v_2, \ldots, v_n]$$

(here $v_1, v_2, \ldots, v_n \in \mathbb{R}$)

or a column vector

$$\underset{\sim}{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

We have:

Vector addition: if $\underset{\sim}{u} = [u_1, u_2, \ldots, u_n]$ and $\underset{\sim}{v} = [v_1, v_2, \ldots, v_n]$ then

$$\underset{\sim}{u} + \underset{\sim}{v} = [u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n]$$

Scalar multiplication if $\lambda \in \mathbb{R}$ then

$$\lambda \underset{\sim}{u} = [\lambda u_1, \lambda u_2, \ldots, \lambda u_n].$$

Examples

In $\mathbb{R}^5$, let $\underset{\sim}{u} = [1, 0, 2, 3, 1]$
$\underset{\sim}{v} = [2, 1, 0, 1, 4]$

Then

$$\underset{\sim}{u} + \underset{\sim}{v} = [1+2, 0+1, 2+0, 3+1, 1+4]$$
$$= [3, 1, 2, 4, 5]$$
$$\underset{\sim}{u} - \underset{\sim}{v} = [1-2, 0-1, 2-0, 3-1, 1-4]$$
$$= [-1, -1, 2, 2, -3]$$

$$2\underset{\sim}{u} = [2, 0, 4, 6, 2].$$

Theorem (Algebraic properties of vectors in $\mathbb{R}^n$)

Let $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w} \in \mathbb{R}^n$ and $\lambda, \mu \in \mathbb{R}$.

Greek letter mu

Then:

(1) $\underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{v} + \underset{\sim}{u}$ (Commutative Law of Addition)

compare Parallelogram Law in 1B

(2) $(\underset{\sim}{u} + \underset{\sim}{v}) + \underset{\sim}{w} = \underset{\sim}{u} + (\underset{\sim}{v} + \underset{\sim}{w})$ (Associate Law of Addition)

(3) $\underset{\sim}{u} + \underset{\sim}{0} = \underset{\sim}{u}$ $\quad \underset{\sim}{0} = [0, 0, \dots, 0]$ zero vector

(4) $\underset{\sim}{u} + (-\underset{\sim}{u}) = \underset{\sim}{0}$

(5) $\lambda(\underset{\sim}{u} + \underset{\sim}{v}) = \lambda\underset{\sim}{u} + \lambda\underset{\sim}{v}$ (Distributive Law of Addition)

(6) $(\lambda + \mu)\underset{\sim}{u} = \lambda\underset{\sim}{u} + \mu\underset{\sim}{u}$ (Distributive Law of Scalar Multiplication)

(7) $\lambda(\mu\underset{\sim}{u}) = (\lambda\mu)\underset{\sim}{u}$

(8) $1\underset{\sim}{u} = \underset{\sim}{u}$.

Proof of (1)

Let $\underset{\sim}{u} = [u_1, u_2, \dots, u_n]$, $\underset{\sim}{v} = [v_1, v_2, \dots, v_n]$.

Then

$$\begin{aligned}
\underset{\sim}{u} + \underset{\sim}{v} &= [u_1, u_2, \dots, u_n] + [v_1, v_2, \dots, v_n] \\
&= [u_1 + v_1, u_2 + v_2, \dots, u_n + v_n] \quad \text{by definition} \\
&= [v_1 + u_1, v_2 + u_2, \dots, v_n + u_n] \quad \text{since addition is commutative in } \mathbb{R} \\
&= [v_1, v_2, \dots, v_n] + [u_1, u_2, \dots, u_n] \\
&= \underset{\sim}{v} + \underset{\sim}{u},
\end{aligned}$$

as required.

Proofs of (2)–(8): exercises (see textbook and/or tut sheet)

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