**16.** Suppose $f$ is integrable on $\mathbb{R}^d$. If $\delta = (\delta_1, \delta_2, \cdots, \delta_d)$ is a $d$-tuple of non-zero real numbers, and

$$f_\delta(x) = f(\delta x) = f(\delta_1 x_1, \delta_2 x_2, \cdots, \delta_d x_d),$$

show that $f^\delta$ is integrable with

$$\int_{\mathbb{R}^d} f^\delta(x) = |\delta_1|^{-1} \ldots |\delta_d|^{-1} \int_{\mathbb{R}^d} f(x)dx.$$

**Answer:** Use Fubini's theorem and the relative invariance of intrgration on $\mathbb{R}$ under dilations. ∎

**17.** Suppose $f$ is defined on $\mathbb{R}^2$ as follows: $f(x,y) = a_n$ if $n \le x < n+1$ and $n \le y < n+1$,$(n \ge 0)$;$f(x,y) = -a_n$ if $n \le x < n+1$ and $n+1 \le y < n+2$,$(n \ge 0)$; while $f(x,y) = 0$ elsewhere. Here $a_n = \sum_{k \le n} b_k$, with $\{b_n\}$ a positive sequence such that $\sum_{k=0}^{\infty} b_k = s < \infty$.
(a) Verify that each slice $f^y$ and $f_x$ is integrable. Also for all $x$, $\int f_x(y)dy = 0$, and hence $\int(\int f(x,y)dy)dx = 0$.
(b) However, $\int f^y(x)dx = a_0$ if $0 \le y < 1$, and $\int f^y(x)dx = a_n - a_{n-1}$ if $n \le y < n+1$ with $n \ge 1$. Hence $y \to \int f^y(x)dx$ is integrable on $(0,\infty)$ and

$$\int(\int f(x,y)dx)dy = s.$$

(c) Note that $\int_{\mathbb{R}\times\mathbb{R}} |f(x,y)|dxdy = \infty$.

**Answer:** (c) Note that

$$\int_{\mathbb{R}\times\mathbb{R}} |f(x,y)|dxdy = \int(\int |f^y(x)|dx)dy = 2\sum_{k=0}^{\infty} a_n > 2\sum_{k=k_0}^{\infty} b_{k_0} = \infty,$$

where $b_{k_0} > 0$. ∎

**20.** The problem that certain slices of measurable sets can be non-measurable may be avoided by restricting attention to Borel measurable functions and Borel sets. In fact, prove the following:
Suppose $E$ is a Borel set in $\mathbb{R}^2$. Then for every $y$, the slice $E_y$ is a Borel set in $\mathbb{R}$.

**Proof.**

Consider the collection $\mathcal{C}$ of subsets $E$ of $\mathbb{R}^2$ with the property that each slice $E^y$ is a Borel set in $\mathbb{R}$. Since $\bigcup\limits_{n=1}^{\infty} E_n^y = (\bigcup\limits_{n=1}^{\infty} E_n)^y$, $(E^c)^y = (E^y)^c$ and $E^y$ is open whenever $E$ is open in $\mathbb{R}^2$, we have $\mathcal{B} \subset \mathcal{C}$. ■

**21.** Suppose that $f$ and $g$ are measurable functions on $\mathbb{R}^d$.

(a) Prove that $f(x-y)g(y)$ is measurable on $\mathbb{R}^{2d}$.

(b) Show that if $f$ and $g$ are integrable on $\mathbb{R}^d$, then $f(x-y)g(y)$ is integrable on $\mathbb{R}^{2d}$.

(c) Recall the definition of the convolution of $f$ and $g$ given by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy.$$

Show that $f * g$ is well defined for a.e. $x$(that is, $f(x-y)g(y)$ is integrable on $\mathbb{R}^d$ for a.e.$x$).

(d) Show that $f * g$ is integrable whenever $f$ and $g$ are integrable, and that

$$\|f * g\|_{L^1(\mathbb{R}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)} \|g\|_{L^1(\mathbb{R}^d)},$$

with equality if $f$ and $g$ are non-negative.

(e) The Fourier transform of an integrable function $f$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\cdot\xi}dx.$$

Check that $f$ is bounded and is a continuous function of $\xi$. Prove that for each $\xi$ one has

$$(\widehat{f * g})(\xi) = \hat{f}(\xi)\hat{g}(\xi).$$

**Proof.**

(1)(2)(3)(4) By Tonelli's theorem and Fubini's theorem.

(5) Since $\left|e^{-2\pi i x\cdot\xi}\right| = 1$ for any $\xi \in \mathbb{R}^d$ and $f$ is integrable, we have

$$\left|\hat{f}(\xi)\right| \leq \int_{\mathbb{R}^d} |f(x)|\, dx \leq \infty.$$

Notice $\left|f(x)e^{-2\pi i x\cdot\xi}\right| = |f(x)|$ and $f$ is integrable, we have

$$\begin{aligned}
\lim_{\xi\to\xi_0} \hat{f}(\xi) &= \lim_{\xi\to\xi_0} \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\cdot\xi}dx \\
&= \int_{\mathbb{R}^d} \lim_{\xi\to\xi_0} f(x)e^{-2\pi i x\cdot\xi}dx \\
&= \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\cdot\xi_0}dx = \hat{f}(\xi_0).
\end{aligned}$$

by MCT. By Fubini's theorem, one can show that $\widehat{(f * g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$. ■