19. Suppose f is integrable on $\mathbb{R}^d$. For each $\alpha > 0$, let $E_\alpha = \{x : |f(x)| > \alpha\}$. Prove that

$$\int_{\mathbb{R}^d} |f(x)|dx = \int_0^\infty m(E_\alpha)d\alpha.$$

Proof.

By Fubini's theorem, we have that

$$\begin{aligned} RHS &= \int_0^\infty \int_{\mathbb{R}^d} \chi_{E_\alpha}(x,\alpha)dxd\alpha \\ &= \int_{\mathbb{R}^d} \int_0^\infty \chi_{E_\alpha}(x,\alpha)d\alpha dx \\ &= \int_{\mathbb{R}^d} |f(x)|dx \\ &= LHS. \end{aligned}$$

Here we are done. ■

22. Suppose that if $f \in {}^1(\mathbb{R}^d)$ and

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\xi},$$

then $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.

Proof.

Set $\xi^{'} = \frac{1}{2}\frac{\xi}{|\xi|^2}$, then we have

$$\begin{aligned} \hat{f}(\xi) &= \frac{1}{2}\int_{\mathbb{R}^d} f(x)e^{-2\pi i x\xi} + f(x-\xi^{'})e^{-2\pi i(x-\xi^{'})\xi}dx \\ &= \frac{1}{2}\int_{\mathbb{R}^d} (f(x) - f(x-\xi^{'}))e^{-2\pi i x\xi}dx, \end{aligned}$$

Then $|\hat{f}(\xi)| \le \frac{1}{2}\|f - f_{\xi^{'}}\|$ hence $\hat{f}(\xi) \to 0$, as $|\xi^{'}| \to 0$, i.e. $|\xi| \to \infty$. ■

23. As an application of the Fourier transform, show that there does not exist a function $I \in L^1(\mathbb{R}^d)$ such that

$$f * I = f \qquad \text{for all } f \in L^1(\mathbb{R}^d).$$

Proof.

Suppose function $I \in L^1(\mathbb{R}^d)$ satisfies $f * I = f$ for all $f \in L^1(\mathbb{R}^d)$, then we have

$$\hat{f}(\xi) = \widehat{f * I}(\xi) = \hat{f}(\xi)\hat{I}(\xi),$$

hence $\hat{I} \equiv 1$ on $\mathbb{R}^d$, which contradicts Exercise 22. ■

24. Consider the convolution

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy.$$

(a) Show that $f * g$ is uniformly continuous when f is integrable and g bounded.

(b) If in addition g is integrable, prove that $(f * g)(x) \to 0$ as $|x| \to \infty$.

Proof.

(a)

$$\begin{aligned}|(f * g)(x) - (f * g)(x-h)| &= \int_{\mathbb{R}^d} |g(y)(f(x-y) - f(x-h-y))|dy \\ &\leq M \int_{\mathbb{R}^d} |f(y) - f(y-h)|dy \\ &= M\|f - f_h\| \to 0 \quad \text{as} \quad h \to 0.\end{aligned}$$

where M is the upper bound of $g(y)$.

(b) Since f and g are integrable on $\mathbb{R}^d$,we have that $f(x-y)g(y)$ is integrable for a.e. x by Fubini's theorem, for any $\epsilon > 0$ there exists $R > 0$ such that

$$\int_{|x|>R} |f(x)|dx < \epsilon,$$

$$\int_{|x|>R} |g(x)|dx < \epsilon,$$

and

$$\int_{|x|>R} |f(x-y)g(y)|dy < \epsilon, \qquad \text{for a.e. } x.$$

Then we have

$$\begin{aligned}&|\int_{\mathbb{R}^d} f(x-y)g(y)dy| \\ \leq&|\int_{|y|>R} |f(x-y)g(y)|dy + |\int_{|y|\leq R} |f(x-y)g(y)|dy| \\ \leq&\epsilon + M \int_{|x|>R} |f(x)|dx \\ \leq&(1+M)\epsilon, \qquad \text{whenever } \ |x| > 2R,\end{aligned}$$

and here we are done. ■

25. Show that for each $\epsilon > 0$ the function $F(\xi) = \frac{1}{(1+|\xi|^2)^\epsilon}$ is the Fourier transform of an L^1 function.

Follow the hint.