$f_n(x)$ measurable functions

收敛类型:

① Pointwise convergence a.e. $f_n(x) \xrightarrow{a.e.} f(x)$

② Convergence in measure $f_n(x) \xrightarrow[E]{m} f(x)$

$\forall \varepsilon > 0.\ \exists N = N(\varepsilon) > 0$ s.t. $\forall n \geq N(\varepsilon)$

$$m(\{x \in E \mid |f_n(x) - f(x)| \geq \varepsilon\}) < \varepsilon$$

Fact: $f_n \xrightarrow{a.e.} f \Rightarrow f_n \xrightarrow{m} f$. 反之不成立. $(m(E) < \infty)$.

Thm:

Assume $f_n \xrightarrow{m} f$ on E. $(m(E) < \infty)$.

Then $\exists$ subsequence $f_{n_k} \xrightarrow{a.e.} f$ on E.

pf:

Since $f_n \xrightarrow{m} f$. $\exists \{n_k\} \subseteq \{n\}$ s.t.

$$m(\{x \in E \mid |f_{n_k}(x) - f(x)| \geq \tfrac{1}{2^k}\}) < \tfrac{1}{2^k}$$

$E_k = \{x \in E \mid |f_{n_k}(x) - f(x)| \geq \frac{1}{2^k}\}$ We know $m(E_k) < \frac{1}{2^k}$

$$\overline{\lim} E_k = \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} E_j$$

Claim: $m(\overline{\lim} E_k) = 0$

$LHS = \lim_{i \to +\infty} m(\bigcup_{j=i}^{\infty} E_j) \leq \lim_{i \to \infty} \sum_{j=i}^{\infty} m(E_j) < \lim_{i \to \infty} \sum_{j=i}^{\infty} \frac{1}{2^j} = \lim_{i \to \infty} \frac{1}{2^{i-1}} = 0$

$\forall x \notin \overline{\lim} E_k \iff x \in (\bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} E_j)^c = \bigcup_{i=1}^{\infty} \bigcap_{j=i}^{\infty} E_j^c$

$\Rightarrow \exists i_0$ s.t. $x \in \bigcap_{j=i_0}^{\infty} E_j^c$. $\Rightarrow \forall j \geq i_0 \quad |f_{n_j}(x) - f(x)| < \frac{1}{2^j}$

In particular $f_{n_k}(x) \to f(x)$

In sum. $f_{n_k}(x) \xrightarrow{a.e.} f(x)$.

Rmk: 可以看到. 在上面的过程中没有用到 $m(E) < +\infty$. 可以略去.

Application Corollary:

$f_n(x) \xrightarrow{m} f(x)$ iff $\forall \{n_k\} \subseteq \{n\}$. $\exists$ a subsequence s.t. $f_{n_j} \xrightarrow{a.e.} f$.

pf:

"$\Rightarrow$" Since $f_n \xrightarrow{m} f$. $f_{n_k} \xrightarrow{m} f$

By the theorem. $\exists \{n_{k_j}\} \subseteq \{n_k\}$ s.t. $f_{n_{k_j}} \xrightarrow{a.e.} f$

"$\Leftarrow$" Prove by contradiction

Assume $f_n \overset{m}{\not\to} f$. Then $\exists\ \varepsilon_0 > 0$ and $\{n_k\}$ s.t.

$m(\{x \in E \mid |f_{n_k}(x) - f(x)| \geq \varepsilon_0\}) \geq \varepsilon_0$

However, $f_{n_{k_j}} \xrightarrow{a.e.} f \Rightarrow f_{n_{k_j}} \xrightarrow{m} f$. Contradiction!

Exercise 1:

Suppose $f_n \xrightarrow{m} f$ on E. Prove $|f_n|^p \xrightarrow{m} |f|^p$ ($\forall p > 0$).

pf:

$\forall \{n_k\} \subseteq \{n\}$ 由上述推论. $\exists \{n_{k_j}\} \subseteq \{n_k\}$ s.t. $f_{n_{k_j}} \xrightarrow{a.e.} f$.

We have $|f_{n_{k_j}}|^p \xrightarrow{a.e.} |f|^p$. Therefore $|f_n|^p \xrightarrow{m} |f|^p$. according to last corollary.

Integration.

① Simple function $f(x) = \sum_{k=1}^{\infty} a_k \chi_{E_k}$. $a_k \in \mathbb{R}$. E_k measurable set.

$$\int f = \sum_{k=1}^{\infty} a_k m(E_k)$$

② Consider f. $|f| \leq M$ and supported on finite E.

φ_k (simple function) $\xrightarrow{a.e.} f$

$$\int f = \lim_{k \to \infty} \int \varphi_k$$

③ $f \geq 0$. $0 \leq g \leq f$. g is of type ②.

$\int f = \sup_g \int g < \infty$ iff f is integrable ($f \in L^1$)

④. $f = f^+ - f^-$ $\quad \int f = \int f^+ - \int f^-$

f integrable $\Leftrightarrow$ f_+, f_- integrable $\Leftrightarrow$ $|f|$ integrable

$L^1 = \{$integrable functions$\}$

$f, g \in L^1$. we define $\|f-g\|_{L^1} = \int |f-g|$

Fact: L^1 is a complete metric space.

① **Monotone Convergence Theorem (MCT):**

$f_n \geq 0$. assume $f_n(x) \leq f_{n+1}(x)$. $\forall n$.

Define $f(x) = \lim_{n \to +\infty} f_n(x)$.

Then $\lim_{n \to \infty} \int f_n = \int f$

② **Fatou's lemma:**

$f_n(x) \geq 0$ and $f_n \xrightarrow{a.e.} f$, $\liminf_{n \to \infty} \int f_n \geq \int f$.

eg. $f_n = n(1+(-1)^n)\chi_{[0,\frac{1}{n}]}$ define on $\mathbb{R}$. $f_n \geqslant 0$.

$\int f_n = n(1+(-1)^n)\int \chi_{[0,\frac{1}{n}]} = 1+(-1)^n = \{0,2,0,2\dots\}$

If $x \neq 0$. $f_n(x) \to 0$. $\Rightarrow f_n(x) \xrightarrow{a.e.} 0$

③. Dominated Convergence Theorem. (DCT).

Assume $|f_n| \leqslant g \in L^1$. If $f_n \xrightarrow{a.e.} f$. then $f \in L^1$ and

$\lim\limits_{n\to\infty} \int |f_n - f| = 0 \Rightarrow \lim\limits_{n\to\infty} \int f_n = \int f$

$f_n \xrightarrow[|f_n|\leqslant g]{a.e.} f \Rightarrow f_n \xrightarrow{L^1} f$

eg. $f_n = \frac{1}{n}\chi_{[0,n]}$ $f_n \to 0$.

$\int f_n = 1 \neq 0$ 因此时不存在 $|f_n| \leqslant g \in L^1$.

Fact:

If $f_n \xrightarrow{L^1} f$, then $f_n \xrightarrow{m} f$.

Corollary:

If $f_n \xrightarrow{L^1} f$, then $\exists \{n_k\} \subseteq \{n\}$. $f_{n_k} \xrightarrow{a.e.} f$

pf of the Fact:

Consider $E_n = \{x \mid |f_n(x) - f(x)| \geqslant \varepsilon\}$

$\varepsilon m(E_n) \leqslant \int_{E_n} |f_n - f| \leqslant \int |f_n - f| \xrightarrow{n\to\infty} 0$

So $\exists N(\varepsilon)$ s.t. $m(E_n) < \varepsilon$. $\forall n \geqslant N$ ∎

Exercise 2.

$f_n \xrightarrow{a.e.} f$ and $f_n, f \in L^1$.

Prove if $\int |f_n| \xrightarrow{n\to\infty} \int |f|$, then $\int |f_n - f| \xrightarrow{n\to\infty} 0$

pf:

$\big||f_n - f| - |f_n|\big| \leqslant |f|$.

$\big||f_n - f| - |f_n|\big| \xrightarrow{a.e.} |f|$.

$\lim\limits_{n\to\infty} \int \big||f_n - f| - |f_n|\big| = \int |f|$.

$\lim\limits_{n\to\infty} \int |f_n| - \lim\limits_{n\to\infty} \int |f_n - f| = \int |f|$. ∎