1 Preliminaries

D $f : I \to \mathbb{R}^d$, where $f(x) = (f_1(x), \cdots f_d(x))$

- f is continuous if **each** f_i is continuous
- f is differentiable if **each** f_i is differentiable
- f is injective if **at least one** f_i is injective
- If f is surjective, **each** f_i is surjective

T Mean Value Let $f : [a,b] \to \mathbb{R}$ be continuous and differentiable. Then $\exists c \in]a,b[$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

2 ODE

2.1 Introduction

D ODE: An equation for an unknown function f

- f is a function of one variable
- The equation relates $f(x)$ to the values of its derivatives at the same point
- Order of ODE: order of the highest derivative

2.2 Linear ODEs

D Linear ODE is an equation of the form

$$y^{(k)} + a_{k-1}y^{(k-1)} + \cdots + a_0 y = b, \text{ where}$$

$y = f(x)$ is the unknown function
$a_{k-1}(x), ..., a_0(x), b(x)$ are continuous functions

D Linear homogeneous ODE: $b(x) = 0$
D Linear inhomogeneous ODE: $b(x) \neq 0$

D Initial Value Problem for ODE: Specifying values of $y, y', ..., y^{(k-1)}$ at an initial point x_0

$$y(x_0) = y_0, y'(x_0) = y_1, ..., y^{(k-1)}(x_0) = y_{k-1}$$

T 2.2.3 $I \subset \mathbb{R}$, linear ODE of order $k \geq 1$

(1) Let S_0 be the set of solutions for $b = 0$. Then is S_0 a vector space of dimension k.

(2) For any initial conditions, there is a unique solution $f \in S_0$, s.t.

$$y(x_0) = y_0, y'(x_0) = y_1, ..., y^{(k-1)}(x_0) = y_{k-1}$$

(3) For an arbitrary b, the set of solutions is $S_b = \{f + f_p | f \in S_0\}$, where f_p is a particular solution

(4) For any initial value problem, there is a unique solution $f \in S_b$

Bem: If $b \neq 0$, then S_b is not a vector space
Bem: If f_1, f_2 are solutions for $b_1(x), b_2(x)$, $f_1 + f_2$ is a solution for $b_1(x) + b_2(x)$

2.3 Linear ODEs of order 1

D Consider linear ODE of order 1: $y' + ay = b$

1. Solve homogeneous equation $y' + ay = 0$

2. Find a solution of inhomogeneous equation, s.t. S_b contains $f_0 + f$ where $f \in S$.

Bem: The solutions are given by $f_0 + zf_1$, where $z \in \mathbb{C}$ and f_1 is a basis of S
Bem: To solve the real value problem $f(x_0) = y_0$, one can solve $f_0(x_0) + zf_1(x_0) = y_0$
Bem: If $a \in \mathbb{R}$, then there exists $f_0, f_1 \in \mathbb{R}$

Procedure: Solving homogeneous equations

$$f_0(x) = z \cdot e^{-A(x)} \text{ for } z \in \mathbb{C}$$

Procedure: Solving inhomogeneous equations

$$f_p(x) = \int b(x) \cdot e^{A(x)} dx \cdot e^{-A(x)}$$

$$f(x) = f_0(x) + f_p(x)$$

$$= z \cdot e^{-A(x)} + \int b(x) \cdot e^{A(x)} dx \cdot e^{-A(x)}$$