Topic 7A: Inverse of a matrix

Consider the equation
$$ax = b$$
where $a, b \in \mathbb{R}$. This has solution
$$x = \frac{b}{a}$$
if $a \neq 0$. We will see that something similar occurs for A an $n \times n$ matrix which has an inverse, $\underset{\sim}{x}, \underset{\sim}{b} \in \mathbb{R}^n$, and the equation
$$\underset{n\times n}{A}\ \underset{n\times 1}{\underset{\sim}{x}} = \underset{n\times 1}{\underset{\sim}{b}}$$

Defn If A is an $n \times n$ matrix, the inverse of A is an $n \times n$ matrix B so that
$$AB = I_n \quad \text{and} \quad BA = I_n.$$
If such a matrix B exists, we say A is invertible.

Examples

1. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$

then

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 7-6 & -2+2 \\ 21-21 & -6+7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I_2$$

and $BA = I_2$ (check!)

so B is an inverse of A, hence A is invertible.

2. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then for any $a, b, c, d \in \mathbb{R}$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} \neq I_2$$

So A is not invertible.

Theorem (Inverses are unique)
If A is invertible, then it has a unique inverse.

Proof Suppose B and C are both inverses of A. Then

$$AB = I = BA \quad \text{and} \quad AC = I = CA.$$

Now

$$B = BI = B(AC) = (BA)C = IC = C$$

so the inverse is unique. $\square$

Since the inverse is unique, we can write the inverse of an invertible matrix A as

$$A^{-1}.$$

(If $a \in \mathbb{R}$, $a \neq 0$, then $aa^{-1} = 1 = a^{-1}a$.)

Warning! Never write $\frac{1}{A}$ for a matrix A. This makes no sense. Should think: multiplying by A^{-1} (not dividing by A).

Theorem (Properties of inverses)
Suppose A is an invertible $n \times n$ matrix. Then:

1. A^{-1} is invertible, and $(A^{-1})^{-1} = A$.
2. if $c \neq 0$ is a scalar, then cA is invertible, with
$$(cA)^{-1} = c^{-1}A^{-1}$$
inverse of a matrix
inverse of a real number
3. If B is $n \times n$ and invertible, then AB is invertible, and
$$(AB)^{-1} = B^{-1}A^{-1}.$$
4. A^T is invertible and
$$(A^T)^{-1} = (A^{-1})^T.$$

5. A^k is invertible for all integers $k \geq 1$ and

$$(A^k)^{-1} = (A^{-1})^k.$$

Proof exercise.

Def^n If A is invertible, for all integers $k \geq 1$ we define

$$A^{-k} = (A^{-1})^k.$$

Example

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$ then $A^2 = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$

$$= \begin{bmatrix} 7 & 16 \\ 24 & 55 \end{bmatrix}$$

We showed above that

$$A^{-1} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$$

(we called this B earlier)

So

$$A^{-2} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 55 & -16 \\ -24 & 7 \end{bmatrix}$$

Check: $A^2 A^{-2} = I = A^{-2} A^2.$

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Topic 7B: Inverses and determinants of 2×2 matrices

Theorem Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $a,b,c,d\in\mathbb{R}$.

If $ad-bc\neq 0$ then A is invertible and

$$A^{-1}=\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad-bc=0$ then A is not invertible.

Proof Suppose $ad-bc\neq 0$. We need to show that

$$A\left(\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right)=I_2$$

and

$$\left(\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right)=I_2.$$

Let $r=ad-bc\neq 0$. Then

$$\begin{aligned}
A\left(\frac{1}{r}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right) &= \frac{1}{r}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\
&= \frac{1}{r}\begin{bmatrix} ad-bc & -ab+ab \\ cd-cd & -bc+ad \end{bmatrix} \\
&= \frac{1}{r}\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= I_2.
\end{aligned}$$

Check: $\left(\frac{1}{r}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right) A = I_2$ as well.

Thus A is invertible, with

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Suppose $ad - bc = 0$. Consider 2 cases: $a = 0$ and $a \neq 0$. Exercise. $\square$.

<u>Def^n</u> If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the <u>determinant</u> of A is

$$\underline{\det(A)} = ad - bc.$$

The theorem above says:

- A is invertible $\Longleftrightarrow$ $\det(A) \neq 0$.
- if A is invertible, then
$$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

<u>Examples</u>

1. $A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$.

Then $\det(A) = 1\times 7 - 2\times 3 = 7 - 6 = 1$.

And

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \frac{1}{1} \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}.$$

2. Find the inverses of

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 5 \\ 2 & 10 \end{bmatrix}.$$

if they exist.

$\det(A) = 4\times 1 - 5\times 2 = 4 - 10 = -6$

Since $\det(A) \neq 0$, A is invertible.

$$A^{-1} = \frac{1}{-6} \begin{bmatrix} 4 & -5 \\ -2 & 1 \end{bmatrix}.$$

CHECK!

$AA^{-1} = I_2 = A^{-1}A.$

$\det(B) = 1\times 10 - 2\times 5 = 10 - 10 = 0$

Since $\det(B) = 0$, B is not invertible.

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Topic 7C: Inverses and elementary row operations

Suppose A is an $n \times n$ matrix.
To determine whether A is invertible, and find A^{-1} if it exists:

1. Augment A with the identity matrix:
$$[A \mid I_n]$$
2. Carry out elementary row operations on this entire augmented matrix, to try to get A in reduced row echelon form.
3. If A has reduced row echelon form I_n, then we have
$$[A \mid I_n] \xrightarrow[\text{row ops}]{} [I_n \mid A^{-1}]$$
4. If you get a matrix on the LHS which can't be reduced to I_n, then A is not invertible.

Examples Find A^{-1}, if it exists.

1. $A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$.

$$[A \mid I_2] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 7 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \mapsto R_2 - 3R_1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & -3 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \mapsto R_1 - 2R_2} \left[\begin{array}{cc|cc} 1 & 0 & 7 & -2 \\ 0 & 1 & -3 & 1 \end{array}\right].$$

So $A^{-1} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$.

2. $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$.

$$[A \mid I] = \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 4 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \mapsto R_2 - 2R_1} \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right]$$

Since the LHS matrix has an all-0 row, its reduced row echelon form will not be I_2

So A is not invertible.

(Note : $\det(A) = 2 \times 2 - 4 \times 1 = 0$.)

3. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \mapsto \frac{1}{3}R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right]$$

$$\xrightarrow[R_2 \mapsto R_2 - R_3]{R_1 \mapsto R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 2 & 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right]$$

$$\xrightarrow{R_2 \mapsto \frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{6} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right]$$

$$\xrightarrow{R_1 \mapsto R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{6} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right].$$

So $$A^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{6} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}.$$ (Check!)

4. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 5 & 7 & 9 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \mapsto R_3 - 5R_1]{R_2 \mapsto R_2 - 4R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -3 & -6 & -5 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \mapsto R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right]$$

So A is not invertible.

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Topic 7D: Inverses and systems of linear equations

Consider the linear equation
$$ax = b \qquad \text{(1 equation in 1 variable)}$$
where $a, b \in \mathbb{R}$.
If $a \neq 0$ then this equation has unique solution given by
$$x = a^{-1}b.$$
added later
for all $b \in \mathbb{R}$

If this equation has unique solution, then $a \neq 0$ and the solution is
$$x = a^{-1}b.$$
(If $a=0$ we get $0x = b$. This has infinitely many solutions if $b=0$, and no solution if $b \neq 0$.)

Now consider a system of linear equations
n equations in n variables:
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad\qquad & \vdots \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

Let A be the matrix of coefficients i.e.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$n \times n$ matrix

Let

$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \underset{\sim}{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Then $\underset{\sim}{x}$ satisfies the equation $A\underset{\sim}{x} = \underset{\sim}{b}$ exactly when the system above is satisfied.

Theorem If A is an invertible $n \times n$ matrix, then $A\underset{\sim}{x} = \underset{\sim}{b}$ has a unique solution given by

$$\underset{\sim}{x} = A^{-1}\underset{\sim}{b}.$$

Example Solve the system

$$x + 5y = 3$$
$$2x + 4y = 1$$

using the inverse of the coefficient matrix.

We have coefficient matrix

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}, \text{ and } \underset{\sim}{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$\det(A) = 4 - 10 = -6 \neq 0$ so A is invertible

Now

$$A^{-1} = \frac{1}{-6}\begin{bmatrix} 4 & -5 \\ -2 & 1 \end{bmatrix}$$

so the unique solution is

$$\underset{\sim}{x} = \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\underset{\sim}{b} = -\frac{1}{6}\begin{bmatrix} 4 & -5 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= -\frac{1}{6}\begin{bmatrix} 7 \\ -5 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{7}{6} \\ \frac{5}{6} \end{bmatrix}. \qquad \text{(Check!)}$$

Proof of Theorem

First we show that $\underset{\sim}{x} = A^{-1}\underset{\sim}{b}$ is a solution:

$$A(A^{-1}\underset{\sim}{b}) = (AA^{-1})\underset{\sim}{b} = I_n\underset{\sim}{b} = \underset{\sim}{b}$$

so $\underset{\sim}{x} = A^{-1}\underset{\sim}{b}$ is a solution to $A\underset{\sim}{x} = \underset{\sim}{b}$.

To show that this solution is unique:

suppose that $A\underset{\sim}{y} = \underset{\sim}{b}$ i.e. $\underset{\sim}{y}$ is a solution.

Then

$$A^{-1}(A\underset{\sim}{y}) = A^{-1}\underset{\sim}{b}$$

$$\Longrightarrow \quad (A^{-1}A)\underset{\sim}{y} = \underset{\sim}{x}$$

$$\Longrightarrow \quad I_n \underset{\sim}{y} = \underset{\sim}{x}$$
$$\Longrightarrow \quad \underset{\sim}{y} = \underset{\sim}{x}.$$

So the only solution is $\underset{\sim}{x} = A^{-1}\underset{\sim}{b}$. $\square$

Theorem

Let A be an $n \times n$ matrix.
The following are equivalent:
(1) A is invertible
(2). $A\underset{\sim}{x} = \underset{\sim}{b}$ has unique solution, for every $\underset{\sim}{b} \in \mathbb{R}^n$.
(3) $A\underset{\sim}{x} = \underset{\sim}{0}$ has unique solution $\underset{\sim}{x} = \underset{\sim}{0}$.
(4) The reduced row echelon form of A is I_n.

Proof Not an exercise.

(1) $\Rightarrow$ (2) previous theorem

(2) $\Rightarrow$ (3) put $\underset{\sim}{b} = \underset{\sim}{0}$

(3) $\Rightarrow$ (4) related to row operations used to go from $[A \mid I] \rightsquigarrow [I \mid A^{-1}]$

(4) $\Rightarrow$ (1) uses elementary matrices (Topic 8A).

Corollary (A 1-sided inverse is a 2-sided inverse)

Let A be an $n \times n$ matrix. If B is $n \times n$ and either

$$AB = I_n \quad \text{or} \quad BA = I_n$$

then A is invertible and $A^{-1} = B$.

Proof Suppose $BA = I_n$. By previous theorem it's enough to show that $A\underset{\sim}{x} = \underset{\sim}{0}$ has a unique solution. Now

$$
\begin{aligned}
& A\underset{\sim}{x} = \underset{\sim}{0} \\
\Rightarrow\ & B(A\underset{\sim}{x}) = B\underset{\sim}{0} \\
\Rightarrow\ & (BA)\underset{\sim}{x} = \underset{\sim}{0} \\
\Rightarrow\ & I_n \underset{\sim}{x} = \underset{\sim}{0} \\
\Rightarrow\ & \underset{\sim}{x} = \underset{\sim}{0}
\end{aligned}
$$

so $A\underset{\sim}{x} = \underset{\sim}{0}$ has unique solution $\underset{\sim}{x} = \underset{\sim}{0}$. Thus A is invertible. To show $B = A^{-1}$ we just need to show $AB = I_n$.

Now

$$
\begin{aligned}
& BA = I_n \\
\Rightarrow\ & (BA)A^{-1} = I_n A^{-1} \\
\Rightarrow\ & B(AA^{-1}) = A^{-1} \\
\Rightarrow\ & BI_n = A^{-1} \\
\Rightarrow\ & B = A^{-1}.
\end{aligned}
$$

Now suppose $AB = I_n$. Rest of proof: exercise. $\square$

added later

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