MATH1002 Linear Algebra

Topic 2A: Linear combinations

A vector $\underset{\sim}{v}$ is a linear combination of vectors $\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_k$ if there are scalars $c_1, \dots, c_k$ so that

$$\underset{\sim}{v} = c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2 + \cdots + c_k \underset{\sim}{v}_k$$

The scalars $c_1, \dots, c_k$ are the coefficients of the linear combination.

Examples

1. $\begin{bmatrix} -1 \\ 2 \\ -11 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \\ 7 \end{bmatrix}$

since

$$\begin{bmatrix} -1 \\ 2 \\ -11 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ 7 \end{bmatrix}.$$

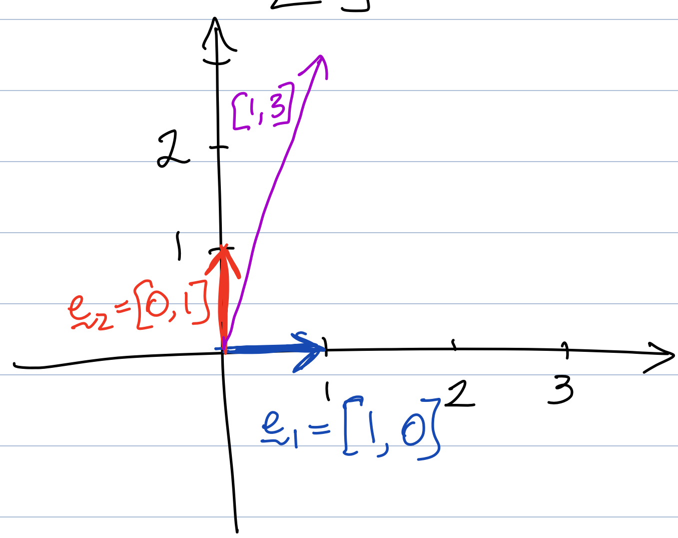
2. In $\mathbb{R}^2$ let $\underset{\sim}{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\underset{\sim}{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Then $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ has coefficients 1 and 3 in the linear combination

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1 \underset{\sim}{e}_1 + 3 \underset{\sim}{e}_2$$

Check:

$$1\underset{\sim}{e}_1 + 3\underset{\sim}{e}_2 = 1\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\end{bmatrix}$$
$$= \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\3\end{bmatrix}$$
$$= \begin{bmatrix}1\\3\end{bmatrix}.$$



3. In $\mathbb{R}^2$, let $\underset{\sim}{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $\underset{\sim}{v}_2 = \begin{bmatrix}-2\\-2\end{bmatrix}$.

Then $\underset{\sim}{v} = \begin{bmatrix}1\\3\end{bmatrix}$ is <u>not</u> a linear combination of $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$.

Why? Try to find c_1, c_2 so that
$$\underset{\sim}{v} = c_1\underset{\sim}{v}_1 + c_2\underset{\sim}{v}_2$$

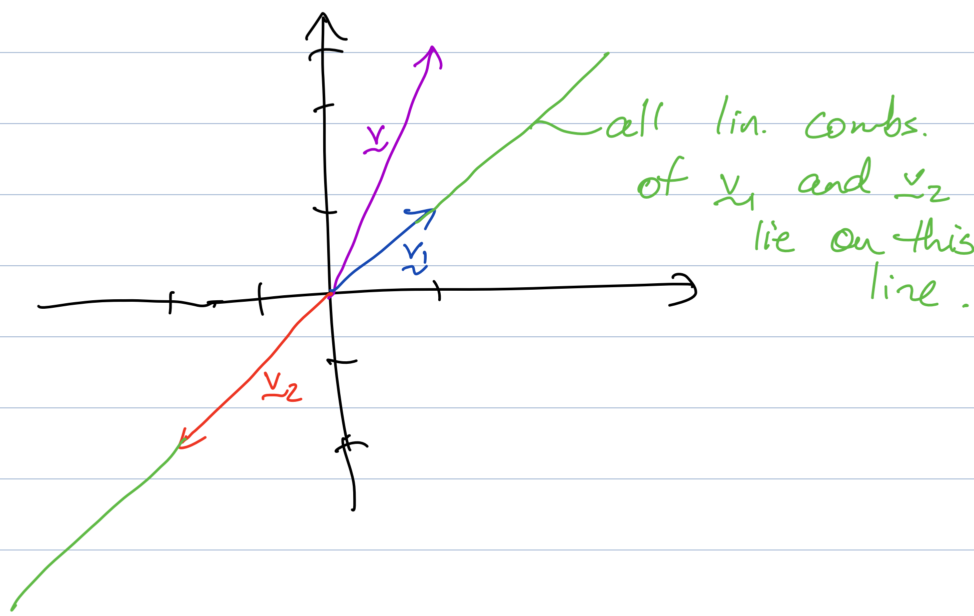
$$\Longleftrightarrow \begin{bmatrix}1\\3\end{bmatrix} = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}-2\\-2\end{bmatrix}$$

$$\iff \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} -2c_2 \\ -2c_2 \end{bmatrix}$$

$$\iff \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} c_1 - 2c_2 \\ c_1 - 2c_2 \end{bmatrix}$$

$$\iff \begin{cases} 1 = c_1 - 2c_2 \\ 3 = c_1 - 2c_2 \end{cases}$$

This set of simultaneous equations has no solution, so $\underset{\sim}{v}$ is <u>not</u> a lin. comb. of $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$.



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MATH1002 Linear Algebra

Topic 2B: Dot Product

If $\underset{\sim}{u} = [u_1, u_2, \ldots, u_n]$ and $\underset{\sim}{v} = [v_1, v_2, \ldots, v_n]$ then the dot product of $\underset{\sim}{u}$ and $\underset{\sim}{v}$ is

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \underbrace{u_1 v_1 + u_2 v_2 + \cdots + u_n v_n}_{\text{a scalar}}$$

don't leave out the dot

Examples

1. If $\underset{\sim}{u} = [1, 2]$ and $\underset{\sim}{v} = [2, -3]$ then

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = 1 \times 2 + 2 \times (-3) = 2 - 6 = -4.$$

2. If $\underset{\sim}{u} = [2, -1, 0]$ and $\underset{\sim}{v} = [1, 4, 9]$ then

$$\begin{aligned} \underset{\sim}{u} \cdot \underset{\sim}{v} &= 2 \times 1 + (-1) \times 4 + 0 \times 9 \\ &= 2 - 4 + 0 \\ &= -2. \end{aligned}$$

Theorem (Properties of the dot product)

Let $\underset{\sim}{u}, \underset{\sim}{v}$ and $\underset{\sim}{w}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then 2 of 3

(1) $\underset{\sim}{u} \cdot \underset{\sim}{v} = \underset{\sim}{v} \cdot \underset{\sim}{u}$ (Commutative Law)

(2) $\underset{\sim}{u} \cdot (\underset{\sim}{v} + \underset{\sim}{w}) = \underset{\sim}{u} \cdot \underset{\sim}{v} + \underset{\sim}{u} \cdot \underset{\sim}{w}$ (Distributive Law)

(3) $(c\underset{\sim}{u}) \cdot \underset{\sim}{v} = c(\underset{\sim}{u} \cdot \underset{\sim}{v})$

if and only if

(4) $\underset{\sim}{u} \cdot \underset{\sim}{u} \geqslant 0$ and $\underset{\sim}{u} \cdot \underset{\sim}{u} = 0 \iff \underset{\sim}{u} = \underset{\sim}{0}$.

Proof of (4)

Let $\underset{\sim}{u} = [u_1, u_2, \dots, u_n]$.

Then $\underset{\sim}{u} \cdot \underset{\sim}{u} = u_1^2 + u_2^2 + \cdots + u_n^2$.

Now $u_1, u_2, \dots, u_n$ are real numbers, so $u_i^2 \geqslant 0$ for $i = 1, 2, \dots, n$.

Hence $u_1^2 + u_2^2 + \cdots + u_n^2 \geqslant 0$.

So $\underset{\sim}{u} \cdot \underset{\sim}{u} \geqslant 0$ as required.

Suppose that $\underset{\sim}{u} \cdot \underset{\sim}{u} = 0$. Then

$$u_1^2 + u_2^2 + \cdots + u_n^2 = 0$$

and $u_i^2 \geqslant 0$ for $1 \leq i \leq n$.

Thus $u_i^2 = 0$ for $1 \leq i \leq n$.

So $u_i = 0$ for $1 \leq i \leq n$.

So $\quad \underset{\sim}{u} = [0, 0, \ldots, 0] = \underset{\sim}{0}$, as required.

$\boxed{3 \text{ of } 3}$

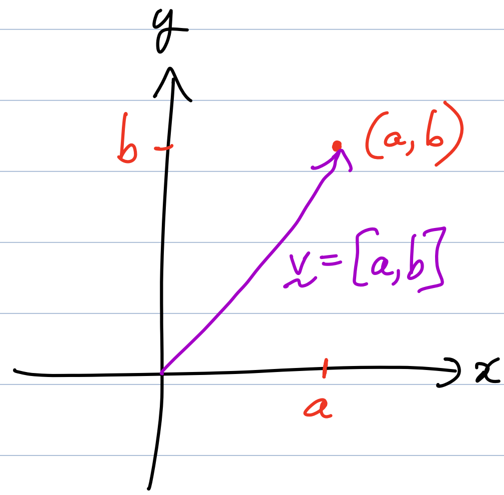
Suppose $\underset{\sim}{u} = \underset{\sim}{0}$. Then

$$\underset{\sim}{u} \cdot \underset{\sim}{u} = \underset{\sim}{0} \cdot \underset{\sim}{0} = [0, 0, \ldots, 0] \cdot [0, 0, \ldots, 0]$$
$$= 0 + 0 + \cdots + 0$$
$$= 0, \quad \text{as required.}$$

end of proof $\rightarrow \square$

Topic 2C: Length of vectors

In $\mathbb{R}^2$:



Length of $\underset{\sim}{v}$ = distance from origin to (a,b)
$= \sqrt{a^2+b^2}$ (by Pythagoras' Theorem)

Now

$$\underset{\sim}{v} \cdot \underset{\sim}{v} = [a,b] \cdot [a,b] = a^2 + b^2$$

so length of $\underset{\sim}{v}$ is $\sqrt{\underset{\sim}{v} \cdot \underset{\sim}{v}}$.

Def^n The length of a vector $\underset{\sim}{v} = [v_1, v_2, \ldots, v_n]$ is given by

$$\|\underset{\sim}{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} = \sqrt{\underset{\sim}{v} \cdot \underset{\sim}{v}}.$$

(So above, we have $\|[a,b]\| = \sqrt{a^2+b^2}$.)

Remarks 1. From 2B, we have $\underset{\sim}{v} \cdot \underset{\sim}{v} \geq 0$, so we can take its square-root.

2. From 2B, we have $\underset{\sim}{v} \cdot \underset{\sim}{v} = 0 \iff \underset{\sim}{v} = \underset{\sim}{0}$ so $\|\underset{\sim}{v}\| = 0 \iff \underset{\sim}{v} = \underset{\sim}{0}$.

Examples

1. If $\underset{\sim}{v} = [1, 3]$ then

$$\|\underset{\sim}{v}\| = \sqrt{1^2 + 3^2} = \sqrt{10}$$

2. If $\underset{\sim}{v} = [-1, -2, 3]$ then

$$\|\underset{\sim}{v}\| = \sqrt{(-1)^2 + (-2)^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}.$$

Theorem (Length and scalar multiplication)

If $\underset{\sim}{v} \in \mathbb{R}^n$ and c is a scalar, then

$$\|c\underset{\sim}{v}\| = |c|\|\underset{\sim}{v}\|.$$

length of vector   absolute value of scalar

Proof

Since $\|c\underset{\sim}{v}\|$ and $|c|\,\|\underset{\sim}{v}\|$ are both non-negative, it's enough to show

that
$$\|c\underset{\sim}{v}\|^2 = |c|^2\,\|\underset{\sim}{v}\|^2.$$
Now
$$\|c\underset{\sim}{v}\|^2 = \left(\sqrt{(c\underset{\sim}{v})\cdot(c\underset{\sim}{v})}\right)^2$$
$$= (c\underset{\sim}{v})\cdot(c\underset{\sim}{v})$$
$$= c^2(\underset{\sim}{v}\cdot\underset{\sim}{v})$$
$$= |c|^2\left(\sqrt{\underset{\sim}{v}\cdot\underset{\sim}{v}}\right)^2$$
$$= |c|^2\,\|\underset{\sim}{v}\|^2$$
as required. $\square$

Unit vectors

A unit vector is a vector of length 1.

Examples

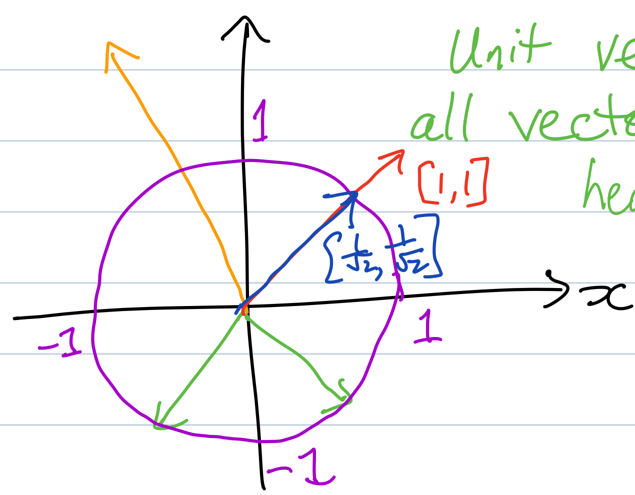
1. In $\mathbb{R}^2$
$$\underset{\sim}{e}_1 = \begin{bmatrix}1\\0\end{bmatrix} \quad \text{and} \quad \underset{\sim}{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$$
are unit vectors.
$$\|\underset{\sim}{e}_1\| = \sqrt{1^2+0^2} = \sqrt{1^2} = 1$$
$$\|\underset{\sim}{e}_2\| = \sqrt{0^2+1^2} = 1.$$

2. In $\mathbb{R}^3$

$$\underset{\sim}{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \underset{\sim}{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \underset{\sim}{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

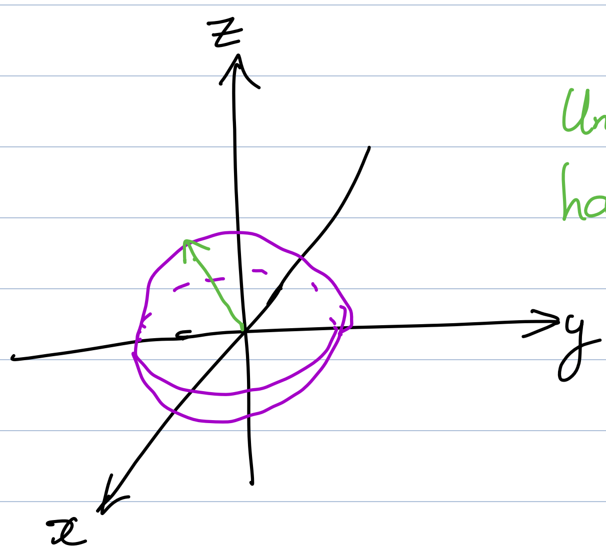
are unit vectors.

In $\mathbb{R}^2$:



Unit vectors are all vectors with heads on unit circle

In $\mathbb{R}^3$:



Unit vectors have heads on unit sphere.

Given any nonzero vector $\underset{\sim}{v} \in \mathbb{R}^n$, we can find a unit vector in the same direction as $\underset{\sim}{v}$ by multiplying by the scalar $\frac{1}{\|\underset{\sim}{v}\|}$.

Example If $\underset{\sim}{v} = [1, 1]$ then $\|\underset{\sim}{v}\| = \sqrt{2}$, so a unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{1}{\sqrt{2}}[1, 1] = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right].$$

Obtaining a unit vector in the same direction as $\mathbf{v}$ is called normalising the vector.

Check that $\frac{1}{\|\mathbf{v}\|}\mathbf{v}$ has length 1:

$$\left\| \frac{1}{\|\mathbf{v}\|}\mathbf{v} \right\| = \left| \frac{1}{\|\mathbf{v}\|} \right| \|\mathbf{v}\| = \frac{1}{\|\mathbf{v}\|} \|\mathbf{v}\| = 1.$$

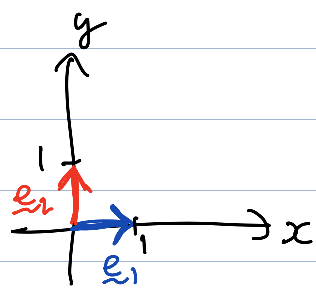
Standard unit vectors

In $\mathbb{R}^n$, the standard unit vectors are

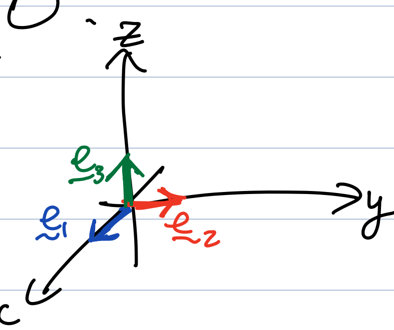
$$\mathbf{e}_1 = [1, 0, \ldots, 0]$$
$$\mathbf{e}_2 = [0, 1, 0, \ldots, 0]$$
$$\vdots$$
$$\mathbf{e}_n = [0, \ldots, 0, 1]$$

ie. $\mathbf{e}_i$ has i^{th} entry 1, all other entries 0.

In $\mathbb{R}^2$:



In $\mathbb{R}^3$:



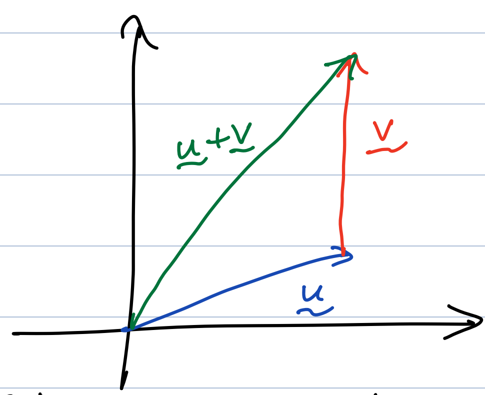
Inequalities involving length

(1) Cauchy - Schwarz Inequality:
For all $\underset{\sim}{u}, \underset{\sim}{v} \in \mathbb{R}^n$

$$|\underset{\sim}{u} \cdot \underset{\sim}{v}| \leq \|\underset{\sim}{u}\| \, \|\underset{\sim}{v}\|.$$

(2) Triangle Inequality: for all $\underset{\sim}{u}, \underset{\sim}{v} \in \mathbb{R}^n$

$$\|\underset{\sim}{u} + \underset{\sim}{v}\| \leq \|\underset{\sim}{u}\| + \|\underset{\sim}{v}\|$$



Proof of (2): Enough to show

$$\|\underset{\sim}{u} + \underset{\sim}{v}\|^2 \leq \left(\|\underset{\sim}{u}\| + \|\underset{\sim}{v}\|\right)^2.$$

Now

$$\begin{aligned}
\|\underset{\sim}{u} + \underset{\sim}{v}\|^2 &= (\underset{\sim}{u} + \underset{\sim}{v}) \cdot (\underset{\sim}{u} + \underset{\sim}{v}) \\
&= \underset{\sim}{u} \cdot (\underset{\sim}{u} + \underset{\sim}{v}) + \underset{\sim}{v} \cdot (\underset{\sim}{u} + \underset{\sim}{v}) \\
&= \underset{\sim}{u} \cdot \underset{\sim}{u} + \underset{\sim}{u} \cdot \underset{\sim}{v} + \underset{\sim}{v} \cdot \underset{\sim}{u} + \underset{\sim}{v} \cdot \underset{\sim}{v}
\end{aligned}$$

by 2 applications of Distributive Law

$$\begin{aligned}
&= \|\underset{\sim}{u}\|^2 + 2(\underset{\sim}{u} \cdot \underset{\sim}{v}) + \|\underset{\sim}{v}\|^2 \\
&\leq \|\underset{\sim}{u}\|^2 + 2|\underset{\sim}{u} \cdot \underset{\sim}{v}| + \|\underset{\sim}{v}\|^2
\end{aligned}$$

$$\leq \|\underset{\sim}{u}\|^2 + 2\|\underset{\sim}{u}\|\,\|\underset{\sim}{v}\| + \|\underset{\sim}{v}\|^2$$

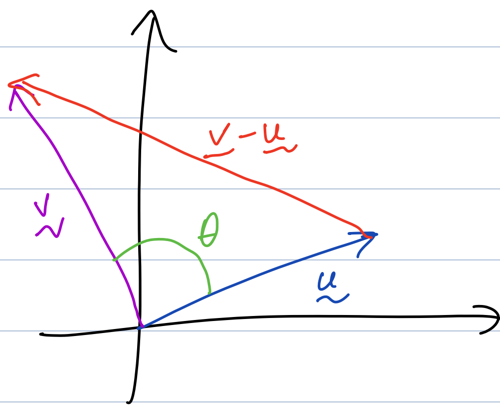
by Cauchy-Schwarz

$$= \left(\|\underset{\sim}{u}\| + \|\underset{\sim}{v}\|\right)^2,$$

as required. $\square$

Topic 2D: Angles between vectors

In $\mathbb{R}^2$:



Angle θ between $\underset{\sim}{u}$ and $\underset{\sim}{v}$

$0 \leq \theta \leq \pi$

Cosine Rule:

$$\|\underset{\sim}{v}-\underset{\sim}{u}\|^2 = \|\underset{\sim}{u}\|^2 + \|\underset{\sim}{v}\|^2 - 2\|\underset{\sim}{u}\|\,\|\underset{\sim}{v}\|\cos\theta$$

Now
$$\|\underset{\sim}{v}-\underset{\sim}{u}\|^2 = (\underset{\sim}{v}-\underset{\sim}{u})\cdot(\underset{\sim}{v}-\underset{\sim}{u})$$
$$= \|\underset{\sim}{v}\|^2 - 2(\underset{\sim}{u}\cdot\underset{\sim}{v}) + \|\underset{\sim}{u}\|^2.$$

So after cancellation, we get:

$$-2(\underset{\sim}{u}\cdot\underset{\sim}{v}) = -2\|\underset{\sim}{u}\|\,\|\underset{\sim}{v}\|\cos\theta$$

so

$$\cos\theta = \frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{\|\underset{\sim}{u}\|\,\|\underset{\sim}{v}\|}.$$

Def^n The angle θ between $\underset{\sim}{u}, \underset{\sim}{v} \in \mathbb{R}^n$ is given by

$$\cos\theta = \frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{\|\underset{\sim}{u}\|\,\|\underset{\sim}{v}\|}. \qquad (0 \leq \theta \leq \pi)$$

Example Find the angle between

$$\underset{\sim}{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \underset{\sim}{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

$$\cos\theta = \frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{\|\underset{\sim}{u}\| \, \|\underset{\sim}{v}\|}$$

$$= \frac{1 \times 1 + 1 \times 1 + 1 \times 0}{\sqrt{3}\,\sqrt{2}}$$

$$= \frac{2}{\sqrt{3}\sqrt{2}}$$

$$= \frac{\sqrt{2}}{\sqrt{3}}$$

$$= \frac{\sqrt{2}\sqrt{3}}{3}$$

all OK

so $\theta = \cos^{-1}\left(\frac{2}{\sqrt{3}\sqrt{2}}\right)$.

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MATH1002 Linear Algebra

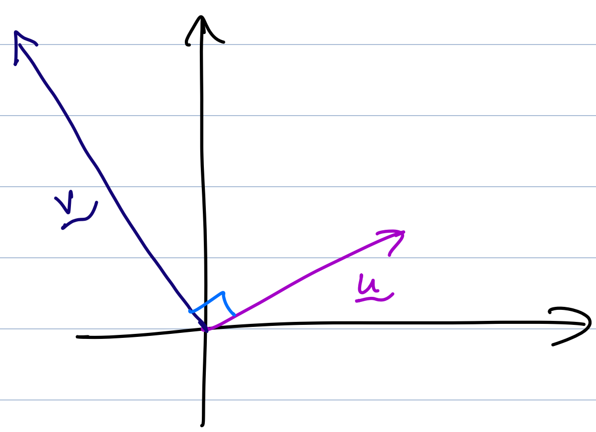
Topic 2E: Orthogonal vectors

upright

angle

Vectors which have angle $\frac{\pi}{2}$ between them.

In $\mathbb{R}^2$: if $\underset{\sim}{u}$ and $\underset{\sim}{v}$ have angle $\frac{\pi}{2}$ between them



then $$\cos\theta = \frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{\|\underset{\sim}{u}\|\,\|\underset{\sim}{v}\|}.$$

So if $\theta = \frac{\pi}{2}$ then $\cos\theta = 0$ so

$$0 = \frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{\|\underset{\sim}{u}\|\,\|\underset{\sim}{v}\|}$$

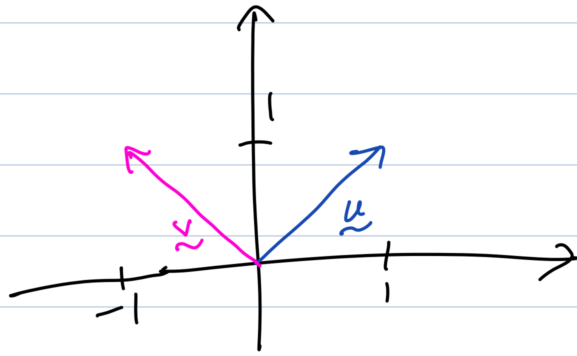
hence $\underset{\sim}{u}\cdot\underset{\sim}{v} = 0$.

Def^n Two vectors $\underset{\sim}{u}, \underset{\sim}{v} \in \mathbb{R}^n$ are **orthogonal** if $\underset{\sim}{u}\cdot\underset{\sim}{v} = 0$.

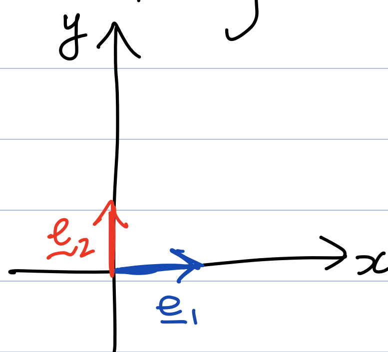
Examples

1. $\underset{\sim}{u} = [1, 1]$ and $\underset{\sim}{v} = [-1, 1]$ are orthogonal as

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = -1 + 1 = 0.$$



2. In $\mathbb{R}^2$, $\underset{\sim}{e}_1 = [1, 0]$ and $\underset{\sim}{e}_2 = [0, 1]$ are orthogonal.



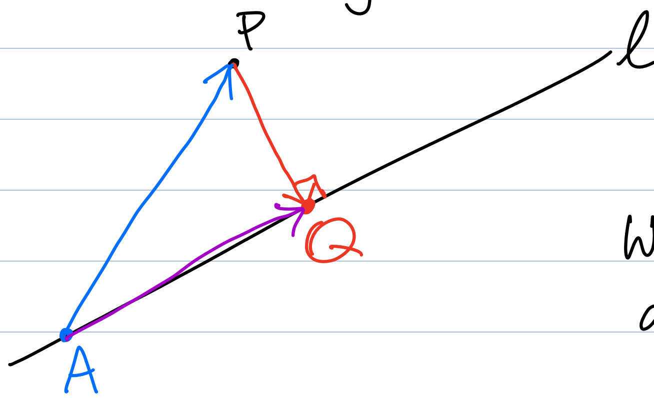
3. In $\mathbb{R}^3$, $\underset{\sim}{e}_1 = [1, 0, 0]$, $\underset{\sim}{e}_2 = [0, 1, 0]$ and $\underset{\sim}{e}_3 = [0, 0, 1]$ are mutually orthogonal.

4. In $\mathbb{R}^n$

$$\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

so $\underset{\sim}{e}_i$ is orthogonal to $\underset{\sim}{e}_j \iff i \neq j$.

Topic 2F: Projections



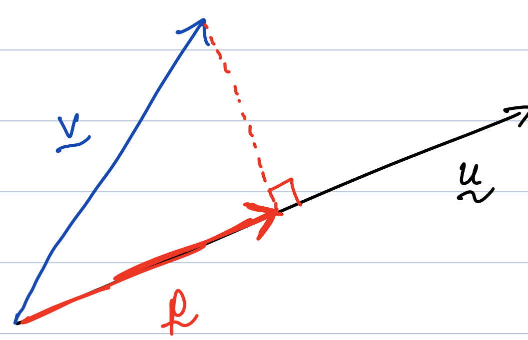
What is the distance from P to ℓ?

What is the length of the vector $\overrightarrow{PQ}$?

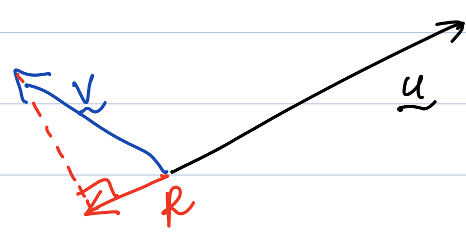
If we know $\overrightarrow{AP}$ and $\overrightarrow{AQ}$, we can find the vector $\overrightarrow{PQ}$.

Enough to know a point A on ℓ, and the projection of $\overrightarrow{AP}$ onto ℓ. This projection is $\overrightarrow{AQ}$.

In terms of vectors:



$\underset{\sim}{p}$ is projection of $\underset{\sim}{v}$ onto $\underset{\sim}{u}$



$\underset{\sim}{p}$ is scalar multiple of $\underset{\sim}{u}$

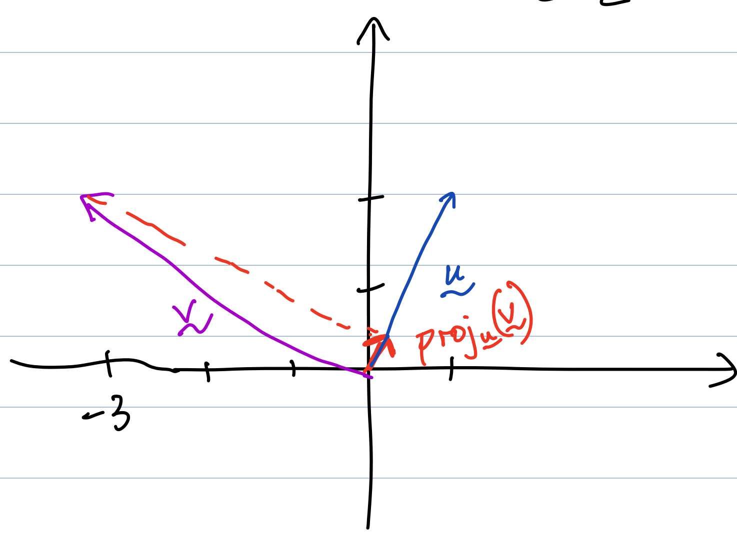
Defn If $\underset{\sim}{u}, \underset{\sim}{v} \in \mathbb{R}^n$ and $\underset{\sim}{u} \neq \underset{\sim}{0}$ then the projection of $\underset{\sim}{v}$ onto $\underset{\sim}{u}$ is

$$\text{proj}_{\underset{\sim}{u}}(\underset{\sim}{v}) = \left(\frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{\underset{\sim}{u} \cdot \underset{\sim}{u}}\right) \underset{\sim}{u}$$

Examples

1. If $\underset{\sim}{u} = [1, 2]$ and $\underset{\sim}{v} = [-3, 2]$ then

$$\text{proj}_{\underset{\sim}{u}}(\underset{\sim}{v}) = \left(\frac{1 \times (-3) + 2 \times 2}{1 \times 1 + 2 \times 2}\right)[1, 2]$$
$$= \frac{1}{5}[1, 2]$$



2. $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are orthogonal $\iff \text{proj}_{\underset{\sim}{u}}(\underset{\sim}{v}) = \underset{\sim}{0}$ $\iff \underset{\sim}{u} \cdot \underset{\sim}{v} = 0$

3. $$\text{proj}_{\underset{\sim}{e}_j}(\underset{\sim}{e}_i) = \begin{cases} \underset{\sim}{0} & \text{if } i \neq j \\ \underset{\sim}{e}_i & \text{if } i = j \end{cases}$$