Lebesque measurable set

Def: $E \in \mathcal{M}$ if $\forall \varepsilon > 0$ $\exists$ an open set $U \supseteq E$
s.t. $m_*(U \setminus E) < \varepsilon$

$\forall \varepsilon_k = \frac{1}{k}$ 由 Def，存在开集 $U_k \supseteq E$ 且 $m_*(U_k \setminus E) < \varepsilon_k$

$\Rightarrow$ We define $G = \bigcap_{k=1}^{\infty} U_k \in G_\delta$ (intersection of countable open sets)

We know $G \supseteq E$ Notice $G \setminus E \subseteq U_k \setminus E$

$m_*(G \setminus E) \leq m_*(U_k \setminus E) < \varepsilon_k \Longrightarrow m_*(G \setminus E) = 0$

① $\mathcal{M}$ is closed under countable intersection / union & complement

② $\forall E \in \mathcal{M}$ $\exists F = \bigcup_{k=1}^{\infty} F_k$ (closed sets) $\subseteq E$
s.t. $m_*(E \setminus F) = 0$ $\quad \in F_\delta$ (union of countable closed sets)

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* In general, given a open set E, if $m_*(E) < \infty$, then $\forall \varepsilon > 0$,
$\exists$ an open set $U \supseteq E$ and $m_*(U) < m_*(E) + \varepsilon$

* For general set E, if $m_*(E) < \infty$, then $\exists$ a set $G \in G_\delta$ s.t. $E \subseteq G$
and $m_*(E) = m_*(G) = m(G)$, G is called a 等测包

Exercise

Ex 1: Suppose $A_1 \subseteq A_2$ and $A_1 \in \mathcal{M}$, $m(A_1) = m_*(A_2) < \infty$
prove $A_2 \in \mathcal{M}$

Hint: Consider the 等测包 of A_2

proof: $\exists A_3 \in \mathcal{M}$ $A_3 \supseteq A_2$ and $m(A_3) = m_*(A_2) = m(A_1)$

$m(A_3 \setminus A_1) = m(A_3) - m(A_1) = 0 \Rightarrow m_*(A_3 \setminus A_2) = m(A_3 \setminus A_1) = 0$

$\Rightarrow A_3 \setminus A_2 \in \mathcal{M} \Rightarrow A_2 = A_3 \setminus (A_3 \setminus A_2) = A_3 \cap (A_3 \setminus A_2)^c \in \mathcal{M}$

Ex 2: E_1, E_2 Asume $m_*(E_1 \cup E_2) < \infty$
s.t. $m_*(E_1 \cup E_2) = m_*(E_1) + m_*(E_2)$
if & only if $\exists E_1 \subseteq F_1$, $E_2 \subseteq F_2$ s.t. $F_1, F_2 \in \mathcal{M}$ and $m(F_1 \cap F_2) = 0$

proof: "$\Longrightarrow$"

F_i = 等测包 of E_i $\quad F_i \supseteq E_i$ and $m_*(E_i) = m_*(F_i)$

$m(F_1 \cup F_2) = m(F_1) + m(F_2) - m(F_1 \cap F_2)$

$m(F_1 \cup F_2) \geq m(F_1) + m(F_2) \Longleftrightarrow m(F_1 \cap F_2) = 0$

$F_1 \cup F_2 \supseteq E_1 \cup E_2 \Longrightarrow m(F_1 \cup F_2) \geq m_*(E_1 \cup E_2) = m_*(E_1) + m_*(E_2)$

$= m(F_1) + m(F_2)$

"$\Longleftarrow$"

Consider F = 可测包 of $E_1 \cup E_2$

We define $F_1' = F \cap F_1$ $F_2' = F \cap F_2 \in \mathcal{M}$

First, we know ① $F_i' \supseteq E_i$ $(i = 1, 2)$

② $m(F_1' \cap F_2') \le m(F_1 \cap F_2) = 0$

③ $m(E_1 \cup E_2) = m(F) \geqslant m(F_1' \cup F_2') \geqslant m(F_1') + m(F_2')$

$\geqslant m_*(E_1) + m_*(E_2)$

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*σ - algebra is a collection of subsets of $\mathbb{R}^n$ which is closed under countable union and complement

$\mathcal{B}$ (Borel sets) is the smallest σ - algebra countaining all open sets.

$$\boxed{\mathcal{M} = \mathcal{B} + \{0\}}$$

* Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous function

$\Longleftrightarrow f^{-1}(u)$ is open if $u \subseteq \mathbb{R}$ is open

Fact: $f^{-1}(E)$ is a Borel set if E is a Borel set.

proof: We define $\mathcal{B}_1 = \{E \subseteq \mathbb{R} \mid f^{-1}(E)$ is a Borel set$\}$

① if $U \subseteq \mathbb{R}$ is open, then $U \in \mathcal{B}_1$

② $\mathcal{B}_1$ is a σ - algebra

$\Big\} \Rightarrow \mathcal{B}(\mathbb{R}) \subseteq \mathcal{B}_1$

$E_i \in \mathcal{B}_1$ $f^{-1}(\bigcup_{i=1}^{\infty} E_i) = \bigcup_{i=1}^{\infty} f^{-1}(E_i) \in \mathcal{B}(\mathbb{R}^n)$

$\bigcap_{i=1}^{\infty} E_i \in \mathcal{B}_1$ $E \in \mathcal{B}_1 \Rightarrow E^c \in \mathcal{B}_1$

Fact: $\mathcal{B}_2$ = σ - algebra generated by $\{(-\infty, a), a \in \mathbb{R}\}$

we have $\mathcal{B}_2 = \mathcal{B}(\mathbb{R})$

$\Uparrow$

Ex

Def (Measurable function): $f: \mathbb{R}^n \to \mathbb{R}$ if $\overset{\{x \mid f(x) < a\}}{f^{-1}(E)}$ is a measurable set.

if $E = (-\infty, a)$, $a \in \mathbb{R}$.

property : If f is a measurable function, then $f^{-1}(E)$ is measurable with E is a Borel set.

Question

1: If f is continuous and E is a Borel set, is it true that $f(E)$ is a Borel set? ✗

2: If f is continuous and E is a Borel set, then $f(E)$ is a measurable set

Def : Recall f is the Cantor function $[0,1] \to [0,1]$

① f is measurable

② $\exists$ non-measurable set $N \subseteq [0,1]$

③ $f(C) = [0,1]$ (C cantor set)

we consider $C_1 = f^{-1}(N) \cap C \subseteq C$

$m_*(C_1) \leq m_*(C) = 0 \Rightarrow C_1$ is measurable, but not a Borel set.