MATH1002 Linear Algebra

Topic 4A: Planes in $\mathbb{R}^3$

The general form for the equation of a plane $\mathcal{P}$ in $\mathbb{R}^3$ is

$$ax + by + cz = d$$

where $a, b, c, d \in \mathbb{R}$ and a, b and c are not all 0.

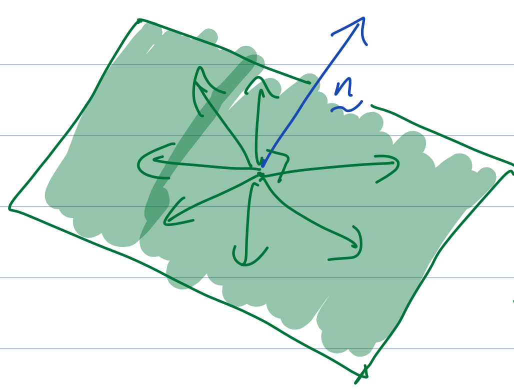
If $\underset{\sim}{n} = [a, b, c]$ and $\underset{\sim}{x} = [x, y, z]$ then this becomes

$$\underset{\sim}{n} \cdot \underset{\sim}{x} = d$$

If $d = 0$ we get

$$\underset{\sim}{n} \cdot \underset{\sim}{x} = 0$$

This is all the points in $\mathbb{R}^3$ which are orthogonal to $\underset{\sim}{n}$.



Infinitely many vectors are orthogonal to $\underset{\sim}{n}$

These vectors form a plane.

The normal form of the equation for a plane $\mathcal{P}$ in $\mathbb{R}^3$ is

$$\underset{\sim}{n} \cdot (\underset{\sim}{x} - \underset{\sim}{p}) = 0$$

where $\underset{\sim}{n}$ is a nonzero vector normal to the plane (i.e. $\underset{\sim}{n}$ is orthogonal to $\mathcal{P}$), $\underset{\sim}{p}$ is a specific point on the plane, and $\underset{\sim}{x} = [x, y, z]$.

Examples 1. Find a normal form for the plane $2x - y + z = 5$.

Here, a normal vector is

$$\underset{\sim}{n} = [2, -1, 1]$$

A point on the plane is

$$P = (0, 0, 5)$$

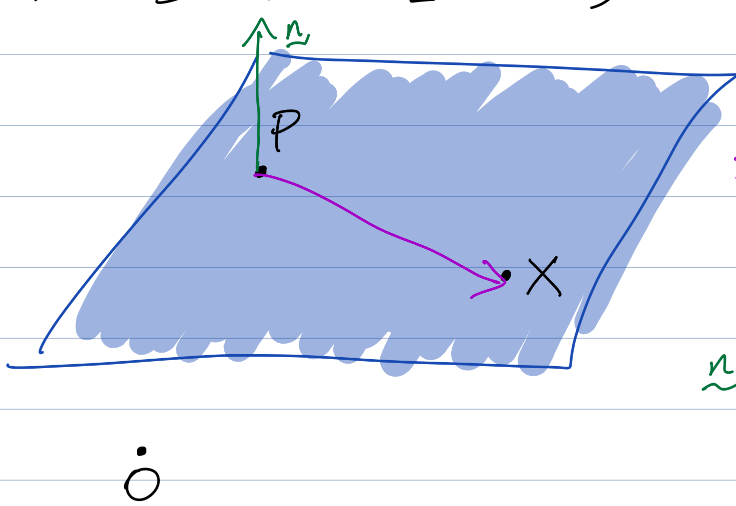
so

$$\underset{\sim}{p} = \overrightarrow{OP} = [0, 0, 5].$$

A normal form is

$$\underset{\sim}{n} \cdot (\underset{\sim}{x} - \underset{\sim}{p}) = 0$$

$$[2, -1, 1] \cdot (\underset{\sim}{x} - [0, 0, 5]) = 0.$$



$\overrightarrow{PX} = \tilde{x} - \tilde{p}$ lies in the plane

$\underset{\sim}{n}$ is orthogonal to the plane.

2. Find a normal form and a general equation of the plane which contains $P=(1,0,6)$ and has normal vector $\underset{\sim}{n}=[1,2,3]$.

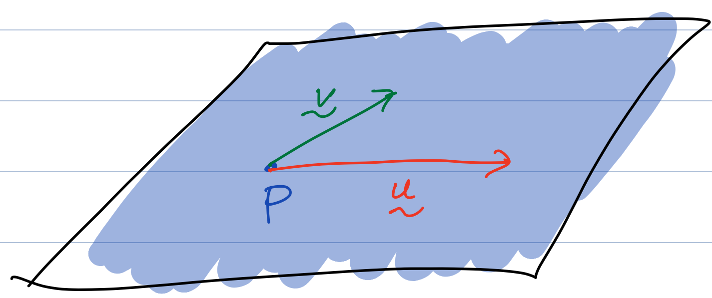
Normal form: $[1,2,3]\cdot(\underset{\sim}{x}-[1,0,6])=0$.

General form:

$$[1,2,3]\cdot[x,y,z]=[1,2,3]\cdot[1,0,6]$$

$$x+2y+3z=19.$$

Vector form and parametric equations



The vector form of the equation of a plane $\mathcal{P}$ in $\mathbb{R}^3$ is

$$\underset{\sim}{x}=\underset{\sim}{p}+s\underset{\sim}{u}+t\underset{\sim}{v}$$

where $\underset{\sim}{p}$ is a specific point on $\mathcal{P}$, $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are direction vectors for $\mathcal{P}$ (with $\underset{\sim}{u}\neq\underset{\sim}{0}$, $\underset{\sim}{v}\neq\underset{\sim}{0}$, $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are not parallel to each other, but they are parallel to $\mathcal{P}$) $s,t\in\mathbb{R}$, and $\underset{\sim}{x}=[x,y,z]$.

The parametric equations for $\mathcal{P}$ are

$$\left.\begin{aligned} x &= p_1 + su_1 + tv_1 \\ y &= p_2 + su_2 + tv_2 \\ z &= p_3 + su_3 + tv_3 \end{aligned}\right\} s,t \in \mathbb{R}$$

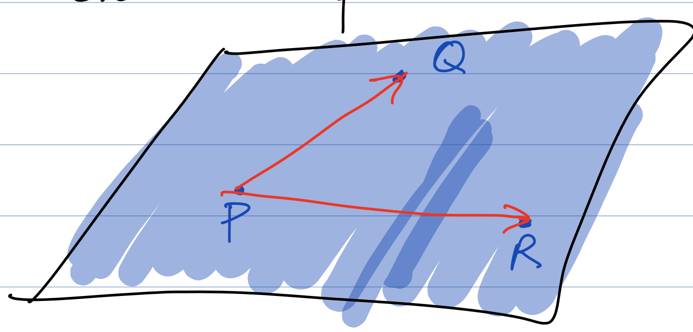
where $\underset{\sim}{p} = [p_1, p_2, p_3]$, $\underset{\sim}{u} = [u_1, u_2, u_3]$, $\underset{\sim}{v} = [v_1, v_2, v_3]$.

Examples Find a vector form and parametric equations for the plane $x + 2y + 3z = 19$.

Find vector form first:

Have point $\underset{\sim}{p} = [1, 0, 6]$

Need two direction vectors: can use $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ where Q and R are two other points on the plane.



To ensure $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are not parallel, need P, Q and R to not lie in the same line

ie. need P, Q and R to be non-collinear.

$x + 2y + 3z = 19$
$P = (1, 0, 6)$

Here, try $Q = (19, 0, 0)$, $R = (0, 8, 1)$

then $\overline{PQ} = (18, 0, -6)$, $\overline{PR} = (-1, 8, -5)$

Vector form:

$$\underset{\sim}{x} = [1, 0, 6] + s[18, 0, -6] + t[-1, 8, -5].$$

$s, t \in \mathbb{R}$

Parametric equations:

$$\left.\begin{aligned} x &= 1 + 18s - t \\ y &= 8t \\ z &= 6 - 6s - 5t \end{aligned}\right\} s, t \in \mathbb{R}.$$

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Topic 4B: Spans

Recall: A vector $\underset{\sim}{v} \in \mathbb{R}^n$ is a *linear combination* of vectors $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \dots, \underset{\sim}{v}_k \in \mathbb{R}^n$ if

$$\underset{\sim}{v} = c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2 + \cdots + c_k \underset{\sim}{v}_k$$

for some scalars $c_1, c_2, \dots, c_k$.

Def^n If $S = \{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \dots, \underset{\sim}{v}_k\}$ is a set of vectors in $\mathbb{R}^n$, then the set of all linear combinations of $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \dots, \underset{\sim}{v}_k$ is called the *span* of $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \dots, \underset{\sim}{v}_k$, and is denoted by

$$\operatorname{span}(\underset{\sim}{v}_1, \underset{\sim}{v}_2, \dots, \underset{\sim}{v}_k) \quad \text{or} \quad \operatorname{span}(S).$$

That is,

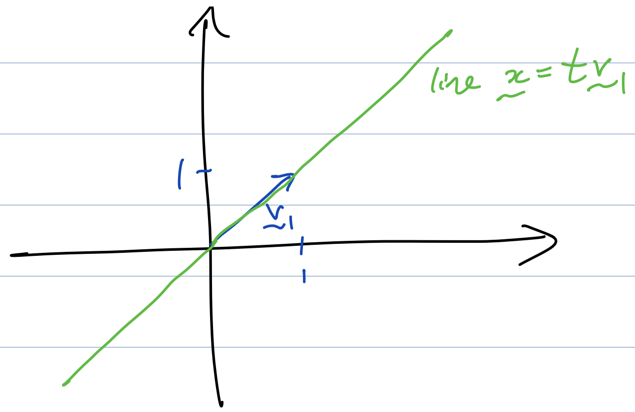
$$\operatorname{span}(S) = \{\underset{\sim}{v} \in \mathbb{R}^n : \underset{\sim}{v} = c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2 + \cdots + c_k \underset{\sim}{v}_k \text{ for some } c_1, c_2, \dots, c_k \in \mathbb{R}\}$$

Examples

1. Let $\underset{\sim}{v}_1 \in \mathbb{R}^2$. Then

$$\operatorname{span}(\underset{\sim}{v}_1) = \{\underset{\sim}{v} \in \mathbb{R}^2 : \underset{\sim}{v} = c_1 \underset{\sim}{v}_1 \text{ for some } c_1 \in \mathbb{R}\}$$

Say $\underset{\sim}{v}_1 = [1, 1]$.



$$\text{span}(\underset{\sim}{v}_1) = \{ t\underset{\sim}{v}_1 : t \in \mathbb{R} \}$$
$$= \{ \underset{\sim}{x} \in \mathbb{R}^2 : \underset{\sim}{x} = t\underset{\sim}{v}_1 \text{ for some } t \in \mathbb{R} \}$$

This is exactly the line in $\mathbb{R}^2$ with vector form $\underset{\sim}{x} = t\underset{\sim}{v}_1$ $t \in \mathbb{R}$.

In $\mathbb{R}^2$, span($\underset{\sim}{v}_1$) (for $\underset{\sim}{v}_1 \neq \underset{\sim}{0}$) is the line through the origin with direction vector $\underset{\sim}{v}_1$.

2. If $\underset{\sim}{v}_1 \in \mathbb{R}^3$, then span($\underset{\sim}{v}_1$) is the line through the origin in $\mathbb{R}^3$ with vector form $\underset{\sim}{x} = t\underset{\sim}{v}_1$, $t \in \mathbb{R}$, ie, line through origin with direction vector $\underset{\sim}{v}_1$.

3. In $\mathbb{R}^2$ or $\mathbb{R}^3$, a line is equal to span($\underset{\sim}{v}_1$) (for some $\underset{\sim}{v}_1 \neq \underset{\sim}{0}$) if and only if that line goes through

the origin.

Why? For any set $S = \{\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_k\}$ in $\mathbb{R}^n$, $\underset{\sim}{0} \in \mathrm{Span}(S)$ as

$$0 = 0\underset{\sim}{v}_1 + 0\underset{\sim}{v}_2 + \cdots + 0\underset{\sim}{v}_k.$$

So $\mathrm{span}(\underset{\sim}{v}_1)$ contains $\underset{\sim}{0}$.

4. Let $\underset{\sim}{v}_1 = [1, 2, 0]$, $\underset{\sim}{v}_2 = [-2, 3, 5]$.

What is $\mathrm{span}(\underset{\sim}{v}_1, \underset{\sim}{v}_2)$?

$$\mathrm{span}(\underset{\sim}{v}_1, \underset{\sim}{v}_2) = \{\underset{\sim}{v} : \underset{\sim}{v} = c_1\underset{\sim}{v}_1 + c_2\underset{\sim}{v}_2, \text{ with } c_1, c_2 \in \mathbb{R}\}$$

$$= \{\underset{\sim}{x} \in \mathbb{R}^3 : \underset{\sim}{x} = s\underset{\sim}{v}_1 + t\underset{\sim}{v}_2,\ s, t \in \mathbb{R}\}$$

This is the plane through the origin with direction vectors $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$. (here $\underset{\sim}{p} = \underset{\sim}{0}$).

5. A plane in $\mathbb{R}^3$ is a span $\Longleftrightarrow$ this plane contains the origin.

6. Is $[8, -5, -15] \in \mathrm{span}([1, 2, 0], [-2, 3, 5])$?

Are there scalars c_1 and c_2 so that

$$[8, -5, -15] = c_1[1, 2, 0] + c_2[-2, 3, 5]?$$

$$[8, -5, -15] = [c_1 - 2c_2, 2c_1 + 3c_2, 5c_2]?$$

Looking at components:
Are there scalars c_1 and c_2 so that

$$\begin{aligned} 8 &= c_1 - 2c_2 \\ -5 &= 2c_1 + 3c_2 \\ -15 &= 0c_1 + 5c_2 \end{aligned}$$

?

Answer in a later Topic.

Def^n Let $S = \{\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_k\}$ be a set of vectors in $\mathbb{R}^n$. If $\text{span}(S) = \mathbb{R}^n$ then we say that S is a spanning set for $\mathbb{R}^n$.

Examples

$\underset{\sim}{e}_1 = [1,0]$
$\underset{\sim}{e}_2 = [0,1]$

1. In $\mathbb{R}^2$, then $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2\}$ is a spanning set.

Why? Let $[x,y] \in \mathbb{R}^2$. We want to show there are scalars $c_1, c_2 \in \mathbb{R}$ so that

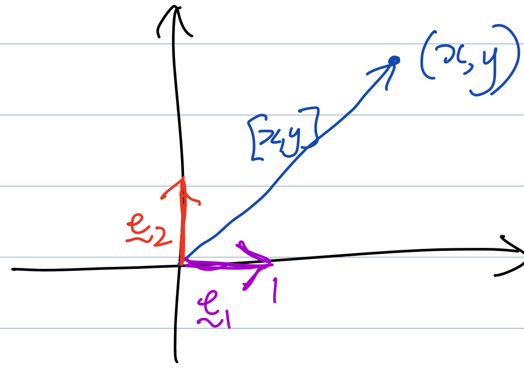
$$[x,y] = c_1 \underset{\sim}{e}_1 + c_2 \underset{\sim}{e}_2.$$

Now

$$\begin{aligned} & [x,y] = c_1[1,0] + c_2[0,1] \\ \iff & [x,y] = [c_1, 0] + [0, c_2] \\ \iff & [x,y] = [c_1, c_2] \\ \iff & x = c_1 \ , \ y = c_2 \end{aligned}$$

So yes, $[x,y] \in \text{span}(\underset{\sim}{e}_1, \underset{\sim}{e}_2)$ as

$$[x,y] = x\underset{\sim}{e}_1 + y\underset{\sim}{e}_2.$$

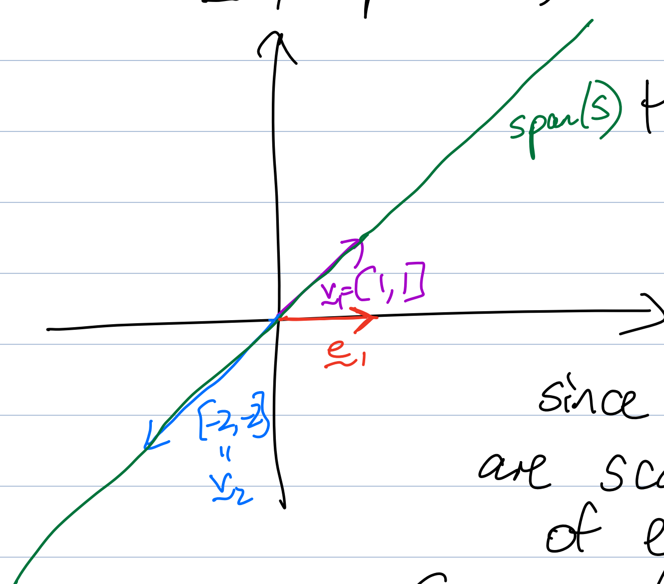


2. In $\mathbb{R}^3$, $\{e_1, e_2, e_3\}$ is a spanning set.
3. In $\mathbb{R}^n$, $\{e_1, e_2, \ldots, e_n\}$ is a spanning set.

exercise.

4. In $\mathbb{R}^2$, $S = \{[1,1], [-2,-2]\}$ is <u>not</u> a spanning set.

To show S does not span $\mathbb{R}^2$, we just need to find one vector $v \in \mathbb{R}^2$ so that $v \notin \text{span}(S)$.



Here,

$$\text{span}(S) = \text{span}(v_1) = \text{span}(v_2)$$

since v_1 and v_2 are scalar multiples of each other.

So span(S) is a line through origin.

To see that $\underset{\sim}{e}_1 \notin \operatorname{span}(S)$, try to solve

$$\underset{\sim}{e}_1 = c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2 \qquad c_1, c_2 \in \mathbb{R}$$

and see that there is no solution.

$$[1,0] = c_1[1,1] + c_2[-2,-2]$$

$$\Leftrightarrow \quad [1,0] = [c_1, c_1] + [-2c_2, -2c_2]$$

$$\Leftrightarrow \quad [1,0] = [c_1 - 2c_2, c_1 - 2c_2]$$

$$\Leftrightarrow \quad \begin{cases} c_1 - 2c_2 = 1 \\ c_1 - 2c_2 = 0 \end{cases}$$

This system has no solution, so $\underset{\sim}{e}_1 \notin \operatorname{span}(S)$.

Hence S is not a spanning set.

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Topic 4C: Linear independence

Def^n A set of vectors $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \ldots, \underset{\sim}{v}_k$ is linearly independent if the only solution to the equation

$$c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2 + \cdots + c_k \underset{\sim}{v}_k = \underset{\sim}{0}$$

with $c_1, c_2, \ldots, c_k \in \mathbb{R}$ is when

$$c_1 = c_2 = \cdots = c_k = 0.$$

A set of vectors is linearly dependent if it is not linearly independent i.e. there exist scalars $c_1, c_2, \ldots, c_k \in \mathbb{R}$ not all zero so that

$$c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2 + \cdots + c_k \underset{\sim}{v}_k = \underset{\sim}{0}.$$

Examples

1. Let $\underset{\sim}{u} = [2,1]$, $\underset{\sim}{v} = [1,-1]$ and $\underset{\sim}{w} = [1,2]$.

Then

$$\underset{\sim}{u} - \underset{\sim}{v} = \underset{\sim}{w}$$

so

$$\underset{\sim}{u} - \underset{\sim}{v} - \underset{\sim}{w} = \underset{\sim}{0}.$$

correction here (was $+\underset{\sim}{w}$ in video)

Since this linear combination has at least one nonzero coefficient, the set $\{\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}\}$ is linearly dependent.

2. The set $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2\}$ in $\mathbb{R}^2$ is linearly independent:

Suppose $c_1 \underset{\sim}{e}_1 + c_2 \underset{\sim}{e}_2 = \underset{\sim}{0}$

Then

$$c_1 [1, 0] + c_2 [0, 1] = [0, 0]$$

$$\Rightarrow \quad [c_1, c_2] = [0, 0]$$

$$\Rightarrow \quad c_1 = c_2 = 0.$$

3. Two vectors $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$ in $\mathbb{R}^n$ are linearly independent $\Leftrightarrow$ they are <u>not</u> scalar multiples of each other.

Equivalently: $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$ are linearly dependent $\Leftrightarrow$ they are scalar multiples of each other.

Suppose $\underset{\sim}{v}_2 = c\underset{\sim}{v}_1$, where $c \in \mathbb{R}$.

Then $c\underset{\sim}{v}_1 - \underset{\sim}{v}_2 = \underset{\sim}{0}$.

Since the coefficient of $\underset{\sim}{v}_2$ is nonzero, the set $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2\}$ is linearly dependent.

Suppose $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2\}$ is linearly dependent. Then there exist $c_1, c_2 \in \mathbb{R}$, not both 0, so that $c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2 = \underset{\sim}{0}$.

Without loss of generality, assume $c_1 \neq 0$.
Then we can divide through by c_1:
(i.e. multiply by $\frac{1}{c_1}$)

$$\underset{\sim}{v}_1 + \left(\frac{c_2}{c_1}\right)\underset{\sim}{v}_2 = \underset{\sim}{0}$$

$$\Longrightarrow \quad \underset{\sim}{v}_1 = \left(-\frac{c_2}{c_1}\right)\underset{\sim}{v}_2$$

so $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$ are scalar multiples of each other.

4. Vector form for planes in $\mathbb{R}^3$:

$$\underset{\sim}{x} = \underset{\sim}{p} + s\underset{\sim}{u} + t\underset{\sim}{v}$$

where $\underset{\sim}{p}$ is a point on plane,
$\underset{\sim}{u}$ and $\underset{\sim}{v}$ are nonzero vectors which are parallel to the plane but are not scalar multiples of each other.
In other words, $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are linearly independent vectors (parallel to the plane).

Theorem A set of vectors $\{\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_k\}$ in $\mathbb{R}^n$ is linearly dependent if and only if at least one of them can be expressed as a linear combination of the others i.e. at least one $\underset{\sim}{v}_i$ linearly depends on the other vectors.

Proof exercise.

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Topic 4D: Systems of Linear Equations

Recall:

- the general equation for a line in $\mathbb{R}^2$ is
$$ax + by = c$$
- the general equation for a plane in $\mathbb{R}^3$ is
$$ax + by + cz = d$$

Def^n A linear equation in the variables $x_1, x_2, \ldots, x_n$ is an equation of the form
$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$
where $a_1, a_2, \ldots, a_n \in \mathbb{R}$ are the coefficients and $b \in \mathbb{R}$ is the constant term.

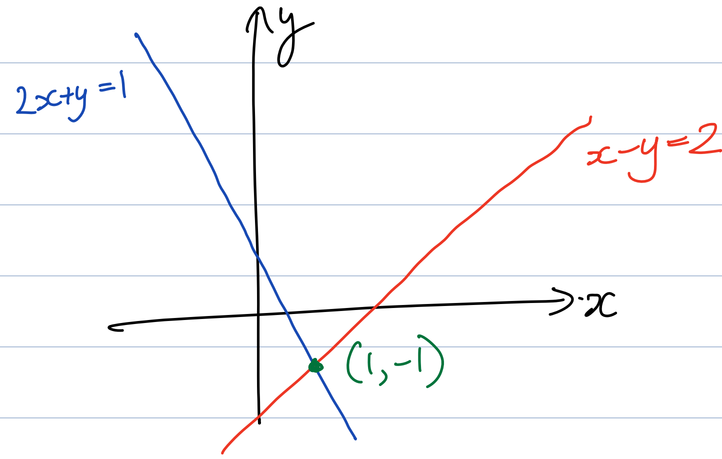
A system of linear equations is a finite set of linear equations, all in the same variables.

Examples

Consider 3 systems of linear equations:

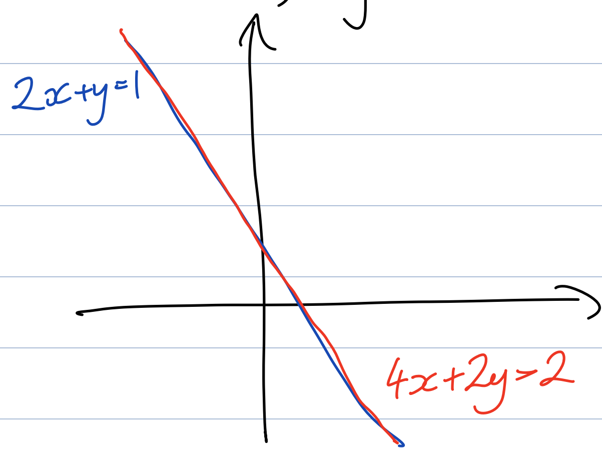
(a) $\begin{cases} 2x + y = 1 \\ x - y = 2 \end{cases}$ (b) $\begin{cases} 2x + y = 1 \\ 4x + 2y = 2 \end{cases}$ (c) $\begin{cases} 2x + y = 1 \\ 2x + y = 0 \end{cases}$

(a) has unique solution $x = 1$, $y = -1$.

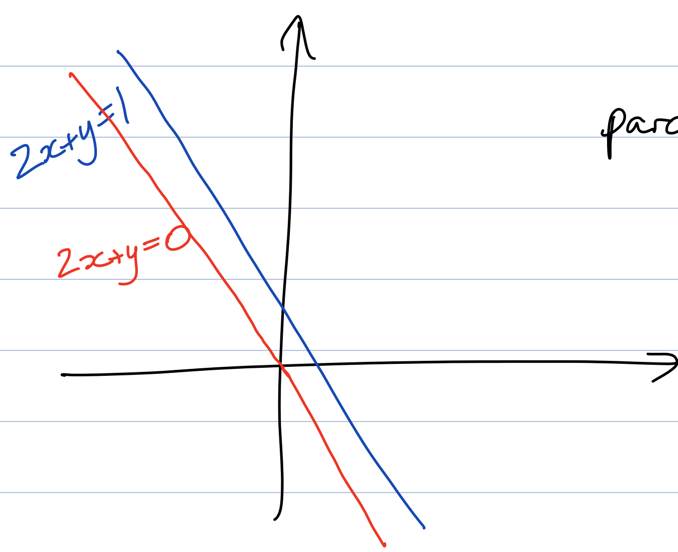


(b) has infinitely many solutions,

$x = \frac{1}{2} - \frac{1}{2}t$, $y = t$ for $t \in \mathbb{R}$.



(c) has no solutions



parallel lines

We will see that a system of linear equations has either

(a) a unique solution
(b) infinitely many solutions
(c) no solution.

$\left.\begin{matrix}\text{(a)}\\ \text{(b)}\end{matrix}\right\}$ consistent

(c) } inconsistent.

The system is consistent if it has at least one solution, inconsistent if it has no solutions.

Notation We can write a general system of m linear equations in n variables $x_1, x_2, \ldots, x_n$ as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \;\; & \;\;\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

This system has augmented matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Examples

1. $$\begin{aligned} 2x + y &= 1 \\ x - y &= 2 \end{aligned}$$ has augmented matrix $\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 1 & -1 & 2 \end{array}\right]$

2. In Topic 4B (Spans) we asked: is $[8,-5,-15]$ in $\mathrm{span}([1,2,0],[-2,3,5])$?

We got system

$$\begin{aligned} c_1 - 2c_2 &= 8 \\ 2c_1 + 3c_2 &= -5 \\ 5c_2 &= -15 \end{aligned} \qquad \text{i.e. } 0c_1 + 5c_2 = -15$$

(Variables here are c_1, c_2.)

Augmented matrix:

$$\left[\begin{array}{cc|c} 1 & -2 & 8 \\ 2 & 3 & -5 \\ 0 & 5 & -15 \end{array}\right].$$

Def^{n} A system of linear equations is homogeneous if the constant term in each equation is 0.

i.e. augmented matrix will be

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ \vdots & \vdots & & \vdots & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 \end{array}\right]$$

Homogeneous systems always have at least one solution i.e. $x_1 = x_2 = \cdots = x_n = 0$.

Example

$$\begin{aligned} 2x - y + z &= 0 \\ x + y - 5z &= 0 \end{aligned}$$

is a homogeneous system. One solution is $x=0,\ y=0,\ z=0.$