Review: Sets, measure, measurable function, integration

1. $A \sim B$ A and B have the same cardinal

$A \xrightarrow{f} B$ f is a bijection

(Cantor-Berstein Thm)

countable set $A \sim \mathbb{Z} \setminus \mathbb{Q}$

uncountable set $\mathbb{R}$ $\mathbb{R}^n$ $\mathbb{R} \setminus \mathbb{Q}$

EX1:

Define a bijection from $\mathbb{R}$ to $\mathbb{R} \setminus \mathbb{Q}$

Proof: Define: $Q = \{r_n\}_{n=1,2,\dots}$ $Q+\sqrt{2} = \{r_n + \sqrt{2}\}_{n=1,2,\dots}$

It is easy to see $Q \cap Q+\sqrt{2} = \phi$

$x \in R$ we define

$$f(x) = \begin{cases} x & x \notin Q \cup Q+\sqrt{2} \\ r_{2n} + \sqrt{2} & x = r_n \in Q \\ r_{2m} + \sqrt{2} & x = r_n + \sqrt{2} \in Q+\sqrt{2} \end{cases}$$

f is a bijection

EX2:

Assume $f: \mathbb{R} \to \mathbb{R}$ is increasing. Prove that $D = \{$discontinuous pts of $f\}$ is at most countable

proof: $\forall r \in D \Leftrightarrow \lim_{x \to r^-} f(x) < \lim_{x \to r^+} f(x)$

we define $h(r) \in Q \cap (\lim_{x \to r^-} f(x), \lim_{x \to r^+} f(x))$

$D \xrightarrow{h} Q$ we need to prove h is injection

$r_1 < r_2 \in D \Rightarrow h(r_1) < h(r_2)$

EX3:

Prove $C + C = [0, 2]$ $C+C = \{x+y \mid x, y \in C\}$

proof: "$\Rightarrow$" since $C \subseteq [0,1]$, $C+C \subseteq [0,2]$

"$\Leftarrow$" $\forall x \in [0,2]$ $x = \sum_{n=1}^{\infty} \frac{2a_n}{3^n}$ $a_n \in \{0, 1, 2\}$

$2a_n = b_n + c_n \Longleftarrow = \sum_{n=1}^{\infty} \frac{b_n}{3^n} + \sum_{n=1}^{\infty} \frac{c_n}{3^n}$

$\begin{cases} b_n = c_n = a_n & \text{if } a_n \in \{0, 2\} \\ b_n = 0,\ c_n = 2 & \text{if } a_n = 1 \end{cases}$ $\in C + C$

Measure: $E \subseteq \mathbb{R}^n$ $m_*(E)$ outermeasure.

(1) $m_*(A \cup B) \le m_*(A) + m_*(B)$ "$=$" holds if $d(A,B)>0$

(2) $\forall A \subseteq \mathbb{R}^n$ $\exists$ a measurable set $B \supseteq A$ and $m(B) = m_*(A)$

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(3) $\exists$ non-measurable set $N \subseteq [0,1]$ equivalence relation

$\forall A \subseteq \mathbb{R}^n$ If $m(A)>0$, then A contains a non-measurable set.

EX4:

$\mathcal{M} = \{\text{measurable sets}\}$

Given $E_1, E_2 \subseteq \mathbb{R}^n$ assume $E_1 \cup E_2 \in \mathcal{M}$ and $m(E_1 \cup E_2) < \infty$

If $m(E_1 \cup E_2) = m_*(E_1) + m_*(E_2)$, then $E_1, E_2 \in \mathcal{M}$

proof: $E_i \in F_i$ s.t. $m(F_i) = m_*(E_i)$ $(F_i \in \mathcal{M})$

$m(F_1) + m(F_2) = m_*(E_1) + m_*(E_2)$

$= m(E_1 \cup E_2) \underset{①}{\le} m(F_1 \cup F_2) \underset{②}{\le} m(F_1) + m(F_2)$

$(\le)_1$ "$=$" $m(F_1 \cup F_2 \setminus (E_1 \cup E_2)) = 0$

$(\le)_2$ "$=$" $m(F_1 \cap F_2) = 0$ $\left(m(F_1) + m(F_2) = m(F_1 \cup F_2) + m(F_1 \cap F_2)\right)$

From $\forall i \in \{1,2\}$ $\underline{F_i \setminus E_i \subseteq (F_1 \cap F_2) \cup (F_1 \cup F_2 \setminus E_1 \cup E_2)}$

$m(F_i \setminus E_i) = 0 \xRightarrow{+F_i \in \mathcal{M}} E_i \in \mathcal{M}.$

Exercise 1.

$\mathcal{M} = \mathcal{B} + \{\text{measure } 0 \text{ sets}\}$

$G_\delta, F_\sigma \in \mathcal{B}$ $\mathcal{B} \setminus \{G_\delta \cup F_\sigma\} \neq \varnothing$

(1) $\{E_n\}_{n \ge 1}$ $E = \bigcup_{n=1}^{\infty} E_n$ If $E_n \subseteq E_{n+1}$, then $m(E) = \lim_{n \to \infty} m(E_n)$

(2) ... $E = \bigcap_{n=1}^{\infty} E_n$ If $E_n \supseteq E_{n+1}$ and $m(E_1) < \infty$

then $m(E) = \lim_{n \to \infty} m(E_n)$

(3) $\overline{\lim} E_n = \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} E_j$ If $m(\bigcup_{i=1}^{\infty} E_i) < \infty$ then $(\overline{\lim} E_n) = \lim_{i \to \infty} m(\bigcup_{j=i}^{\infty} E_j)$

Measurable function $f \in MF$ $f^{-1}(A) \in \mathcal{M}$ $\forall A \in \mathcal{B}$ $A = (-\infty, a)$

$f \in MF$ defined on E

$f_n \xrightarrow{a.e.} f$ on E $\xrightarrow[m(E)<\infty]{\text{Egorov}}$ $f_n \to f$ nearly uniformly on E
$\forall \varepsilon\ \exists A_\varepsilon \subseteq E$ s.t. $m(E \setminus A_\varepsilon) < \varepsilon$ and $f_n \to f$ uniformly on A_ε.

$f_n \xrightarrow{a.e.} f$ on E $\xrightarrow{m(E)<\infty}$ $f_n \to f$ in measure $\Rightarrow f_{n_k} \xrightarrow{a.e.} f$

$f_n \xrightarrow{a.e.} f$ on E $\xrightarrow{|f_n| \le g \in L^1(E)}$ $f_n \xrightarrow{L^1} f$ $\left(\lim_{n\to\infty} \int_E |f_n - f| = 0 \Rightarrow \lim_{n\to\infty} \int_E f_n = \int_E f\right)$

$E_n(\varepsilon) = \{x \in E \mid |f_n(x) - f(x)| \ge \varepsilon\}$

(A) $f_n \xrightarrow{a.e.} f$ on $E \iff \forall \varepsilon > 0,\ m(\overline{\lim}\, E_n(\varepsilon)) = 0$

(B) $f_n \to f$ in measure on $E \iff \forall \varepsilon > 0\ \lim_{n\to\infty} m(E_n(\varepsilon)) = 0$

(C) $f_n \to f$ nearly uniformly on $E \iff \forall \varepsilon > 0\ \lim_{n\to\infty} \left(\bigcup_{i=n}^{\infty} E_i(\varepsilon)\right) = 0$.

$m(E) < \infty$

EX5.

Assume $f \in L^1(E)$ and $f > 0$ on E $m(E) < \infty$

prove $\lim_{n\to\infty} \int_E (f(x))^{\frac{1}{n}} = m(E)$

proof: $|(f(x))^{\frac{1}{n}}| \le g(x) \in L^1(E) \Rightarrow \lim_{n\to\infty} \int f^{\frac{1}{n}} = \int \lim_{n\to\infty} f^{\frac{1}{n}} = m(E)$

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$f(x)^{\frac{1}{n}} \le f(x) + 1 \in L^1(E)$

Ex6:

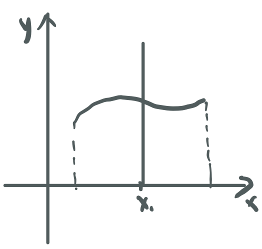
$f: \mathbb{R}^n \to \mathbb{R}$ a measurable function. Define $\Gamma = \{(x,y) \in \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1} \mid y = f(x)\}$

prove Γ is measurable set and $m(\Gamma) = 0$

We define $F(x,y) = f(x)$ $\forall A \in \mathcal{B}$ $F^{-1}(A) = f^{-1}(A) \times \mathbb{R}$

$\pi(x,y) = y$

$\Gamma = \{(x,y) \mid F(x,y) - \pi(x,y) = 0\}$ is a measurable set



$\mathbb{R}^1 \supseteq \Gamma^{x_0} = \{(x,y) \in \Gamma \mid x = x_0\} = \{(x_0, f(x_0))\}$

$m(\Gamma) = \int_{\mathbb{R}^n} m(\Gamma^x)\, dx = \int_{\mathbb{R}^n} 0\, dx = 0$

$\int_{\mathbb{R}^n \times \mathbb{R}^1} \chi_\Gamma(x,y)\, dx\, dy$