11. Prove that if f is integrable on $\mathbb{R}^d$, real-valued, and $\int_E f(x)dx \geq 0$ for every measurable E, then $f(x) \geq 0$ a.e. x. As a result, if $\int_E f(x)dx = 0$ for every measurable E, then $f(x) = 0$ a.e.

Proof.

Set $E = \{f < 0\}$, then $\int_E f(x)dx \geq 0$ implies that $m(\{f < 0\}) = 0$. Similarly, $\int_E f(x)dx = 0$ for every measurable E implies $f(x) = 0$ a.e.. ■

12. Show that there are $f \in L^1(\mathbb{R}^d)$ and a sequence$\{f_n\}$ with $f_n \in L^1(\mathbb{R}^d)$ such that

$$\|f - f_n\|_{L^1} \to 0,$$

but $f_n(x) \to f(x)$ for no x.

Answer: Set $f(x) \equiv 0$ and $f_{2^n+k} = \chi_{[\frac{k}{2^n}, \frac{k+1}{2^n}]}$, where $n \geq 0$ and $0 \leq k \leq 2^n - 1$. ■

14. In Exercise 6 of the previous chapter we saw that $m(B) = v_d r^d$, whenever B is a ball of radius r in $\mathbb{R}^d$ and $v_d = m(B_1)$, with B_1 the unit ball. Here we evaluate the constant v_d.

(a) For $d = 2$, prove using Corollary 3.8 that

$$v_2 = 2\int_{-1}^{1} (1 - x^2)^{1/2} dx,$$

and hence by elementary calculus, that $v_2 = \pi$.

(b) By similar mathods, show that

$$v_d = 2v_{d-1} \int_0^1 (1 - x^2)^{(d-1)/2} dx.$$

(c) The rusult is

$$v_d = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)}.$$

Proof.

By Fubini's theorem,

$$\begin{aligned} v_d &= \int_{B_1^d(0)} 1dx \\ &= 2\int_0^1 \int_{B^{d-1}_{\sqrt{1-r^2}}(0)} 1dxdr \\ &= 2v_{d-1} \int_0^1 (1 - r^2)^{(d-1)/2} dr \end{aligned}$$

15. Consider the function defined over $\mathbb{R}$ by

$$f(x) = \begin{cases} x^{-1/2} & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

For a fixed enumeration $\{r_n\}_{n=1}^{\infty}$ of the rationals $\mathbb{Q}$, let

$$F(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n).$$

Prove that F is integrable, hence the series defining F converges for almost every $x \in \mathbb{R}$. However, observe that this series is unbounded on every interval, and in fact, any function $\tilde{F}$ that agrees with F a.e. is unbounded in any interval.

Proof.
(1) By MCT, $\int F(x)dx = \sum\limits_{n=1}^{\infty} \int 2^{-n} f(x - r_n)dx = \sum\limits_{n=1}^{\infty} 2^{-n} \cdot 2 = 2 < \infty$.
(2) For every internal (a, b) of $\mathbb{R}$, there exists $r_n \in \mathbb{Q} \cap (a, b)$, hence

$$\begin{aligned} & m(\{F(x) > M\} \cap (a, b)) \\ \geq & m(\{x : f(x - r_n) > M \cdot 2^n\} \cap (a, b)) \\ = & m(\left(r_n, r_n + \frac{1}{(M \cdot 2^n)^2} \right) \cap (a, b)) > 0 \end{aligned}$$

for every $M > 0$. ■