Linear Algebra
Problem Set 2 - Solutions

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۱) - برای ماتریس A تجزیه QR را با استفاده از فرایند گرام- اشمیت بدست آورید.

الف) $A=\begin{array}{c}\begin{bmatrix}3 & -6\\ 4 & -8\\ 0 & 1\end{bmatrix}\\ \uparrow a_1 \quad \uparrow a_2\end{array}$

ب) $A=\begin{bmatrix}2 & 1 & 3\\ -1 & 0 & 7\\ 0 & -1 & -1\end{bmatrix}$ B

$A: \tilde{q}_1 = a_1 \rightarrow q_1 = \frac{1}{5}\begin{bmatrix}3\\4\\0\end{bmatrix}$

$\tilde{q}_2 = \vec{a}_2 - (q_1^T a_2)\vec{q}_1$

$= \begin{bmatrix}-6\\-8\\1\end{bmatrix} + 10\begin{bmatrix}3/5\\4/5\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$\rightarrow q_2 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$A = QR = \begin{bmatrix}3/5 & 0\\ 4/5 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}5 & -10\\ 0 & 1\end{bmatrix}$

$B: \tilde{q}_1 = \begin{bmatrix}2\\-1\\0\end{bmatrix} \rightarrow q_1 = \frac{1}{\sqrt{5}}\begin{bmatrix}2\\-1\\0\end{bmatrix}$

$\tilde{q}_2 = \begin{bmatrix}1\\0\\-1\end{bmatrix} - \frac{2}{\sqrt{5}}\begin{bmatrix}2/\sqrt{5}\\-1/\sqrt{5}\\0\end{bmatrix} = \begin{bmatrix}1/5\\-2/5\\-1\end{bmatrix} \rightarrow q_2 = \sqrt{\frac{5}{6}}\begin{bmatrix}1/5\\2/5\\-1\end{bmatrix}$

$\tilde{q}_3 = \vec{a}_3 - (q_2^T a_3)\vec{q}_2 - (q_1^T a_3)\vec{q}_1 = \begin{bmatrix}3\\7\\-1\end{bmatrix} - \underbrace{\sqrt{\frac{5}{6}}\cdot\frac{22}{5}\cdot\sqrt{\frac{5}{6}}}_{11/3}\begin{bmatrix}1/5\\2/5\\-1\end{bmatrix} + \frac{1}{5}\begin{bmatrix}2\\-1\\0\end{bmatrix} = \begin{bmatrix}8/3\\16/3\\8/3\end{bmatrix}$

$q_3 = \frac{3}{8\sqrt{6}}\begin{bmatrix}8/3\\16/3\\8/3\end{bmatrix} = \begin{bmatrix}1/\sqrt{6}\\2/\sqrt{6}\\1/\sqrt{6}\end{bmatrix}$

$\rightarrow B = QR = \begin{bmatrix}2/\sqrt{5} & 8/3 & 1/\sqrt{6}\\ -1/\sqrt{5} & 16/3 & 2/\sqrt{6}\\ 0 & 8/3 & 1/\sqrt{6}\end{bmatrix}\begin{bmatrix}\sqrt{5} & 2/\sqrt{5} & -1/\sqrt{5}\\ 0 & \sqrt{6}/5 & \frac{22}{5}\cdot\sqrt{5/6}\\ 0 & 0 & 3/8\sqrt{6}\end{bmatrix}$

۲) فضای پوچ و بعد فضای پوچ ماتریس **A** را بدست آورید

الف) $A=\begin{bmatrix}1 & 2 & 0\\ -1 & 0 & 1\\ 0 & 2 & 1\\ 3 & 8 & 1\end{bmatrix}$

ب) $A=\begin{bmatrix}1 & 5 & 7\\ 0 & 0 & 9\end{bmatrix}$ B

$Ax = 0 \rightarrow \begin{bmatrix}1 & 2 & 0\\ -1 & 0 & 1\\ 0 & 2 & 1\\ 3 & 8 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0$

$\begin{bmatrix}x_1 & 2x_2 & 0\\ -x_1 & 0 & x_3\\ 0 & 2x_2 & x_3\\ 3x_1 & 8x_2 & x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \rightarrow \begin{matrix}x_1 = 2x_2\\ x_3 = -2x_2\end{matrix}$

$\rightarrow x = x_2\begin{bmatrix}-2\\1\\-2\end{bmatrix} \rightarrow \text{null space}(A) = \left[\begin{bmatrix}-2\\1\\-2\end{bmatrix}\right]$

$\rightarrow \text{rank}(A) = 2, \ \text{nullity}(A) = 1$

$Bx = 0 \rightarrow \begin{bmatrix}1 & 5 & 7\\ 0 & 0 & 9\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \rightarrow \begin{matrix}x_1 + 5x_2 + 7x_3 = 0 \rightarrow x_1 = -5x_2\\ 0 + 0 + 9x_3 = 0 \rightarrow x_3 = 0\end{matrix}$

$x = x_2\begin{bmatrix}-5\\1\\0\end{bmatrix} \rightarrow \text{nullity}(A) = 1$

6.1 *Matrix and vector notation.* Suppose $a_1, \ldots, a_n$ are m-vectors. Determine whether each expression below makes sense (*i.e.*, uses valid notation). If the expression does make sense, give its dimensions.

(a) $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \longrightarrow$ stacking n m-vector $\longrightarrow$ makes a $nm \times 1$-matrix

(b) $\begin{bmatrix} a_1^T \\ \vdots \\ a_n^T \end{bmatrix} \longrightarrow$ stacking n $1 \times m$-matricies $\longrightarrow$ it makes $n \times m$

(c) $\begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} \longrightarrow$ concatenating n m-vectors $\longrightarrow$ $n \times m$ - matrix

(d) $\begin{bmatrix} a_1^T & \cdots & a_n^T \end{bmatrix} \longrightarrow$ concatenating n m-row vectors $\longrightarrow$ $1 \times nm$-matrix

6.2 *Matrix notation.* Suppose the block matrix

$$\begin{bmatrix} \overset{p\times q}{A} & \overset{p \times p}{I} \\ \underset{q\times q}{I} & \underset{q \times p}{C} \end{bmatrix}$$

makes sense, where A is a $p \times q$ matrix. What are the dimensions of C? $\longrightarrow q \times p$

6.3 *Block matrix.* Assuming the matrix

$$K = \begin{bmatrix} \overset{n\times n}{I} & \overset{n\times m}{A^T} \\ \underset{m\times n}{A} & \underset{m\times m}{0} \end{bmatrix}$$

makes sense, which of the following statements must be true? ('Must be true' means that it follows with no additional assumptions.)

(a) K is square. $\longrightarrow$ T $\longrightarrow$ K is $(m+n)\times(m+n)$

(b) A is square or wide. $\longrightarrow$ A can be tall

(c) K is symmetric, *i.e.*, $K^T = K$. if A is symmetric

(d) The identity and zero submatrices in K have the same dimensions. nope

(e) The zero submatrix is square. T

6.11 Let A and B be two $m \times n$ matrices. Under each of the assumptions below, determine whether $A = B$ must always hold, or whether $A = B$ holds only sometimes.

(a) Suppose $Ax = Bx$ holds for all n-vectors x. $A = B$ always

(b) Suppose $Ax = Bx$ for some nonzero n-vector x. $\longrightarrow$ sometimes

6.12 *Skew-symmetric matrices.* An $n \times n$ matrix A is called *skew-symmetric* if $A^T = -A$, *i.e.*, its transpose is its negative. (A symmetric matrix satisfies $A^T = A$.)

(a) Find all 2×2 skew-symmetric matrices.

$C^T = -C \rightarrow \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix} = \begin{bmatrix} -C_{11} & -C_{21} \\ -C_{12} & -C_{22} \end{bmatrix}$

$\Rightarrow C_{11} = -C_{11} \rightarrow C_{11} = 0$ and $C_{12} = -C_{21} \rightarrow C = \alpha \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$C_{22} = -C_{22} \rightarrow C_{22} = 0$

(b) Explain why the diagonal entries of a skew-symmetric matrix must be zero.

since $C^T = -C \rightarrow \underbrace{(C^T)_{ii}}_{\text{equal to } C_{ii}} = -C_{ii} \rightarrow C_{ii} = -C_{ii} \rightarrow C_{ii} = 0$

(c) Show that for a skew-symmetric matrix A, and any n-vector x, $(Ax) \perp x$. This means that Ax and x are orthogonal. *Hint.* First show that for any $n \times n$ matrix A and n-vector x, $x^T(Ax) = \sum_{i,j=1}^n A_{ij}x_ix_j$.

$\rightarrow x^T(Ax) = \sum_{i=1}^n x_i (Ax)_i = \sum_{i=1}^n x_i \sum_{j=1}^n A_{ij} x_j = \sum_{i,j=1}^n A_{ij} x_i x_j$

$\rightsquigarrow$ since A is a skew-symmetric $\longrightarrow$ $A_{ii} = 0$, $A_{ij} = -A_{ji}$ $\rightarrow \sum_{i,j=1}^n A_{ij} x_i x_j = 0$

$\rightsquigarrow$ so $x^T(Ax) = 0 \rightarrow x \perp Ax$

(d) Now suppose A is any matrix for which $(Ax) \perp x$ for any n-vector x. Show that A must be skew-symmetric. *Hint.* You might find the formula

$$(e_i + e_j)^T(A(e_i + e_j)) = A_{ii} + A_{jj} + A_{ij} + A_{ji},$$

valid for any $n \times n$ matrix A, useful. For $i = j$, this reduces to $e_i^T(Ae_j) = A_{ii}$.

$Ax \perp x \Rightarrow A_{ii} = e_i^T A_i e_i = 0 \rightarrow$ all diagonal entries are 0

$\longrightarrow x = e_i + e_j \rightarrow 0 = (e_i + e_j)^T A (e_i + e_j) = A_{ij} + A_{ji}$

6.16 *Columns of difference matrix.* Are the columns of the difference matrix D, defined in (6.5), linearly independent?

since D is $(n-1) \times n$ matrix $\rightarrow$ from dimentions-independence inequality D is linearly dependent.

6.17 *Stacked matrix.* Let A be an $m \times n$ matrix, and consider the stacked matrix S defined by

$$S_{(m+n)\times n} \quad S = \begin{bmatrix} A \\ I \end{bmatrix}. \quad Sx = \begin{bmatrix} Ax \\ x \end{bmatrix} = 0 \quad \begin{array}{l} Ax = 0 \\ x = 0 \end{array}$$

$m \times n$, $n \times n$

When does S have linearly independent columns? When does S have linearly independent rows? Your answer can depend on m, n, or whether or not A has linearly independent columns or rows.

→ S always has linearly independent column

↳ $Sx = 0 = (Ax, x) \rightarrow x = 0$

→ S always has linearly dependent rows

↳ S has $(m+n)$ rows

each row has dimention n

so by independence-dimention inequality → is dependent

10.2 *Ones matrix.* There is no special notation for an $m \times n$ matrix all of whose entries are one. Give a simple expression for this matrix in terms of matrix multiplication, transpose, and the ones vectors $\mathbf{1}_m$, $\mathbf{1}_n$ (where the subscripts denote the dimension).

$(1_m 1_n)^T$ → $m \times n$ - matrix that all the entries are one.

$(m \times 1)$ $(1 \times n)$

10.3 *Matrix sizes.* Suppose A, B, and C are matrices that satisfy $A + BB^T = C$. Determine which of the following statements are necessarily true. (There may be more than one true statement.)

(a) A is square. True

(b) A and B have the same dimensions. not, necessarily

(c) A, B, and C have the same number of rows. True

(d) B is a tall matrix. not, necessarily

$B_{\boxed{m}\times n} \, B^T_{n \times m} = \ldots_{\boxed{m}\times m}$

since $A + \ldots_{\boxed{m}\times m} = C_{m \times m}$

↳ A is $(m \times m)$

10.4 *Block matrix notation.* Consider the block matrix

$$A = \begin{bmatrix} I & B & 0 \\ B^T & 0 & 0 \\ 0 & 0 & BB^T \end{bmatrix},$$

I: 10×10; B^T: 5×10; 0 (row 3, col 1): 10×10; B: 10×5; 0 (row 1, col 3): 10×10; 0 (row 2, col 2): 5×5; 0 (row 3, col 2): 10×5; 0 (row 2, col 3): 5×10; BB^T: 10×10

where B is 10×5. What are the dimensions of the four zero matrices and the identity matrix in the definition of A? What are the dimensions of A?

10.6 *Product of rotation matrices.* Let A be the 2×2 matrix that corresponds to rotation by θ radians, defined in (7.1), and let B be the 2×2 matrix that corresponds to rotation by ω radians. Show that AB is also a rotation matrix, and give the angle by which it rotates vectors. Verify that $AB = BA$ in this case, and give a simple English explanation.

$$AB = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos w & -\sin w \\ \sin w & \cos w \end{bmatrix} = \begin{bmatrix} \cos\theta\cos w - \sin\theta\sin w & -\sin w\cos\theta - \sin\theta\cos w \\ \cos w\sin\theta + \cos\theta\sin w & -\sin w\sin\theta + \cos w\cos\theta \end{bmatrix}$$

AB is rotation by $(\theta + w)$ $\leftarrow$
$$= \begin{bmatrix} \cos(\theta + w) & -\sin(w+\theta) \\ \sin(\theta + w) & \cos(\theta + w) \end{bmatrix}$$

$$BA = \begin{bmatrix} \cos w & -\sin w \\ \sin w & \cos w \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos w\cos\theta - \sin w\sin\theta & -\cos w\sin\theta - \sin w\cos\theta \\ \sin w\cos\theta + \sin w\cos\theta & -\sin w\sin\theta + \cos w\cos\theta \end{bmatrix}$$

$BA = AB$ $\leftarrow$
$$= \begin{bmatrix} \cos(w+\theta) & -\sin(w+\theta) \\ \sin(w+\theta) & \cos(w+\theta) \end{bmatrix}$$

10.17 *Patients and symptoms.* Each of a set of N patients can exhibit any number of a set of n symptoms. We express this as an $N \times n$ matrix S, with

$$S_{ij} = \begin{cases} 1 & \text{patient } i \text{ exhibits symptom } j \\ 0 & \text{patient } i \text{ does not exhibit symptom } j. \end{cases}$$

$$\begin{array}{cc} & \text{symptom} \\ \text{patients} & \begin{bmatrix} & & \end{bmatrix} \end{array}$$

Give simple English descriptions of the following expressions. Include the dimensions, and describe the entries.

(a) $S\mathbf{1}$. $\rightarrow$ is an N-vector that contains the num of symptoms patient has.

(b) $S^T\mathbf{1}$. $\rightarrow$ is an n-vector that contains the num of patients exhibit symptoms.

(c) S^TS. $\rightarrow$ is an $n \times n$-matrix, i,j entry contains the number of patients exhibit both symptoms i and j.

(d) SS^T. $\rightarrow$ is an $N \times N$-matrix, i,j entry $\hookrightarrow$ contains the num of symptoms which patients i and j have.

$$\begin{array}{cc} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{S} \end{array} \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{S^T} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

10.24 *Matrix power identity.* A student says that for any square matrix A,

$$(A + I)^3 = A^3 + 3A^2 + 3A + I.$$

Is she right? If she is, explain why; if she is wrong, give a specific counterexample, *i.e.*, a square matrix A for which it does not hold.

$$(A+I)^3 = (A+I)(A+I)(A+I) = (A^2 + IA + AI + I^2)(A+I)$$
$$= (A^2 + 2A + I)(A+I) = A^3 + 3A^2 + 3A + I \rightarrow \text{she is right.}$$

10.34 Choose one of the responses *always*, *never*, or *sometimes* for each of the statements below. 'Always' means the statement is always true, 'never' means it is never true, and 'Sometimes' means it can be true or false, depending on the particular values of the matrix or matrices. Give a brief justification of each answer.

(a) An upper triangular matrix has linearly independent columns. sometimes

(b) The rows of a tall matrix are linearly dependent. always $\rightarrow$ independent - dimention inequality

(c) The columns of A are linearly independent, and $AB = 0$ for some nonzero matrix B. never

$\hookrightarrow$ so $Ax = 0 \rightarrow x = 0$

10.37 *Orthogonal* 2×2 *matrices.* In this problem, you will show that every 2×2 orthogonal matrix is either a rotation or a reflection (see §7.1).

(a) Let

$$Q = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be an orthogonal 2×2 matrix. Show that the following equations hold:

$$a^2 + c^2 = 1, \qquad b^2 + d^2 = 1, \qquad ab + cd = 0.$$

$$\rightsquigarrow Q^TQ = I \rightsquigarrow \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) Define $s = ad - bc$. Combine the three equalities in part (a) to show that

$$|s| = 1, \qquad b = -sc, \qquad d = sa. \qquad a^2b^2 \quad c^2d^2 \quad ab = -cd$$

$$s^2 = (ad - bc)^2 = a^2d^2 + c^2b^2 - abcd - abcd$$
$$= a^2d^2 + c^2b^2 + a^2b^2 + c^2d^2 = (a^2+c^2)(b^2+d^2) = 1$$

$$-sc = -(ad - bc)c = -adc + bc^2 = (a^2+c^2)b = b$$
$$sa = (ad - bc)a = a^2d - abc = (a^2+c^2)d = d$$

(c) Suppose $a = \cos\theta$. Show that there are two possible matrices Q: A rotation (counterclockwise over θ radians), and a reflection (through the line that passes through the origin at an angle of $\theta/2$ radians with respect to horizontal).

$$\rightsquigarrow c = \pm\sqrt{1 - a^2} = \pm \sin\theta, \quad s = \pm 1$$

$$\left\{ \begin{array}{l} Q_1 = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}, \quad Q_2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\ \\ Q_3 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & -\cos\theta \end{bmatrix}, \quad Q_4 = \begin{bmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \end{array} \right\}$$

if $\cos\theta\sin\theta \neq 0$

$\hookrightarrow Q_3, Q_4$ not orthogonal

if $\cos\theta\sin\theta = 0$

$\hookrightarrow Q_3, Q_4$ same form as Q_1, Q_2

$\hookrightarrow \sin\theta = 0 \rightarrow Q_3 = Q_1, Q_4 = Q_2$

$\hookrightarrow \cos\theta = 0 \rightarrow Q_4, Q_3$ same form as Q_2, Q_1

$\longrightarrow Q_1, Q_2$ are orthogonal

Q_1: reflection, Q_2: rotation

11.4 *Transpose of orthogonal matrix.* Let U be an orthogonal $n \times n$ matrix. Show that its transpose U^T is also orthogonal.

$U \to$ orthogonal $\to$ it square and $U^TU = I$ $\to$ U^T is inverse of U
and $UU^T = I$ $\to$ since $(U^T)^T = U$ $\to$ U^T is orthogonal too.

11.18 *Tall-wide product.* Suppose A is an $n \times p$ matrix and B is a $p \times n$ matrix, so $C = AB$ makes sense. Explain why C cannot be invertible if A is tall and B is wide, *i.e.*, if $p < n$. *Hint.* First argue that the columns of B must be linearly dependent.

$\to$ $A_{n \times p}$, $B_{p \times n}$, $C_{n \times n}$

if $n > p$ $\to$ by independence - dimention inequality B is linearly dependent

$\to$ $Bx = 0$ $\to$ $ABx = 0$ $\to$ $A(\underbrace{Bx}_{0}) = 0$ $\to$ $AB = C$ $\to$ linearly dependent

at least a none zero vecto

not inveetable

Upper Triangular Matrix

```
import numpy as np
n = 5
max_element = 9
min_element = 0
mat = np.random.randint(min_element, max_element + 1, size=(n, n))
for row in range(n):
    for col in range(n):
        if row > col:
            mat[row][col] = 0
print(mat)
```

Computes the transpose of a given matrix

```
import torch
def transpose(mat):
    # PyTorch tensor transposition
    return mat.T
# Matrix dimensions
n = 7
m = 3
# Create a random tensor of integers between 0 and 9
test_mat = torch.randint(0, 1000, (n, m))
print("Original Matrix:")
print(test_mat)
transposed_mat = transpose(test_mat.cuda())
print("\nTransposed Matrix:")
print(transposed_mat)
```

Visualizing a Matrix and Its Submatrix

This code demonstrates how to create and manipulate matrices in NumPy, extracting a submatrix from a larger matrix, and visualizing them using Matplotlib. The original 10x10 matrix is constructed using np.arange and reshaped into a 2D array. A 5x5 submatrix is then extracted from the top-left corner of the original matrix. Both matrices are visualized as grayscale images, with the original matrix including dashed lines to highlight the boundaries of the submatrix. Text labels are added to both matrices to display their values.

```
import numpy as np
import matplotlib.pyplot as plt
# Create the matrix
C = np.arange(100).reshape((10,10))
block_shifted = C.copy()
# extract submatrix
```

```
block_shifted[5:10:1,5:10:1], block_shifted[0:5:1,0:5:1] = C[0:5:1,0:5:1].copy(),
C[5:10:1,5:10:1].copy()
block_shifted[0:5:1,5:10:1], block_shifted[5:10:1,0:5:1] = C[5:10:1,0:5:1].copy(),
C[0:5:1,5:10:1].copy()
# here's what the matrices look like
print(C), print(' ')
print(block_shifted)
# visualize the matrices as maps
_,axs = plt.subplots(1,2,figsize=(10,5))
axs[0].imshow(C,cmap='gray',origin='upper',vmin=0,vmax=np.max(C))
axs[0].plot([4.5,4.5],[-.5,9.5],'w--')
axs[0].plot([-.5,9.5],[4.5,4.5],'w--')
axs[0].set_title('Original matrix')
# text labels
for (j,i),num in np.ndenumerate(C):
 axs[0].text(i,j,num,color=[.8,.8,.8],ha='center',va='center')
axs[1].imshow(block_shifted,cmap='gray',origin='upper',vmin=0,vmax=np.max(C))
axs[1].set_title('Block Shifted')
# text labels
for (j,i),num in np.ndenumerate(block_shifted):
 axs[1].text(i,j,num,color=[.8,.8,.8],ha='center',va='center')
plt.savefig('Figure_05_06.png',dpi=300)
plt.show()
```

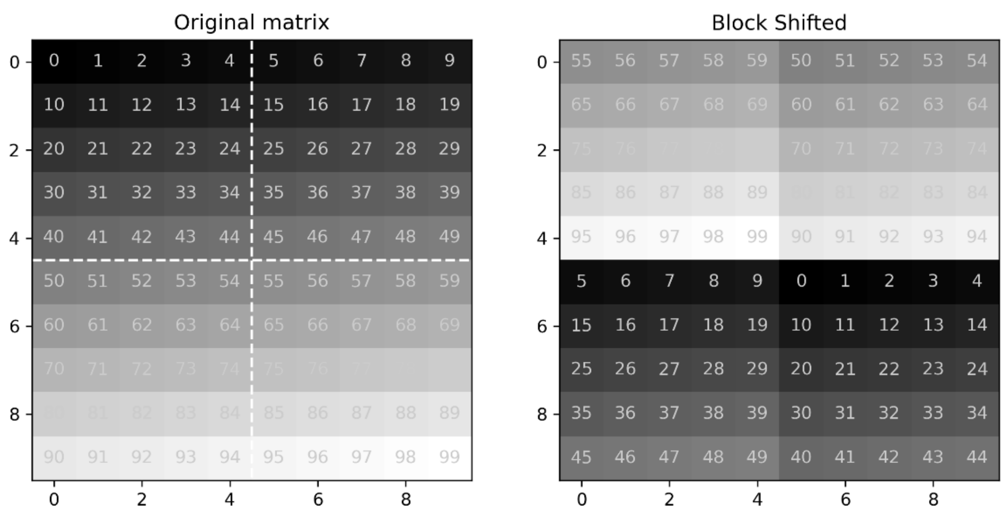

Figure 1- Output