COMP2022|2922
Models of Computation

Introduction to Predicate Logic

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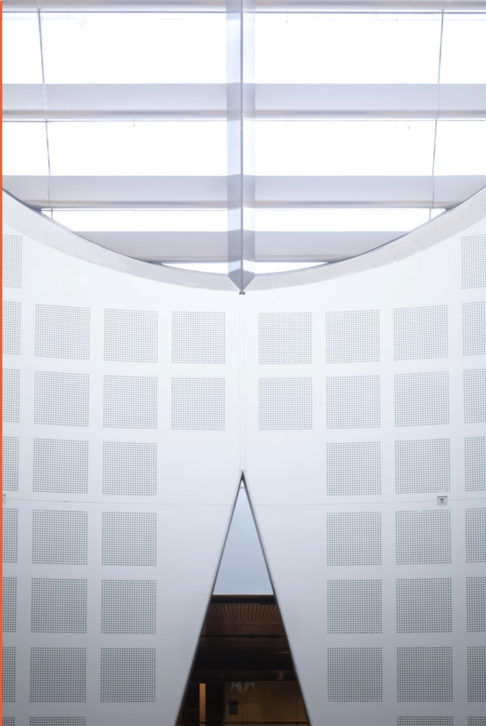
THE UNIVERSITY OF SYDNEY



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Equivalences in Predicate Logic

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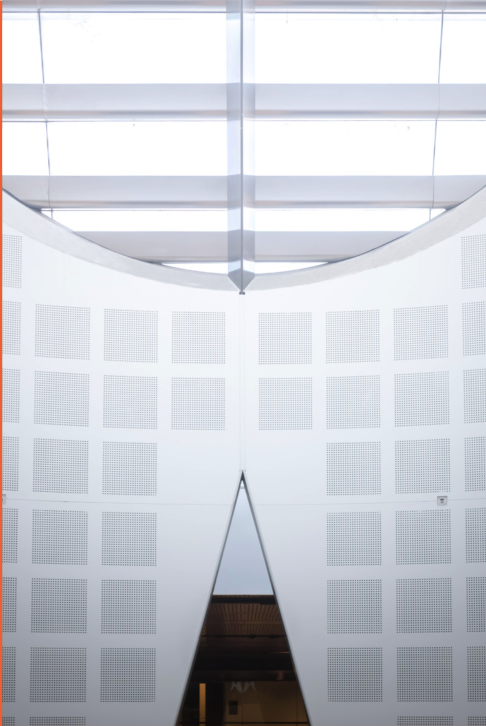

Formulas that “mean the same thing” are called equivalent. We now study common equivalences, also called laws.

Equivalences

F and G are logically equivalent ($F \equiv G$) if the truth value of F under α equals the truth value of G under α, for all assignments, and all domains and predicates.

All equivalences for propositional logic also hold for predicate logic.

Example (De Morgan's Law)

$\neg(\exists x P(x) \wedge Q(y))$ and $\neg\exists x P(x) \vee \neg Q(y)$ are equivalent.

Equivalences involving quantifiers

For all formulas F, G:

(Q. Negation)
$$\neg \forall x F \equiv \exists x \neg F$$
$$\neg \exists x F \equiv \forall x \neg F$$

(Q. Unification)
$$(\forall x F \wedge \forall x G) \equiv \forall x (F \wedge G)$$
$$(\exists x F \vee \exists x G) \equiv \exists x (F \vee G)$$

(Q. Transposition)
$$\forall x \forall y F \equiv \forall y \forall x F$$
$$\exists x \exists y F \equiv \exists y \exists x F$$

(Q. Extraction) if $x \notin \mathrm{Free}(G)$:
$$(\forall x F \wedge G) \equiv \forall x (F \wedge G)$$
$$(\forall x F \vee G) \equiv \forall x (F \vee G)$$
$$(\exists x F \wedge G) \equiv \exists x (F \wedge G)$$
$$(\exists x F \vee G) \equiv \exists x (F \vee G)$$

Equivalences

Here are informal reasons behind some of these equivalences:[1]

1. $\neg\forall x F \equiv \exists x \neg F$
   - the LHS says that not all x satisfy F,
   - which means the same thing as some x doesn't satisfy F,
   - which means that some x does satisfy $\neg F$,
   - which is what the RHS says.
2. $(\forall x F \wedge \forall x G) \equiv \forall x(F \wedge G)$
   - the LHS says that F holds for every x and G holds for every x,
   - which is the same as saying both F and G hold for every x,
   - which is what the RHS says.
3. $\forall x \forall y F \equiv \forall y \forall x F$
   - Both sides say that F holds for all values of the listed variables.
4. $(\forall x F \wedge G) \equiv \forall x(F \wedge G)$ if $x \notin \mathrm{Free}(G)$
   - LHS says F holds for every x, and G holds.
   - RHS says F and G hold for every x; but G doesn't depend on the value of x.

[1] To prove them formally, use the inductive definition of truth-value.

Equivalences

Example

Show that $\neg(\exists x P(x,y) \vee \forall z \neg R(z)) \equiv \forall x \exists z(\neg P(x,y) \wedge R(z))$

$$
\begin{aligned}
&\neg(\exists x\, P(x,y) \vee \forall z \neg R(z)) \\
&\qquad \equiv (\neg\exists x P(x,y) \wedge \neg\forall z \neg R(z)) && \text{DeMorgan's Laws} \\
&\qquad \equiv (\forall x \neg P(x,y) \wedge \exists z \neg\neg R(z)) && \text{Quantifier Negation} \\
&\qquad \equiv (\forall x \neg P(x,y) \wedge \exists z R(z)) && \text{Double Negation} \\
&\qquad \equiv \forall x(\neg P(x,y) \wedge \exists z R(z)) && \text{Quantifier Extraction} \\
&\qquad \equiv \forall x(\exists z R(z) \wedge \neg P(x,y)) && \text{Comm } \wedge \\
&\qquad \equiv \forall x \exists z(R(z) \wedge \neg P(x,y)) && \text{Quantifier Extraction} \\
&\qquad \equiv \forall x \exists z(\neg P(x,y) \wedge R(z)) && \text{Comm. } \wedge
\end{aligned}
$$

$$\forall x \exists z (\neg P(x, y) \wedge R(z))$$

This formula has a very nice shape... all the quantifiers are out the front! this can make it easier to understand and manipulate.

Normal Forms

Definition
A formula F is in negation normal form (NNF) if negations only occur immediately infront of atomic formulas.

$\neg P(x) \to Q(y)$ is in NNF

$\neg(P(x) \to Q(y))$ is not in NNF

Normal Forms

Theorem

For every formula F there is an equivalent formula in NNF.

Algorithm ("push negations inwards by applying Q. Negation and DM")

Substitute in F every occurence of a subformula of the form $\neg\neg G$ by G, and

$$\neg\forall x F \text{ by } \exists x \neg F \qquad \neg\exists x F \text{ by } \forall x \neg F$$
$$\neg(G \land H) \text{ by } (\neg G \lor \neg H) \qquad \neg(G \lor H) \text{ by } (\neg G \land \neg H)$$

until no such subformulas occur, and return the result.

Normal Forms

Definition
A formula F is in prenex normal form (PNF) if it has the form

$$Q_1 x_1 Q_2 x_2 \cdots Q_n x_n F$$

where each $Q_i \in \{\exists, \forall\}$ is a quantifier symbol, the x_is are variables, $n \geq 0$ (so, there may be no quantifiers in the prefix), and F does not contain a quantifier.

$\forall x \exists y (P(x) \vee L(x, y))$ is in PNF.
$\forall x (P(x) \vee \exists y L(x, y))$ is not in PNF.

Normal Forms

Theorem

For every formula F there is an equivalent formula in PNF.

Algorithm ("pull quantifiers out the front by applying Q. Extraction")

1. Put F in NNF, call the result F'.
2. Substitute in F' every occurence of a subformula of the form

$$\begin{gathered}(\forall x F \wedge G) \text{ by } \forall x(F \wedge G)\\(\forall x F \vee G) \text{ by } \forall x(F \vee G)\\(\exists x F \wedge G) \text{ by } \exists x(F \wedge G)\\(\exists x F \vee G) \text{ by } \exists x(F \vee G)\end{gathered}$$

   until no such subformulas occur (use commutativity to handle $(G \wedge \forall x F)$, etc.), and return the result.

   NB. To apply these equivalences we need that $x \notin \mathrm{Free}(G)$.
   This can always be achieved by renaming the bound variable x.

Logical consequence

Definition

A sentence F is a logical consequence of the set $E_1, \cdots, E_k$ of sentences if for every domain, predicates, assignments, if all of the $E_1, \cdots, E_k$ are true, also F is true. In this case we write

$$E_1, \cdots, E_k \models F$$

Example

- $\forall x\, R(x,x)$ is a logical consequence of $\forall x \forall y\, R(x,y)$.
- $P(c)$ is a logical consequence of $Q(c), \forall x\, (Q(x) \rightarrow P(x))$.

In case $\models F$ we say that F is valid.

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ND for predicate logic

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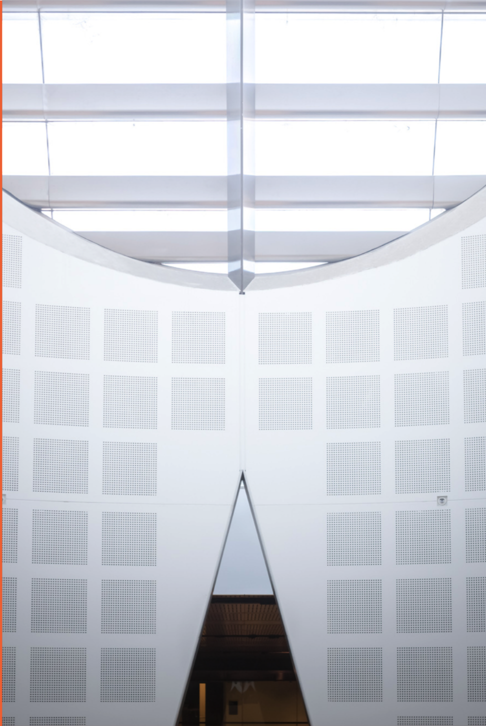

Deductive systems are a syntactic mechanism for deriving validities as well as logical consequences from assumptions

Natural deduction

- We extend ND for propositional logic with rules to handle quantifiers.
- Each quantifier symbol $\exists, \forall$ has two types of rules:
    1. Introduction rules introduce the quantifier
    2. Elimination rules remove the quantifier

Replacing free variables by constants.

Definition

For a formula F, variable x, constant c, we can obtain a formula

$$F[c/x]$$

by simultaneously replacing all free occurrences of x in F by c.

The idea is that whatever F said about x, now $F[c/x]$ says about c.

$\forall$ elimination

$(\forall E)$	$\dfrac{\forall x F}{F[c/x]}$

$(\forall E)$ formalises the reasoning

> *If we know that F holds for every x, then it must hold, in particular, taking $x = c$*

$\forall$ introduction

$(\forall I)$	$\dfrac{F[c/x]}{\forall x F}$
	where c is a constant, not occuring in F, nor in any of the assumptions of $F[c/x]$.

$(\forall I)$ formalises the reasoning

> *Let c be any element ... (insert proof of $F[c/x]$). Since c was arbitrary, deduce F holds for all x.*

That c is arbitrary is captured by requiring that c is not in the assumptions used to prove $F[c/x]$, and so c is not constrained in any way.

$\forall x \forall y P(x,y) \vdash \forall y \forall x P(x,y)$

Plan: instantiate the variables to new constants, then introduce them in the reverse order.

Line	Assumptions	Formula	Justification	References
1	1	$\forall x \forall y P(x,y)$	Asmp. I	
2	1	$\forall y P(c,y)$	$\forall$ E	1
3	1	$P(c,d)$	$\forall$ E	2
4	1	$\forall x P(x,d)$	$\forall$ I *	3
5	1	$\forall y \forall x P(x,y)$	$\forall$ I **	4

* the constant c does not occur in F (i.e., $P(x,d)$), nor in the formula of its assumption (in line 1).

** the constant d does not occur in F (i.e., $\forall x P(x,y)$), nor in the formula of its assumption (in line 1).

$$\forall x(P(x) \land Q(x)) \vdash \forall x P(x) \land \forall x Q(x)$$

Plan: instantiate the variable to a new constant, split, then introduce the variables back.

Line	Assumptions	Formula	Justification	References
1	1	$\forall x(P(x) \land Q(x))$	Asmp. I	
2	1	$P(c) \land Q(c)$	$\forall$ E	1
3	1	$P(c)$	$\land$ E	2
4	1	$Q(c)$	$\land$ E	2
5	1	$\forall x P(x)$	$\forall$ I *	3
6	1	$\forall x Q(x)$	$\forall$ I *	5
7	1	$\forall x P(x) \land \forall x Q(x)$	$\land$ I	4,6

* c does not appear in $P(x)$ nor in the assumption 1

What is wrong with the following "proof" of $P(c) \vdash \forall x P(x)$?

Line	Assumptions	Formula	Justification	References
1	1	$P(c)$	Asmp. I	
2	1	$\forall x P(x)$	$\forall$ I	1

What is wrong with the following "proof" of $P(c) \vdash \forall x P(x)$?

Line	Assumptions	Formula	Justification	References
1	1	$P(c)$	Asmp. I	
2	1	$\forall x P(x)$	$\forall$ I	1

$(\forall I)$	$\dfrac{F[c/x]}{\forall x F}$ where c is a constant, not occuring in F, nor in any of the assumptions of $F[c/x]$.

- $F = P(x)$
- $F[c/x] = P(c)$
- The assumption of $F[c/x]$ is $P(c)$.

$\exists$ Introduction

$(\exists I)$	$\dfrac{F[c/x]}{\exists x F}$

$(\exists I)$ formalises the reasoning

> *If we know that F holds for a specific constant c, then we know it holds for some x.*

$\forall x P(x) \vdash \exists x P(x)$

Plan: instantiate x arbitrarily, then introduce x existentially.

Line	Assumptions	Formulas	Just.	Ref.
1	1	$\forall x P(x)$	Asmp. I	
2	1	$P(c)$	$\forall$ E	1
3	1	$\exists x P(x)$	$\exists$ I	2

$\neg\exists x \neg P(x) \vdash \forall x P(x)$

Plan: take a fresh constant c, assume $\neg P(c)$, get a contradiction, deduce $P(c)$, and then that $\forall x P(x)$ since c was arbitrary.

Line	Asmp.	Form.	Just.	Ref.
1	1	$\neg\exists x \neg P(x)$	Asmp. I	
2	2	$\neg P[c/x]$	Asmp. I	
3	2	$\exists x \neg P(x)$	$\exists I$	2
4	1,2	$\perp$	$\perp I$	1,3
5	1	$P[c/x]$	$\neg E$	2,4
6	1	$\forall x P(x)$	$\forall I$*	5

* c does not appear in $P(x)$ nor in the assumption 1

$\exists$ Elimination

$(\exists E)$	$$\frac{\exists x F \qquad F[c/x] \vdash G}{G}$$ where c is a constant symbol, not occuring in F, nor in G, nor in any assumption used in the proof of G except for $F[c/x]$

$(\exists E)$ formalises the reasoning

> *From $\exists x F$ we know there is an x that satisfies F, so we take one and call it c. If c is new, and has not been used so far, and we manage to derive G, then we can deduce G from that weaker assumption that there is some x that satisfies F (even if we don't know which one).*

$\exists$ Elimination

$(\exists E)$	$$\frac{\exists x F \qquad F[c/x] \vdash G}{G}$$ where c is a constant symbol, not occuring in F, nor in G, nor in any assumption used in the proof of G except for $F[c/x]$

How to use $(\exists E)$?

1. Assume $F[c/x]$ ensuring that c does not occur in F.
2. Derive G making sure that c is not in the assumption set of G except for $F[c/x]$.
3. Cancel the assumption $F[c/x]$, and conclude G.

$\forall x(Q(x) \to P(y)), \exists x Q(x) \vdash P(y)$

Line	Assumptions	Formulas	Just.	Ref.
1	1	$\forall x(Q(x) \to P(y))$	Asmp. I	
2	2	$\exists x Q(x)$	Asmp. I	
3	1	$Q(c) \to P(y)$	$\forall$ E	1
4	4	$Q(c)$	Asmp. I	
5	1,4	$P(y)$	$\to$ E	3,4
6	1,2	$P(y)$	$\exists$ E	2,4,5

What is wrong with the following "proof" of $\exists x P(x) \vdash \forall x P(x)$?

Line	Asmp.	Form.	Just.	Ref.
1	1	$\exists x P(x)$	Asmp. I	
2	1	$P(c)$	$\exists$ E	1
3	1	$\forall x P(x)$	$\forall$ I	2

Here is the faulty argument in natural language:

1. *We are given that P is satisfied by some x.*
2. *Let c be such an x.*
3. *Since c was chosen arbitrarily (?!), conclude that every x satisfies P.*

Wrapping up

ND for predicate logic.

- Allows a machine to check if a given proof is correct.
- It is sound and complete.

$$E_1, \cdots, E_k \models F \text{ if and only if } E_1, \cdots, E_k \vdash F$$

- However, unlike Propositional Logic, the problem of checking if F is a logical consequence of $E_1, \cdots, E_k$ is undecidable.
- This means that finding proofs of Predicate Logic cannot be fully automated.