COMP2022|2922
Models of Computation

Deduction in Propositional Logic

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September 17, 2022

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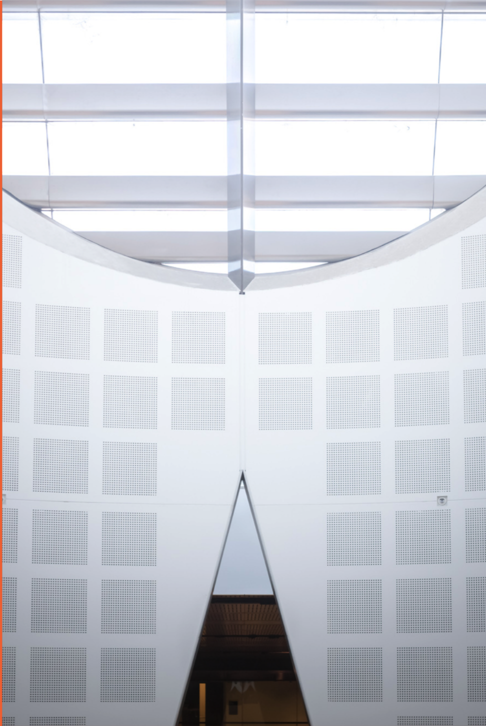

Agenda

1. Natural deduction and proof strategies

Why are we interested in Deduction?

Deduction is about proving statements from assumptions. Why is this useful or interesting?

1. It can be used by humans to improve their **reasoning skills**, e.g., to check that an argument is logically correct.
2. It forms the basis of an approach to software reliability called **Formal Methods**.
3. Logic programming languages like **Prolog** work by proving things.
4. It is used in automated reasoning systems, **automated theorem provers**

Self test

A certain Company has Directors.

1. Every Director holds either Bonds or Shares; but no Director holds both.
2. Every Bondholder is a Director.

On mentimeter's Q+A...

Write one new fact ($<$ 10 words) that you can deduce from these facts, or upvote all you agree with.

Deduction: Formal proofs

A deductive system gives you rules for constructing formal proofs.

Formal proofs:

1. Allow one to be sure that the proven statement is true (as long as the assumptions are true).
2. A highly disciplined way of reasoning (good for computers)
3. A sequence of formulas where each step is an assumption, or a deduction based on earlier steps
4. Based entirely on rewriting formulas – no semantics.

Deduction: motivation

1. The most famous deductive system is surely found in Euclid's *Elements* for deducing facts in elementary geometry (and number theory).
2. Euclid made some assumptions about points and lines.
   - "A straight line may be drawn between any two points."
3. . . . and used logical rules to produce mathematical proofs of statements about geometry.
   - "The angles in a triangle sum to 180 degrees."

We study a deductive system called Natural Deduction (ND) for reasoning about propositions. Let's look at some of its rules of inference.

Rules of Inference

Consider the following argument. Does it make logical sense?

1. It is raining and it is hot.
2. So, it is raining.

Rules of Inference

Consider the following argument. Does it make logical sense?

1. It is raining and it is hot.
2. So, it is raining.

Sure!

The premise (1.) says it is raining and it is hot, so, if we forget some of the information we can deduce the conclusion (2.) that it is hot.

This is formalised as rule $\wedge$ Elim read Conjunction Elimination

$$(\wedge\ \text{Elim})\ \frac{A \wedge B}{A} \qquad\qquad (\wedge\ \text{Elim})\ \frac{A \wedge B}{B}$$

Rules of Inference

Does the following argument make logical sense?

1. It is raining.
2. It is hot.
3. So, it is raining and it is hot.

Rules of Inference

Does the following argument make logical sense?

1. It is raining.
2. It is hot.
3. So, it is raining and it is hot.

Sure!

Line 1 says it is raining, line 2 says it is hot, so, if we put this information together, we can deduce line 3 that it is raining and it is hot.

This is formalised as rule $\wedge$ Intr read Conjunction Introduction

$$(\wedge\ \mathrm{Intr})\ \frac{A \qquad B}{A \wedge B}$$

Proofs

Does the following argument make logical sense?

1. It is hot and it is raining.
2. So, it is raining and it is hot.

Proofs

Does the following argument make logical sense?

1. It is hot and it is raining.
2. So, it is raining and it is hot.

Sure!

It reflects the commutativity of $\wedge$.

- From $H \wedge R$ we can deduce $R \wedge H$.
- But we won't introduce a rule for it. Instead, we will prove it from the existing rules.

Proofs

$$(\wedge\ \text{Elim})\ \frac{A \wedge B}{A} \qquad (\wedge\ \text{Elim})\ \frac{A \wedge B}{B} \qquad (\wedge\ \text{Intr})\ \frac{A \qquad B}{A \wedge B}$$

Idea of the proof that from $H \wedge R$ we can conclude $R \wedge H$:

Line 1. Assume $H \wedge R$.

Line 2. Deduce H (by the $\wedge$ Elim rule applied to Line 1)

Line 3. Deduce R (by the $\wedge$ Elim rule applied to Line 1)

Line 4. Deduce $R \wedge H$ (by the $\wedge$ Intr rule applied to Lines 3, 2).

So, from $H \wedge R$ we have deduced $R \wedge H$.

Proofs

$$(\wedge \text{ Elim}) \ \frac{A \wedge B}{A} \qquad (\wedge \text{ Elim}) \ \frac{A \wedge B}{B} \qquad (\wedge \text{ Intr}) \ \frac{A \qquad B}{A \wedge B}$$

Here is the same idea, written as a formal proof

Line	Assumptions	Formula	Justification	References
1	1	$H \wedge R$	Asmp. Intr	
2	1	H	$\wedge$ Elim	1
3	1	R	$\wedge$ Elim	1
4	1	$R \wedge H$	$\wedge$ Intr	3,2

The only new rule is Asmp. Intr read Assumption Introduction.

Deduction: Formal proofs

- A formal proof usually starts with some assumptions.
- Each step creates another formula which is justified by applying an inference rule to formulas from previous steps.
- If we manage to prove a formula F from assumptions[1] $E_1, \cdots, E_k$, then we will write

  $$E_1, \cdots, E_k \vdash F$$

  read $E_1, \cdots, E_k$ proves F, which is called a deductive consequence.
- The name of the symbol $\vdash$ is called "turnstile".
- If there are no assumptions we write $\vdash F$.

In the previous slide we showed: $H \wedge R \vdash R \wedge H$.

[1] Actually, this refers to the assumptions that have not been cancelled... this will be explained later.

$\models$ is a semantic notion

$\vdash$ is a syntactic notion

Natural deduction

We use a deductive system called Natural Deduction (ND).[2]

1. It uses the connectives $\wedge, \vee, \neg, \rightarrow$, and the constant $\perp$.
2. Every connective has two types of inference rules:
   - Introduction rules introduce the connective
   - Elimination rules remove the connective
   - So there are 8 rules involving the connectives.
3. There is also a rule to introduce assumptions.
4. ...
5. ...

[2] Developed by Gerhard Gentzen (20th century German logician), and Stanisław Jaśkowski (20th century Polish logician).

Natural deduction

Each line of the proof means the following:

> If we know the *assumptions* hold, then we conclude the *formula* holds, because it can be *justified* by applying the rule to the *referenced* lines above it.

Line	Assumptions	Formula	Justification	References
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
8	1,2,4	$\neg C$	$\wedge$ Elim	6
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Natural deduction

Each line of the proof means the following:

> If we know the *assumptions* hold, then we conclude the *formula* holds, because it can be *justified* by applying the rule to the *referenced* lines above it.

Line	Assumptions	Formula	Justification	References
1	1	$H \wedge R$	Asmp. Intr	
2	1	H	$\wedge$ Elim	1
3	1	R	$\wedge$ Elim	1
4	1	$R \wedge H$	$\wedge$ Intr	3,2

Natural deduction: rules involving $\to$

The the following argument make logical sense?

1. If it is hot then it is raining.
2. It is hot.
3. So, it is raining.

Sure!

This is formalised as rule $\to$ Elim read Implication Elimination

$$(\to \text{Elim}) \; \frac{A \to B \qquad A}{B}$$

Natural deduction: proofs involving $\rightarrow$

Does the following argument make logical sense?

1. If it is hot then it is raining.
2. If it is raining then it is wet.
3. So, it if it is hot then it is wet.

Let's try prove: $(H \rightarrow R), (R \rightarrow W) \vdash (H \rightarrow W)$

Line	Assumptions	Formula	Justification	References
1	1	$H \rightarrow R$	Asmp. Intr	
2	2	$R \rightarrow W$	Asmp. Intr	
3	3	H	Asmp. Intr	
4	1,3	R	$\rightarrow$ Elim	1,3
5	1,2,3	W	$\rightarrow$ Elim	2,4

But this only allows us to deduce: $(H \rightarrow R), (R \rightarrow W), H \vdash W$

Natural deduction: rules involving $\rightarrow$

$$
(\rightarrow \mathsf{Intr})\ \dfrac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \rightarrow B}
$$

The rule $\rightarrow$ Intr read Implication Introduction

- formalises that if we have a proof of B by assuming A, then we can get "A then B" **without the assumption** A
- has a new feature: we assume A, use it to prove B, and then cancel the assumption when deducing $A \rightarrow B$.

Prove $(H \to R), (R \to W) \vdash (H \to W)$ in ND

Line	Assumptions	Formula	Justification	References
1	1	$H \to R$	Asmp. Intr	
2	2	$R \to W$	Asmp. Intr	
3	3	H	Asmp. Intr	
4	1,3	R	$\to$ Elim	1,3
5	1,2,3	W	$\to$ Elim	2,4
6	1,2	$H \to W$	$\to$ Intr	3,5

In line 6, when we apply ($\to$ Intr):

- we do not have line 3 (corresponding to formula H) in the Assumptions of line 6.

This is called cancelling (aka discharging) the assumption.

Natural deduction

We use a deductive system called Natural Deduction (ND).

1. ...
2. ...
3. ...
4. Some of the rules cancel (aka discharge) assumptions.
5. If we collect all the assumptions $E_1, \cdots, E_k$ that have not been cancelled at the end of the derivation, and if F is the last formula in the proof, then we write

$$E_1, \cdots, E_k \vdash F$$

Natural deduction: rules involving $\vee$

$$(\vee\ \mathsf{Intr})\ \frac{A}{A \vee B} \qquad\qquad (\vee\ \mathsf{Intr})\ \frac{A}{B \vee A}$$

The $\vee I$ rule formalises the following reasoning:

- if we have A, then we can get $A \vee B$ (as well as $B \vee A$), no matter what B is.

Natural deduction: rules involving $\vee$

$$
(\vee\ \text{Elim})\ \frac{A \vee B \qquad \begin{matrix}[A] \\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[B] \\ \vdots \\ C\end{matrix}}{C}
$$

The ($\vee$ Elim) rule formalises "reasoning by cases":

– if we have $A \vee B$, and

– a proof of C by assuming A, and

– a proof of C by assuming B,

– then we can get C.

Self test

Which of the following is an example of "reasoning by cases"?

Vote now! (on mentimeter)

1. If it is hot, I wear a hat. If it is cold, I wear a hat. Conclude, I wear a hat.
2. If it is hot, I wear a hat. If it is cold, I wear a hat. It is hot or cold. Conclude, I wear a hat.

Prove $(A \vee B), (A \to C), (B \to C) \vdash C$ in ND

Proof strategy: get C by assuming A, get C by assuming B, apply ($\vee$ Elim) to $A \vee B$.

Line	Assumptions	Formula	Justification	References
1	1	$A \vee B$	Asmp. Intr	
2	2	$A \to C$	Asmp. Intr	
3	3	$B \to C$	Asmp. Intr	
4	4	A	Asmp. Intr	
5	2,4	C	$\to$ Elim	2,4
6	6	B	Asmp. Intr	
7	3,6	C	$\to$ Elim	3,6
8	1,2,3	C	$\vee$ Elim	1,4,5,6,7

In line 8 we use ($\vee$ Elim):

- References: L1 ($A \vee B$), L4,L5 ($A \vdash C$), L6,L7 ($B \vdash C$)

Natural deduction: rules involving $\neg$

$$(\neg\ \text{Elim})\ \frac{A \qquad \neg A}{\perp}$$

$$\begin{array}{c} [A] \\ \vdots \\ (\neg\ \text{Intr})\ \dfrac{\perp}{\neg A} \end{array}$$

- The ($\neg$ Elim) rule formalises that $\perp$ follows from any contradiction.
- The ($\neg$ Intr) rule formalises that if we find a proof of $\perp$ by assuming A, then we get $\neg A$.

Prove $F \vdash \neg\neg F$ in ND

Proof strategy: Assume $\neg F$, get $\perp$, and apply ($\neg$ Intr) to get $\neg\neg F$.

Line	Assumptions	Formula	Justification	References
1	1	F	Asmp. Intr	
2	2	$\neg F$	Asmp. Intr	
3	1,2	$\perp$	$\neg$ Elim	1,2
4	1	$\neg\neg F$	$\neg$ Intr	2,3

Natural deduction: two more rules...

$$(\perp)\ \frac{\perp}{A}$$

The $(\perp)$ rule formalises that from a false assumption, any formula can be derived.

Let's prove : $A, (A \to \neg A) \vdash B$

Line	Assumptions	Formula	Justification	References
1	1	A	Asmp. Intr	
2	2	$A \to \neg A$	Asmp. Intr	
3	1,2	$\neg A$	$\to$ Elim	1,2
4	1,2	$\perp$	$\neg$ Elim	1,3
5	1,2	B	$\perp$	4

Natural deduction: last rule

$$
\dfrac{\begin{array}{c}[\neg A] \\ \vdots \\ \bot\end{array}}{A}\ (\text{RA})
$$

- Read Reduction to the Absurd.
- Formalises the method of proof by contradiction: if assuming A is not true leads to a contradiction, then A must be true.
- Note the symmetry between (RA) and ($\neg$ Intr)
  - in (RA) we assume $\neg A$, prove $\bot$, and deduce A.
  - in ($\neg$ Intr) we assume A, prove $\bot$, and deduce $\neg A$.

Natural deduction: proofs

$$\neg\neg F \vdash F$$

Proof strategy: apply (RA) by assuming $\neg F$, deducing $\bot$, and concluding $\neg\neg F$.

Line	Assumptions	Formula	Justification	References
1	1	$\neg\neg F$	Asmp. Intr	
2	2	$\neg F$	Asmp. Intr	
3	1,2	$\bot$	$\neg$ Elim	1,2
4	1	F	RA	2,3

Natural Deduction

Writing formal proofs in ND takes practice!

- The creative part is coming up with the proof strategy.
- The technical part is writing down the correct assumptions and references.

You will use a tool for writing and checking proofs called Logic Tutor. See Ed for details.

Proof strategies

What assumptions should I introduce?

- Obviously you should introduce the antecedents of whatever consequence you are proving (not all may be needed).

But what additional assumptions should I introduce? You should introduce an assumption only if you have a strategy (aka plan) on how to cancel it (you might even label your assumptions with how you plan to cancel it in the future).

- E.g., if you are trying to prove A you could assume $\neg A$ with a plan to cancel it using (RA).
- E.g., if you are trying to prove $\neg A$ you could assume A with a plan to cancel it using ($\neg$ Intr).
- E.g., if you are trying to prove $A \to B$ you could assume A with a plan to cancel it with ($\to$ Intr).

Summary slides

($\wedge$ Intr) $\dfrac{A \qquad B}{A \wedge B}$	($\wedge$ Elim) $\dfrac{A \wedge B}{A}$	($\wedge$ Elim) $\dfrac{A \wedge B}{B}$
($\vee$ Intr) $\dfrac{A}{A \vee B}$	($\vee$ Intr) $\dfrac{A}{B \vee A}$	($\rightarrow$ E) $\dfrac{A \rightarrow B \qquad A}{B}$
($\vee$ Elim) $\dfrac{A \vee B \qquad \begin{array}{c}[A] \\ \vdots \\ C\end{array} \qquad \begin{array}{c}[B] \\ \vdots \\ C\end{array}}{C}$		($\rightarrow$ Intr) $\dfrac{\begin{array}{c}[A] \\ \vdots \\ B\end{array}}{A \rightarrow B}$

(¬ Elim) $\dfrac{A \qquad \neg A}{\bot}$	(⊥) $\dfrac{\bot}{A}$
(¬ Intr) $\begin{array}{c}[A]\\ \vdots\\ \dfrac{\bot}{\neg A}\end{array}$	(RA) $\begin{array}{c}[\neg A]\\ \vdots\\ \dfrac{\bot}{A}\end{array}$

What line numbers do I put under the “references” column?

– Look above the horizontal line of the rule being used:

– Every formula above the line gets a reference, including the formulas that were assumed.

– Eg.

$$(\wedge\ \mathsf{Intr})\ \frac{A \qquad B}{A \wedge B}$$

gets two references, one for where A was proved and one for where B was proved.

– Eg.

$$\begin{array}{c} [\neg A] \\ \vdots \\ (\mathsf{RA})\ \dfrac{\bot}{A} \end{array}$$

gets two references, one where $\bot$ was proved and one where $\neg A$ is assumed.

What line numbers do I put under the "assumptions" column?

- Only line numbers of (Asmp. Intr) rules.
- If a rule deduces F, then collect all the assumptions used to derive the formulas above the horizontal line, but do not include cancelled assumptions.
- Eg.

$$(\wedge\ \text{Intr})\ \frac{A \qquad B}{A \wedge B}$$

  put the line numbers of the assumptions used to prove A and used to prove B.

- Eg.

$$\begin{array}{c}[\neg A] \\ \vdots \\ (\text{RA})\ \dfrac{\bot}{A}\end{array}$$

  put the line numbers of the assumptions used to prove all the formulas above the line, except $\neg A$.