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目录

Part I

Riemannian Geometry

Chapter 1

Introduction

1 Preliminaries

1.1 Topological manifolds

Definition 1.1 (拓扑流形). • *局部欧几里得*

- *Hausdorff*
- *第二可数*

Remark. • *Hausdorff 保证 Cauchy 列收敛到唯一点*

- *局部欧几里得保证不了分离性*
- *任意的一个拓扑空间不一定可以度量化，因此要满足第二可数性*
- *参见 Spivak 的第一册的 459 页附录 A，拓扑空间满足前两条的话，可度量化当且仅当第二可数当且仅当 paracompact*
- *paracompact: 每一个开覆盖都有一个局部有限的细化，refinement 的意思是取的新的开覆盖要么是原来的开集要么是原来的子集，*
- *paracompactness 告诉我们单位分解的存在性*

1.2 Smooth manifolds

Definition 1.2. *An atlas $\{(U_\alpha, x_\alpha)\}$ on a manifold is called **differentiable** if all chart transitions*

$$x_\beta \circ x_\alpha^{-1} : x_\alpha(U_\alpha \cap U_\beta) \to x_\beta(U_\alpha \cap U_\beta)$$

are differentiable of class C^∞ in case of $U_\alpha \cap U_\beta \neq \varnothing$.

*A **maximal** differentiable atlas is called a **differentiable structure**.*

*A **differentiable manifold** of dim d is a manifold of dim d with a differentiable structure.*

Remark. *dim$\leqslant 3$ differentiable structure is unique.*

Milnor 1956 exotic 7 sphere.

1.3 Partition of unity

Lemma 1.3. *Let M be a smooth manifold, $(U_\alpha)_{\alpha\in A}$ an open covering. Then $\exists$ a partition of unity subordinate to (U_α). That is $\exists$ a locally finite refinement $(V_\beta)_{\beta\in B}$ of $(U_\alpha)_{\alpha\in A}$ and C_0^∞ functions $\varphi_\beta : M \to \mathbb{R}$ such that*

(1) $\operatorname{supp}\varphi_\beta \subset V_\beta, \forall\ \beta \in B$

(2) $0 \leqslant \varphi_\beta(x) \leqslant 1, \forall\ x \in M, \forall\ \beta \in B$

(3) $\displaystyle\sum_{\beta\in B} \varphi_\beta(x) = 1, \forall\ x \in M$

1.4 Tangent vector

Smooth curve $\gamma : (a, b) \to M$
$x \in \Omega \subset \mathbb{R}^d$
$x = (x^1, \cdots, x^d)$
$T_x\Omega = \left\{ v^i \dfrac{\partial}{\partial x^i} = (v^1, \cdots, v^d), v^i \in \mathbb{R} \right\}$

2 Riemannian metric

$\gamma : (a, b) \to M$

$\int_a^b |\gamma'(t)| \mathrm{d}t = length(\gamma)$

Hilbert space$\Longrightarrow$ Riemannian geometry

Banach space$\Longrightarrow$ Finsler geometry

Just for the purpose of

$\gamma'(t), (v, x)$

$\|\gamma'(t)\|^2 = g_{ij} v^i v^j = (v^1, \cdots, v^d) \begin{pmatrix} g_{ij} \end{pmatrix} \begin{pmatrix} v^1 \\ \vdots \\ v^d \end{pmatrix}$ bilinear form, (g_{ij}) positive definite,symmetric matrix

(U, y) $w^i \dfrac{\partial}{\partial y^i} = w^i \dfrac{\partial x^j}{\partial y^i} \dfrac{\partial}{\partial x^j}$

$h_{ij}(y(p)) = g_{kl}(x(p)) \dfrac{\partial x^k}{\partial y^i} \dfrac{\partial x^l}{\partial y^j}$

(g_{ij}) $(0, 2)$ tensor! And we assume its coefficients are smooth on $x(U)$

Definition 2.1. *A Riemannian metric g on a smooth manifold M is a smooth $(0, 2)$-tensor satisfying*

$$g(X, Y) = g(Y, X), \quad g(X, X) \geqslant 0 \& g_p(X, X) = 0 \Longleftrightarrow X(p) = 0$$

for any smooth tangent vector field X, Y.

A Riemannian manifold is a smooth manifold with a Riemannian metric.

Example 2.2. $\mathbb{R}^n$

- $(g_{ij}) = (\delta_{ij})$
- 球面几何 $(g_{ij}) = \dfrac{4}{(1 + \sum_{i=1}^n (x^i)^2)^2} (\delta_{ij})$
- 双曲几何 $(g_{ij}) = \dfrac{4}{(1 - \sum_{i=1}^n (x^i)^2)^2} (\delta_{ij})$

2.1 Existence of Riemannian metric

Theorem 2.3. *A smooth manifold has a Riemannian metric.*

Extrinsic proof. Whitney embedding

$f : M^n \to N^{n+k}$ smooth immersion ($\mathrm{d}f_p$ is injective)

Let (N, g_N) be a Riemannian metric

Pull-back metric $f^* g_N$ on M

$$(f^* g_N)_p(X_p, Y_p) = g_N(\mathrm{d}f_p(X_p), \mathrm{d}f_p(Y_p))$$

□

Intrinsic proof. U_p coordinate neighborhood.$\{U_p, p \in M\}$ open cover.

paracompact$\Longrightarrow$ WLOG,let $\{U_\alpha\}$ be a locally finite covering of M by coordinate neighborhood.

Partition of unity $\{\varphi_\alpha\}$ subordinate to $\{U_\alpha\}$.

$x\colon U_\alpha \to x(U_\alpha) \subset \mathbb{R}^n$

$g_p(X,Y) = \sum_\alpha \varphi_\alpha(p)(g_\alpha)_p(X,Y).$ $\square$

Definition 2.4. *Let $(M, g_M), (N, g_N)$ be two Riemannian manifolds. $\varphi\colon M \to N$ is called an* ***isometry*** *if φ is a diffeomorphism and $\varphi^* g_N = g_M$.*

2.2 Curves and length

Suppose (M, g) is a Riemannian manifold.

Definition 2.5. *We say a smooth curve* $\gamma\colon I = [a, b] \to M$ *is regular if* $\|\gamma'(t)\| \neq 0, \forall\ t \in I$.

Definition 2.6. *If* $\gamma\colon I \to M$ *is a smooth regular curve and if* $p\colon I' \to I$ *is a smooth map with non-zero derivative, then we say that* $\gamma \circ p\colon I' \to M$ *is a reparametrization of* $\gamma\colon I \to M$.

It is easy to check that any reparametrization of a smooth regular curve is still a smooth regular curve and this defines an equivalent relationship on the space of all smooth regular curves to M.

We will use **parametrized curve** to refer to a smooth regular curve and **curve without parametrization** to refer to an equivalent class of smooth regular curves under reparametrization.

Let $\gamma\colon [a, b] \to M$ be a parametrized curve, we can define its length

$$L(\gamma) = \int_a^b \|\gamma'(t)\| \mathrm{d}t := \int_a^b \sqrt{g(\gamma'(t), \gamma'(t))} \mathrm{d}t.$$

It is easy to check that

Lemma 2.7. *If* $\gamma \circ p\colon I' \to M$ *is a reparametrization of* $\gamma\colon I \to M$*, then* $L(\gamma \circ p) = L(\gamma)$.

So we can actually define length for curves without parametrization.

Arclength parametrization

There is always a canonical representative element for any equivalent class of smooth regular curves under reparametrization.

Propostion 2.8. *Suppose* $\gamma\colon I \to M$ *is a parametrized curve.*

(1) $p\colon I \to [0, L(\gamma)], t \mapsto \int_a^t \|\gamma'(s)\| \mathrm{d}s$ *is a smooth map with non-zero derivative.*

(2) *Suppose* $\gamma \sim \gamma'$*, then* $\gamma \circ p^{-1} = \gamma' \circ p'^{-1}$ *as maps from* $[0, L(\gamma)]$ *to* M.

We call $\gamma \circ p^{-1}$ *the* ***arclength reparametrization*** *of* γ.

Propostion 2.9. $\gamma\colon I \to M$ *is parametrized with arclength iff* $\|\gamma'(t)\| \equiv 1$.

2.3 Metric induced by Riemannian metric tensor

Definition 2.10. *A function $d\colon M \times M \to \mathbb{R}$ is called a metric if*

(i) $d(p,q) \geqslant 0$, *and* $d(p,q) = 0 \Longleftrightarrow p = q$.

(ii) $d(p,q) = d(q,p)$.

(iii) $d(p,q) \leqslant d(p,r) + d(r,q)$, $\forall\ r \in M$.

Let (M,g) be a Riemannian manifold, for any $p,q \in M$, consider
$C_{p,q} = \{\gamma : [a,b] \to M \mid \gamma$ piecewise smooth regular curve with $\gamma(a) = p, \gamma(b) = q\}$.
Define $d(p,q) = \inf\{Length(\gamma) \mid \gamma \in C_{p,q}\}$.
The following questions are immediate

(1) Is $C_{p,q}$ empty?

(2) Is $d(p,q) < +\infty$?

(3) Is d a metric?

(4) Can the infimum be attained?

Let $E_p = \{q \in M : p,q$ can be connected by a curve $\in C_{p,q}\}$. It is easy to show by connectedness argument that $E_p = M$. So $C_{p,q}$ could not be empty.

Take $\gamma \in C_{p,q}$, we can cover it by finite coordinate charts. So we just need to show any piecewise smooth curve contained in a coordinate chart has finite length.

$$Length(\gamma) = \int_a^b \sqrt{g_{ij}\frac{\partial x^i \circ \gamma}{\partial t}\frac{\partial x^j \circ \gamma}{\partial t}}\mathrm{d}t$$

Lemma 2.11.

Next we show $d(p,q)$ is a metric. It is obvious from definition that $d(p,q) \geqslant$ and $d(p,q) = d(q,p)$. Because we consider piesewise smooth curve, triangle inequality is also easy. If $p \neq q$, we can find a coordinate chart U of p such that $q \notin U$.

Chapter 2

Geodesics

1 Looking for shortest curves

1.1 例子

Euclidean geometry

(r, θ)
$g = \mathrm{d}r \otimes \mathrm{d}r + r^2 \mathrm{d}\theta \otimes \mathrm{d}\theta$
$\gamma : [a, b] \to M, \gamma(a) = p, \gamma(b) = q$
$Length(\gamma) = \int_a^b \sqrt{g(\gamma'(t), \gamma'(t))}\mathrm{d}t$
$(r(t), \theta(t)), r'(t)\frac{\partial}{\partial t} + \theta'(t)\frac{\partial}{\partial \theta}$

$$\begin{aligned} Length(\gamma) &= \int_a^b \sqrt{g(\gamma'(t), \gamma'(t))}\mathrm{d}t \\ &= \int_a^b \sqrt{r'(t)^2 + r(t)^2\theta'(t)^2}\mathrm{d}t \\ &\geqslant \int_a^b |r'(t)|\mathrm{d}t \\ &\geqslant \left| \int_a^b r'(t)\mathrm{d}t \right| \\ &= |r(b) - r(a)| \end{aligned}$$

$=$ holds iff $\theta'(t) \equiv= 0, \gamma(t)$ monotonic.

$S^2 \subset \mathbb{R}^3$

$\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2}), \theta \in (0, 2\pi)$
$\left\{(\varphi, \theta) \mid \varphi \in (-\frac{\pi}{2}, \frac{\pi}{2}), \theta \in (0, 2\pi)\right\}$
$g = \mathrm{d}\varphi \otimes \mathrm{d}\varphi + \cos^2 \varphi \mathrm{d}\theta \otimes \mathrm{d}\theta$

1.2 问题的化归

Consider the length functional $L\colon C_{p,q} \to \mathbb{R}$.

我要找 L 的最小值点. 一个简单但关键的观察是：如果 γ 是连接 p 和 q 的最短线，那么它也是连接其上 p, q 之间任意两点的最短线. 因此我们可以将问题局部化！

下一个观察是，作为 L 的我要找 L 的最小值点，首先找 L 的极小值点.

假设 $\gamma_0 \in C_{p,q}$ 是 L 的极小值点，那么对于任意一族曲线 $\gamma_\varepsilon\colon (-\delta,\delta) \to C(p,q)$，都应有

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} L(\gamma_\varepsilon) = 0, \quad \left.\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\right|_{\varepsilon=0} L(\gamma_\varepsilon) \geqslant 0.$$

Remark. γ_ε 上得附加可微性吧？不然 $L(\gamma_\varepsilon)$ 怎么可导？

Localizable

Suppose γ is the shortest curve connecting p and q, then it is also the shortest curve connecting any two points on γ between p and q. WLOG, we can suppose p, q are in one coordinate chart.

Remark. 但这里是不是还需要说明我们不需要考虑那些跑出 p, q 落在的坐标卡的那些曲线，只考虑包含在坐标卡里的那些曲线.

Energy functional

$$L(\gamma_\varepsilon) = \int_a^b \sqrt{g_{ij}(x\circ\gamma_\varepsilon(t))\frac{\mathrm{d}x^i\circ\gamma_\varepsilon(t)}{\mathrm{d}t}\frac{\mathrm{d}x^j\circ\gamma_\varepsilon(t)}{\mathrm{d}t}}\mathrm{d}t$$

要对它求导太麻烦，为此我们考虑能量泛函 $E(\gamma) = \dfrac{1}{2}\displaystyle\int_a^b g(\gamma'(t),\gamma'(t))\mathrm{d}t$.

Lemma 1.1. *$\forall\ \gamma \in C_{p,q}$, $\gamma : [a,b] \to M$, we have*

$$L(\gamma)^2 \leqslant 2(b-a)E(\gamma).$$

and "$=$" holds iff $\|\gamma'(t)\| \equiv \mathrm{const}$.

证明.

$$L(\gamma) = \int_a^b \|\gamma'(t)\|\mathrm{d}t \leqslant \left(\int_a^b 1^2\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_a^b \|\gamma'(t)\|^2\mathrm{d}t\right)^{\frac{1}{2}} = \sqrt{b-a}\sqrt{2E(\gamma)}.$$

□

容易验证 $E(\gamma)$ 只能对于参数化曲线 $\gamma\colon [a,b] \to M$ 定义，这与长度泛函是不同的.

If γ is arclength parametrized, then $L(\gamma)^2 = 2L(\gamma)\cdot E(\gamma) \Longrightarrow L(\gamma) = 2E(\gamma)$.

Let us fix some notations. Suppose

$$\begin{aligned} \gamma : [a,b] &\longrightarrow U \subset M^n \xrightarrow{x} x(U) \subset \mathbb{R}^n \\ t &\longmapsto \gamma(t) \in U \longmapsto x(\gamma(t)) =: x(t) \end{aligned}$$

where $\gamma : [a,b] \to M$ is a parametrized curve and (U,x) is a chart.

Given $y\colon [a,b] \to \mathbb{R}^n$ a parametrized curve such that $y(a) = y(b) = 0$, define $\gamma_\varepsilon(t) = x(t) + \varepsilon y(t)$. You can belive that for sufficient small δ, γ_ε is contained in $x(U)$, $\forall\ \varepsilon \in (-\delta,\delta)$.

Remark. 一个问题是这样构造出来的 γ_ε 是否把所有的这种扰动找全了.

Remark. 流形上没有线性结构，搬到 $\mathbb{R}^n$ 上去加！

Propostion 1.2 (光滑 + 最短线 + 平行弧长参数 $\Longrightarrow$ 能量泛函临界点)**.** *If γ is a C^∞ shortest curve from p to q.*（前一句话与参数化无关，但后一句话给定了一个参数化）*Then γ with a parametrization $\gamma : [a, b] \to U \subset M$ s.t. $\|\gamma'(t)\| \equiv \text{const}$ is a critical point of E,i.e.,*$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0}E(\gamma_\varepsilon) = 0$.

Remark.

- 原则上来说最短线是在所有分段光滑的曲线中找的，以后会说明最短线一定是光滑的.
- 在不担心这个额外的光滑性假定的条件下，上面的命题告诉我们，最短线赋予平行于弧长的参数一定是能量泛函的临界点.

  因此如果我们去找能量泛函的临界点，是不会漏掉最短线的.

证明. γ shortest $\Longrightarrow L(\gamma) \leqslant L(\gamma_\varepsilon)$

$L(\gamma) = \sqrt{2(b-a)E(\gamma)}$

$L(\gamma_\varepsilon) \leqslant \sqrt{2(b-a)E(\gamma_\varepsilon)}$

$\Longrightarrow E(\gamma) \leqslant E(\gamma_\varepsilon)$

$\Longrightarrow \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0}E(\gamma_\varepsilon) = 0.$ □

最短线加弧长参数是临界点，临界点如果都不是弧长参数就完了，没听懂.

1.3 能量泛函的临界点

任给 $y(t)$ 满足 $y(a)=y(b)=0$，

$$2E(\gamma_\varepsilon)=\int_a^b g_{ij}(x(t)+\varepsilon y(t))\frac{\mathrm{d}}{\mathrm{d}t}(x^i(t)+\varepsilon y^i(t))\frac{\mathrm{d}}{\mathrm{d}t}(x^j(t)+\varepsilon y^j(t))\mathrm{d}t$$

$$0=\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0}2E(\gamma_\varepsilon)=\int_a^b g_{ij,k}(x)y^k\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}\mathrm{d}t+\int_a^b g_{ij}(x)\frac{\mathrm{d}y^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}\mathrm{d}t+\int_a^b g_{ij}(x)\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}y^j}{\mathrm{d}t}\mathrm{d}t$$

$$\int_a^b g_{ij}(x)\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}y^j}{\mathrm{d}t}\mathrm{d}t=-\int_a^b\frac{\mathrm{d}}{\mathrm{d}t}\left(g_{ij}(x)\frac{\mathrm{d}x^i}{\mathrm{d}t}\right)y^j\mathrm{d}t=-\int_a^b g_{ij,k}(x)\frac{\mathrm{d}x^k}{\mathrm{d}t}\frac{\mathrm{d}x^i}{\mathrm{d}t}y^j\mathrm{d}t-\int_a^b g_{ij}(x)\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}y^j\mathrm{d}t$$

$$\int_a^b g_{ij}(x)\frac{\mathrm{d}y^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}\mathrm{d}t=-\int_a^b\frac{\mathrm{d}}{\mathrm{d}t}\left(g_{ij}(x)\frac{\mathrm{d}x^j}{\mathrm{d}t}\right)y^i\mathrm{d}t=-\int_a^b g_{ij,k}(x)\frac{\mathrm{d}x^k}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}y^i\mathrm{d}t-\int_a^b g_{ij}(x)\frac{\mathrm{d}^2x^j}{\mathrm{d}t^2}y^i\mathrm{d}t$$

$$0=\int_a^b\left(g_{ij,k}(x)\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}-g_{ik,j}(x)\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^i}{\mathrm{d}t}-g_{kj,i}(x)\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}-2g_{ik}(x)\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}\right)y^k\mathrm{d}t$$

$$g_{ij,k}(x)\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}-g_{ik,j}(x)\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^i}{\mathrm{d}t}-g_{kj,i}(x)\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}-2g_{lk}(x)\frac{\mathrm{d}^2x^l}{\mathrm{d}t^2}$$

$$2g_{lk}(x)\frac{\mathrm{d}^2x^l}{\mathrm{d}t^2}+(g_{ik,j}+g_{kj,i}-g_{ij,k})\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}=0$$

$$\frac{\mathrm{d}^2x^l}{\mathrm{d}t^2}+\frac{1}{2}g^{kl}(g_{ik,j}+g_{kj,i}-g_{ij,k})\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}=0$$

Definition 1.3. 设 (M,g) 是黎曼流形，(U,x) 是一个坐标卡，g 在 (U,x) 下的分量表示为 (g_{ij})，定义 U 上的一族函数 $\Gamma_{ij}^k=\frac{1}{2}g^{kl}(g_{jl,i}+g_{il,j}-g_{ij,l})$，称作第二类 *Christoffel* 符号.

Propostion 1.4.

(1) $\Gamma_{ij}^k=\Gamma_{ji}^k$

(2) $g_{ij,l}=g_{kj}\Gamma_{il}^k+g_{ik}\Gamma_{jl}^k$

Propostion 1.5. $\widetilde{\Gamma}_{ij}^k=\Gamma_{\alpha\eta}^\gamma\frac{\partial x^\alpha}{\partial\tilde{x}^i}\frac{\partial x^\eta}{\partial\tilde{x}^j}\frac{\partial\tilde{x}^k}{\partial x^\gamma}+\frac{\partial\tilde{x}^k}{\partial x^\gamma}\frac{\partial^2x^\gamma}{\partial\tilde{x}^i\partial\tilde{x}^j}$

Propostion 1.6. $\frac{\mathrm{d}^2x^k}{\mathrm{d}t^2}+\Gamma_{ij}^k\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}=0$ 是定义在流形上的方程.

Definition 1.7. *A parametrized curve* $\gamma\colon[a,b]\to M$ *satisfies the equation above is called a geodesic.*

Propostion 1.8. *Geodesics are parametrited proportionally by arclength*

证明.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(g_{ij}(x(t))\frac{\mathrm{d}x^i(t)}{\mathrm{d}t}\frac{\mathrm{d}x^j(t)}{\mathrm{d}t}\right)&=g_{ij,l}\frac{\mathrm{d}x^l}{\mathrm{d}t}\frac{\mathrm{d}x^i}{\mathrm{d}t}+2g_{ij}\frac{\mathrm{d}^2x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}\\&=g_{ij,l}\frac{\mathrm{d}x^l}{\mathrm{d}t}\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}+2g_{ij}\left(-\Gamma_{kl}^i\frac{\mathrm{d}x^k}{\mathrm{d}t}\frac{\mathrm{d}x^l}{\mathrm{d}t}\right)\frac{\mathrm{d}x^j}{\mathrm{d}t}\\&=\left(g_{ij,l}-2g_{kj}\Gamma_{il}^k\right)\frac{\mathrm{d}x^l}{\mathrm{d}t}\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}\end{aligned}$$

Claim $g_{ij,l}=g_{kj}\Gamma_{il}^k+g_{ik}\Gamma_{jl}^k$

$$\begin{aligned}RHS&=\frac{1}{2}g_{kj}g^{kp}(g_{pl,i}+g_{ip,l}-g_{il,,p})+\frac{1}{2}g_{ik}g^{kp}(g_{pl,j}+g_{jp,l}-g_{jl,p})\\&=\frac{1}{2}(g_{il,i}+g_{ij,l}-g_{il,j})+\frac{1}{2}(g_{il,j}+g_{ji,l}-g_{jl,i})=g_{ij,l}\end{aligned}$$

□

Theorem 1.9. $\forall p \in M, \exists \mathcal{U}_{V,\delta} = \{(q,v) \mid p,q \in V \subset M\, open v \in T_qM, \|v\| < \delta, \delta > 0\}$
and a $\varepsilon > 0$ *and* C^∞ *map* $\gamma : (-\varepsilon, \varepsilon) \times \mathcal{U}_{V,\delta} \to M$ s.t. $\forall (q,v) \in \mathcal{U}_{V,\delta}$, *the curve* $t \mapsto \gamma(t,q,v)$ *is the unique geodesic satisfying* $r(0,q,v) = q, r'(0,q,v) = v \in T_qM$

3 月 4 日 22 分 27 秒

Lemma 1.10 (Homogeneity of geodesic)**.** *If the geodesic* $\gamma(t,q,v)$ *is defined on* $t \in (-\varepsilon, \varepsilon)$*, then the geodesic* $\gamma(t,q,\lambda v)$,$\lambda \in \mathbb{R}^+$ *is defined on the interval* $t \in (-\dfrac{\varepsilon}{\lambda}, \dfrac{\varepsilon}{\lambda})$ *and*

$$\gamma(t,q,\lambda v) = \gamma(\lambda t, q, v).$$

1.4 Exponential Map

要根据一点附近的测地线的性质，来确定一个坐标系，使得测地线在这个坐标映射下投到欧氏区域后是直线.

其实拿切空间来做坐标区域应该是个挺自然的想法，毕竟切空间是该处的一阶线性近似

$$\begin{aligned} \exp_p : T_pM &\longrightarrow M \\ v &\longmapsto \gamma(1,p,v) \end{aligned}$$

- 选取 1 能够使测地线走的长度等于 $\|v\|_g$.

指数映射的定义域

3 月 4 日 52 分 30 秒

$V_p := \{v \in T_pM \mid \text{the geodesic } \gamma(t,p,v) \text{ is defined on } [0,1]\}$.

为了 $\exp_p$ 成为坐标映射，我们希望 V_p 至少包含以 O 为心的一个开球！

3 月 4 日 55 分 45 秒

Propostion 1.11.

(1) V_p *is star-shaped aroud* $O \in T_pM$*, i.e.*$\forall\ v \in V_p, \forall\ \lambda \in [0,1]$*, then* $\lambda v \in V_p$*.*

(2) $\forall\ p, \exists\ \varepsilon = \varepsilon(p)$, s.t. $\gamma(t,p,v)$ *is defined on* $[0,1]$ *once* $\|v\| < \varepsilon$*.*

3 月 4 日 1 小时 1 分 0 秒，反函数定理

3 月 4 日 1 小时 5 分 2 秒

Propostion 1.12. $\mathrm{d}\exp_p = \mathrm{Id}_{T_pM}$.

由逆映射定理，存在 p 点的一个邻域 U 使得 $\exp_p^{-1} : U \to T_pM$ 是微分同胚.

距离 $\exp_p^{-1}$ 成为坐标映射只差 T_pM 到 $\mathbb{R}^n$ 的一个同构，任取 T_pM 的一组基即可.

Propostion 1.13. $\Gamma^k_{ij}(p) = 0$.

Propostion 1.14. 选取 T_pM 的一组基 $\{v_1, \cdots, v_n\}$. 断言 g 在坐标映射 $\exp_p^{-1} : U \to T_pM \cong \mathbb{R}^n$ 下的分量在 O 处的取值 $g_{ij}(O) = g(v_i, v_j)$.

Definition 1.15. 选取 T_pM 的一组标准正交基，此时的 $(\exp_p^{-1}, U)$ 称为 p 的一个法坐标.

3 月 4 日 1 小时 17 分 45 秒

证明. $0 = \dfrac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + \Gamma^i_{jk}(x(t)) \dfrac{\mathrm{d}x^j}{\mathrm{d}t} \dfrac{\mathrm{d}x^k}{\mathrm{d}t}$ □

1.5 极坐标

A curve $c(t) = (r(t), \varphi^1(t), \cdots, \varphi^{n-1}(t))$
$c'(t) = (\frac{\mathrm{d}r}{\mathrm{d}t}, \frac{\mathrm{d}\varphi^1}{\mathrm{d}t}, \cdots, \frac{\mathrm{d}\varphi^{n-1}}{\mathrm{d}t}) =: (v^1, v^2, \cdots, v^n)$
$\|c'(t)\| = g_{ij}(c(t))v^i v^j = \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \underbrace{\sum_{i,j=1}^{n-1} g_{\varphi^i\varphi^j} \frac{\mathrm{d}\varphi^i}{\mathrm{d}t} \frac{\mathrm{d}\varphi^j}{\mathrm{d}t}}_{\geqslant 0}$

3 月 8 日第二段 11 分 20 秒

Corollary 1.16. *For any $p \in M$,$\exists\ \rho > 0$ s.t. $\forall\ q$ with $d(p, q) = \rho$, there exists a unique shortest curve $\in C_{p,q}$.*

证明. $\exists\ \rho > 0$ s.t. $B(p, 2\rho)$ lies in a Riemannian polar coordinate neighborhood.

For any curve $c \in C_{p,q}$

$$c : [0, T] \to M, c(0) = p, c(T) = q$$

□

2 一致邻域

3 月 8 日 25 分 20 秒

3 月 8 日 28 分 56 秒

Definition 2.1. *totally normal neighborhood.*

$\forall\ p \in M$, if $W \ni p$, W is a normal neighborhood of every point $q \in W$, then W is cassled a **totally normal neighborhood**.

2.1 totally normal neighborhood 的存在性

3 月 8 日第二段 35 分 4 秒

Lemma 2.2.

$$\mathrm{d}\exp(p, 0_p) : T_{p,0_p}(TM) \longrightarrow T_{p,p}(M \times M)$$

is non sigular.

3 月 8 日第二段 1 小时 3 分 54 秒

Theorem 2.3. *For any $p \in M$, $\exists$ a neighborhood W of p, and a $\delta > 0$ such that $\forall q \in W$, $\exp_q$ is a diffeomorphism on $B(0_q, \delta) \subset T_qM$ and*

3 月 8 日第二段 1 小时 14 分 7 秒

Corollary 2.4.

3 Cut locus 1

3 月 8 日第二段 1 小时 25 分 32 秒，总结

测地线的最大存在区间的右端点是开的

3 月 8 日第二段 1 小时 28 分 5 秒

测地线是最短线的最大区间相对测地线的最大存在区间是闭的

3 月 8 日第二段 1 小时 31 分 7 秒

记 $S_p = \{v \in T_pM \mid \|v\|_g = 1\}$.

Definition 3.1.

- *Cut point*
- *Cut locus*
- *Define a map* $\tau : S_p \to \mathbb{R} \cup \{\infty\}$

$$\forall v \in S_p, \tau(v) = \begin{cases} a & \textit{if } \exp_p(av) \textit{is a cut point ofp} \\ \infty & \textit{ifphas no cut point alongt} \mapsto \exp_p(tv) \end{cases}$$

$$E(p) = \{tv \mid v \in S_p, 0 \leqslant t < \tau(v)\}$$

$$\tilde{C}(p) = \{tv \mid v \in S_p, t = \tau(v)\}$$

$$C(p) = \{\textit{cut points ofp}\} = \exp_p(\tilde{C}(p))$$

$[0, b)$ is the maximal interval on which $t \mapsto \exp_p tv$ is defined.

Propostion 3.2. $\forall\ p, q \in M, \exists$ *two shortest curve connecting* p *and* q*,*

Corollary 3.3. $\exp_p : E(p) \to \exp_p(E(P)) \subset M$ *is injective.*

证明. Suppose $\exists V, W \in E(p)$ s.t. $\exp_p(V) = \exp_p(W) = q$.

$$t \mapsto \exp_p\left(t\frac{v}{\|v\|}\right)$$

$$t \mapsto \exp_p\left(t\frac{w}{\|w\|}\right)$$

Contradiction. □

Corollary 3.4. $\exp_p(E(P)) \cap C(p) = \varnothing$.

证明. Suppose $\exists\ v \in \tilde{C}(p), W \in E(p)$ s.t. $\exp_p V = \exp_p W = q$

Contradiction. □

Question: $\exp_p(E(p)) \cup C(p) = M$?

$\mathbb{R}^2 \setminus \{0\}$

$\forall q \in \exp_p(E_p) \cup C(p) = M$?

4 Hopf-Rinow Theorem

3 月 11 日 27 分 14 秒

任给 $p_0, q \in M, d(p_0, q) = r_0$. 我们想要找 p_0, q 之间的最短线.

我们知道局部上总是可以做的，问题是 p_0, q 可能离得很远.

思路是一步一步走.

选取以 p_0 为中心的一个 normal ball $B(p_0, \rho_0)$，若 $q \in B(p_0, \rho_0)$，结束.

若 $q \notin B(p_0, \rho_0)$，假设 p_0, q 之间存在最短线 γ，易知

- $\gamma \cap \partial B(p_0, \rho_0) = \{pt\} =: \{p_1\}$.
- $d(p_0, q) = \min\limits_{p \in \partial B(p_0, \rho_0)} d(p, q)$.

从 p_1 出发，我们可以找一个 normal ball $B(p_1, \rho_1)$，并重复上述操作.

问题是：(1) p_0 到 p_2 的分段曲线是最短的吗？(2) 最终能达到 q 吗？

- $d(p_1, q) = r_0 - \rho_0$
  - 假如 $d(p_1, q) < r_0 - \rho_0$，那么可以找到一条连接 p_0, q 的长度小于 r_0 的曲线，矛盾.
  - 假如 $d(p_1, q) > r_0 - \rho_0$. 任选连接 p_0, q 的曲线 γ，$Length(\gamma) \geqslant \rho_0 + d(p_1, q)$.
    取下确界，得 $r_0 \geqslant \rho_0 + d(p_1, q) > r_0$，矛盾.
- $d(p_0, p_2) = \rho_0 + \rho_1$
  - $d(p_0, p_2) \leqslant d(p_0, p_1) + d(p_0, p_2) = \rho_0 + \rho_1$.
  - $d(p_0, p_2) \geqslant d(p_0, q) - d(p_2, q) = r - (r - \rho_0 - \rho_1) = \rho_0 + \rho_1$.

因此，走了 n 步之后，p_0 和 p_n 之间的连线仍是最短的.

3 月 11 日 55 分 24 秒名场面：方向决定道路，道路决定命运.

容易举出一些例子使得 (2) 不成立，为此我们附加一些额外的条件.

3 月 11 日 59 分 19 秒

Definition 4.1.

- *injective radius at* $p \in M$: $i(p) = \sup\left\{\rho > 0 \mid \exp_p\big|_{B(O,\rho)} \text{ is a diffeomorphism}\right\}$.
- *injective radius of* M : $i(M) = \inf\limits_{p \in M} i(p)$.

M compact $\Longrightarrow i(M) > 0$.

3 月 11 日 1 小时 4 分 52 秒

Given $p \in M$,

1. Assumption I: $\overline{B_p(r)}$ is compact($\Longleftrightarrow$All closed bounded subsets of M is compact).
2. Assumption II: (M, g) is a complete metric space.
3. Assumption III: $\exp_p(p)$ is defined on the whole space T_pM.

这三个条件都可以保证 (2). 下面用 Assupmtion III 推 (2).

3 月 21 日 1 小时 11 分 43 秒

证明. $p, V \in T_pM$ $c(t) = \exp_p tV$

Aim:$c(r) = \exp_p(rV) = q$

Consider the set $I := \{t \in [0, r] \mid d(c(t), q) = r - t\}$

□

1 小时 24 分 33 秒

事实上，上面几种假定是等价的，这就是 Hopf-Rinow 定理.

3 月 15 日 2 分 31 秒

Theorem 4.2 (Hopf-Rinow,1931)**.** *Let (M, g) be a Riemannian manifold, TFAE*

(1) *(M, d_g) is a complete metric space.*

(2) *All closed bounded subsets of M is compact.*

(3) *$\exists\ p \in M$, $\exp_p$ is defined on the whole T_pM.*

(4) *$\forall\ p \in M$, $\exp_p$ is defined on the whole T_pM.*

Moreover, each of the statements $(1) - (4)$ implies

(5) *$\forall\ p, q \in M$ can be joined by a shortest curve.*

Remark. 原始论文：*Ueber den Begriff der vollständigen differentialgeometrischen Fläche.*

证明.

- $(3) \implies (2)$

  Claim:$\forall\ r > 0$,$\overline{B(p, r)}$ is compact.

  For any bounded closed subset K, $\exists\ r_k$ such that $K \subset \overline{B(p, r_k)}$.

  FACT:$\overline{B(p, r)} = \exp_p(\overline{B(Op, r)})$

  - $\exp_p(\overline{B(O_p, r)}) \subset \overline{B(p, r)}$

    $\forall\ v \in \overline{B(O_p, r)}, d(p, \exp_p V) \leqslant r \implies \exp_p V \in \overline{B(p, r)}$
  - $\forall q \in \overline{B(p, r)}$,

- $(2) \implies (1)$

- $(1) \implies (4)$

  Suppose $\exists\ p \in M$ and $v \in T_pM$ such that the geodesic $t \mapsto \exp_p tv$ is defined on the maximal interval $[a, b), b < \infty$.

  For any $\{t_n\} \subset [a, b)$ such that $t_n \to b$, $d(\exp_p t_n v, \exp_p t_m v) \leqslant \|v\|_g |t_n - t_m|$ and then $\{\exp_p t_n v\}$ is a Cauchy sequence.

  $\exists\ p_0 \in M$, $\lim\limits_{n \to +\infty} \exp_p t_n v = p_0$,i.e. $\forall \delta > 0, \exists N$ such that $\exp_p t_n v \in B(p_0, \delta), \forall n \geqslant N$.

□

Lemma 4.3. 内容...

5 Cut locus 2

3 月 15 日 39 分 24 秒

Theorem 5.1. *Let (M, g) be a complete Riemannian manifold, then*

$$M = \exp_p(E(p)) \sqcup c(p).$$

Theorem 5.2. *Let (M, g) be a complete Riemannian manifold.*

Let $\gamma\colon [a, b] \to M$ be a normal geodesic with $p = \gamma(0)$, v

证明. Choose a sequence of parameters

$$a_1 > a_2 > a_3 > \cdots, \quad \lim_{i\to+\infty} a_i = a.$$

By completeness, $\exists v_i \in T_pM, \|v_i\| = 1$ such that

$$\gamma_i(t) = \exp_p tv_i, t \in [0, b_i]$$

is a shortest curve form p to $\gamma(a_i)$, where $b_i = d(p, \gamma(a_i))$.

Notice that $v_i \neq v$.

$\lim\limits_{i\to+\infty} p_i =$ □

6 Existence of shortest curves in given homotopy class

3 月 15 日 1 小时 26 分 15 秒

Theorem 6.1. *Let (M, g) be compact.*

Then every homotopy class of closed curves in M contains a curve which is shortest in its homotopy class and a geodesic.

Lemma 6.2. *Let (M, g) be compact, $\exists \rho_0 > 0$ such that for any $\gamma_0, \gamma_1 \colon S^1 \to M$ be closed curves with $d(\gamma_0(t), \gamma_1(t)) \leqslant \rho_0, \forall\ t \in S_1$ we have γ_0 and γ_1 are homotopic.*

Lemma 6.3. *A shortest curve in a homotopy class is geodesic.*

证明. Let $(\gamma)n_{n\in\mathbb{N}}$ is a minimizing sequence for length in the homotopy class.

All are parametrized proportional to arc length.

We can find $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 2\pi$ with the property that

$$Length(\gamma_n \mid_{t_i, t_{i+1}}) \leqslant \frac{\rho_0}{2}$$

□

7 title

7.1 前情回顾

Riemannian Covering map

$\pi : (\tilde{M}, \pi^* g) \to (M, g)$ smooth map

locally Riemannian isometry

isometry $\varphi \colon (M, g_M) \to (N, g_N)$ is called an isometry if φ is diffeomorphism and $g_M = \varphi^* g_N$

locally isometry $\varphi \colon (M, g_M) \to (N, g_N)$ smooth map, $\forall p \in M \exists U \in p$such that $\varphi\big|_U \colon U \to \varphi(U)$ is an isometry 问题：对 $\varphi(U)$ 有没有要求

locally Riemannian isometry $\varphi \colon (M, g_M) \to (N, g_N)$ smooth map, $\forall\ p \in M, \mathrm{d}\varphi_p : T_p M \to T_{\varphi(p)} N$ is a linear isometry.

Propostion 7.1. *Let $\varphi \colon (M, g_M) \to (N, g_N)$ be a locally Riemannian isometry.*

(1) *φ maps geodesics to geodesics.*

(2) *For any $\tilde{p}, \tilde{v} \in T_{\tilde{p}} M$, we have*

$$\varphi \circ (\exp_{\tilde{p}} \tilde{v}) = \exp_{\varphi(\tilde{p})}(\mathrm{d}\varphi_{(\tilde{p})}(\tilde{v})).$$

$$\begin{array}{ccc} T_{\tilde{p}}M & \xrightarrow{\mathrm{d}\varphi(\tilde{p})} & T_{\varphi(\tilde{p})N} \\ \downarrow & & \downarrow \\ M & \xrightarrow{\varphi} & N \end{array}$$

(3) *φ is distance non-increasing.*

$$\forall\ \tilde p,\tilde q, d_N(\varphi(\tilde p),\varphi(\tilde q)) \leqslant d_M(\tilde p,\tilde q)$$

(4) *φ is bijective, then it is distance preserving.*

Theorem 7.2. *(M, g_M) complete Riemannian manifold, $p, q \in M$. Every homotopy class of paths from p to q contains a shortest curve.*

证明. Assume that (M, g_M) complete$\Longrightarrow (\tilde M, \pi^* g)$ is complete.

□

Propostion 7.3. *Let $\pi\colon (\tilde M,\tilde g) \to (M, g)$ is a Riemannian covering map, then (M, g) complete iff $(\tilde M,\tilde g)$ complete.*

证明.

ete $\Longrightarrow (\tilde M,\tilde g)$complete $\forall \tilde p \in \tilde M, \tilde v \in T_{\tilde p \tilde M}, t \mapsto \exp_{\tilde p} t\tilde v$

$p = \pi(\tilde p), v = \mathrm{d}\pi(\tilde p)(\tilde v)$

geodesic $t \mapsto \exp_p tv$ is defined on $[0,\infty)$

path liftying, $\exists \tilde\gamma$ a path in $\tilde M$ such that $\tilde{\gamma(0)} = \tilde p, \pi\circ\tilde\gamma = \gamma$

$\frac{\mathrm{d}\tilde\gamma}{\mathrm{d}t}\Big|_{t=0} = \tilde v$

ete $\Longrightarrow (M, g)$complete $\forall p \in M, \forall v \in T_pM, t \mapsto \exp_p tv$

□

Propostion 7.4. *Let $\pi\colon (\tilde M,\tilde g) \to (M, g)$ is a local Riemannian isometry. Suppose $(\tilde M,\tilde g)$ complete. Then (M, g) is complete and π is a Riemannian covering map.*

证明. (1) π is surjective.

$\forall \tilde p \in \tilde M, p = \pi(\tilde p) \in M$

$\forall q \in M$,$\exists$ a shortest geodesic γ from p to q.

Let $\tilde\gamma$ be the lifting of γ starting at $\tilde p = \tilde\gamma(0)$

$\pi\circ\tilde\gamma = \gamma, q = \gamma(t_0), \pi\circ\tilde\gamma(t_0) = \gamma(t_0) = q$

(2) evenly covered

$p \in U, \pi^{-1}(U) = \bigsqcup_{\alpha\in\Lambda} \tilde U_\alpha$

$\pi\colon \tilde U_\alpha \to U$ diffeomorphism

Normall ball $B(p,\varepsilon)$

$\tilde U_\alpha = B(\tilde p_\alpha, \varepsilon)$ metric ball

(a) $\tilde{U}_\alpha \cap \tilde{U}_\beta = \varnothing, \forall \alpha \neq \beta$

$d(\tilde{p}_\alpha, \tilde{p}_\beta) \geqslant 2\varepsilon$

(b) $\pi^{-1}(U) = \bigcup_{\alpha \in \Lambda} \tilde{U}_\alpha$

- $\forall \tilde{q} \in \tilde{U}_\alpha$ for some $\alpha \in \Lambda$
  $\exists$ a geodesic $\tilde{\gamma}$ of length $< \varepsilon$ from

$\square$

(U, x)

$$\frac{\mathrm{d}^2 x^i(t)}{\mathrm{d}t^2} + \Gamma^i_{jk}(x(t)) \frac{\mathrm{d}x^j}{\mathrm{d}t} \frac{\mathrm{d}x^k}{\mathrm{d}t} = 0, i = 1, \cdots, n$$

Chapter 3

Connections,Parallelism, and Covariant derivative

1 Affine connection

Aim: Given $A \in \Gamma(\bigotimes^{r,s} TM)$, define the derivative of A along $X_p \in T_pM$ which lies in $\bigotimes^{r,s} T_pM$.

$(0,0)$ tensor field

3 月 22 日 10 分 27 秒
Given $f \in C^\infty(M), X_p \in T_pM$. Suppose $\gamma\colon (-\varepsilon, \varepsilon) \to M$ satisfies $\gamma(0) = p, \gamma'(0) = X_p$. Then
$X_p f = \lim\limits_{t\to 0} \dfrac{f(\gamma(t)) - f(\gamma(0))}{t} \xlongequal{(U,x)} \lim\limits_{t\to 0} \dfrac{f(x^1(t), \cdots, x^n(t)) - f(x^1(0), \cdots, x^n(0))}{t} = \dfrac{\partial f}{\partial x^i}\Big|_{x=p} \dfrac{\mathrm{d}x^i}{\mathrm{d}t}\Big|_{t=0}$.

$(1,0)$ tensor field

3 月 22 日 14 分 0 秒
$Y \in \Gamma(TM)$
$D_{X(p)} Y = \lim\limits_{t\to 0} \dfrac{Y(\gamma(t)) - Y(\gamma(0))}{t}, \gamma(0) = p, \gamma'(0) = X(p)$
$M = \mathbb{R}^n$, directional derivative

(1) $D_{\alpha v} Y = \alpha D_v Y, \forall\ \alpha \in \mathbb{R}$

(2) $D_{v_1+v_2} Y = D_{v_1} Y + D_{v_2} Y$

(3) $D_v(Y_1 + Y_2) = D_v Y_1 + D_2 Y_2$

(4) $D_v(fY) = v(f) Y_f D_v Y$ （Leibniz）

3 月 22 日 21 分 36 秒

Definition 1.1 (Affine connection). *An affine connection ∇ on a smooth manifold M is a map:*

$$\begin{aligned}\nabla\colon \Gamma(TM)\times\Gamma(TM) &\longrightarrow \Gamma(TM)\\ (X,Y) &\longmapsto \nabla(X,Y)=\nabla_X Y\end{aligned}$$

satisfying the following properties

(1) $\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z, \forall\ f,g\in C^{\infty(M)}, \forall\ X,Y,Z\in\Gamma(TM)$*,i.e.* ∇ *is tensorial in* X.

(2) $\nabla_X(Y+Z)=\nabla_X Y+\nabla_X Z$*,i.e.* ∇ *is* $\mathbb{R}$ *linear in* Y

(3) $\nabla_X(fY)=X(f)Y+f\nabla_X Y$

The vector field $\nabla_X Y$ *is called the covariant derivative*（协变导数）*of* Y *along* X *with respect to the connection* ∇.

3 月 22 日 30 分 7 秒

Remark.

(1) *covariant differentiation*（协变微分）

$$\begin{aligned}\nabla\colon \Gamma(TM) &\longrightarrow \Gamma\otimes\Gamma(T^*M)\\ Y &\longmapsto \nabla Y\colon \Gamma(T^*M)\times\Gamma(TM)\longrightarrow\mathbb{R}\\ (\alpha,X) &\longmapsto \nabla Y(\alpha,X)=\alpha(\nabla_X Y)\end{aligned}$$

(2) $\mathbb{R}^n$, *directional derivation*

(3) $M,(U_\alpha,X_\alpha)$

$\nabla_X Y(p) \xlongequal{p\in U_\alpha}$ *directional derivative*

$p\in(U,x),(V,y)$

$Y=f^i\dfrac{\partial}{\partial x^i}$ *in* (U,x)

$\nabla_X Y(p)=D_{X(p)}f^i\dfrac{\partial}{\partial x^i}$

$Y=g^j\dfrac{\partial}{\partial y^j}$

$\nabla_X Y(p)=D_{X(p)}g^j\dfrac{\partial}{\partial y^j}$

$$\begin{aligned}
D_{X(p)}f^i\frac{\partial}{\partial x^i} &= D_{x(p)}g^j\frac{\partial x^k}{\partial y^j}\frac{\partial}{\partial x^k}\\
&= D_{X(p)}\left(f^k\frac{\partial y^j}{\partial x^k}\right)\frac{\partial x^k}{\partial y^j}\frac{\partial}{\partial x^k}\\
&= D_{X(p)}(f^k)\frac{\partial y^i}{\partial x^k}\frac{\partial x^k}{\partial y^i}\frac{\partial}{\partial x^k}+f^kD_{X(p)}\left(\frac{\partial y^j}{\partial x^l}\right)\frac{\partial}{\partial x^k}\\
&= D_{X(p)}f^k\frac{\partial}{\partial x^k}+f^l\frac{\partial x^k}{\partial y^j}D_{X(p)}\left(\frac{\partial y^j}{\partial x^l}\right)\frac{\partial}{\partial x^k}
\end{aligned}$$

$$= f^l(D_{X(P)}\left(\frac{\partial x^k}{\partial y^i}\frac{\partial y^j}{\partial x^l}\right) - D_{X(p)}()$$
$$= -f^l\frac{\partial y^j}{\partial x^l}D_{X(p)}\left(\frac{\partial x^k}{\partial y^j}\right)\frac{\partial}{\partial x^k}$$

Existence

3 月 22 日 45 分 33 秒

$M, (U_\alpha)_{\alpha\in A}$

partition of unity $(V_\beta)_{\beta\in B}$

locally finite refinemetn $(\varphi_\beta)_{\beta\in B}$

$\nabla_X Y(p) := \sum_{\beta\in B}\varphi_\beta(D_X^{V_\beta}Y)$

$\nabla_X(fY) = \sum_{\beta\in B}\varphi_\beta(p)\left(X(f)Y + fD_X^{V_\beta}Y\right)$

$= \sum_{\beta\in B}\varphi_\beta(p)X(f)(p)Y(p) + f\nabla_X Y = Xf\cdot Y + f\nabla_X Y$

Lemma 1.2. *If $\nabla^{(1)}, \cdots, \nabla^{(k)}$ are affine connection on M and $f_1, \cdots, f_k \in C^\infty(M)$ such that $\sum_{i=1}^k f_i(x) = 1, \forall\, x\in M$. Then $\sum_{i=1}^k f_k\nabla^{(i)}$ is an affine connection on M.*

Locality

3 月 22 日 1 小时 0 分 58 秒

Propostion 1.3. *For any open $U\subset M$, if $X\big|_U = \tilde{X}\big|_U$ and $Y\big|_U = \tilde{Y}\big|_U$ Then*

$$\nabla_X Y\big|_U = \nabla_{\tilde{X}}\tilde{Y}\big|U$$

证明. Aim $\nabla_X Y\big|_U = \nabla_{\tilde{X}}Y\big|_U = \nabla_{\tilde{X}}\tilde{Y}\big|_U$

(1) Claim If $X\big|_U \equiv 0$, then $\nabla_X Y\big|_U = 0$.

$\forall\, p\in U$, $\exists$ compact $V\subset U$ such that $f\in C_0^\infty(U), f = 1$ on V

We have $(1-f)X = X$

$\nabla_X Y(p) = \nabla_{(1-f)X}Y(p) = 1 - f(p)\nabla_X Y(p)$.

(2) Claim If $Y\big|_U = 0$, then $\nabla_X Y\big|_U = 0$

$(1-f)Y = Y$

$\nabla_X Y(p) = \nabla_X(1-f)Y(p) = X(1-f)(p)Y(p) - (1-f(p))\nabla_X Y(p) = 0$

□

$\nabla_X Y(p), p\in(U,x), X = X^i\frac{\partial}{\partial x^i}, Y = Y^j\frac{\partial}{\partial x^j}$

$\nabla_X Y(p) = \nabla_{X^i\frac{\partial}{\partial x^i}}\left(Y^j\frac{\partial}{\partial x^j}\right)(p)$

$= X^i\nabla_{\frac{\partial}{\partial x^i}}\left(Y^j\frac{\partial}{\partial x^j}\right) = X^i\frac{\partial Y^j}{\partial x^i}\frac{\partial}{\partial x^j} + X^i(p)Y^j(p)\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}(p)$

$$= \left(X^i \frac{\partial Y^k}{\partial x^i} X^i Y^j \alpha_{ij}^k \right) \frac{\partial}{\partial x^k}$$

Propostion 1.4. *If $X(p) = \tilde{X}(p)$, then $\nabla_X Y(p) = \nabla_{\tilde{X}} Y(p)$.*

$\forall\ v \in T_p M, \nabla_v Y(p) = \nabla_X Y(p)$ such that $X(p) = v$. 问题：如果 Y 在某点处为零，对他求协变导数是否不依赖于联络的信息.

Propostion 1.5. *Let $\gamma\colon (-\varepsilon, \varepsilon) \to M$ be a C^∞ curve on M with $\gamma(0) = p, \gamma'(0) = v$. Suppose $Y(\gamma(t)) = \tilde{Y}(\gamma(t)), t \in (-\varepsilon, \varepsilon)$, then $\nabla_v Y(p) = \nabla_v \tilde{Y}(p)$.*

2 Parallelism

Consider a C^∞ curve $c\colon [a,b] \to M$
A vector field along c is a map

$$[a,b] \longrightarrow TM, \quad t \longmapsto V(t) \in T_{c(t)}M$$

A C^∞ vector field V along c, $\forall\ f \in C^\infty(m)$, then function $t \mapsto V(t)(f)$ is C^∞
(U,x)
$V(t) = \sum_{i=1}^{n} v^i(t) \frac{\partial}{\partial x^i}\big|_{c(t)}$
V is $C^\infty \iff v^i(t) \in C^\infty$.
$V \in \Gamma(TM)$
$V(c(t)) = \sum_{i=1}(c(t)) \frac{\partial}{\partial x^i}$
好微妙.
Covariant derivative of V along c $\frac{DV}{\mathrm{d}t}$
如果 V 是诱导下来的，他就是 $\nabla_{c'(t)}X$

Propostion 2.1. (M, ∇). *There exists a unique map from C^∞ vector fields along C^∞ curve to C^∞ vector fields along c: $V \longmapsto \frac{DV}{\mathrm{d}t}$ such that*

(1) $\frac{D(V+W)}{\mathrm{d}t} = \frac{DV}{\mathrm{d}t} + \frac{DW}{\mathrm{d}t}$

(2) $\frac{D(fV)}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}t}V + f\frac{DV}{\mathrm{d}t}, f \in C^\infty([a,b])$

(3) *If $V\big|_{c(t)} = Y\big|_{c(t)}$ for some Y C^∞ vector field defined in a neightborhood of c, then* $\frac{DV}{\mathrm{d}t} = \nabla_{c'(t)}Y$.

证明. Suppose existence.
locality
$\frac{DV}{\mathrm{d}t}(t_0)$
$V(t), t \in (t_0 - \varepsilon, t_0 + \varepsilon)$
$c(t_0) \in (U,x)$
$V(t) = \sum v^j(t) \frac{\partial}{\partial x^j}\big|_{c(t)}$
$\frac{DV}{\mathrm{d}t}(t_0) = \frac{D}{\mathrm{d}t}\left(\sum_{j=1}^{n} v^j(t)\frac{\partial}{\partial x^j}\big|_{c(t)}\right) = \sum_{j=1}^{n}\frac{D}{\mathrm{d}t}\left(v^j(t)\frac{\partial}{\partial x^j}\big|_{c(t)}\right) = \sum_{j=1}^{n}\frac{\mathrm{d}v^j}{\mathrm{d}t}\frac{\partial}{\partial x^j}\big|_{c(t)} + v^j(t)\frac{D}{\mathrm{d}t}\frac{\partial}{\partial x^j}\big|_{c(t)}$
$= \sum_{i=1}^{n}\left(\frac{\mathrm{d}v^j}{\mathrm{d}t}\frac{\partial}{\partial x^j}\big|_{c(t)} + v^j(t)\nabla_{c'(t)}\frac{\partial}{\partial x^j}(c(t))\right)$
$\longrightarrow$ uniqueness

existence
$t_0, \frac{DV}{\mathrm{d}t(t_0)}$
Choose (U,x)

$$\frac{DV}{\mathrm{d}t}(t_0) := \sum_{j=1}^{n}\left(\frac{\mathrm{d}v^j}{\mathrm{d}t}\frac{\partial}{\partial x^j} + v^j\nabla_{c'(t)}\frac{\partial}{\partial x^j}\right)$$

$c(t_0) \in (\tilde{U}, y)$

dose not depend on the choices of charts □

$V \in \Gamma(TM)$

$V = V^i(x)\dfrac{\partial}{\partial x^i}$

$V(t) = V^i(c(t))$

Definition 2.2 (Parallelism)**.** (M, Δ). *A vector field V along $c\colon [a,b] \to M$ is called parallel if* $\dfrac{DV}{\mathrm{d}t} = 0, \forall\ t \in [a,b]$

Propostion 2.3. (M, Δ) *Let $c\colon [a,b] \to MC^\infty$ curve. Let $V_0 \in T_{c(t_0)}M, t_0 \in I$*

Then $\exists!$ parallel vector field V along c such that $V(t_0) = V_0$

证明. $c(I) \subset (U, x)$ □

Remark. 线性方程组，跟测地线很不同的地方，因此可以保证全局存在

Propostion 2.4. *Let c be a C^∞ curve with $c(0) = p, c'(0) = X(p)$*

Let $Y \in \Gamma(TM)$

Then $\nabla_{X(p)}Y = \lim\limits_{h\to 0}\dfrac{P^{(}_{c,0,h}Y(c(h))) - Y(c(0))}{h}$

证明. Let $V_1, \cdots, V_n$ be parallel vector fields along c which is linearly independent.

$Y(c(t)) = f^i(t)V_i(t)$，这是一件非常方便的事情

$RHS = \lim\limits_{h\to 0}\dfrac{f_i(h)V_i(0) - f^i(0)V_i(0)}{h} = \dfrac{\mathrm{d}f^i}{\mathrm{d}h}\big|_{h=0}V_i(0)$

$= \dfrac{D}{\mathrm{d}t}(f^i(t)V_i(t))\big|_{t=0}$

$= \dfrac{DY}{\mathrm{d}t}(0) = \nabla_{\frac{\mathrm{d}c}{\mathrm{d}t}(0)}Y = \nabla_{X(p)}Y$ □

前情补充

(1) 仿射联络：外蕴观点

(2) (M, Δ)

$\dfrac{DV}{\mathrm{d}t}$,V vector field along a curve c

induced connection

$c\colon (-\varepsilon, \varepsilon) \to M$

Let $\varphi\colon N \to M$ C^∞ map.

A C^∞ vector field along φ.

$x \in N \mapsto V(x) \in T_{\varphi(x)}M$

$\varphi(x) \in M$, frame field E_i in a neighborhood

$V(x) = \sum V^i(x)E_i(\varphi(x))$ 其中 V^i 看作 N 上的函数

Given $u \in T_xN$, $\tilde{\nabla}_u V = \sum u(V^i)E_i(\varphi(x)) + V^i(x)\nabla_{\mathrm{d}\varphi(x)(u)}E_i(\varphi(x))$

induced connection

(3) Lie derivative

$X, Y \in \Gamma(TM)$

$\mathcal{L}_x Y = \lim_{t \to 0} \dfrac{Y - (\varphi_t)_*(Y)}{t}$

$\mathcal{L}_X Y = [X, Y]$

没有办法用来定义平行移动，李导数联系的是对称性

导数的基本精神是将不同空间的东西变成同一个空间的东西

3 Covariant derivatives of tensor fields

前情回顾

(1) $f \in C^\infty(M), \nabla_X f = Xf$

(2) $Y \in \Gamma(TM), \nabla_X Y$

(3) $A \in \Gamma(\bigotimes^{r,s} TM), \nabla_X A?$

希望是前两者的 extension

Theorem 3.1. *$(M, \nabla), \nabla\colon \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$. There is a unique map*

$$\nabla\colon \Gamma(TM) \times \Gamma(\bigotimes^{r,s}(TM)) \to \Gamma(\bigotimes^{r,s} TM)$$

that satisfies

(1) $\nabla_{fX+fY} A = f\nabla_X A + g\nabla_Y A$

(2) $\nabla_X(A_1 + A_2) = \nabla_X A_1 + \nabla_X A_2$

(3) $\nabla_X(fA) = X(f)A + f\nabla_X A$

and

(4) ∇ *conincide with the given connection on* $\Gamma(TM), C^\infty(TM)$

(5) $\nabla_X(A_1 \otimes A_2) = (\nabla_X A_1) \otimes A_2 + A_1 \otimes \nabla_X A_2$

(6) $C(\nabla_X A) = \nabla_X(CA)$, *where* $C\colon \Gamma(\bigotimes^{r,s} TM) \to \Gamma(\bigotimes^{r-1,s-1} TM)$

证明. $A \in \Gamma(\bigotimes^{r,s} TM)$

$A = A^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} Y_{i_1} \otimes Y_{i_2} \otimes \cdots \otimes Y_{i_r} \otimes \omega^{j_1} \otimes \cdots \otimes \omega^{j_s}$

$\nabla_X A = \sum \nabla_X$

$= \sum X(A^{i_1 \cdots i_r}_{j_1 \cdots j_s}) Y_{i_1}$

线性，Leibniz

唯一的问题是如何对微分 1 形式求导

$\omega \in \Omega^1(M) = \Gamma(T^*M)$

$\nabla_X \omega?$

$\forall Y \in \Gamma(TM), \omega(Y) \in C^\infty(TM)$

$X(\omega(Y)) = \nabla_X(\omega(Y)) = \nabla_X(C(\omega \otimes Y)) = C(\nabla_X(\omega \otimes Y))$

$C(\nabla_X \omega \otimes Y + \omega \otimes \nabla_X Y)$

$\nabla_X(\omega)Y + \omega(\nabla_X Y)$

$(\nabla_X \omega) = X(\omega(Y)) - \omega(\nabla_X Y)$

$\implies$ uniqueness □

Remark. (1) *is a consequence of the other assumptions.*

不是那么令人惊讶，这是说在这里是多余的，而不是在仿射联络的最初定义中也是多余的

$\forall X, Y, Z \in \Gamma(TM)$

$f, g \in C^{\infty}(TM), \omega \in \Gamma(T^*M)$

证明. $(fX + gY)\omega(Z) = \nabla_{fX+gY}\omega(Z) + \omega(\nabla_{fX+gY})Z$

$= fX(\omega(Z)) + gY(\omega(Z))$ □

Corollary 3.2. $\forall\ A \in \Gamma(\bigotimes^{r,s} TM), \omega_\alpha \in \Gamma(T^*M), \alpha = 1, 2, \cdots, r, Y_j \in \Gamma(TM), j = 1, \cdots, s$

We have $(\nabla_X A)(\omega_1, \cdots, \omega_s; Y_1, \cdots, Y_s)$

$= A(\omega_1, \cdots, \omega_r, Y_1, \cdots, Y_s)$

locality

$\nabla_X A(p)$ only depends on X at p and Y in $U \ni p$.

(M, ∇)

$\varphi\colon V \to W$ isomorphism

$\varphi^*\colon W^* \to V^*$ isomorphism ,$\alpha \mapsto \varphi^*(\alpha)$

$\forall v \in V, \varphi^*(\alpha)(v) := \alpha(\varphi(v))$

$P_{c,0,t}\colon T_{c(0)}M \to T_{c(t)}M$

$\longrightarrow \tilde{P}_{c,0,t}\colon \bigotimes^{r,s} T_{c(0)}M \to \bigotimes^{r,s} T_{c(t)}M$

$v_1 \otimes \cdots \otimes v_r \otimes \omega^1 \otimes \cdots \otimes \omega^r \mapsto P_{c,0,t}(v_1) \otimes \cdots \otimes$

Define $\nabla_{X(p)} A := \lim\limits_{h \to 0} \dfrac{\tilde{P}}{}$

Definition 3.3. *A tensor field is called prarllel if* $\nabla_X A = 0$, $\forall X \in \Gamma(TM)$.

$c(t) = (c^1(t), \cdots, c^n(t))$

$$\frac{Dc'(t)}{\mathrm{d}t} = \frac{D}{\mathrm{d}t}\left(\frac{\mathrm{d}c^i(t)}{\mathrm{d}t}\frac{\partial}{\partial x^i}\right)$$

$$= \frac{\mathrm{d}^2 c^i(t)}{\mathrm{d}t^2}\frac{\partial}{\partial^i} + \frac{\mathrm{d}c^i(t)}{\mathrm{d}t}\nabla_{\frac{\mathrm{d}c^j}{\mathrm{d}t}\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i}$$

$$= \left(\frac{\mathrm{d}^2 c^k(t)}{}\right)\frac{\partial}{\partial x^k}$$

4 Levi-Civita connection

Definition 4.1. *An affine connection ∇ on (M, g) is called a Levi-Civita connection if*

(1) *it is torsion free.*

(2) *it is compatible with g.*

Theorem 4.2 (The fundamental theorem of Riemannian geometry)**.** *On any Reimannian manifold (M, g), there exists a unique Levi-Civita connection.*

Proof without coordinates. Suppose existence.

Given $X, Y \in \Gamma(TM)$, we can determine $\nabla_X Y$ by determine $\langle \nabla_X Y, Z\rangle$ for any $Z \in \Gamma(TM)$.

$$\begin{aligned}
\langle \nabla_X Y, Z\rangle &\overset{(2)}{=} X\langle Y, Z\rangle - \langle Y, \nabla_X Z\rangle \\
&\overset{(1)}{=} X\langle Y, Z\rangle - \langle Y, \nabla_Z X + [X, Z]\rangle \\
&= X\langle Y, Z\rangle - \langle Y, [X, Z]\rangle - \langle Y, \nabla_Z X\rangle \\
&\overset{(2)}{=} X\langle Y, Z\rangle - \langle Y, [X, Z]\rangle - Z\langle Y, X\rangle + \langle \nabla_Z Y, X\rangle \\
&\overset{(1)}{=} X\langle Y, Z\rangle - \langle Y, [X, Z]\rangle - Z\langle Y, X\rangle + \langle \nabla_Y Z + [Z, Y], X\rangle \\
&= X\langle Y, Z\rangle - \langle Y, [X, Z]\rangle - Z\langle Y, X\rangle + \langle X, [Z, Y]\rangle + \langle \nabla_Y Z, X\rangle \\
&\overset{(2)}{=} X\langle Y, Z\rangle - \langle Y, [X, Z]\rangle - Z\langle Y, X\rangle + \langle X, [Z, Y]\rangle + Y\langle Z, X\rangle - \langle Z, \nabla_Y X\rangle \\
&\overset{(1)}{=} X\langle Y, Z\rangle - \langle Y, [X, Z]\rangle - Z\langle Y, X\rangle + \langle X, [Z, Y]\rangle + Y\langle Z, X\rangle - \langle Z, \nabla_X Y + [Y, X]\rangle \\
2\langle \nabla_X Y, Z\rangle &= X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle X, [Y, Z]\rangle + \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle
\end{aligned}$$

□

Lemma 4.3. *Let $c\colon (a, b) \to M$ C^∞*

Remark.

Propostion 4.4. *Let M be C^∞ manifold with an affine connection ∇. Then ∇ is compatible with g iff any parallel transport is an isometry.*

证明. $c\colon [a, b] \to M$ curve

$\mathcal{P}_{c,a,t}\colon T_{c(a)}M \to T_{c(t)}M$

-
- 任意 $X, Y, Z \in \Gamma(TM), \forall p \in M$

□

Propostion 4.5. *Let ∇ be a torsion-free connection of M. Let $s\colon \mathbb{R}^2 \to M$ be C^∞ map.*

Let $V(x, y) \in T_{s(x,y)}M$. V vector field along s.

For convinence, we denote $\mathrm{d}s\left(\dfrac{\partial}{\partial s}\right) =: \dfrac{\partial s}{\partial x}$

Then $\tilde{\nabla}_{\frac{\partial}{\partial x}} \dfrac{\partial s}{\partial y} = \tilde{\nabla}_{\frac{\partial}{\partial y}} \dfrac{\partial s}{\partial x}$

5 second variation formulae

应用到曲线的变分 $F\colon [a,b] \times (-\varepsilon, \varepsilon) \to M$

证明. 内容... □

Definition 5.1. *Let* $c\colon [a,b] \to M$ *be a smooth curve.*

A variation of c *is a smooth map*

$$F\colon [a,b] \times (-\varepsilon, \varepsilon) \to M$$

with $F(t,0) = c(t), \forall t \in [a,b]$.

We call the vector field along c

6 Curvature tensor

Chapter 4

1 title

如何理解内蕴这个词？

- 有时指不依赖于坐标系的选取
- 有时指不依赖于标架的选取？
- 有时指仅依赖于第一基本形式，也就是黎曼度量 g.