$f \in AC([a,b])$

$\forall \varepsilon > 0 \quad \exists \delta > 0$ s.t. $I_k = (a_k, b_k) \subseteq [a,b]$ $k = 1, 2, \cdots N$
satisfy $I_k \cap I_s = \phi$ If $\sum_{k=1}^{N} |I_k| = \sum_{k=1}^{N} (b_k - a_k) < \delta$, then $\sum_{k=1}^{N} |f(b_k) - f(a_k)| < \varepsilon$

Rmk: N can be finite or $+\infty$

E.g ① M-Lipschitz i.e. $|f(x) - f(y)| \le M|x-y|$ $\left(\begin{matrix} \text{Given } \varepsilon \\ \delta = \frac{\varepsilon}{M} \end{matrix} \right)$

② $\forall g \in L^1([a,b])$ we define $f(x) = \int_a^x g(y)\, d(y)$ $(a \le x \le b)$

$\forall$ disjoint $I_k \subseteq [a,b]$

$$\sum_{k=1}^{N} |f(b_k) - f(a_k)| = \sum_{k=1}^{N} \left| \int_{I_k} g \right| \le \underbrace{\int_{\cup I_k} |g|}_{m(\cup I_k) < \delta} < \varepsilon$$

Conversly if $f \in AC([a,b])$ then $f(b) - f(a) = \int_a^b f'(x)\,dx$ $\left\{ \begin{matrix} f(x) \text{ exists a.e. } x \\ f'(x) \in L^1([a,b]) \end{matrix} \right.$

Comparison if $f \in BV([a,b])$ then $f'(x)$ exist a.e. x
and $\int_a^b |f'(x)|\,dx \le T_a^b(f) < \infty$ Furthermore, if $f \in AC$
then $\int_a^b |f'(x)|\,dx = T_a^b(f)$

Property if $f \in AC([a,b])$, then $m(f(e)) = 0$ for any $m(e) = 0$

Property 2 if Z is measurable, then $f(Z)$ is measurable

proof $Z \in G_\delta \setminus 0\text{-set}$ $Z \in F_\sigma \cup 0\text{-set} \Longrightarrow Z \in K_\sigma \cup 0\text{-set}$

$Z = \bigcup_{n=1}^{\infty} \underset{\text{compact}}{K_n} \cup \underset{m(N)=0}{N} \Longrightarrow f(Z) = \bigcup_{n=1}^{\infty} \underset{\text{compact}}{f(K_n)} \cup \underset{m(f(N))=0}{f(N)}$ so $f(Z)$ is measurable

EX1: Given a $f \in AC([a,b])$, prove there exist increasing functions

$f_1, f_2 \in AC([a,b])$ such that $f = f_1 - f_2$

$f_1(x) = \frac{1}{2} T_a^x(f) + \frac{1}{2} f(x)$ $\quad f_2(x) = \frac{1}{2} T_a^x(f) - \frac{1}{2} f(x)$

$T_a^x(f) = \int_a^x |f'(y)|\,dy \in AC$ since $|f'| \in L^1$

Q1: If $\forall m(Z) = 0$, $m(f(Z)) = 0$, can we prove $f \in AC([a,b])$?

Counter example $f = \operatorname{sgn} x$ $\quad f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{Q}^c \end{cases}$

Q2: If $\forall m(Z) = 0 \Rightarrow m(f(Z)) = 0$ & $f \in C([a,b])$
can we prove $f \in AC([a,b])$?

e.g. $f(x) = x \sin \frac{1}{x}$ $x \in [0,1]$ $\quad f' \notin L^1$ $\quad f \notin BV$

Theorem If $m(f(Z))=0$ $\forall m(Z)=0$ and $f\in BV([a,b])\cap C([a,b])$ then $f\in AC([a,b])$

Lemma: Given a measurable set $E\subseteq[a,b]$. Suppose $f'(x)$ exists for any $x\in E$. Then $m_*(f(E))\leq\int_E|f'|$

proof: f' exists a.e. and $f'\in L'([a,b])$

consider disjoint $I_k=[a_k,b_k]\subseteq[a,b]$ $k=1,2,\cdots,N$

$I_k=A_k\sqcup N_k$ s.t. f' exists on A_k and $m(N_k)=0$

From $f(I_k)=f(A_k)\cup f(N_k)$ & Lemma, $m(f(I_k))=m(f(A_k))\leq\int_{A_k}|f'|=\int_{I_k}|f'|$

$|f(b_k)-f(a_k)|\leq m(f(I_k))$ $\quad f(I_k)=[m_k,M_k]$

Therefore $\sum_{k=1}^{N}|f(b_k)-f(a_k)|\leq\sum_{k=1}^{N}m(f(I_k))\leq\sum_{k=1}^{N}\int_{I_k}|f'|=\int_{\cup I_k}|f'|$

$\forall\varepsilon.\ \exists\delta$ s.t. if $m(\cup I_k)<\delta$, then $\int_{\cup I_k}|f'|<\varepsilon$

Sublemma: Given a measurable set $E\in[a,b]$ Suppose $f'(x)$ exists on E and $|f'(x)|\leq M$ $(\forall x\in E)$. Then $m_*(f(E))\leq Mm(E)$

proof: Consider $E_n=\{x\in E\mid \frac{|f(y)-f(x)|}{|y-x|}\leq M+\varepsilon,\ \forall|y-x|<\frac{1}{n}\}$

It is easy to see $E_n\subseteq E_{n+1}$ and $\bigcup_{n=1}^{\infty}E_n=E$

$\forall n$ $\exists I_{n,k}$ open internals s.t. ① $E_n\subset\bigcup_{k=1}^{\infty}I_{n,k}$

② $\bigcup_{k=1}^{\infty}|I_{n,k}|\leq m(E_n)+\varepsilon$

③ $|I_{n,k}|<\frac{1}{n}$

We consider $E_n\cap I_{n,k}$ $\forall s,t\in E_n\cap I_{n,k}$, we have $\frac{|f(s)-f(t)|}{|s-t|}\leq M+\varepsilon$

and $|f(s)-f(t)|\leq(M+\varepsilon)|s-t|\leq(M+\varepsilon)|I_{n,k}|$

$m(f(E_n\cap I_{n,k}))\leq\operatorname{diam}(f(E_n\cap I_{n,k}))\leq(M+\varepsilon)|I_{n,k}|$

so $m_*(f(E_n))\leq\sum_{k=1}^{\infty}m(f(E_n\cap I_{n,k}))\leq(M+\varepsilon)\sum_{k=1}^{\infty}|I_{n,k}|\leq(M+\varepsilon)(m(E_n)+\varepsilon)$

Let $n\to+\infty$, $m_*(f(E))\leq(M+\varepsilon)(m(E)+\varepsilon)\xrightarrow{\varepsilon\to0}m_*(f(E))\leq M\,m(E)$

proof of lemma: We consider $E_n\subseteq E$

$E_n=\{x\in E\mid n\varepsilon<|f'(x)|\leq(n+1)\varepsilon\}$ $n=0,1,2,\cdots$

We know $\bigcup_{n=0}^{\infty}E_n=\{x\in E\mid |f'(x)|>0\}$

According to the Sublemma, $m_*(f(E_n)) \le (n+1)\varepsilon\, m(E_n) = n\varepsilon\, m(E_n) + \varepsilon\, m(E_n)$

$$\le \int_{E_n} |f'| + \varepsilon\, m(E_n)$$

$$m_*(f(E)) = \sum_{n=0}^{\infty} m(f(E_n)) \le \sum_{n=0}^{\infty} \left(\int_{E_n} |f'| + \varepsilon\, m(E_n) \right) = \int_E |f'| + \varepsilon\, m(E)$$

$\varepsilon \to 0 \Rightarrow m_*(f(E_n)) \le \int_E |f'|$