**Recall:**

Lebesgue measure  Outer measure

$m_*(E) = \inf\{\sum_{i=1}^{\infty}|Q_i| : E \subseteq \bigcup_{i=1}^{\infty} Q_i\}$  $E \subseteq \mathbb{R}^n$.

Q: why cube?

A: 简便起见. 事实上. 换成 Balls 也可 (见 Ch 3 Ex 26).

$Q(\text{cube}) = Q_r(x_0) = \{x \mid |x^i - x_0^i| \le \frac{r}{2}, 1 \le i \le n\}$ → closed

$\mathring{Q}_r(x_0) = \{x \mid |x^i - x_0^i| < \frac{r}{2}, 1 \le i \le n\}$

$|Q_r(x_0)| = |\mathring{Q}_r(x_0)| = r^n$ ($r$ is the size of $Q$)

**Some properties:**

① Nonnegativity: $m_*(E) \geq 0$ & $m_*(\phi) = 0$

② Monotonicity: $E_1 \subseteq E_2 \Rightarrow m_*(E_1) \le m_*(E_2)$

③ Countable sub-additivity $E = \bigcup_{i=1}^{\infty} E_i \Rightarrow m_*(E) \le \sum_{i=1}^{\infty} m_*(E)$

③'. $\forall A, B \subseteq \mathbb{R}^n$. $m_*(A \cup B) \le m_*(A) + m_*(B)$

special case: If $d(A, B) > 0$, then $m_*(A \cup B) = m_*(A) + m_*(B)$.

**Fact:**

1. If $E = \bigcup_{i=1}^{\infty} E_i$ and $m_*(E_i) = 0$, then $m_*(E) = 0$

2. $m_*(E) = \inf\{\sum_{j=1}^{\infty}|\mathring{Q}_j| \mid E \subseteq \bigcup_{j=1}^{\infty} \mathring{Q}_j\} =: m_*'(E)$

pf: Need: $m_*(E) = m_*'(E)$

If $E \subseteq \bigcup_{i=1}^{\infty} \mathring{Q}_i$, then $E \subseteq \bigcup_{i=1}^{\infty} Q_i$.

$\forall \varepsilon > 0$. If $E \subseteq \bigcup_{i=1}^{\infty} Q_i$. then $E \subseteq \bigcup_{i=1}^{\infty} (1+\varepsilon)\mathring{Q}_i$

From these 2 statements, we obtain

① $m_*(E) \le m_*'(E)$

$\forall \varepsilon>0$. $\exists E \subseteq \bigcup_{i=1}^{\infty} \mathring{Q}_i$

$m_*'(E) \le \sum_{i=1}^{\infty} |\mathring{Q}_i| \le m_*'(E) + \varepsilon$. we know $E \subseteq \bigcup_{i=1}^{\infty} Q_i$

$m_*(E) \le \sum_{i=1}^{\infty} |Q_i| = \sum_{i=1}^{\infty} |\mathring{Q}_i| \le m_*'(E) + \varepsilon$

Let $\varepsilon \to 0$, $m_*(E) \le m_*'(E)$.

② Similarly, $m_*'(E) \le (1+\varepsilon)^n m_*(E) \xRightarrow{\varepsilon\to 0} m_*'(E) \le m_*(E)$ #

3. If $m_*(A)=0$, then $\forall B$.

$m_*(B\setminus A) = m_*(A\cup B) = m_*(B)$.

pf: $B\setminus A \subseteq B$. $m_*(B\setminus A) \le m_*(B)$

Notice $B = (B\setminus A) \cup (A\cap B)$

$m_*(B) \le m_*(B\setminus A) + m_*(A\cap B) = m_*(B\setminus A)$.

Similarly, $m_*(A\cup B) = m_*(B)$ #

课堂练习:

Ex 1: prove that for $\forall \delta>0$

$m_*(E) = \inf\{ \sum_{i=1}^{\infty} |Q_i| \mid E \subseteq \bigcup_{i=1}^{\infty} Q_i$ and the size of $Q_i < \delta \}$

pf: Let RHS be $m$. obviously $m_*(E) \le m$.

If $\forall \varepsilon>0$. $\exists E \subseteq \bigcup_{i=1}^{\infty} Q_i$ and $m_*(E) \le \sum_{i=1}^{\infty} |Q_i| \le m_*(E) + \varepsilon$

Let $Q_i = \bigcup_{k=1}^{N} Q_{ik}$ s.t. size of $Q_{ik} < \delta$ and

$|Q_i| = \sum_{k=1}^{N} |Q_{ik}| \Rightarrow m \le m_*(E) + \varepsilon \Rightarrow m \le m_*(E)$ #

Ex 2: Suppose $E$. for any $x\in E$. $\exists$ cube $Q_x$ centered at $x$ s.t.

$m_*(E\cap Q_x) = 0$, prove that $m_*(E) = 0$

Hint: Define $E_k = \left\{ x\in E \;\middle|\; \begin{array}{l} m_*(E\cap Q)=0 \text{ for any cube } Q \\ \text{with size} < \frac{1}{k} \text{ and centered at } x \end{array} \right\}$

① $E = \bigcup_{k=1}^{\infty} E_k$

②. $m_*(E_k) = 0$

pf of ②: $m_*(E_k) = \inf \{ \sum_{i=1}^{\infty} |Q_i| \mid E_k \subseteq \bigcup_{i=1}^{\infty} Q_i$ and size of $Q_i < \frac{1}{100k} \}$

WLOG, $Q_i \cap E_k \neq \phi$. $\exists x \in Q_i \cap E_k$.

Notice $E_k \subseteq \bigcup_{i=1}^{\infty} E_k \cap Q_i$

We define $Q_i' = Q_{\frac{1}{2k}}(x)$

We know $Q_i \subset Q_i'$ since $x \in E_k$. $m_*(E \cap Q_i') = 0$

$m_*(E_k \cap Q_i) \leq m_*(E \cap Q_i) \leq m_*(E \cap Q_i') = 0$

$\Rightarrow m_*(E_k) = 0$. #

## Recall

Def: $E$ is measurable if $\forall \varepsilon > 0$. $\exists$ an open set $U$ s.t. $E \subseteq U$ and $m_*(U \setminus E) < \varepsilon$.

Notation: $E$ measurable : $E \in \mathcal{M}$

## Some properties:

1. open / closed sets $\in \mathcal{M}$
2. If $E_i \in \mathcal{M}$, then $\bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$. $\bigcap_{i=1}^{\infty} E_i \in \mathcal{M}$
3. If $E \in \mathcal{M}$, then $E^c \in \mathcal{M}$.
4. Countable additivity:

   If $E = \bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$ and $E_i \cap E_j = \phi$.

   then $m(E) = \sum_{i=1}^{\infty} m(E_i)$

Corollary of 4: Principle of inclusion-exclusion

If $A, B \in \mathcal{M}$ then

$$m(A\cup B) = m(A) + m(B) - m(A\cap B)$$

Corollary:

If $m(A\cup B) < \infty$, then

$$m(A\cup B) = m(A) + m(B) \text{ iff } m(A\cap B) = 0$$

$\forall E \subseteq \mathbb{R}^n$. $m_*(E) = \inf \left\{ \sum_{i=1}^{\infty} |\mathring{Q}_i| \;\middle|\; E \subseteq \bigcup_{i=1}^{\infty} \mathring{Q}_i \right\}$

If we set $U_\varepsilon = \bigcup_{i=1}^{\infty} \mathring{Q}_i \supseteq E$ and

$$m(E) \le m(U_\varepsilon) \le \sum_{i=1}^{\infty} |\mathring{Q}_i| \le m_*(E) + \varepsilon$$

Define $U = \bigcap_{k=1}^{\infty} U_{\frac{1}{k}}$. then $E \subseteq U$. $m(U) = m_*(E)$

对任一个集合.可找到一个可测集 测度 = 其外测度.

$U$ 即为集合$E$的等测包.