12. Theorem 1.3 states that every open set in $\mathbb{R}$ is the disjoint union of open intervals. The analogues in $\mathbb{R}^d, d \geq 2$, is generally false. Prove the following:
(a) An open disc in $\mathbb{R}^2$ is not the disjoint union of open rectangles.
(b) An open connected set Ω is the disjoint union of open rectangles if and only if Ω is itself an open rectangle.

Answer: Notice that a connected set in $\mathbb{R}^d$ cannot be the disjoint union of two open sets. ■

15. At the start of the theory, one might define the outer measure by taking coverings by rectangles instead of cubes. More precisely, we define

$$m_*^{\mathcal{R}}(E) = inf \sum_{j=1}^{\infty} |R_j| \,,$$

where the inf is now taken over all countable coverings $E \subset \bigcup_{j=1}^{\infty} R_j$ by (closed) rectangles.
Show that this approach gives rise to the same theory of measure developed in the text, by proving that $m_*(E) = m_*^{\mathcal{R}}(E)$ for every set E of $\mathbb{R}^d$.

Proof.
For any set E of $\mathbb{R}^d$, notice that cubes are rectangles, we now have $m_*(E) \geq m_*^{\mathcal{R}}(E)$. However, suppose $m_*^{\mathcal{R}}(E) < \infty$, by defintion for any $\epsilon > 0$ we choose a covering $\{R_j\}$ of E such that $\sum_{j=1}^{\infty} |R_j| \leq m_*^{\mathcal{R}}(E) + \epsilon$. By lemma 1.1, we choose cubes $\{C_j, k\}$ to cover each R_j such that $\sum_{k=1}^{\infty} |C_j, k| \leq |R_j| + \epsilon/2^{-j}$ for $j = 1, 2, \cdots$. Now we have $\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |C_{j,k}| \leq m_*^{\mathcal{R}} + 2\epsilon$, by taking inf and letting $\epsilon \to 0$ we have $m_*(E) \leq m_*^{\mathcal{R}}(E)$. ■

16. The Borel-Cantelli lemma. Suppose $\{E_k\}_{k=1}^{\infty}$ is a countable family of measurable subsets of $\mathbb{R}^d$ and that

$$\sum_{k=1}^{\infty} m(E_k) < \infty.$$

Let

$$E = \{x \in \mathbb{R}^d : x \in E_k, \text{for infinitely many } k\} = \limsup_{k \to \infty} E_k$$

Show that E is measurable and $m(E) = 0$.

Proof.

Since $E = \limsup_{k\to\infty} E_k = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} E_j$, E is the countable intersection of countable union of measurable and hence measurable. And $m(E) = 0$ follows that $m(\bigcup_{j=1}^{\infty} E_j) \leq \sum_{j=1}^{\infty} m(E_j) \leq \infty$ and $m(E) = \lim_{n\to 0} m(\bigcup_{j=n}^{\infty} E_j) = 0$. ■

26. Suppose $A \subset E \subset B$, where A and B are measurable sets of finite measure. Prove that if $m(A) = m(B)$, then E is measurable.

Proof.

Since $m_*(E - A) \leq m(B - A) = 0$, $E = (E - A) \cup A$ is the union of measurable set and set of exterior measure 0 and hence measurable. ■

28. Let E be a subset of $\mathbb{R}$ with $m_*(E) > 0$. Prove that for each $0 < \alpha < 1$, there exists an open interval I so that

$$m_*(E \cap I) \geq \alpha m_*(I).$$

Proof.

By outer regularity and $m_*(E) > 0$, there exists open set $O \supset E$ such that $\alpha m(O) < m_*(E)$. Notice that open set in $\mathbb{R}$ can be written by the union of disjoint intervals $\{I_j\}$. We claim that there exists $j \geq 1$ such that $m_*(E \cup I_j) \geq \alpha m_*(I_j)$. If not, then we have $\alpha m(O) < m_*(E) \leq \sum_{j=1} m_*(E \cap I_j) < \sum_{j=1} \alpha m(I_j) = \alpha m(O)$ which is a contradiction. ■