Recall:

$f \in L^1$. $Mf(x) = \sup_{x \in B} ⨍_B |f|$ $\quad \left(⨍_B = \frac{1}{m(B)} \int_B \cdot \right)$

(i). $Mf \notin L^1(\mathbb{R}^n)$

(ii). If $|f| \leq \lambda$ a.e. then $|Mf(x)| \leq \lambda$ $\forall x \in \mathbb{R}^n$.

$\quad (f \in L^\infty \Rightarrow Mf \in L^\infty)$

(iii) weak (1,1) type $\quad m\{Mf > \lambda\} \leq \frac{A}{\lambda} \|f\|_{L^1}$ $\quad (\forall \lambda > 0.\ A = 3^n)$

Today:

1. $\forall \lambda > 0.$ $\quad m(\{Mf(x) > 2\lambda\}) \leq \frac{A}{\lambda} \int_{|f| > \lambda} |f|$

pf: $\quad f = f \chi_{\{|f| \leq \lambda\}} + f \chi_{\{|f| > \lambda\}} = f_1 + f_2$

$\quad (|f_1| \leq \lambda)$

We know

$$Mf(x) \leq Mf_1(x) + Mf_2(x) \leq \lambda + Mf_2(x).$$

If $Mf(x) > 2\lambda$. then $Mf_2(x) > \lambda$

Therefore $\quad m(\{Mf(x) > 2\lambda\}) \leq m(\{Mf_2(x) > \lambda\}).$

From (1,1)-type,

$$m(\{Mf_2(x) > \lambda\}) \leq \frac{A}{\lambda} \int |f_2| \qquad \blacksquare$$

Lebesgue density point

$\forall E \subseteq \mathbb{R}^n$. a.e. $x \in E$. we have

$$\lim_{\substack{x \in B \\ m(B) \to 0}} ⨍_B \chi_E = 1 = \chi_E(x)$$

$$\Leftrightarrow \quad \lim_{r \to 0^+} \frac{m(B(x,r) \cap E)}{m(B(x,r))} = 1$$

In this case, x is called a density point.

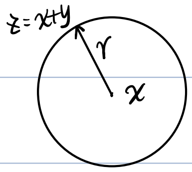
Exercise 2.

Given a closed set $F \subseteq \mathbb{R}^n$. Prove a.e. $x \in F$.

$$\lim_{|y| \to 0} \frac{d(x+y, F)}{|y|} = 0$$

pf:

We set $|y| = r$. $x+y$ represents any point z s.t. $d(z, x) = r$



$$\lim_{r \to 0} \frac{d(z, F)}{r} = 0 \quad \forall\, d(z, x) = r$$

Choose x to be a density point.

Set

$$\frac{m(B(x,r) \cap F^c)}{\omega_n r^n} = \delta(r) \xrightarrow{r \to 0} 0$$

and $s = (100\,\delta(r))^{\frac{1}{n}}\, r$

Claim: $B(z, s) \cap F \neq \phi$.

If true. $d(z, F) < s = (100\,\delta)^{\frac{1}{n}} r \Rightarrow \lim_{r \to 0} \frac{d(z,F)}{r} = 0$.

Exercise : $s \ll r \Rightarrow m(B(x, r) \cap B(z, s)) \geq \frac{1}{50} \omega_n s^n$

$$= 2\,\delta(r)\,\omega_n r^n.$$

If the claim is false, then

$$m(B(x,r) \cap F^c) \geq m(B(x,r) \cap B(z,s)) \geq 2\delta(r)\,\omega_n r^n.$$

Contradiction ! ∎

Ex. 1.

Can we find a measurable set $E \subseteq \mathbb{R}$ s.t. the density points of E are $\mathbb{R} \setminus \{0\}$?

pf: Since 0 is not a density point. $\exists$ an interval $I_1 \ni 0$ s.t.

$$\frac{m(I_1 \cap E)}{m(I_1)} < 1 - \varepsilon_0$$

Therefore. $m(I_1 \cap E^c) > 0$ and $m(I_1 \setminus E \cup \{0\}) > 0$.

$F := I_1 \setminus E \cup \{0\}$. $\exists$ a density point $y \in F$ of F.

Hence

$$\frac{m(I_2 \cap F)}{m(I_2)} \to 1 \quad \text{if } m(I_2) \to 0$$

where $y \in I_2$.

Since $F \cap E = \Phi$. y can't be a density point of E!

Exercise 17:

$F(x) = \int K_\delta(x-y) f(y)\, dy = K_\delta * f$

(i). $|K_\delta(x)| \le \frac{A\delta}{|x|^{n+1} + \delta^{n+1}} \quad \forall x \in \mathbb{R}^n$

prove $|K_\delta * f| \le CMf$

Rmk:

$\forall p > 1$. $\|Mf\|_{L^p} \le C\|f\|_{L^p} \Rightarrow \|K_\delta * f\|_{L^p} \lesssim$.

pf:

$\forall x \in \mathbb{R}^n$.

$$K_\delta * f(x) = \int_{\mathbb{R}^n} K_\delta(y) f(x-y)\, dy$$

$$= \underbrace{\int_{|y| \le \delta} K_\delta(y) f(x-y)\, dy}_{I_0} + \sum_{k=1}^{\infty} \underbrace{\int_{2^{k-1}\delta \le |y| < 2^k \delta} K_\delta(y) f(x-y)\, dy}_{I_k}$$

$$|I_0| \le \int_{|y| \le \delta} |K_\delta(y)|\,|f(x-y)|\, dy$$

$$\le A\delta^{-n} \int_{|y| \le \delta} |f(x-y)|\, dy$$

$$\le A w_n Mf(x)$$

$$|I_k| \le \int_{2^{k-1}\delta \le |y| < 2^k \delta} |K_\delta(y)|\,|f(x-y)|\, dy$$

$$\le A\delta\,(2^{k-1}\delta)^{-n-1}\int_{|y|<2^k\delta}|f(x-y)|\,dy$$

$$\le A\delta\,(2^{k-1}\delta)^{-n-1}\,Mf(x)\,(2^k\delta)^n\,\omega_n$$

$$= A\omega_n\,2^{n+1-k}\,Mf(x)$$

$$\Rightarrow |K_\delta * f(x)| \le I_0 + \sum_{k=1}^{\infty} I_k$$

$$\le \left(A\omega_n + \sum_{k=1}^{\infty} A\omega_n 2^{n+1-k}\right) Mf(x)$$

$$= A\omega_n(2^n+1)\,Mf(x) \qquad \blacksquare$$

Functions of bdd variation (BV)

$f \in BV([a,b]) \Leftrightarrow T_a^b = \sup \sum_{j=1}^{N} |f(x_j)-f(x_{j-1})|$ $(a=x_0<x_1<\cdots<x_N=b)$

Total variation

Fact:

$\forall f \in BV([a,b])$ can be expressed as $f = f_1 - f_2$ s.t.

f_1 and f_2 are increasing and bdd

More precisely, $f_1(x) = \frac{1}{2}f(x) + \frac{1}{2}T_a^x$ (Total variation of f on $[a,x]$)

$f_2(x) = \frac{1}{2}T_a^x - \frac{1}{2}f(x)$

Fact: f_1 & f_2 are increasing

Eg 1.

$T_a^b(f) = 0 \Leftrightarrow f$ is a constant function.

$\forall a \le x < y \le b.\quad T_a^b(f) \ge |f(y)-f(x)|$

Eg2.

$f \in BV([a,b]) \Rightarrow |f| \in BV([a,b])$

since $\big||f(x_j)| - |f(x_{j-1})|\big| \le |f(x_j) - f(x_{j-1})|$

$T_a^b(|f|) \le T_a^b(f).$

反之不对, counter example:

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \in \mathbb{Q}^c \end{cases}$$

$$T_a^b(f) = \infty$$