1. Given a collection of sets $F_1, F_2, \cdots, F_n$, construct another collection $F_1^*, F_2^*, \cdots, F_N^*$, with $N = 2^n - 1$, so that $\bigcup_{k=1}^n F_k = \bigcup_{j=1}^N F_j^*$; the collection F_j^* is disjoint; also $F_k = \bigcup_{F_j^* \subset F_k} F_j^*$, for every k.

Answer: Consider the 2^n sets $F_1' \cap F_2' \cap \cdots \cap F_n'$ where each F_k' is either F_k or F_k^c. ■

6. Integrability of f on $\mathbb{R}$ does not necessarily imply the convergence of $f(x)$ to 0 as $x \to \infty$.

(a) There exists a positive continuous function f on $\mathbb{R}$ so that f is integrable on $\mathbb{R}$, but yet $\limsup_{x\to\infty} f(x) = \infty$.

(b) However, if we assume that f is uniformly continuous on $\mathbb{R}$ and integrable, then $\lim_{|x|\to\infty} f(x) = 0$.

Proof.

(a) Set

$$f(x) = \begin{cases} 1 & \text{if } x \in \left[n, n + \frac{1}{n^3}\right] \text{ for } n \leq 2, \\ -n^3 x + n^4 + 2 & \text{if } x \in \left[n + \frac{1}{n^3}, n + \frac{2}{n^3}\right] \text{ for } n \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Use integrability and uniformly continuity of $|f|$. ■

8. If f is integrable on $\mathbb{R}$, show that $F(x) = \int_{-\infty}^x f(t)dt$ is uniformly continuous.

Answer: By absolutely continuity of integrable functions. ■

9. Tchebychev inequality. Suppose $f \geq 0$, and f is integrable. If $\alpha > 0$, and $E_\alpha = \{x : f(x) > \alpha\}$, prove that

$$m(E_\alpha) \leq \frac{1}{\alpha} \int f.$$

Proof.

Note that $\alpha m(E_\alpha) \leq \int_{E_\alpha} f(x)dx \leq \int f$, and here we are done. ■

10. Suppose $f \geq 0$, and let $E_k = \{x : f(x) > 2^k\}$ and $F_k = \{x : 2^k < f(x) \leq 2^{k+1}\}$. If f is finite almost everywhere, then

$$\bigcup_{k=-\infty}^{\infty} F_k = \{f(x) > 0\},$$

and the sets F_k are disjoint.

Prove that f is integrable if and only if

$$\sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty, \qquad \text{if and only if} \qquad \sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) < \infty.$$

Use this result to verify the following assertions. Let

$$f(x) = \begin{cases} |x|^{-a} & \text{if} |x| \le 1, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} |x|^{-b} & \text{if} |x| > 1, \\ 0 & \text{otherwise,} \end{cases}$$

Then f is integrable on $\mathbb{R}^d$ if and only if $a < d$; also g is integrable on $\mathbb{R}^d$ if and only if $b > d$.

Proof.
1) $\bigcup\limits_{k=-\infty}^{\infty} F_k = \bigcup\limits_{k=-\infty}^{\infty} f^{-1}\big((2^k, 2^{k+1}]\big) = f^{-1}\big(\bigcup\limits_{k=-\infty}^{\infty} (2^k, 2^{k+1}]\big) = \{f(x) > 0\}$.
2) Note that

$$\begin{aligned} \frac{1}{2}\int f(x)dx &= \frac{1}{2}\sum_{k=-\infty}^{\infty} \int_{F_k} f(x)dx \\ &\le \sum_{k=-\infty}^{\infty} 2^k m(F_k) \\ &\le \sum_{k=-\infty}^{\infty} \int_{F_k} f(x)dx = \int f(x)dx \end{aligned}$$

and

$$\begin{aligned} \sum_{k=-\infty}^{\infty} 2^k m(F_k) &\le \sum_{k=-\infty}^{\infty} 2^k m(E_k) \\ &\le \sum_{k=-\infty}^{\infty} 2^k (\sum_{n=k}^{\infty} m(F_n)) \\ &= \sum_{n=-\infty}^{\infty} (\sum_{k=-\infty}^{n} 2^k m(F_n)) = 2\sum_{n=-\infty}^{\infty} 2^n m(F_n). \end{aligned}$$