习题课4.

不可测集

· 作业题 (Stein ch1, 32)　$G \subseteq \mathbb{R}$　$m^*(G)>0$，pf：G有不可测子集

pf：不妨设G可测

define $G_k = G \cap [k,k+1]$　$k \in \mathbb{Z}$　, we have $G = \bigcup G_k$

$\Rightarrow m(G) = \sum_{k\in\mathbb{Z}} m(G_k) > 0 \Rightarrow \exists k_0$ s.t. $m(G_{k_0})>0$.

(由平移) 不妨设 $k_0=0$，i.e. $m(G_0)=m(G\cap[0,1])>0$

define $x \sim y \Leftrightarrow x-y \in \mathbb{Q}$.　$x,y \in G_0$

从每个等价类中选出一个元素，构成 $\mathcal{N}$　$\mathcal{N} \subset G_0$

记 $\mathcal{N}_k = \mathcal{N} + r_k$　$r_k \in [-1,1]\cap\mathbb{Q}$　$\mathcal{N}_k \cap \mathcal{N}_j = \phi$　$\forall k \neq j$

& $G_0 \subseteq \bigcup_{k=1}^{\infty} \mathcal{N}_k \subseteq [-1,2]$

if $\mathcal{N}$ 可测 $\Rightarrow m(G_0) \le \sum_{k=1}^{\infty} m(\mathcal{N}_k) = \sum_{k=1}^{\infty} m(\mathcal{N}) \le 3 \Rightarrow m(\mathcal{N})=0$

$\Rightarrow m(G_0)=0$ 矛盾.

· [Ex] Given a set $G \subseteq \mathbb{R}^2$　$m^*(G)>0$，prove there are exist a non-measurable set contained in G.

pf：· 类似，不妨设 $G \subset [0,1]\times[0,1]$

· define $x \sim y \Leftrightarrow x-y \in \mathbb{Q}\times\mathbb{Q}$

$\mathcal{N} = \{$每个等价类中选取一个代表元$\}$

· $\mathcal{M}$ 可测集.　G_δ 开集的可数交　F_σ 闭集的可数并　$\mathcal{B}$ Borel set

$G_\delta \subset \mathcal{B} \subseteq \mathcal{M} = \begin{cases} G_\delta \setminus 0测集 \\ F_\sigma \cup 0测集 \end{cases}$

$F_\sigma \subset$

Thm $\mathbb{Q}$ is not a G_δ set ($\mathbb{Q}$ is a F_σ-set)

Lemma $U_k \overset{\text{dense}}{\subset} \mathbb{R}$, open $\Rightarrow \bigcap_{k=1}^{\infty} U_k \neq \phi$

· pf of thm: if $\mathbb{Q} = \bigcap_{k=1}^{\infty} U_k$ U_k open

假设 $\mathbb{Q} = \{r_k\}_{k=1}^{\infty}$, define $U_k' = U_k \setminus \{r_k\}$ $\overset{\text{dense}}{\subset} \mathbb{R}$ and open

claim $\bigcap_{k=1}^{\infty} U_k' = \phi$ $(\bigcap_{k=1}^{\infty} U_k' \subset \mathbb{Q}, \mathbb{Q}^c)$ $\xRightarrow{\text{by Lemma}}$ 矛盾

· pf of Lemma: Consider U_1 open $\Rightarrow \exists I_1 = (a_1, b_1) \subset U_1$

define $c_1 = \frac{a_1+b_1}{2}$, since U_2 dense, $\exists I_2 = (a_2, b_2)$ s.t. $\bar{I}_2 \subset I_1 \cap U_2$

以此类推, 得到 $\{I_k\}$ $\bar{I}_k \subset I_{k-1}$, $\bar{I}_k \subset U_k$

由闭区间套 $\Rightarrow \exists q \in \bigcap_{k=1}^{\infty} \bar{I}_k \in \bigcap_{k=1}^{\infty} U_k$

Rmk: · In fact $\bigcap_{k=1}^{\infty} U_k \overset{\text{dense}}{\subset} \mathbb{R}$ (Baire 纲定理)

· In fact $\mathbb{R}$ 完备 $\Rightarrow$ Baire 纲

Ex2. consider a function $f: \mathbb{R} \to \mathbb{R}$, 记 f 的连续点集为 C. prove C is a G_δ-set

pf. f 在 x 点处连续 $\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0$. 对 $\forall |y-x| < \delta$, 有 $|f(y) - f(x)| < \varepsilon$

$\Leftrightarrow \forall k \in \mathbb{N}, \exists \delta > 0$ s.t. $\forall |y-x| < \delta$, 有 $|f(y) - f(x)| < 1/k$

define $C_k = \{x \mid \exists \delta > 0$ s.t. $|f(y) - f(x)| < 1/k$ for $\forall y \in (x-\delta, x+\delta)\}$

check 1. C_k open $(x \in C_k \Rightarrow$ $x-\delta$ x $x+\delta$ $\Rightarrow (x-\delta, x+\delta) \subset C_k)$

2. $\bigcap_{k=1}^{\infty} C_k = C$

Cor: 不存在只在 $\mathbb{Q}$ 上连续的函数

可测函数

f 可测 $\Leftrightarrow \forall$ Borel 集的原像是可测集

Rmk: 1. $f: \mathbb{R} \to \mathbb{R}$ or $E \to \mathbb{R}$ E 可测

2. $f: E \to \mathbb{R} \cup \{\pm\infty\}$, 一般地, 对 $m(\{x \mid |f| = \infty\}) = 0$ i.e. f 几乎处处有限

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$\mathcal{MF} ::= \{$可测函数$\}$

1. $f, g \in \mathcal{MF} \Rightarrow f \pm g,\ f \cdot g,\ \sup\{f, g\},\ \inf\{f, g\} \in \mathcal{MF}$
2. $f_n \in \mathcal{MF} \Rightarrow \lim\limits_{n\to\infty} f_n,\ \sup\limits_n\{f_n\},\ \inf\limits_n\{f_n\} \in \mathcal{MF}$

consider $f_\alpha(x) \in \mathcal{MF}$ $\alpha \in A$ Q: $\sup\limits_{\alpha\in A}\{f_\alpha\} \in \mathcal{MF}$? $(\times)$

反例: consider $\chi_N(x) = \begin{cases} 1 & x \in N \\ 0 & x \notin N \end{cases}$ 不可测. $\chi_{\{\alpha\}}(x)$ $\alpha \in N$ 可测

$\Rightarrow \sup\limits_{\alpha\in N}\{\chi_{\{\alpha\}}\} = \chi_N$