THE UNIVERSITY OF SYDNEY

MATH1002 LINEAR ALGEBRA

Semester 1 **Tutorial Exercises for Week 5 — Solutions** 2022

1. Find a normal and a general form of the equation for the plane containing P having normal vector $\mathbf{n}$ in each of the following cases:

(i) $P = (0, 0, 0)$, $\quad \mathbf{n} = [3, 1, 4]$

(ii) $P = (7, 5, -3)$, $\quad \mathbf{n} = [2, 0, 1]$

(iii) $P = (0, 0, -9)$, $\quad \mathbf{n} = [1, 1, 1]$

(iv) $P = (-6, 5, 6)$, $\quad \mathbf{n} = [0, 1, 0]$

Solution:

(i) Normal form: $[3, 1, 4] \cdot \mathbf{x} = 0$. General form: $3x + y + 4z = 0$.

(ii) Normal form: $[2, 0, 1] \cdot \mathbf{x} = 11$. General form: $2x + z = 11$.

(iii) Normal form: $[1, 1, 1] \cdot \mathbf{x} = -9$. General form: $x + y + z = -9$.

(iv) Normal form: $[0, 1, 0] \cdot \mathbf{x} = 5$. General form: $y = 5$.

2. In the following cases determine whether $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. If so, find a vector form and parametric equations for the plane containing P with direction vectors $\mathbf{u}$ and $\mathbf{v}$. If not, find a vector form and parametric equations of the corresponding line.

(i) $P = (1, -1, 4)$, $\quad \mathbf{u} = [2, 1, -1]$, $\quad \mathbf{v} = [1, 7, -3]$

(ii) $P = (5, 0, 2)$, $\quad \mathbf{u} = [3, 0, 0]$, $\quad \mathbf{v} = [1, 0, 3]$

(iii) $P = (0, 0, 0)$, $\quad \mathbf{u} = [1, 2, 3]$, $\quad \mathbf{v} = [6, 0, -1]$

(iv) $P = (-5, 3, 6)$, $\quad \mathbf{u} = [1, -2, 3]$, $\quad \mathbf{v} = [-3, 6, -9]$

Solution:

(i) $\mathbf{u}$ and $\mathbf{v}$ are not parallel so they are linearly independent.
Vector form: $\mathbf{x} = [1, -1, 4] + s[2, 1, -1] + t[1, 7, -3]$ where $s, t \in \mathbb{R}$.
Parametric equations:

$$x = 1 + 2s + t, \quad y = -1 + s + 7t, \quad z = 4 - s - 3t, \quad \text{where } s, t \in \mathbb{R}.$$

(ii) $\mathbf{u}$ and $\mathbf{v}$ are linearly independent as they are not parallel. Vector form: $\mathbf{x} = [5, 0, 2] + s[3, 0, 0] + t[1, 0, 3]$ where $s, t \in \mathbb{R}$.
Parametric equations:

$$x = 5 + 3s + t, \quad y = 0, \quad z = 2 + 3t, \quad \text{where } s, t \in \mathbb{R}.$$

(iii) $\mathbf{u}$ and $\mathbf{v}$ are linearly independent as they are not parallel.
Vector form: $\mathbf{x} = s[1, 2, 3] + t[6, 0, -1]$ where $s, t \in \mathbb{R}$.
Parametric equations:

$$x = s + 6t, \quad y = 2s, \quad z = 3s - t, \quad \text{where } s, t \in \mathbb{R}.$$

(iv) $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent as $\mathbf{u}$ and $\mathbf{v}$ are parallel (we have $\mathbf{v} = -3\mathbf{u}$).
Vector form for the line is $\mathbf{x} = [-5, 3, 6] + t[1, -2, 3]$ where $t \in \mathbb{R}$.
Parametric equations:

$$x = -5 + t, \; y = 3 - 2t, \; z = 6 + 3t.$$

3. Let $\mathcal{P}$ be the plane containing the points $P = (2, 0, 0)$, $Q = (0, 1, 3)$ and $R = (1, -3, 5)$.

(i) Find a vector form of the equation for $\mathcal{P}$.
(ii) Find a normal form of the equation for $\mathcal{P}$.
(iii) Find a general form of the equation for $\mathcal{P}$.

Solution:

(i) We find direction vectors $\overrightarrow{PQ} = [-2, 1, 3]$ and $\overrightarrow{PR} = [-1, -3, 5]$. So a vector form is $\mathbf{x} = [2, 0, 0] + s[-2, 1, 3] + t[-1, -3, 5]$ where $s, t \in \mathbb{R}$.
(ii) A normal vector to this plane is $\overrightarrow{PQ} \times \overrightarrow{PR} = [14, 7, 7]$. A normal form is $[14, 7, 7] \cdot \mathbf{x} = 28$.
(iii) $14x + 7y + 7z = 28$. (You should check that each of the points P, Q and R satisfy this equation.)

4. * The planes $x + y + z = 2$ and $x - y + 3z = 0$ intersect in a line ℓ.

(i) Find a point on ℓ.
(ii) Use cross products to find a vector $\mathbf{d}$ pointing in the direction of ℓ.
(iii) Write down parametric equations for ℓ.

Solution:

(i) $(1, 1, 0)$ (found by trial and error).
(ii) A vector pointing in the direction of ℓ will be orthogonal to the normal vectors for both these planes. Hence $\mathbf{d}$ can be found by taking the cross product of these normal vectors:

$$\mathbf{d} = [1, 1, 1] \times [1, -1, 3] = [4, -2, -2].$$

(iii) $x = 1 + 4t$, $y = 1 - 2t$, $z = -2t$, where $t \in \mathbb{R}$.

5. The plane $\mathcal{P}$ has equation $2x - y + 3z = 5$. In each of the following cases, is the given plane parallel to $\mathcal{P}$, perpendicular to $\mathcal{P}$, or neither?

(i) $x + 2y = 4$
(ii) $2x - y + 3z = 0$
(iii) $4x - 2y + 6z = 1$
(iv) $3x + y - z = 7$

Solution:

(i) Perpendicular, since the normal vector $\mathbf{n} = [2, -1, 3]$ to $\mathcal{P}$ and the normal vector $[1, 2, 0]$ to this plane are orthogonal:

$$[2, -1, 3] \cdot [1, 2, 0] = 0.$$

(ii) Parallel, since this plane has normal vector $[2, -1, 3]$ the same as $\mathbf{n}$.
(iii) Parallel, since this plane has normal vector $[4, -2, 6]$ parallel to $\mathbf{n}$.
(iv) Neither, since the normal vector $[3, 1, -1]$ to this plane is neither parallel to $\mathbf{n}$ nor orthogonal to $\mathbf{n}$.

6. Let $\mathbf{u} = [1, 2, 1], \mathbf{v} = [5, 2, -4]$ and $\mathbf{w} = [-3, 2, 6]$ be vectors in $\mathbb{R}^3$.

(i) Is $\mathbf{w}$ in the span of $\mathbf{u}$ and $\mathbf{v}$?
(ii) Hence, determine whether $\mathbf{w}$ is parallel to the plane given by $\mathbf{x} = [1, 3, -2] + s\mathbf{u} + t\mathbf{v}$ for $s, t \in \mathbb{R}$.

Solution:

(i) If $\mathbf{w} \in \operatorname{span}(\mathbf{u}, \mathbf{v})$, then there are scalars c_1 and c_2 such that

$$\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v}.$$

Comparing the components in each side gives us three equations

$$\begin{aligned} -3 &= c_1 + 5c_2, \\ 2 &= 2c_1 + 2c_2, \\ 6 &= c_1 - 4c_2. \end{aligned}$$

This system has a single solution $c_1 = 2, c_2 = -1$. Hence, $\mathbf{w} \in \operatorname{span}(\mathbf{u}, \mathbf{v})$.

(ii) $\mathbf{w}$ is parallel to the plane.

7. * Find the vector form of the equation for the line in $\mathbb{R}^3$ that passes through $P = (1, 4, 2)$ and is perpendicular to the plane with general equation $2x - y + 5z = 1$.

Solution: A direction vector for this line is the normal vector $[2, -1, 5]$ for this plane. Hence a vector form for this line is $\mathbf{x} = [1, 4, 2] + t[2, -1, 5]$ where $t \in \mathbb{R}$.

8. * Find a normal form of the equation for the plane $\mathcal{P}$ in $\mathbb{R}^3$ that passes through $P = (0, 2, 3)$ and is parallel to the plane with general equation $4x + y - 3z = 2$.

Solution: A normal vector for this plane is $[4, 1, -3]$ since $\mathcal{P}$ is parallel to the given plane. Hence $\mathcal{P}$ has normal form $[4, 1, -3] \cdot \mathbf{x} = -7$.

9. Let $\mathbf{u} = [1, 0, -1], \mathbf{v} = [0, 1, 0]$ and $\mathbf{w} = [2, 0, 2]$. Suppose that we have the following equation $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = \mathbf{0}$, where $c_1, c_2, c_3 \in \mathbb{R}$.

(i) Write down a system of linear equations for c_1, c_2, c_3.

(ii) Solve this system and deduce that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent.

Solution:

(i) We have $c_1[1, 0, -1] + c_2[0, 1, 0] + c_3[2, 0, 2] = [0, 0, 0]$. Comparing the components in each side gives us the following linear system for c_1, c_2 and c_3

$$c_1 + \qquad 2c_3 = 0, \tag{1}$$
$$c_2 \qquad = 0, \tag{2}$$
$$-c_1 + 2c_3 \qquad = 0. \tag{3}$$

(ii) Using equation (2), we get $c_2 = 0$. Solving (1) and (3) gives us $c_1 = c_3 = 0$. Thus, the system in (i) has only trivial solution $c_1 = c_2 = c_3 = 0$. Hence the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent.

10. A cubic polynomial in x takes the value -2 when $x = 1$ and the value -10 when $x = -1$. Its derivative takes the value 0 when $x = 1$ and the value 12 when $x = -1$. Set up a system of linear equations representing this information, and write down the augmented matrix corresponding to this system.

Solution: Call the polynomial

$$p(x) = ax^3 + bx^2 + cx + d,$$

where a, b, c, d are constants to be determined, so that

$$p'(x) = 3ax^2 + 2bx + c.$$

Since

$$p(1) = -2, \quad p(-1) = -10, \quad p'(1) = 0, \quad p'(-1) = 12,$$

the following system represents the given information:

$$\begin{array}{rcrcrcrcr} a & + & b & + & c & + & d & = & -2 \\ -a & + & b & - & c & + & d & = & -10 \\ 3a & + & 2b & + & c & & & = & 0 \\ 3a & - & 2b & + & c & & & = & 12. \end{array}$$

This system has augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -2 \\ -1 & 1 & -1 & 1 & -10 \\ 3 & 2 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 & 12 \end{array}\right].$$

11. The Leith family consists of Amie and Brad and their two children Clara and Damien. Their current ages add up to 70. Brad is three times as old as the present total age of Clara and Damien. In 10 years' time, Amie's age will be 20 less than twice the total of the ages that Clara and Damien will be. Four years ago, Clara's age was equal to Amie's age minus Brad's age. Set up a system of linear equations representing this information, and write down the augmented matrix corresponding to this system.

Solution: If we call the ages of the family members A, B, C, D, respectively, then the information given in the question tells us that $A+B+C+D=70$, $B=3(C+D)$, $A+10=2((C+10)+(D+10))-20$, and $C-4=(A-4)-(B-4)$. This becomes the following system of equations

$$\begin{array}{ccccccccc} A & + & B & + & C & + & D & = & 70 \\ & & B & - & 3C & - & 3D & = & 0 \\ A & & & - & 2C & - & 2D & = & 10 \\ A & - & B & - & C & & & = & -4, \end{array}$$

with augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 70 \\ 0 & 1 & -3 & -3 & 0 \\ 1 & 0 & -2 & -2 & 10 \\ 1 & -1 & -1 & 0 & -4 \end{array}\right].$$

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