MATH1002 Linear Algebra

Topic 10A: Eigenvalues of triangular matrices

Recall: An $n \times n$ matrix is

- upper-triangular if all its entries below the diagonal are 0s
- lower-triangular " " " " above
- triangular if it's either upper- or lower-triangular

If A is triangular then

$$\det(A) = a_{11} a_{22} \cdots a_{nn}$$

is the product of its diagonal entries.

Recall: For an $n \times n$ matrix A, its eigenvalues are the roots of its characteristic polynomial $\det(A - \lambda I_n)$.

$\lambda \in \mathbb{R}$ s.t. $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$ for some $\underset{\sim}{x} \neq \underset{\sim}{0}$

Theorem If A is a triangular $n \times n$ matrix, the eigenvalues of A are its diagonal entries $a_{11}, a_{22}, \ldots, a_{nn}$.

Proof Since A is triangular, the matrix $A-\lambda I$ is also triangular, so $\det(A-\lambda I)$ is the product of the diagonal entries of $A-\lambda I$. Now the diagonal entries of A are

$$a_{11},\ a_{22},\ \ldots,\ a_{nn}$$

so the diagonal entries of $A-\lambda I$ are

$$a_{11}-\lambda,\ a_{22}-\lambda,\ \ldots,\ a_{nn}-\lambda.$$

Hence

$$\det(A-\lambda I) = (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)$$

This has roots

$$a_{11},\ a_{22},\ \ldots,\ a_{nn}$$

which are the diagonal entries of A. $\square$

Examples

1. Find the eigenvalues and corresponding eigenspaces for

$$A = \begin{bmatrix} 5 & 2 & 4 \\ 0 & 3 & 1 \\ 0 & 0 & -2 \end{bmatrix}.$$

Eigenvalues are $\lambda_1 = 5$, $\lambda_2 = 3$, $\lambda_3 = -2$.

To find E_5:

$$[A-5I \mid \underset{\sim}{0}] = \left[\begin{array}{ccc|c} 0 & 2 & 4 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -7 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \mapsto R_2+R_1} \left[\begin{array}{ccc|c} 0 & 2 & 4 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & -7 & 0 \end{array}\right] \xrightarrow[R_2 \mapsto \frac{1}{5}R_2]{R_1 \mapsto \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -7 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \mapsto R_3+7R_2} \left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Put $x = t$, $t \in \mathbb{R}$.
Then $z = 0$ and $y + 2z = 0$
so $y = 0$.
Eigenspace for $\lambda_1 = 5$ is

$$E_5 = \left\{ \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\} = \left\{ t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\}$$

Now for E_3:

$$[A - 3I \,|\, \underset{\sim}{0}] = \left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -5 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 \mapsto R_3+5R_2]{R_1 \mapsto \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Put $y = t$. Then $z = 0$ and
$x + y + 2z = 0$

$$\Rightarrow \quad x + t = 0$$
$$\Rightarrow \quad x = -t.$$
$$E_3 = \left\{ t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\}$$

For $\lambda_3 = -2$ we have

$$[A - (-2)I \mid \underset{\sim}{0}] = [A + 2I \mid \underset{\sim}{0}]$$
$$= \left[\begin{array}{ccc|c} 7 & 2 & 4 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Put $z = t$ then $5y + z = 0$
$$\Rightarrow y = -\tfrac{1}{5}t$$

And
$$7x + 2y + 4z = 0$$
$$\Rightarrow \quad 7x - \tfrac{2}{5}t + 4t = 0$$
$$7x = \tfrac{-18}{5}t$$
$$x = -\tfrac{18}{35}t.$$

So
$$E_{-2} = \left\{ t\begin{bmatrix} -\frac{18}{35} \\ -\frac{1}{5} \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

2. Find eigenvalues and eigenspaces for

$$B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 17 \end{bmatrix}.$$

Eigenvalues: $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = 17$.

Eigenspaces:

$$[B - 2I \,|\, \underset{\sim}{0}] = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 15 & 0 \end{array}\right]$$

$$= \ldots$$

$$E_2 = \left\{ t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\} \text{ (check!)}$$

$$= \{ t\underset{\sim}{e}_1 : t \in \mathbb{R} \}.$$

Also

$$E_1 = \{ t\underset{\sim}{e}_2 : t \in \mathbb{R} \} \text{ (check!)}$$

$$E_{17} = \{ t\underset{\sim}{e}_3 : t \in \mathbb{R} \}. \text{ (check!)}$$

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Topic 10B: Multiplicities of eigenvalues

Recall: The eigenvalues of an $n \times n$ matrix A are the roots of its characteristic polynomial $\det(A - \lambda I)$.

Def^n The algebraic multiplicity of an eigenvalue is its multiplicity as a root of the characteristic polynomial.

Examples

1. If $\det(A - \lambda I) = (\lambda - 2)^1 (\lambda - 3)^{2}$ (algebraic multiplicity)
 then the eigenvalue 2 has algebraic multiplicity 1, and the eigenvalue 3 has algebraic multiplicity 2.

2. If $\det(A - \lambda I) = \lambda(\lambda - 1)^2(\lambda + 4)^5$
 then eigenvalue 0 has alg. mult. 1
 " 1 " " " 2
 " -4 " " " 5.

Recall: If λ is an eigenvalue of an $n \times n$ matrix A then the corresponding eigenspace is

$$E_\lambda = \{ \underset{\sim}{x} \in \mathbb{R}^n : A\underset{\sim}{x} = \lambda \underset{\sim}{x} \}$$

To find E_λ: solve $[A - \lambda I \mid \underset{\sim}{0}]$.

<u>Defn</u> The geometric multiplicity of an eigenvalue λ is the number of distinct parameters appearing in its eigenspace.

Examples

1. For $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ we found

$$\det(A - \lambda I) = (\lambda - 3)(\lambda - 1).$$

So the eigenvalues are $\lambda_1 = 3$, $\lambda_2 = 1$ and these both have algebraic multiplicity 1.

We found that for $\underset{\sim}{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

$$A\underset{\sim}{x} = 3\underset{\sim}{x}$$

In fact

$$E_3 = \left\{ t \begin{bmatrix} 1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\} \text{ (check!)}$$

We computed

$$E_1 = \left\{ t \begin{bmatrix} 1 \\ -1 \end{bmatrix} : t \in \mathbb{R} \right\}.$$

Thus both eigenvalues 3 and 1 have geometric multiplicity 1.

2. $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

Then $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix}$

$= (2-\lambda)^2$

The eigenvalue λ has algebraic multiplicity 2.

Now

$[A - 2I \mid \underset{\sim}{0}] = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

Put $x = s$, $y = t$.

Eigenspace for $\lambda = 2$ is:

$$E_2 = \left\{ \begin{bmatrix} s \\ t \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} s \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ t \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

$$= \left\{ s \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

The eigenvalue $\lambda = 2$ has geometric multiplicity 2.

3. $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Here we found $\det(A - \lambda I) = \lambda^2(1 - \lambda)$

So the eigenvalue $\lambda_1 = 0$ has alg. mult 2

" $\lambda_2 = 1$ " " " 1.

We found

$$E_0 = \left\{ t \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

$$E_1 = \left\{ t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

So the eigenvalues 0 and 1 both have geometric multiplicity 1.

4.

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} \overset{+}{-1-\lambda} & \overset{-}{0} & 1 \\ 3 & \overset{+}{-\lambda} & -3 \\ 1 & 0 & -1-\lambda \end{vmatrix}$$

$$= (-\lambda) \begin{vmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix}$$

$$= (-\lambda)\left[(-1-\lambda)^2 - 1\right]$$

$$= -\lambda \left[1 + 2\lambda + \lambda^2 - 1\right]$$

$$= -\lambda \left[2\lambda + \lambda^2\right]$$ correction here

correction here $$= -\lambda^2 (2 + \lambda).$$

The eigenvalues are:

$\lambda_1 = 0$ with alg. mult. 2

$\lambda_2 = -2$ " " " 1.

correction here

We find eigenspaces:

$$[A - 0I \mid \underset{\sim}{0}] = \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 3 & 0 & -3 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 \mapsto R_3 + R_1]{R_2 \mapsto R_2 + 3R_1} \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Put $y = s$, $z = t$ $\quad s, t \in \mathbb{R}$

Then

$$-x + z = 0$$

$$\Longrightarrow \quad x = z = t.$$

$$E_0 = \left\{ \begin{bmatrix} t \\ s \\ t \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

$$= \left\{ s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

So the eigenvalue 0 has geometric multiplicity 2.

Also

$$E_{-2} = \left\{ t \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

so the eigenvalue -2 has geometric multiplicity 1.

Theorem For any $n \times n$ matrix A, and any eigenvalue λ of A, the algebraic multiplicity of λ is greater than or equal to its geometric multiplicity.

Proof Not an exercise!

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Topic 10C: Eigenvalues, trace and determinants

Def^n The trace of an $n \times n$ matrix $A = (a_{ij})$ is

$$\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

ie. sum of its diagonal entries.

Examples

If
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & -1 \\ 5 & 4 & 7 \end{bmatrix} \quad \text{then } \mathrm{tr}(A) = 1 + 4 + 7 = 12$$

If
$$B = \begin{bmatrix} 0 & 1 & 1 \\ 3 & 5 & 2 \\ 4 & 9 & -5 \end{bmatrix} \quad \text{then } \mathrm{tr}(B) = 0 + 5 + -5 = 0.$$

Theorem Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (including repetition).

Then

$$\mathrm{tr}(A) = \underbrace{\lambda_1 + \lambda_2 + \cdots + \lambda_n}_{\text{sum of all eigenvalues}}$$

and

$$\det(A) = \underbrace{\lambda_1 \lambda_2 \cdots \lambda_n}_{\text{product of all eigenvalues}}$$

Example If A has characteristic polynomial

$$\det(A-\lambda I) = (\lambda-2)^2(\lambda-5).$$

Then its eigenvalues including multiplicity are

$$2,\ 2,\ 5$$

so

$$\operatorname{tr}(A) = 2+2+5 = 9$$
$$\det(A) = 2\times 2\times 5 = 20.$$

Proof in some special cases.

1. A is triangular. Then the eigenvalues of A, including multiplicity, are its diagonal entries $a_{11}, a_{22}, \dots, a_{nn}$.
   So its eigenvalues are
   $$\lambda_1 = a_{11},\ \lambda_2 = a_{22},\ \dots,\ \lambda_n = a_{nn}.$$
   Now
   $$\operatorname{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$$
   $$= \lambda_1 + \lambda_2 + \cdots + \lambda_n.$$
   For a triangular matrix, its determinant is the product of its diagonal entries. So
   $$\det(A) = a_{11}\, a_{22} \cdots a_{nn}$$
   $$= \lambda_1 \lambda_2 \cdots \lambda_n.$$

2. 2×2 matrices proof:

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Suppose eigenvalues of A, with multiplicity, are λ_1, λ_2.

Now

$$\det(A-\lambda I) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix}$$
$$= (a-\lambda)(d-\lambda) - bc$$
$$= \lambda^2 - a\lambda - d\lambda + ad - bc$$
$$= \lambda^2 - (a+d)\lambda + (ad-bc).$$

Also

$$\operatorname{tr}(A) = a + d$$

and

$$\det(A) = ad - bc.$$

So:

$$\underbrace{\det(A-\lambda I)}_{\text{characteristic polynomial}} = \lambda^2 - \operatorname{tr}(A)\lambda + \det(A)$$

Also since λ_1 and λ_2 are the roots of $\det(A-\lambda I)$,

$$\det(A-\lambda I) = (\lambda-\lambda_1)(\lambda-\lambda_2)$$
$$= \lambda^2 - (\lambda_1+\lambda_2)\lambda + \lambda_1\lambda_2$$

By comparing coefficients, we get (4 of 4

$$tr(A) = \lambda_1 + \lambda_2$$

$$\det(A) = \lambda_1 \lambda_2 .$$

$\square$