The University of Sydney

MATH1002 Linear Algebra

Semester 1 **Tutorial Exercises for Week 6 — Solutions** 2022

1. Solve the following systems of equations. Give a geometric interpretation of your solution in each case.

(i)
$$\begin{aligned} 2x + 3y &= 2 \\ 4x - 5y &= 0 \end{aligned}$$

(ii)
$$\begin{aligned} 2x + 3y + z &= 1 \\ x + y + z &= 4 \\ 2x - 2y + 7z &= 40 \end{aligned}$$

Solution:

(i) We have

$$\left[\begin{array}{cc|c} 2 & 3 & 2 \\ 4 & -5 & 0 \end{array}\right] \xrightarrow{R_2 \mapsto R_2 - 2R_1} \left[\begin{array}{cc|c} 2 & 3 & 2 \\ 0 & -11 & -4 \end{array}\right],$$

so that $y = \frac{4}{11}$, and $x = \frac{5}{11}$.

The equations $2x + 3y = 2$ and $4x - 5y = 0$ describe two non-parallel lines in $\mathbb{R}^2$. The solution of the system gives the coordinates of the intersection of these two lines.

(ii) We have

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 1 & 1 & 1 & 4 \\ 2 & -2 & 7 & 40 \end{array}\right] \xrightarrow{R_2 \mapsto R_2 - \frac{1}{2}R_1,\ R_3 \mapsto R_3 - R_1} \left[\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & \frac{7}{2} \\ 0 & -5 & 6 & 39 \end{array}\right]$$

$$\xrightarrow{R_3 \mapsto R_3 - 10R_2} \left[\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & \frac{7}{2} \\ 0 & 0 & 1 & 4 \end{array}\right],$$

so that $z = 4$, $y = -3$, and $x = 3$.

Each equation in the system describes a plane in $\mathbb{R}^3$. These three planes intersect at one point whose coordinates are the solution of the system.

2. Find parametric equations for the line of intersection of the two planes in each of the following cases:

(i)
$$\begin{aligned} x + y + z &= 2 \\ x - y + 3z &= 0 \end{aligned}$$

(ii)
$$\begin{aligned} -3x + 2y + 7z &= 1 \\ 5x - 3y - 2z &= -2 \end{aligned}$$

Solution:

(i)
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 2 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right],$$

so that, by back substitution, $z = t,\ y = 1 + t\,,\ x = 1 - 2t\,.$

(ii)
$$\left[\begin{array}{ccc|c} -3 & 2 & 7 & 1 \\ 5 & -3 & -2 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 6 & -4 & -14 & -2 \\ 5 & -3 & -2 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & -12 & 0 \\ 5 & -3 & -2 & -2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & -12 & 0 \\ 0 & 2 & 58 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 17 & -1 \\ 0 & 1 & 29 & -1 \end{array}\right],$$

so that, by back substitution, $z = t,\ y = -1 - 29t\,,\ x = -1 - 17t\,.$

3. Solve the following homogeneous systems of equations:

(i)
$$\begin{aligned} x + 2y + 3z &= 0 \\ 3x + 2y + z &= 0 \end{aligned}$$

(ii)
$$\begin{aligned} -x + y + z - w &= 0 \\ 2x + z + w &= 0 \\ x - 2y + z + 3w &= 0 \end{aligned}$$

Solution:

(i) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$,

so that, by back substitution, $z = t,\ y = -2t\,,\ x = t\,.$

(ii) $\left[\begin{array}{cccc|c} -1 & 1 & 1 & -1 & 0 \\ 2 & 0 & 1 & 1 & 0 \\ 1 & -2 & 1 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & 3 & -1 & 0 \\ 0 & -1 & 2 & 2 & 0 \end{array}\right]$

$\sim \left[\begin{array}{cccc|c} 1 & 0 & -3 & -1 & 0 \\ 0 & 1 & -2 & -2 & 0 \\ 0 & 0 & 7 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{2}{7} & 0 \\ 0 & 1 & 0 & -\frac{8}{7} & 0 \\ 0 & 0 & 1 & \frac{3}{7} & 0 \end{array}\right]$,

so that, by back substitution, $w = t,\ z = -\frac{3}{7}t\,,\ y = \frac{8}{7}t\,,\ x = -\frac{2}{7}t\,.$

4. Give a very brief reason why a homogeneous system can never be inconsistent.

Solution: If we assign zero to each variable then each equation is satisfied, so that there is at least one solution of any homogeneous system. Hence all homogeneous systems are consistent.

5. Let $\mathbf{u} = [3, 2, -1]$ and $\mathbf{v} = [1, 3, 1]$. Find all vectors $\mathbf{w} = [x, y, z]$ such that $\mathbf{u} \cdot \mathbf{w} = 1$ and $\mathbf{v} \cdot \mathbf{w} = 5$.

Solution: Since $\mathbf{u} \cdot \mathbf{w} = 1$ we have $3x + 2y - z = 1$; since $\mathbf{v} \cdot \mathbf{w} = 5$ we have $x + 3y + z = 5$.

$\left[\begin{array}{ccc|c} 3 & 2 & -1 & 1 \\ 1 & 3 & 1 & 5 \end{array}\right] \overset{R_2 \to 3R_2 - R_1}{\sim} \left[\begin{array}{ccc|c} 3 & 2 & -1 & 1 \\ 0 & 7 & 4 & 14 \end{array}\right] \overset{R_1 \to R_1 - \frac{2}{7}R_2}{\sim} \left[\begin{array}{ccc|c} 3 & 0 & -\frac{15}{7} & -3 \\ 0 & 7 & 4 & 14 \end{array}\right]$

$\overset{\substack{R_1 \to \frac{1}{3}R_1 \\ R_2 \to \frac{1}{7}R_2}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{5}{7} & -1 \\ 0 & 1 & \frac{4}{7} & 2 \end{array}\right]$

By back substitution, $x = \frac{5}{7}t - 1, y = \frac{4}{7}t + 2, z = t$. So all such vectors $\mathbf{w}$ have the form $[\frac{5}{7}t - 1, -\frac{4}{7}t + 2, t]$.

6. Determine whether the following systems of equations are consistent or inconsistent.

(i) $\begin{array}{rcrcr} x & + & y & = & 5 \\ 2x & + & 2y & = & 8 \end{array}$ (ii) $\begin{array}{rcrcr} x & + & 3y & = & 2 \\ 5x & + & 6y & = & -12 \end{array}$ (iii) $\begin{array}{rcrcr} 4x & - & 6y & = & 2 \\ 6x & - & 9y & = & 3 \end{array}$

(iv) $\begin{array}{rcrcrcr} 3x & + & 3y & + & z & = & -1 \\ x & + & 2y & + & 2z & = & -3 \\ 2x & + & y & - & z & = & 2 \end{array}$ (v) $\begin{array}{rcrcrcr} x & - & y & - & 2z & = & 1 \\ 2x & - & 3y & - & 4z & = & -2 \\ 3x & - & 2y & - & 6z & = & -3 \end{array}$

Solution:

(i) $\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 2 & 2 & 8 \end{array}\right] \overset{R_2 \to R_2 - 2R_1}{\sim} \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 0 & -2 \end{array}\right]$,

inconsistent (no solutions).

(ii) $\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 5 & 6 & -12 \end{array}\right] \overset{R_2 \to R_2 - 5R_1}{\sim} \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & -9 & -22 \end{array}\right]$,

consistent (one solution).

(iii) $\left[\begin{array}{cc|c} 4 & 6 & 2 \\ 6 & 9 & 3 \end{array}\right] \overset{R_2 \to R_2 - \frac{3}{2}R_1}{\sim} \left[\begin{array}{cc|c} 4 & -6 & 2 \\ 0 & 0 & 0 \end{array}\right]$,

consistent (infinitely many solutions).

(iv) $\left[\begin{array}{ccc|c} 3 & 3 & 1 & -1 \\ 1 & 2 & 2 & -3 \\ 2 & 1 & -1 & 2 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 2 & -3 \\ 3 & 3 & 1 & -1 \\ 2 & 1 & -1 & 2 \end{array}\right]$

$\overset{\substack{R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1}}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 2 & -3 \\ 0 & -3 & -5 & 8 \\ 0 & -3 & -5 & 8 \end{array}\right]$

$\overset{R_3 \to R_3 - R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 2 & -3 \\ 0 & -3 & -5 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right]$,

consistent (infinitely many solutions).

(v) $\left[\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ 2 & -3 & -4 & -2 \\ 3 & -2 & -6 & -3 \end{array}\right] \overset{\substack{R_2\to R_2-2R_1 \\ R_3\to R_3-3R_1}}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ 0 & -1 & 0 & -4 \\ 0 & 1 & 0 & -6 \end{array}\right]$

$\overset{R_3\to R_3+R_2}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 0 & -10 \end{array}\right]$,

inconsistent (no solutions).

7. Solve the following systems of equations by first row reducing to reduced row echelon form.

(i)
$$\begin{aligned} 2x + y &= 3 \\ 4x + y &= 7 \\ 2x + 5y &= -1 \end{aligned}$$

(ii)
$$\begin{aligned} x + 2y + 7z &= 5 \\ x + y + 4z &= 3 \\ 2x + 3y + 11z &= 7 \end{aligned}$$

(iii)
$$\begin{aligned} x - 3y - 2z &= 0 \\ -x + 2y + z &= 0 \\ 2x + 4y + 6z &= 0 \end{aligned}$$

(iv)
$$\begin{aligned} x + 2y + z - w &= 4 \\ 2x + 4y - z + 4w &= -1 \\ -x - 2y + 2z - 5w &= 5 \end{aligned}$$

Solution:

(i) $\left[\begin{array}{cc|c} 2 & 1 & 3 \\ 4 & 1 & 7 \\ 2 & 5 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0.5 & 1.5 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$

so that $y = -1$ and $x = 2$.

(ii) $\left[\begin{array}{ccc|c} 1 & 2 & 7 & 5 \\ 1 & 1 & 4 & 3 \\ 2 & 3 & 11 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 7 & 5 \\ 0 & -1 & -3 & -2 \\ 0 & -1 & -3 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 7 & 5 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$,

so that the system is inconsistent; that is, it has no solution.

(iii) $\left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ -1 & 2 & 1 & 0 \\ 2 & 4 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 10 & 10 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim$

$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$,

so that $x = -t$, $x = -t$ and $z = t$ for $t \in \mathbb{R}$.

(iv) $\left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 4 \\ 2 & 4 & -1 & 4 & -1 \\ -1 & -2 & 2 & -5 & 5 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 4 \\ 0 & 0 & -3 & 6 & -9 \\ 0 & 0 & 3 & -6 & 9 \end{array}\right]$

$\sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 4 \\ 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$,

so that, by back substitution, $w = t\,,\ z = 3 + 2t\,,\ y = s\,,\ x = 1 - 2s - t\,.$

8. (i) Show that $\mathbb{R}^2 = \operatorname{span}([3, -2], [0, 1])$.

(ii) Show that $\mathbb{R}^3 = \operatorname{span}([1, 1, 0], [0, 1, 1], [1, 0, 1])$.

Solution:

(i) We show that an arbitrary vector $[x, y]$ can be written as a linear combination of the vectors $[3, -2]$ and $[0, 1]$. So we need to find $a, b \in \mathbb{R}$ such that

$$\begin{aligned} 3a &= x \\ -2a + b &= y. \end{aligned}$$

We see that $a = x/3$ and $b = y + 2x/3$ is a solution. Hence for any vector $[x, y] \in \mathbb{R}^2$ we have

$$[x, y] = \frac{x}{3}[3, -2] + \Big(y + \frac{2x}{3}\Big)[0, 1].$$

Conversely, let $\mathbf{v}$ be an arbitrary vector in $\text{span}([3,-2],[0,1])$, then there are $s,t \in \mathbb{R}$ such that $\mathbf{v} = s[3,-2]+t[0,1]$. This implies that $\mathbf{v} = [3s,-2s+t]$, which is certainly in $\mathbb{R}^2$.
Hence $\mathbb{R}^2 = \text{span}([3,-2],[0,1])$.

(ii) We show that an arbitrary vector $[x,y,z]$ can be written as a linear combination of the vectors $[1,1,0]$, $[0,1,1]$, and $[1,0,1]$. So we need to find $a,b,c \in \mathbb{R}$ such that

$$\begin{aligned} a+c&=x\\ a+b&=y\\ b+c&=z. \end{aligned}$$

Row reducing gives

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & x\\ 1 & 1 & 0 & y\\ 0 & 1 & 1 & z \end{array}\right] \sim \left[\begin{array}{ccc|r} 1 & 0 & 1 & x\\ 0 & 1 & -1 & y-x\\ 0 & 0 & 2 & z-(y-x) \end{array}\right],$$

and we see that the solution is

$$a = \frac{1}{2}(x+y-z),\ b = \frac{1}{2}(z+y-x),\ c = \frac{1}{2}(z+x-y).$$

Hence for any vector $[x,y,z] \in \mathbb{R}^3$ we have

$$[x,y,z] = \frac{1}{2}(x+y-z)[1,1,0] + \frac{1}{2}(z+y-x)[0,1,1] + \frac{1}{2}(z+x-y)[1,0,1].$$

Conversely, let $\mathbf{v} \in \text{span}([1,1,0],[0,1,1],[1,0,1])$, then there are scalars $\lambda_1,\lambda_2,\lambda_3 \in \mathbb{R}$ such that

$$\mathbf{v} = \lambda_1[1,1,0]+\lambda_2[0,1,1]+\lambda_3[1,0,1].$$

This gives us $\mathbf{v} = [\lambda_1+\lambda_3, \lambda_1+\lambda_2, \lambda_2+\lambda_3]$ which is certainly in $\mathbb{R}^3$.
Hence $\mathbb{R}^3 = \text{span}([1,1,0],[0,1,1],[1,0,1])$.

9. Determine whether the sets of vectors are linearly independent. For any sets that are linearly dependent, find a dependence relation among the vectors.

(i) $[0,1,2]$, $[2,1,3]$, $[2,0,1]$.
(ii) $[3,4,5]$, $[6,7,8]$, $[0,0,0]$.
(iii) $[1,-1,1,0]$, $[-1,1,0,1]$, $[1,0,1,-1]$, $[0,1,-1,1]$.

Solution:

(i) We see by inspection that $[2,0,1] = [2,1,3]-[0,1,2]$. So

$$[2,0,1]+[0,1,2]-[2,1,3] = \mathbf{0},$$

and hence these vectors are linearly dependent.

(ii) Since the zero vector is one of the vectors, then we have

$$0[3,4,5]+0[6,7,8]+a\mathbf{0} = \mathbf{0}$$

for any nonzero $a \in \mathbb{R}$, and hence the vectors are linearly dependent.

(iii) We want to find $a,b,c,d \in \mathbb{R}$ such that

$$a[1,-1,1,0]+b[-1,1,0,1]+c[1,0,1,-1]+d[0,1,-1,1] = \mathbf{0}.$$

So we want to solve the system

$$\begin{aligned} a-b+c&=0\\ -a+b+d&=0\\ a+c-d&=0\\ b-c+d&=0. \end{aligned}$$

Row reducing gives

$$\left[\begin{array}{rrrr|r} 1 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \sim$$
$$\left[\begin{array}{rrrr|r} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{array}\right],$$

and hence we see that $a = b = c = d = 0$ is the solution. This means the vectors are linearly independent.

10. A florist sells three types of flowers in three bouquet sizes. A small bouquet has one rose, three tulips, and three irises. A medium bouquet has two roses, four tulips, and six irises. A large bouquet has four roses, eight tulips, and six irises. If the florist uses a total of 24 roses, 50 tulips, and 48 irises in their bouquets, how many of each bouquet is made?

Solution: If we call the number of small, medium and large bouquets used S, M and L, respectively, then the information in the question tells us that

$$\begin{aligned} S + 2M + 4L &= 24 \\ 3S + 4M + 8L &= 50 \\ 3S + 6M + 6L &= 48. \end{aligned}$$

Row reducing gives
$$\left[\begin{array}{rrr|r} 1 & 2 & 4 & 24 \\ 3 & 4 & 8 & 50 \\ 3 & 6 & 6 & 48 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 2 & 4 & 24 \\ 0 & 1 & 2 & 11 \\ 0 & 0 & 1 & 4 \end{array}\right].$$
Hence $S = 2, M = 3, L = 4$ is a solution.

11. * Find A, B, C and D such that

$$\frac{x^3}{(x-1)^4} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{(x-1)^4}.$$

Solution: We have $x^3 = A(x-1)^3 + B(x-1)^2 + C(x-1) + D$ for all x (by continuity). Putting $x = 1$ gives $D = 1$ immediately. Putting $x = 2$, 0, -1, respectively, yields the system

$$\begin{array}{rcrcrcr} A & + & B & + & C & = & 7 \\ -A & + & B & - & C & = & -1 \\ -8A & + & 4B & - & 2C & = & -2 \end{array}$$

with augmented matrix

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 7 \\ -1 & 1 & -1 & -1 \\ -8 & 4 & -2 & -2 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 1 & 1 & 7 \\ 0 & 2 & 0 & 6 \\ 0 & 12 & 6 & 54 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 6 & 18 \end{array}\right],$$

yielding $A = 1\,,\ B = 3\,,\ C = 3\,,\ D = 1\,.$

12. * Use row reduction on augmented matrices to show that the lines $ax+by = k$ and $cx+dy = l$ intersect in a single point if and only if $ad - bc \neq 0$.

Solution: If $a \neq 0$ then the lines intersect if and only if the augmented matrix

$$\left[\begin{array}{cc|c} a & b & k \\ c & d & \ell \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & b/a & k/a \\ c & d & \ell \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & b/a & k/a \\ 0 & d - cb/a & \ell - ck/a \end{array}\right]$$

corresponds to a consistent system with a unique solution, which occurs if and only if $d - cb/a = \dfrac{ad-bc}{a} \neq 0$, that is, $ad - bc \neq 0$. If $a = 0$ then the lines intersect if and only if the augmented matrix

$$\left[\begin{array}{cc|c} a & b & k \\ c & d & \ell \end{array}\right] \sim \left[\begin{array}{cc|c} c & d & \ell \\ 0 & b & k \end{array}\right]$$

corresponds to a consistent system with a unique solution, which occurs if and only if $c \neq 0$ and $b \neq 0$, that is, $ad - bc = -bc \neq 0$. In all cases, the lines intersect if and only if $ad - bc \neq 0$.

13. * Find the values of λ such that the following system (i) is inconsistent; (ii) has infinitely many solutions; (iii) has a unique solution:

$$\begin{array}{rcrcrcr} x & & & - & 3z & = & -3 \\ -2x & - & \lambda y & + & z & = & 2 \\ x & + & 2y & + & \lambda z & = & 1 \end{array}$$

Solution: $\left[\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ -2 & -\lambda & 1 & 2 \\ 1 & 2 & \lambda & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ 0 & -\lambda & -5 & -4 \\ 0 & 2 & \lambda+3 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ 0 & 1 & \frac{\lambda+3}{2} & 2 \\ 0 & -\lambda & -5 & -4 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ 0 & 1 & \frac{\lambda+3}{2} & 2 \\ 0 & 0 & -5 + \frac{\lambda(\lambda+3)}{2} & -4+2\lambda \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ 0 & 2 & \lambda+3 & 4 \\ 0 & 0 & (\lambda+5)(\lambda-2) & 4(\lambda-2) \end{array}\right]$$

(i) For the system to be inconsistent, we require that $(\lambda+5)(\lambda-2) = 0$ and $4(\lambda-2) \neq 0$, so that $\lambda = -5$.

(ii) For the system to have infinitely many solutions, we require that $(\lambda+5)(\lambda-2) = 0 = 4(\lambda-2)$, so that $\lambda = 2$.

(iii) For the system to have a unique solution, we require both (i) and (ii) to fail; that is, $\lambda \neq 2, -5$.

14. Solve the following system of linear equations:

$$\begin{array}{rcr} x - y - z + u - v & = & -1 \\ -x - y + z - u + v & = & -1 \\ -x + y - z + u + v & = & 1 \\ x - y + z + u - v & = & 1 \\ -x + y + z - u - v & = & -1 \end{array}$$

Solution:

$$\left[\begin{array}{ccccc|c} 1 & -1 & -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 & -1 & -1 \end{array}\right] \underset{\substack{R_5 \to R_5 + R_1 \\ R_4 \to R_4 - R_1}}{\overset{\substack{R_2 \to R_2 + R_1 \\ R_3 \to R_3 + R_1}}{\sim}} \left[\begin{array}{ccccc|c} 1 & -1 & -1 & 1 & -1 & -1 \\ 0 & -2 & 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & -2 & -2 \end{array}\right]$$

$$\overset{R_4 \to R_4 + R_3}{\sim} \left[\begin{array}{ccccc|c} 1 & -1 & -1 & 1 & -1 & -1 \\ 0 & -2 & 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & -2 & -2 \end{array}\right] \underset{\substack{R_5 \to \frac{R_5}{2} \\ R_4 \to \frac{R_4}{2}}}{\overset{\substack{R_2 \to \frac{R_2}{2} \\ R_3 \to \frac{R_3}{2}}}{\sim}} \left[\begin{array}{ccccc|c} 1 & -1 & -1 & 1 & -1 & -1 \\ 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{array}\right].$$

By back substitution:

$$v = 1, \quad u = 1, \quad z = 1, \quad y = 1 \quad \text{and} \quad x = 1.$$

15. * Find all integer values $n \geq 3$ such that the following system is inconsistent:

$$\begin{array}{rcl} x_1 + x_n & = & 1 \\ x_1 + x_2 & = & 2 \\ x_2 + x_3 & = & 3 \\ x_3 + x_4 & = & 4 \\ \cdots\cdots\cdots & & \cdots \\ x_{n-1} + x_n & = & n \end{array}$$

Solution:

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 0 & \ldots & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & \ldots & 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & \ldots & 0 & 0 & 0 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 1 & 0 & n-1 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 1 & n \end{array}\right]$$

$$\overset{R_2 \to R_2 - R_1}{\sim} \left[\begin{array}{ccccccc|c} 1 & 0 & 0 & \ldots & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & \ldots & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & \ldots & 0 & 0 & 0 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 1 & 0 & n-1 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 1 & n \end{array}\right]$$

$$\overset{R_3 \to R_3 - R_2}{\sim} \left[\begin{array}{ccccccc|c} 1 & 0 & 0 & \ldots & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & \ldots & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & \ldots & 0 & 0 & 1 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 1 & 0 & n-1 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 1 & n \end{array}\right]$$

By repeating the substitution $R_i \to R_i - R_{i-1}$ we get

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 0 & \ldots & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & \ldots & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & \ldots & 0 & 0 & 1 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 & (-1)^n & \lfloor \frac{n}{2} \rfloor \\ 0 & 0 & 0 & \ldots & 0 & 1 & 1 & n \end{array}\right]$$

where $\lfloor \frac{n}{2} \rfloor$ is the the floor function of $\frac{n}{2}$, that is, the largest integer less than or equal to $\frac{n}{2}$.
From the last two rows we conclude that there are no solutions (inconsistent system) if n is even, and one solution (consistent system) if n is odd.