1 Preliminaries

D $f: I \to \mathbb{R}^d$, where $f(x) = (f_1(x), \cdots f_d(x))$

- f is continuous if **each** f_i is continuous
- f is differentiable if **each** f_i is differentiable
- f is injective if **at least one** f_i is injective
- If f is surjective, **each** f_i is surjective

T Mean Value Let $f: [a, b] \to \mathbb{R}$ be continuous and differentiable. Then $\exists c \in]a, b[$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

2 ODE

2.1 Introduction

D ODE: An equation for an unknown function f

- f is a function of one variable
- The equation relates $f(x)$ to the values of its derivatives at the same point
- Order of ODE: order of the highest derivative

2.2 Linear ODEs

D Linear ODE is an equation of the form

$$y^{(k)} + a_{k-1} y^{(k-1)} + \cdots + a_0 y = b, \text{ where}$$

$y = f(x)$ is the unknown function
$a_{k-1}(x), ..., a_0(x), b(x)$ are continuous functions

D Linear homogeneous ODE: $b(x) = 0$
D Linear inhomogeneous ODE: $b(x) \neq 0$

D Initial Value Problem for ODE: Specifying values of $y, y', ..., y^{(k-1)}$ at an initial point x_0

$$y(x_0) = y_0, y'(x_0) = y_1, ..., y^{(k-1)}(x_0) = y_{k-1}$$

T 2.2.3 $I \subset \mathbb{R}$, linear ODE of order $k \geq 1$

(1) Let S_0 be the set of solutions for $b = 0$. Then is S_0 a vector space of dimension k.

(2) For any initial conditions, there is a unique solution $f \in S_0$, s.t.

$$y(x_0) = y_0, y'(x_0) = y_1, ..., y^{(k-1)}(x_0) = y_{k-1}$$

(3) For an arbitrary b, the set of solutions is $S_b = \{f + f_p | f \in S_0\}$, where f_p is a particular solution

(4) For any initial value problem, there is a unique solution $f \in S_b$

Bem: If $b \neq 0$, then S_b is not a vector space
Bem: If f_1, f_2 are solutions for $b_1(x), b_2(x)$, $f_1 + f_2$ is a solution for $b_1(x) + b_2(x)$

2.3 Linear ODEs of order 1

D Consider linear ODE of order 1: $y' + ay = b$

1. Solve homogeneous equation $y' + ay = 0$
2. Find a solution of inhomogeneous equation, s.t. S_b contains $f_h + f$ where $f \in S$.

Bem: The solutions are given by $f_h + z f_1$, where $z \in \mathbb{C}$ and f_1 is a basis of S
Bem: To solve the real value problem $f(x_0) = y_0$, one can solve $f_h(x_0) + z f_1(x_0) = y_0$
Bem: If $a \in \mathbb{R}$, then there exists $f_h, f_1 \in \mathbb{R}$

Procedure: Solving homogeneous equations

$$f_h(x) = z \cdot e^{-A(x)} \text{ for } z \in \mathbb{C}$$

Procedure: Solving inhomogeneous equations

$$f_p(x) = \int b(x) \cdot e^{A(x)} dx \cdot e^{-A(x)}$$

$$\begin{aligned} f(x) &= f_h(x) + f_p(x) \\ &= z \cdot e^{-A(x)} + \int b(x) \cdot e^{A(x)} dx \cdot e^{-A(x)} \end{aligned}$$

2.4 Linear ODE with constant coefs.

The equation takes the form: Let $a_{k-1}, ..., a_0 \in \mathbb{C}$

$$y^{(k)} + a_{k-1} y^{(k-1)} + ... + a_0 y = b(x)$$

Procedure: Solving homogeneous equations
We look for solutions of the form $f(x) = e^{\alpha x}, \alpha \in \mathbb{C}$

$$\begin{aligned} 0 &= y^{(k)} + a_{k-1} y^{(k-1)} + ... + a_0 y \\ &= e^{\alpha x}(\alpha^k + a_{k-1}\alpha^{k-1} + \cdots + a_1 \alpha + a_0) \\ &= e^{\alpha x} P(\alpha) \end{aligned}$$

T f is a solution if and only if $P(\alpha) = 0$.
Bem: According to the Fundamental Theorem of Algebra, there are k roots for P in $\mathbb{C}$.
Bem: $P(\alpha)$ is the **characteristic polynomial** and the roots are called **eigenvalues**

Case 1: k distinct solutions for $P(\alpha) = 0$
$f_j(x) = e^{\alpha_j x}$ are linearly independent.

Every solution for the ODE is of the form:

$$f(x) = z_1 e^{\alpha_1 x} + \cdots + z_k e^{\alpha_k x}, \text{ with } z_1, ..., z_2 \in \mathbb{C}$$

Case 2: $\exists \alpha$, which is a root of order $2 \leq j \leq k$

$$f_{\alpha,0}(x) = x^0 e^{\alpha x}, \cdots, f_{\alpha,j-1}(x) = x^{j-1} e^{\alpha x}$$

Taking the union of the functions $f_{\alpha,j}$ for all roots of P, each with its multiplicity, gives a basis of the space of solutions.

Procedure: Solving inhomogeneous equations:
Find a solution of inhomogeneous equation, s.t. S_b contains $f_h + f$ where $f \in S$.

Procedure: Variation of Constants
Let $(f_1, f_2, \cdots, f_k)$ be a basis for the f_h
$f_p = z_1(x) f_1(x) + \cdots + z_k(x) f_k(x)$, where

$$\begin{cases} z_1'(x) f_1(x) + \cdots + z_k'(x) f_k(x) = 0 \\ z_1'(x) f_1'(x) + \cdots + z_k'(x) f_k'(x) = 0 \\ \cdots \\ z_1'(x) f_1^{(k-2)}(x) + \cdots + z_k'(x) f_k^{(k-2)}(x) = 0 \end{cases}$$

Trick: Guess the particular solution

$b(x)$	$f_p(x)$
$ae^{\alpha x}$	$ke^{\alpha x}$
$a\sin(\beta x)$ $a\cos(\beta x)$	$k_1 \sin(\beta x) + k_2 \cos(\beta x)$
$ae^{\alpha x}\sin(\beta x)$ $ae^{\alpha x}\cos(\beta x)$	$e^{\alpha x}[k_1 \sin(\beta x) + k_2 \cos(\beta x)]$

Bem: works also for $a = a(x)$, then $k = k(x)$