Linear Algebra
Problem Set 5 - Solutions

Abolfazl Ranjbar

Course Instructors: Dr. Elham Ghasrodashti and Dr. Peyman Adibi

Computer Engineering Department
University of Isfahan

Fall Semester 2024

1. Using singular value decomposition, find a pseudo-inverse for each of the following matrices:

a) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ SVD decomposes A into $A = U \Sigma V^T$

$$A^TA = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

eigenvalues: $\det(A^TA - \lambda I) = 0$

$$\det\left(\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}\right) = (2-\lambda)^2 - 1 = 0$$

$\rightsquigarrow$ singular values are: $\sigma_1 = \sqrt{3}$, $\sigma_2 = \sqrt{1} = 1$

The eigenvectors of A^TA corresponds to the columns of V.

for $\lambda = 3$: $(A^TA - 3I) = 0 \quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$x_1 = x_2 \rightarrow V_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

for $\lambda = 1$: $(A^TA - I) = 0 \rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$x_1 = -x_2 \rightarrow V_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\rightsquigarrow V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$U_i = \frac{1}{\sigma_i} A v_i$$

for $\sigma_1 = \sqrt{3} \rightarrow U_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

for $\sigma_2 = 1 \rightarrow U_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$

U_3 is orthogonal to U_1 and U_2.

$\rightarrow U_3 = U_3^T U_1 = 0 \rightarrow U_3 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$

$$U = \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{6} & 1/\sqrt{2} & 2/\sqrt{6} \\ 2/\sqrt{6} & 0 & 1/\sqrt{6} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \rightsquigarrow \Sigma^+ = \begin{bmatrix} 1/\sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A^+ = V\Sigma^+U^T$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{6} & 1/\sqrt{2} & 2/\sqrt{6} \\ 2/\sqrt{6} & 0 & 1/\sqrt{6} \end{bmatrix}$$

b) $A = \begin{bmatrix} 2 & 2 & -2 \\ 4 & 1 & 2 \\ -4 & 2 & 1 \end{bmatrix}$ $\quad A^TA = \begin{bmatrix} 36 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

eigenvalues: its a diagonal matrix so $\rightsquigarrow$ $\lambda_1 = 36$, $\lambda_2 = 9$, $\lambda_3 = 9$
$\sigma_1 = 6$, $\sigma_2 = 3$, $\sigma_3 = 3$

$$\Sigma = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \rightsquigarrow \Sigma^+ = \begin{bmatrix} 1/6 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

The eigenvectors of A^TA corresponds to the column of V.

$$A^TA - \lambda I = \begin{bmatrix} 36-\lambda & 0 & 0 \\ 0 & 9-\lambda & 0 \\ 0 & 0 & 9-\lambda \end{bmatrix}$$

for $\lambda_1 = 36 \rightsquigarrow V_1 \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & -27 & 0 \\ 0 & 0 & -27 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad V_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

for $\lambda_2 = 9 \rightsquigarrow V_2 \Rightarrow \begin{bmatrix} 27 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad V_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $V_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$U_i = \frac{1}{\sigma_i} A v_i$$

for $\sigma_1 = 6 \rightarrow U_1 = \begin{bmatrix} 1/3 & 2/3 & -2/3 \end{bmatrix}^T$

for $\sigma_2 = 3 \rightarrow U_2 = \begin{bmatrix} 2/3 & 1/3 & 2/3 \end{bmatrix}^T$

for $\sigma_3 = 3 \rightarrow U_3 = \begin{bmatrix} -2/3 & 2/3 & 1/3 \end{bmatrix}^T$

$$U^T = \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$$

$$A^+ = V\Sigma^+U^T$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1/6 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}\begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$$

2. Prove the following statements:

a) For a real symmetric matrix, eigenvalues are real, and eigenvectors corresponding to distinct eigenvalues are orthogonal. Verify this property for the following matrix:

$$A = \begin{bmatrix} 8 & 2 & -5 \\ 2 & 11 & -2 \\ -5 & -2 & 8 \end{bmatrix} \quad A^T = \begin{bmatrix} 8 & 2 & -5 \\ 2 & 11 & -2 \\ -5 & -2 & 8 \end{bmatrix} \to \text{symmetric matrix}$$

$$A - \lambda I = \begin{bmatrix} 8-\lambda & 2 & -5 \\ 2 & 11-\lambda & -2 \\ -5 & -2 & 8-\lambda \end{bmatrix} \to |A - \lambda I| = 0$$

$$\hookrightarrow -\lambda^3 + 27\lambda^2 - 207\lambda + 405 = 0$$

$$\hookrightarrow \lambda_1 = 3,\ \lambda_2 = 9,\ \lambda_3 = 15$$

$$\text{if } \lambda_1 = 3 \to v_1 \Rightarrow \begin{bmatrix} 5 & 2 & -5 \\ 2 & 8 & -2 \\ -5 & -2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \to v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{if } \lambda_2 = 9 \to v_2 \Rightarrow \begin{bmatrix} -1 & 2 & -5 \\ 2 & 2 & -2 \\ -5 & -2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \to v_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$\text{if } \lambda_3 = 15 \to v_3 \Rightarrow \begin{bmatrix} -7 & 2 & -5 \\ 2 & -4 & -2 \\ -5 & -2 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \to v_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$v_1^T v_2 = 0$

$v_1^T v_3 = 0 \to v_1, v_2, v_3$ are orthogonal

$v_2^T v_3 = 0$

b) All eigenvalues of a positive definite matrix are positive.

A is positive definite, so $x^T A x > 0,\ \forall x \in R^n,\ x \neq 0$

since A is symmetric $\to A^T A = I \to A = Q \Lambda Q^T$

$$\to x^T A x = x^T Q \Lambda Q^T x \xrightarrow{y = Q^T x} y^T \Lambda y = \sum_{i=1}^{n} \lambda_i y_i^2 > 0$$

is symmetric

positive

all eigenvalues are positive since matrix was positive definite

3. Find the characteristic equation, eigenvalues, and eigenvectors of the following matrix:

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 4 \\ 0 & -1 & 0 \end{bmatrix}$$

characteristic equation:

$$|A - \lambda I| = 0 \to \begin{vmatrix} -1-\lambda & 0 & 0 \\ 0 & -4-\lambda & 4 \\ 0 & -1 & -\lambda \end{vmatrix} = 0$$

$$(-1-\lambda)(\lambda+2)^2 = 0$$

$$\lambda_1 = -1,\ \lambda_2 = -2,\ \lambda_3 = -2$$

eigenvectors:

$$\text{for } \lambda_1 = -1 \to v_1 \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \to v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{for } \lambda_2 = -2 \to v_2 \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 4 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \to v_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

4. For the following matrix, use the Cayley-Hamilton theorem to compute A^k:

$$A = \begin{bmatrix} -2 & 2 \\ 1 & -3 \end{bmatrix} \qquad A^k = \beta_1 A + \beta_0 I$$

$$|A - \lambda I| = 0 \rightsquigarrow \begin{vmatrix} -2-\lambda & 2 \\ 1 & -3-\lambda \end{vmatrix} \rightsquigarrow \lambda^2 + 5\lambda + 4 = 0 \rightarrow \lambda_1 = -1,\ \lambda_2 = -4$$

$$\left\{ \begin{array}{l} \lambda_1 = -1 \rightarrow (-1)^k = -\beta_1 + \beta_0 \\ \lambda_2 = -4 \rightarrow (-4)^k = -4\beta_1 + \beta_0 \end{array} \right\} \quad \beta_0 = \frac{4(-1)^k - (-4)^k}{3} \qquad \beta_1 = \frac{(-1)^k - (-4)^k}{3}$$

$$A^k = \frac{(-1)^k - (-4)^k}{3} A + \frac{4(-1)^k - (-4)^k}{3} I$$

5. Find the diagonalized form of the matrix A**:**

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -1 & -6 & -2 \\ 5 & 0 & 0 \end{bmatrix} \quad |A - \lambda I| = 0 \rightarrow \begin{vmatrix} 4-\lambda & 0 & 1 \\ -1 & -6-\lambda & -2 \\ 5 & 0 & -\lambda \end{vmatrix} \rightsquigarrow (\lambda+1)(-\lambda^2 - \lambda + 30) = 0$$

$$\lambda_1 = -6,\ \lambda_2 = -1,\ \lambda_3 = 5$$

$$\text{for } \lambda_1 = -6 \rightsquigarrow v_1 \Rightarrow \begin{bmatrix} 10 & 0 & 1 \\ -10 & 0 & -2 \\ 5 & 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \rightsquigarrow v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{for } \lambda_2 = -1 \rightsquigarrow v_2 \Rightarrow \begin{bmatrix} 5 & 0 & 1 \\ -10 & -5 & -2 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \rightsquigarrow v_2 = \begin{bmatrix} 5 \\ 9 \\ -25 \end{bmatrix}$$

$$\text{for } \lambda_3 = 5 \rightsquigarrow v_3 \Rightarrow \begin{bmatrix} -1 & 0 & 1 \\ -10 & -11 & -2 \\ 5 & 0 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \rightarrow v_3 = \begin{bmatrix} 11 \\ -3 \\ 11 \end{bmatrix}$$

$$\rightsquigarrow Q = \begin{bmatrix} 0 & 5 & 11 \\ 1 & 9 & -3 \\ 0 & -25 & 11 \end{bmatrix} \qquad Q^{-1} \rightsquigarrow |Q| \text{ "expand along the first column"}$$

$$-(55 + 25 \times 11) = -330$$

$$\text{cofactor: } \begin{bmatrix} 24 & -11 & -25 \\ -330 & 0 & 0 \\ -11\varepsilon & 11 & -5 \end{bmatrix} \xrightarrow{T} \begin{bmatrix} 24 & -330 & -11\varepsilon \\ -11 & 0 & 11 \\ -25 & 0 & -5 \end{bmatrix} \xrightarrow[-330]{\text{divide by}} Q^{-1}$$

$$\Lambda = Q^{-1} A Q = \frac{-1}{330} \begin{bmatrix} 24 & -330 & -11\varepsilon \\ -11 & 0 & 11 \\ -25 & 0 & -5 \end{bmatrix} \begin{bmatrix} 4 & 0 & 1 \\ -1 & -6 & -2 \\ 5 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 5 & 11 \\ 1 & 9 & -3 \\ 0 & -25 & 11 \end{bmatrix}$$

Singular Value Decomposition (SVD) Visualization

This code illustrates the Singular Value Decomposition (SVD) of a random 4x6 matrix A using NumPy. The SVD decomposes the matrix into three components: the left singular vectors (U), the singular values (Σ), and the right singular vectors (V). These components are visualized side-by-side as grayscale images, with each subplot labeled to indicate its corresponding matrix. The diagonal matrix Σ is constructed from the singular values using NumPy's fill_diagonal function. The figure is formatted for clarity and saved as a high-quality vector image for publication purposes.

```
import numpy as np
import matplotlib.pyplot as plt
# NOTE: these lines define global figure properties used for publication.
import matplotlib_inline.backend_inline
matplotlib_inline.backend_inline.set_matplotlib_formats('svg') # display figures in vector
format
plt.rcParams.update({'font.size':14}) # set global font size
A = np.random.randn(4,6)
# its SVD
U,s,Vt = np.linalg.svd(A)
# create Sigma from sigma's
S = np.zeros(np.shape(A))
np.fill_diagonal(S,s)
# show the matrices
_,axs = plt.subplots(1,4,figsize=(10,6))
axs[0].imshow(A,cmap='gray',aspect='equal')
axs[0].set_title('$\mathbf{A}$\nThe matrix')
axs[1].imshow(U,cmap='gray',aspect='equal')
axs[1].set_title('$\mathbf{U}$\n(left singular vects)')
axs[2].imshow(S,cmap='gray',aspect='equal')
axs[2].set_title('$\mathbf{\Sigma}$\n(singular vals)')
axs[3].imshow(Vt,cmap='gray',aspect='equal')
axs[3].set_title('$\mathbf{V}$\n(right singular vects)')
plt.tight_layout()
plt.savefig('Figure_14_02.png',dpi=300)
plt.show()
```

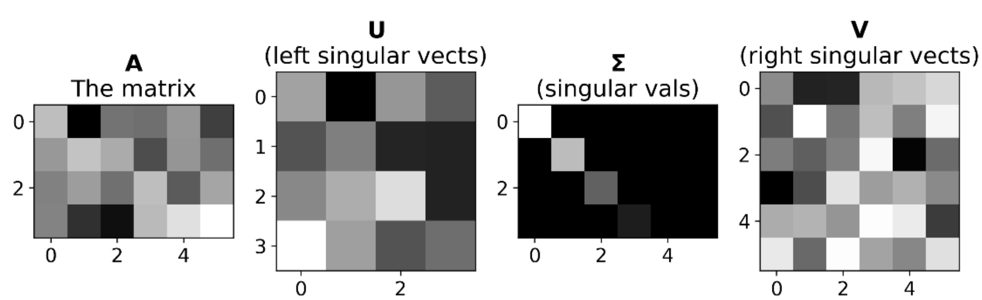


Figure 1-Output

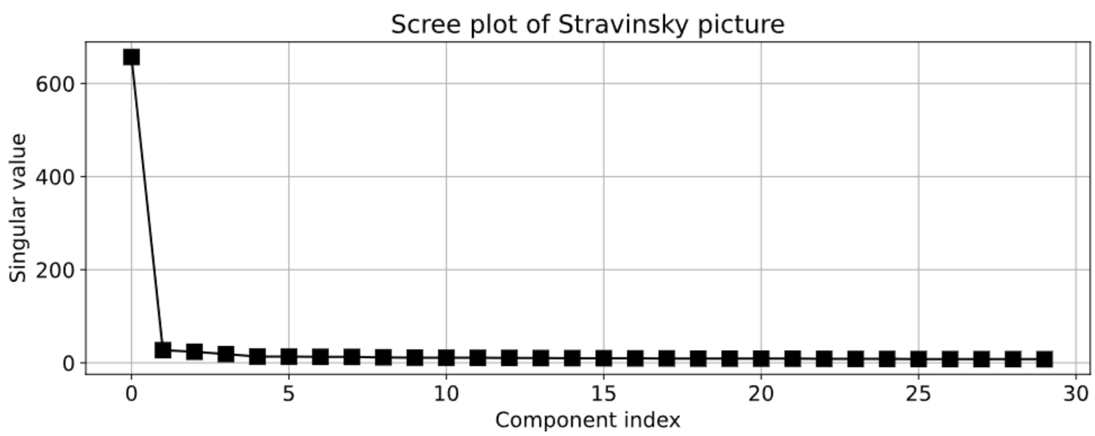
SVD Analysis and Reconstruction of Stravinsky's Portrait

This code demonstrates the application of Singular Value Decomposition (SVD) to compress and reconstruct an image. A portrait of Igor Stravinsky, drawn by Pablo Picasso, is loaded from a URL, converted to grayscale, and analyzed. The SVD decomposes the image matrix into its singular values and vectors. A scree plot is created to visualize the distribution of the singular values, revealing the most significant components for reconstruction.

Using the first 80 components, the image is reconstructed, and the original, reconstructed, and squared error matrices are displayed side by side for comparison. This example highlights the power of SVD in image compression by showing how a low-rank approximation retains much of the visual quality with reduced data. The figure is saved as a high-resolution image for documentation.

```
import numpy as np
import matplotlib.pyplot as plt
from matplotlib.gridspec import GridSpec # for the subplots
import pandas as pd
import seaborn as sns
# NOTE: these lines define global figure properties used for publication.
import matplotlib_inline.backend_inline
matplotlib_inline.backend_inline.set_matplotlib_formats('svg') # display figures in vector
format
plt.rcParams.update({'font.size':14}) # set global font size
from skimage import io,color
url='https://upload.wikimedia.org/wikipedia/en/2/26/Igor_Stravinsky_as_drawn_by_Pablo_Picasso_
31_Dec_1920_-_Gallica.jpg'
# import picture and downsample to 2D
strav = io.imread(url) / 255
#strav = color.rgb2gray(strav)
plt.figure(figsize=(8,8))
plt.imshow(strav,cmap='gray')
plt.title(f'Matrix size: {strav.shape}, rank: {np.linalg.matrix_rank(strav)}')
plt.show()
# SVD
U,s,Vt = np.linalg.svd(strav)
S = np.zeros_like(strav)
np.fill_diagonal(S,s)
# show scree plot
plt.figure(figsize=(12,4))
plt.plot(s[:30],'ks-',markersize=10)
plt.xlabel('Component index')
plt.ylabel('Singular value')
plt.title('Scree plot of Stravinsky picture')
plt.grid()
plt.show()
          # Reconstruct based on first k layers
```

```python
# number of components
k = 80
# reconstruction
stravRec = U[:,:k] @ S[:k,:k] @ Vt[:k,:]
# show the original, reconstructed, and error
_,axs = plt.subplots(1,3,figsize=(15,6))
axs[0].imshow(strav,cmap='gray',vmin=.1,vmax=.9)
axs[0].set_title('Original')
axs[1].imshow(stravRec,cmap='gray',vmin=.1,vmax=.9)
axs[1].set_title(f'Reconstructed (k={k}/{len(s)})')
axs[2].imshow((strav-stravRec)**2,cmap='gray',vmin=0,vmax=1e-1)
axs[2].set_title('Squared errors')
plt.tight_layout()
plt.savefig('Figure_15_10.png',dpi=300)
plt.show()
```



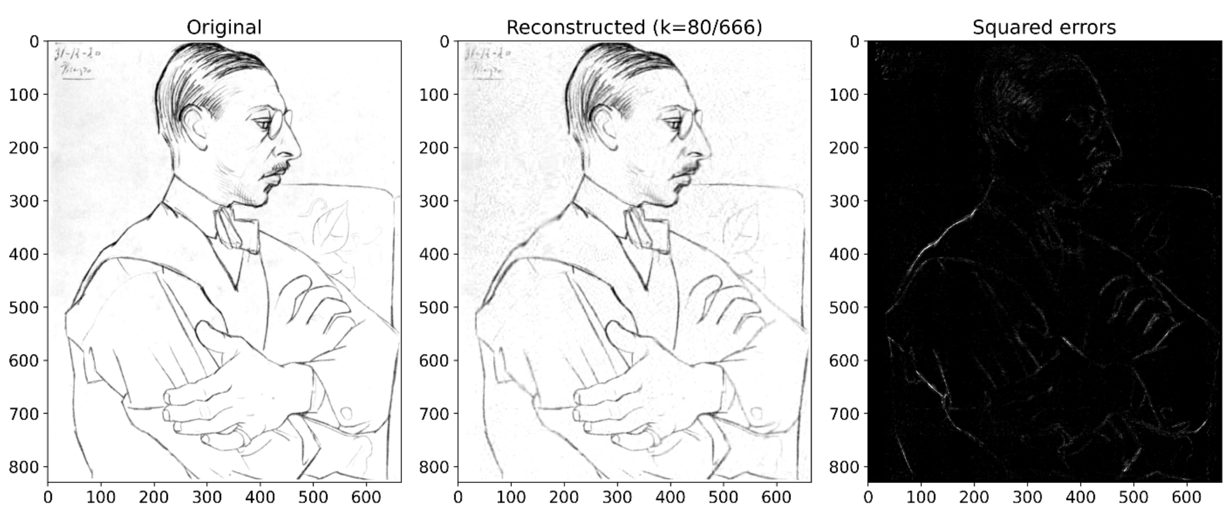


Figure 2-Outputs

One important property of an orthogonal matrix (such as the left and right singular vector matrices) is that it rotates but does not scale a vector. This means that the magnitude of a vector is preserved after multiplication by an orthogonal matrix. Prove that $\|\boldsymbol{uw}\| = \|\boldsymbol{w}\|$. Then, demonstrate this empirically in Python by using a singular vector matrix from the SVD of a random matrix and a random vector.

$$\|v\| = \sqrt{v^T v} \rightarrow \|uw\| = \sqrt{\underbrace{(uw)^T(uw)}_{w^T u\, u^T w}}$$

since u is orthogonal $\rightarrow u^T u = I \rightarrow \|uw\| = \sqrt{w^T w} = \|w\|$

```python
import numpy as np
# Generate a random matrix A
A = np.random.rand(5, 5)
# Perform Singular Value Decomposition
U, S, Vt = np.linalg.svd(A)
# Generate a random vector w
w = np.random.rand(5)
# Compute the magnitudes of the original and transformed vectors
magnitude_original = np.linalg.norm(w)
magnitude_transformed = np.linalg.norm(U.dot(w))
# Display results
print(f"Original magnitude: {magnitude_original:.6f}")
print(f"Transformed magnitude: {magnitude_transformed:.6f}")
print(f"Magnitudes are equal? {'Yes' if np.isclose(magnitude_original, magnitude_transformed) else 'No'}")
```