交换代数

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目录

1 概论

(1) What is CA?Basic object?

Study the category of commutative ring, and modules on them.

(2) What is necessary to attend the course?

(a) The theory ring,module,fields(category is not necessary, homological algebra is not needed.)

(b) It is better if you know manifold.(toplogical space)

(3) Why commutative algebra?

The application of the theory of commutative algebra is MORE important than the theory itself.

Motivation(I):Fermat's Last thm

$$x^p + y^p = z^p, x, y, z \in \mathbb{Z}, p \in \mathbb{Z}\text{prime} \Longrightarrow xyz = 0$$

- $p = 2$. Gauss introduce $\mathbb{Z}[\mathrm{i}]$. 需要用到 $\mathbb{Z}[\mathrm{i}]$ 的 UFD 性质.

  注记. 自己试一下！

- $p = 3$ Euler $\mathbb{Z}[\zeta_3] = \{a + \zeta_3 b \mid a, b \in \mathbb{Z}\}, (\zeta_3)^3 = 1$. $\mathbb{Z}[\zeta_3]$ 也有 UFD 性质，所以证明可以推广. 但接下来对一般的 p 不对.

- p regular prime. Kummer $\mathbb{Z}[\zeta_p]$. FLT holds for regular prime. 有更广泛的一个分解，Kummer 分解. 这是本门课的目的之一.

  注记. $(*_p) : x^p + y^p = z^p, LHS = \prod_{k=0}^{p-1}(x + \zeta_p^k y), \zeta_p = \mathrm{e}^{\frac{2\pi\mathrm{i}}{p}}$

- $p > 2$ Faltings:$(*_p)$ has only finite solutions in the sense of $(x, y, z) \sim (\lambda x, \lambda y, \lambda z)$
  Algebraic Geometry, Grothendieck 将交换代数融入到代数几何中

- $p > 2$ A.Wiles (Algebraic Geometry and representation theory)

References:

- Introduction to commutative algebra by Atiyah and Macdonald
- Commutative algebra by H.Matsumura 不推荐上来就看
- Commutative Ring theory by H.Matsumura 不涉及代数几何
- 数论 I：Fermat 的梦想和类域论（加藤和也）与本门课 topic 相关，起点低，非常推荐
- 古今数学思想（克莱因）

Geometry

1. Line: $ax + by = c \subset \mathbb{R}^2$
2. circle: $x^2 + y^2 = r^2$

3. $ax^2 + bxy + cy^2 + dx + ey + f = 0$

   思考：对称矩阵的合同标准型给出二次曲面的分类.

4. What about $Z(f) = \left\{x \in \mathbb{R}^2 \mid f(x) = 0\right\}$ if $\deg(f) = 3$?

   Newton: there are 72 kinds of plane curves of deg $= 3$.

   Q:What if $\deg(f) > 3$?

   What if $x \in \mathbb{R}^n$ or $\mathbb{C}^n$?

   What if $Z(f_1, \cdots, f_r)$?

Classical Algebraic Geometry
Let $f_1, \cdots, f_r \in \mathbb{C}[x_1, \cdots, x_n]$, How can we study

$$Z(f_1, \cdots, f_r) := \{x \in \mathbb{C}^n \mid f_1(x) = \cdots = f_r(x) = 0\}.$$

Arithmetic Algebraic Geometry(Grothendick)

$$Z(f_1, \cdots, f_r) := \{x \in \mathbb{Q}^n \mid f_1(x) = \cdots = f_r(x) = 0\}.$$

Cauculus differential and integration
Assume $\mathcal{U} \subset \mathbb{R}^n$ open
differential: local
integration/cohomology: global
Key problem: $Z(f_1, \cdots, f_r)$ may NOT be a manifold, such as $xy = 0$